Title: Models of Interacting Topological Phases

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Abstract: We present a set of models which realize interacting topological phases. The models are constructed in 2 dimensions for a system with U(1)xU(1) symmetry. We demonstrate that the models are topological by measuring their Hall conductivity, and demonstrating that they have gapless edge modes. We have also studied the models numerically.

# Models of Interacting Topological Phases

SCOTT GERAEDTS

**PERIMETER INSTITUTE, JUNE 25 2013**

Olexei Motrunich

# Background: Why Interacting Topological Phases?

- Some goals of condensed matter physics:
  - Classify all the phases of matter
  - Realize as many of them as possible
- There was a time when we thought symmetry (+ dimension) was all you needed to classify states of matter
- We now realize you also need information about the topology of the system
- The discovery of topological phases shows that there are many new phases to be classified/realized
- Topological phases can be either short-ranged entangled or long-range entangled
- In short-ranged entangled case, you *need* symmetry to get a topological phase, "symmetry protected topological phase" (SPT)

Chen, Liu, Gu, Wen, Science 338 1604 (2012)

# Some much-studied SPTs

| Name | | Dimension | Symmetry | Entanglement |
|---|---|---|---|---|
| IQHE | Free fermions | 2 | U(1) | Short-ranged |
| Quantum Spin Hall | Free fermions | 2 | Time-reversal, U(1) | Short-ranged |
| 3D Topological Insulator | Free fermions | 3 | Time-reversal, U(1) | Short-ranged |

Much research on SPTs of free fermions

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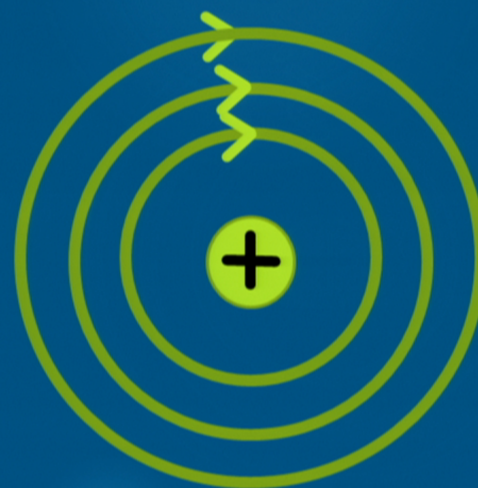
# What about interacting topological phases?

- Much progress on classification
- How can we make them?
- What are their properties?
- Models which realize topological phases can help to answer these questions
- Also useful to study phases numerically

# U(1) models in (2+1) dimensions

- Same symmetry/dimension as IQHE
- Classification: Also integer number of phases [1]
- Can be described by K-matrix theory [2]
- For interacting phases, convenient to study bosons so we don't have statistics
- IQHE relies on statistics, which leads to a phase upon interchange of particles
- 'flux attachment' on bosons to realize similar interchange effects [3]




[1] Chen, Gu, Liu, Wen, Science **338** 1604 (2012)
[2] Lu and Vishwanath, Phys. Rev. B **86**, 125119 (2012)
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- Bosons → 2D Quantum rotor model → 3D Classical action of integer valued, closed current loops

