Title: Dynamic and Thermodynamic Stability of Black Holes and Black Branes

Date: Jun 13, 2013 01:00 PM

URL: http://pirsa.org/13060014

Abstract: I describe recent work with with Stefan Hollands that establishes a new criterion for the dynamical stability of black holes in $D \geq 4$ spacetime dimensions in general relativity with respect to axisymmetric perturbations: Dynamic stability is equivalent to the positivity of the canonical energy, $\mathcal E$, on a subspace of linearized solutions that have vanishing linearized ADM mass, momentum, and angular momentum at infinity and satisfy certain gauge conditions at the horizon. We further show that $\mathcal E$ is related to the second order variations of mass, angular momentum, and horizon area by $\mathcal E = \delta^2 M - \sum_i \Omega_i \delta^2 J_i - (\kappa/8\pi) \delta^2 A$, thereby establishing a close connection between dynamic stability and thermodynamic stability.

Thermodynamic instability of a family of black holes need not imply dynamic instability because the perturbations towards other members of the family will not, in general, have vanishing linearized ADM mass and/or angular momentum. However, we prove that all black branes corresponding to
thermodynmically unstable black holes are dynamically unstable, as conjectured by Gubser and Mitra. We also prove that positivity of $\mathcal E$ is equivalent to the satisfaction of a ``local Penrose inequality," thus showing that satisfaction of this local Penrose inequality is necessary and sufficient for dynamical stability.

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However, even in the very simplest cases—such as the Schwarzschild black hole (Regge-Wheeler, Zerilli) and the Schwarzschild black string (Gregory-Laflamme)—it is quite nontrivial to carry out the decoupling of equations and the fixing of gauge needed to determine stability or instability directly from the equations of motion.

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Thermodynamic Stability

Consider a homogeneous thermodynamic system, whose entropy, S, is a function of energy, E, and other extensive state parameters X_i, $S = S(E, X_i)$. The condition for thermodynamic instability is that the Hessian matrix

$$\mathbf{H}_S = \begin{pmatrix} \frac{\partial^2 S}{\partial E^2} & \frac{\partial^2 S}{\partial X_i \partial E} \\ \frac{\partial^2 S}{\partial E \partial X_i} & \frac{\partial^2 S}{\partial X_i \partial X_j} \end{pmatrix} .$$

admit a positive eigenvalue. If this happens, then one can increase total entropy by exchanging E and/or X_i between different parts of the system. For the case of E, this corresponds to having a negative heat capacity.

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A positive eigenvalue of the Hessian is equivalent to finding a homogeneous perturbation of the system such that

$$\delta^2 E - T\delta^2 S - \sum_i Y_i \delta^2 X_i < 0$$

where $Y_i = (\partial E/\partial X_i)_S$.

For a general (inhomogeneous) thermodynamic system, the condition that S is *not* a maximum (to second order) at fixed E and X_i is equivalent to finding an *inhomogeneous* perturbation of the system for which $\delta E = \delta X_i = 0$ and for which

$$\delta^2 E - T\delta^2 S - \sum_i Y_i \delta^2 X_i < 0$$

Hessian matrix

$$\mathbf{H}_A = \begin{pmatrix} \frac{\partial^2 A}{\partial M^2} & \frac{\partial^2 A}{\partial J_i \partial M} \\ \frac{\partial^2 A}{\partial M \partial J_i} & \frac{\partial^2 A}{\partial J_i \partial J_j} \end{pmatrix} .$$

admits a positive eigenvalue. This is equivalent to finding a perturbation within the black hole/brane family for which

$$\delta^2 M - \frac{\kappa}{8\pi}\delta^2 A - \sum_i \Omega_i \delta^2 J_i < 0 .$$

One might expect that this condition for thermodynamic instability might imply dynamical instability. However, this is clearly false: The Schwarzschild black hole has negative heat capacity ($A = 16\pi M^2$, so $\partial^2 A/\partial M^2 > 0$) but is well known to be stable. Nevertheless, the Schwarzschild black string *is* unstable. The Gubser-Mitra conjecture states that the above thermodynamic criterion for instability is a valid criterion dynamical instability for black branes.

We will prove that the fundamental criterion for the axisymmetric stability of black holes and black branes is

$$\delta^2 M - \frac{\kappa}{8\pi}\delta^2 A - \sum_i \Omega_i \delta^2 J_i < 0\,.$$

for arbitrary axisymmetric perturbations with $\delta M = \delta J_i = 0$. The Gubser-Mitra conjecture follows as a consequence of this criterion.

Local Penrose Inequality

Suppose one has a family of stationary, axisymmetric black holes parametrized by M and angular momenta $J_1, \ldots, J_N$. Consider a one-parameter family $g_{ab}(\lambda)$ of axisymmetric spacetimes, with $g_{ab}(0)$ being a member of this family with surface gravity $\kappa > 0$. Consider initial data on a hypersurface Σ passing through the bifurcation surface B. By the linearized Raychauduri equation, to first order in λ, the event horizon coincides with the apparent horizon on Σ. They need not coincide to second order in λ, but since B is an extremal surface in the background spacetime, their areas must agree to second order. Let $\mathcal{A}$ denotes the area of the apparent horizon of the perturbed spacetime, $\bar{A}$ denotes the the event horizon area of the stationary black hole with the same mass and angular momentum as the perturbed spacetime. Suppose that to second order, we have

$$\delta^2 \mathcal{A} > \delta^2 \bar{A}$$

Since (i) the area of the event horizon can only increase with time (by cosmic censorship), (ii) the final mass of the black hole cannot be larger than the initial total mass (by positivity of Bondi flux), (iii) its final angular momenta must equal the initial angular momenta (by axisymmetry), and (iv) $\bar{A}(M, J_1, \ldots, J_N$ is an increasing function of M at fixed J_i (by the first law of black hole the perturbed spacetime, $\bar{A}$ denotes the the event horizon area of the stationary black hole with the same mass and angular momentum as the perturbed spacetime. Suppose that to second order, we have

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Our Results

We consider perturbations γ of a static or stationary-axisymmetric black hole or black brane with bifurcate Killing horizon and consider the *canonical energy* of the perturbation defined by

$$\mathcal{E} = \int_\Sigma \omega(g; \gamma, \pounds_t \gamma)$$

where Σ extends from the bifurcation surface to infinity.

We show that the necessary and sufficient condition for stability of a black hole (or black brane) with respect to axisymmetric perturbations is positivity of $\mathcal{E}$ on a Hilbert space, $\mathcal{V}$, of perturbations with vanishing perturbed mass, angular momentum, and linear momentum.

$\delta M = \delta J_i = \delta P_i = 0.$

We will also show that for axisymmetric perturbations

$$\mathcal{E} = \delta^2 M - \frac{\kappa}{8\pi}\delta^2 A - \sum_i \Omega_i \delta^2 J_i$$

In other words, dynamical stability is equivalent to thermodynamic stability *for perturbations with* $\delta M = \delta J_i = \delta P_i = 0.$ The "change of mass" perturbation of Schwarzschild—for which $\mathcal{E} < 0$—does not "count" for testing stability because, obviously, $\delta M \neq 0$.

However, if a black hole has a perturbation with $\mathcal{E} < 0$ with $\delta M \neq 0$ and/or $\delta J_i \neq 0$, we prove that there exists a sufficiently long wavelength perturbation of any

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corresponding black brane for which $\mathcal{E} < 0$ but $\delta M = \delta J_i = 0$. This proves the Gubser-Mitra conjecture. Thus, for example, the calculation that $\partial^2 A / \partial M^2 = 32\pi > 0$ for Schwarzschild tells one nothing about the stability of Schwarzschild black hole, but it proves the instability of the Schwarzshild black string to sufficiently long wavelength perturbations.

Finally, we prove that if one can find a perturbation of a black hole for which $\delta^2 \mathcal{A} > \delta^2 \bar{A}$, if and only if one can find a perturbation for which $\delta M = \delta J_i = \delta P_i = 0$ and $\mathcal{E} < 0$. This proves that satisfaction of the local Penrose inequality is equivalent to dynamical stability.

Variational Formulas

Lagrangian for vacuum general relativity:

$$L_{a_1 \dots a_D} = \frac{1}{16\pi} R \, \epsilon_{a_1 \dots a_D} \, .$$

First variation:

$$\delta L = E \cdot \delta g + d\theta \, ,$$

with

$$\theta_{a_1 \dots a_{d-1}} = \frac{1}{16\pi} g^{ac} g^{bd} (\nabla_d \delta g_{bc} - \nabla_c \delta g_{bd}) \epsilon_{c a_1 \dots a_{d-1}} \, .$$

Symplectic current ($(D-1)$-form):

$$\omega(g; \delta_1 g, \delta_2 g) = \delta_1 \theta(g; \delta_2 g) - \delta_2 \theta(g; \delta_1 g) \, .$$

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Symplectic form:

$$W_\Sigma(g;\delta_1 g,\delta_2 g) \equiv \int_\Sigma \omega(g;\delta_1 g,\delta_2 g)$$
$$= -\frac{1}{32\pi}\int_\Sigma (\delta_1 h_{ab}\delta_2 p^{ab} - \delta_2 h_{ab}\delta_1 p^{ab})\,,$$

with

$$p^{ab} \equiv h^{1/2}(K^{ab} - h^{ab}K)\,.$$

Noether current:

$$\mathcal{J}_X \equiv \theta(g,\pounds_X g) - X\cdot L$$
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Fundamental variational identity:

$$\omega(g;\delta g, \pounds_X g) = X \cdot [E(g) \cdot \delta g] + X \cdot \delta C + d\,[\delta Q_X(g) - X \cdot \theta(g;\delta g)]$$