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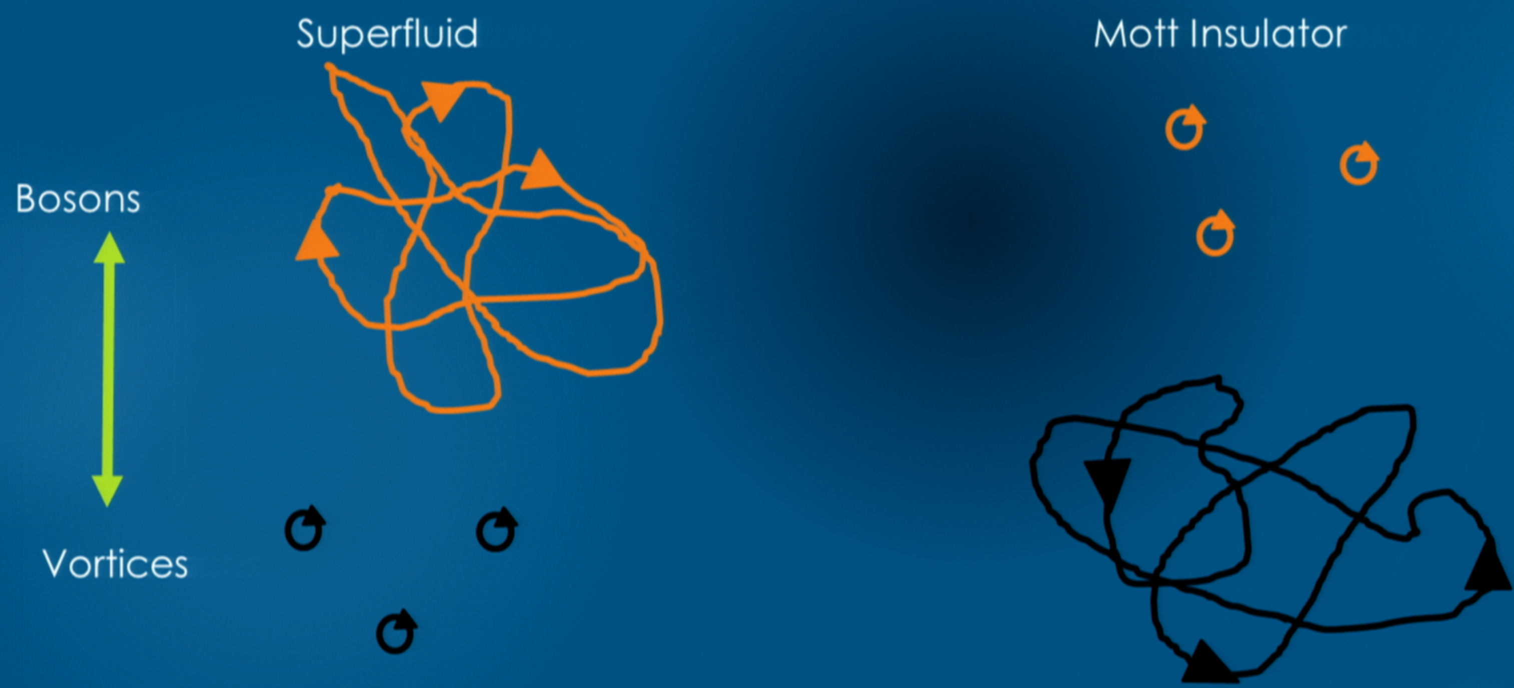
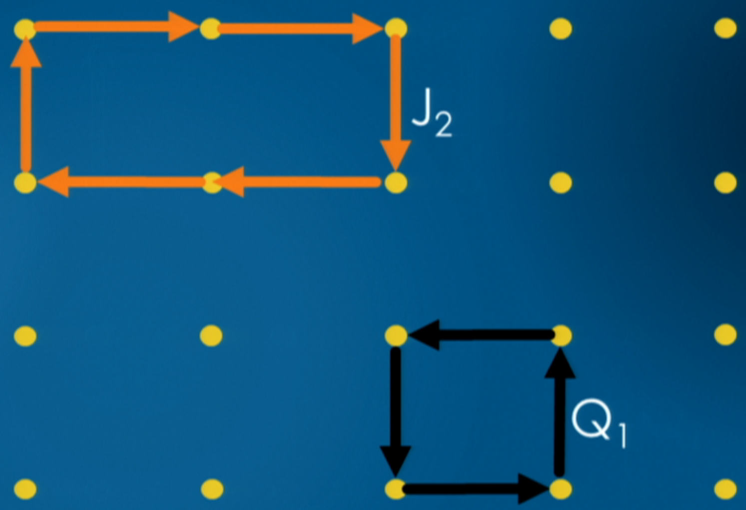


Action in terms of "ordinary" bosons which might realize this binding:

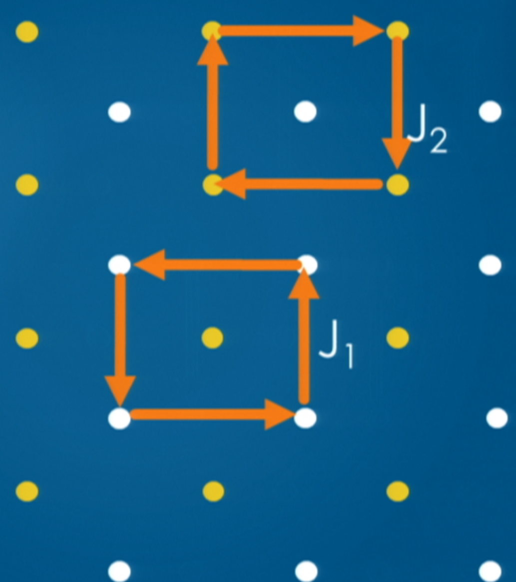
$$S[\vec{Q}_1, \vec{J}_2] = \sum_{r,r'} \Big[ v_Q(r-r')\vec{Q}_1(r)\vec{Q}_1(r') + v_J(r-r')\vec{J}_2(r)\vec{J}_2(r') + v_{QJ}(r-r')\vec{Q}_1(r)\vec{J}_2(r') \Big]$$