ADM conserved quantities:

$$\delta H_X = \int_\infty [\delta Q_X(g) - X \cdot \theta(g;\delta g)]$$

For a stationary black hole, choose X to be the horizon Killing field

$$K^a = t^a + \sum \Omega_i \phi_i^a$$

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Horizon Gauge Conditions

Consider stationary black holes with surface gravity $\kappa > 0$, so the event horizon is of "bifurcate type," with bifurcation surface B. Consider an arbitrary perturbation $\gamma = \delta g$. Gauge condition that ensures that the location of the horizon does not change to first order:

$$\delta\vartheta|_B = 0\,.$$

Additional gauge condition that we impose:

$$\delta\epsilon|_B = \frac{\delta A}{A}\epsilon\,.$$

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Canonical Energy

Define the *canonical energy* of a perturbation $\gamma = \delta g$ by

$$\mathcal{E} \equiv W_\Sigma\left(g; \gamma, \pounds_t \gamma\right)$$

The second variation of our fundamental identity then yields (for axisymmetric perturbations)

$$\mathcal{E} = \delta^2 M - \sum_i \Omega_i \delta^2 J_i - \frac{\kappa}{8\pi} \delta^2 A \,.$$

More generally, can view the canonical energy as a bilinear form $\mathcal{E}(\gamma_1, \gamma_2) = W_\Sigma(g; \gamma_1, \pounds_t \gamma_2)$ on perturbations. $\mathcal{E}$ can be shown to satisfy the following properties:

- $\mathcal{E}$ is conserved, i.e., it takes the same value if evaluated on another Cauchy surface Σ' extending from infinity to B.
- $\mathcal{E}$ is symmetric, $\mathcal{E}(\gamma_1, \gamma_2) = \mathcal{E}(\gamma_2, \gamma_1)$
- When restricted to perturbations for which $\delta A = 0$ and $\delta P_i = 0$ (where P_i is the ADM linear momentum), $\mathcal{E}$ is gauge invariant.
- When restricted to the subspace, $\mathcal{V}$, of perturbations for which $\delta M = \delta J_i = \delta P_i = 0$ (and hence, by the first law of black hole mechanics $\delta A = 0$), we have $\mathcal{E}(\gamma', \gamma) = 0$ for all $\gamma' \in \mathcal{V}$ if and only if γ is a perturbation towards another stationary and

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axisymmetric black hole.

Thus, if we restrict to perturbations in the subspace, $\mathcal{V}'$, of perturbations in $\mathcal{V}$ modulo perturbations towards other stationary black holes, then $\mathcal{E}$ is a non-degenerate quadratic form. Consequently, on $\mathcal{V}'$, either (a) $\mathcal{E}$ is positive definite or (b) there is a $\psi \in \mathcal{V}'$ such that $\mathcal{E}(\psi) < 0$. If (a) holds, we have stability.

Flux Formulas

Let δN_{ab} denote the perturbed Bondi news tensor at null infinity, $\mathcal{I}^+$, and let $\delta\sigma_{ab}$ denote the perturbed shear on the horizon, $\mathcal{H}$. If the perturbed black hole were to "settle down" to another stationary black hole at late times, then $\delta N_{ab} \to 0$ and $\delta\sigma_{ab} \to 0$ at late times. We show that—for axisymmetric perturbations—the change in canonical energy would then be given by

$$\Delta\mathcal{E} = -\frac{1}{16\pi}\int_{\mathcal{I}} \delta\tilde{N}_{cd}\delta\tilde{N}^{cd} - \frac{1}{4\pi}\int_{\mathcal{H}} (K^a\nabla_a u)\,\delta\sigma_{cd}\delta\sigma^{cd} \le 0\,.$$

Thus, $\mathcal{E}$ can only decrease. Therefore if one has a perturbation $\psi \in \mathcal{V}'$ such that $\mathcal{E}(\psi) < 0$, then ψ cannot "settle down" to a stationary solution at late times

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because $\mathcal{E} = 0$ for stationary perturbations with $\delta M = \delta J_I = \delta P_i = 0$. Indeed, $|\mathcal{E}(\psi)|$ should grow without bound, i.e., in case (b) we have instability.

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Instability of Black Branes

Theorem: Suppose a family of black holes parametrized by (M, J_i) is thermodynamically unstable at (M_0, J_{0i}), i.e., there exists a perturbation within the black hole family for which $\mathcal{E} < 0$. Then, for any black brane corresponding to (M_0, J_{0i}) one can find a sufficiently long wavelength perturbation for which $\tilde{\mathcal{E}} < 0$ and $\delta\tilde{M} = \delta\tilde{J}_i = \delta\tilde{P}_i = \delta\tilde{A} = \delta\tilde{T}_i = 0$.

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Conclusion

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Thus, the remarkable relationship between the laws of black hole physics and the laws of thermodynamics extends to dynamical stability.

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