In this talk, figures will usually represent 3D Classical models

## After dualizing to 'particle-particle' description:

$$S[\vec{J}_1, \vec{J}_2] = \sum_{R,R'} v_{J1}(R-R')\vec{J}_1(R)\cdot\vec{J}_1(R') + \sum_{r,r'} v_{J2}(r-r')\vec{J}_2(r)\cdot\vec{J}_2(r') + i\sum_{r} w(r)(\vec{\nabla}\times\vec{J}_1)(r)\cdot\vec{J}_2(r)$$




$$v_{J1/2}(k) = \frac{\lambda_{1/2}}{\lambda_1\lambda_2 + \frac{|f_k|^2}{(2\pi)^2}}$$

$$w(k) = -\frac{1}{2\pi}\frac{1}{\lambda_1\lambda_2 + \frac{|f_k|^2}{(2\pi)^2}}$$

- This action is the path integral of a local Hamiltonian

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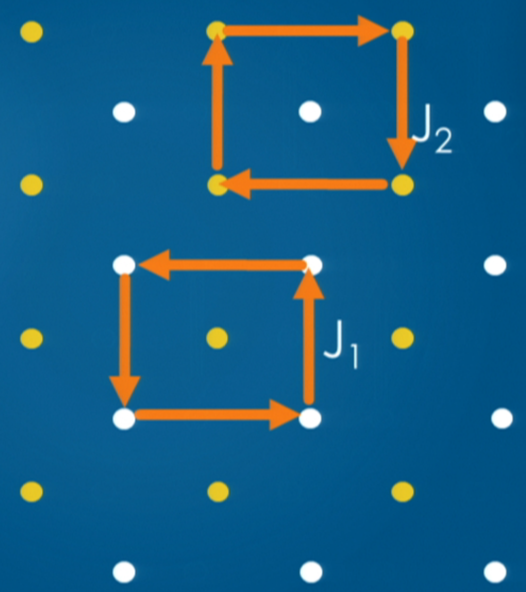


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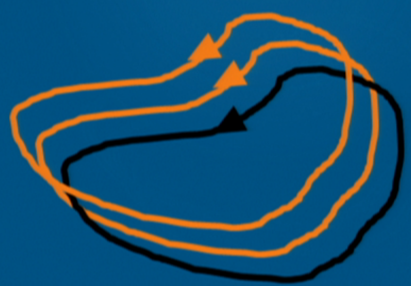
# More general models

- Would like to bind different numbers of particles and vortices
- A change of variables makes things easier

SG and O. Motrunich, Phys. Rev. B 86, 245121 (2012)

$$S[J_1, J_2] = v_{J1}J_1^2 + v_{J2}J_2^2 + iw(\nabla \times J_1)J_2$$

$$S[J_1, J_2] \rightarrow S[Q_1, J_2]$$



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$$S[J_1, J_2] \to S[Q_1, J_2]$$

New current loop variables:

$$G_2 = dJ_2 - cQ_1 \qquad a, b, c, d \in \mathbb{Z}$$

$$F_1 = bJ_2 - aQ_1 \qquad ad - bc = 1$$

$$S[Q_1, J_2] \to S[F_1, G_2] \to S[G_1, G_2]$$

- G action has same form as J action
- When the G variables are gapped, we have binding
- For a given physical action, get 'binding' phase if there is a [a,b,c,d] for which the G variables are gapped

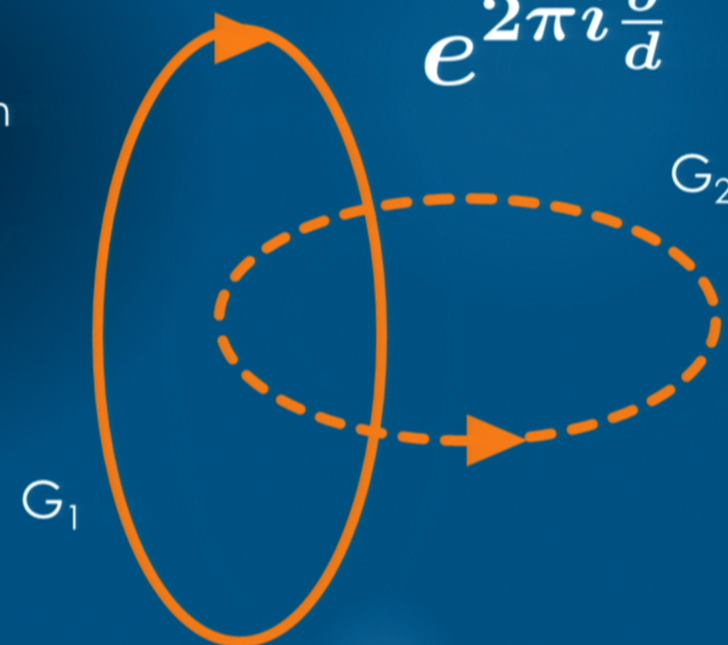
# Action in terms of G variables

$$S[G_1, G_2] = \frac{1}{\lambda_1} G_1^2 + \frac{1}{\lambda_2} G_2^2 + i\frac{2\pi b}{d} G_1 a_{G2}$$

$$\vec{G}_2 = \vec{\nabla} \times a_{G2}$$

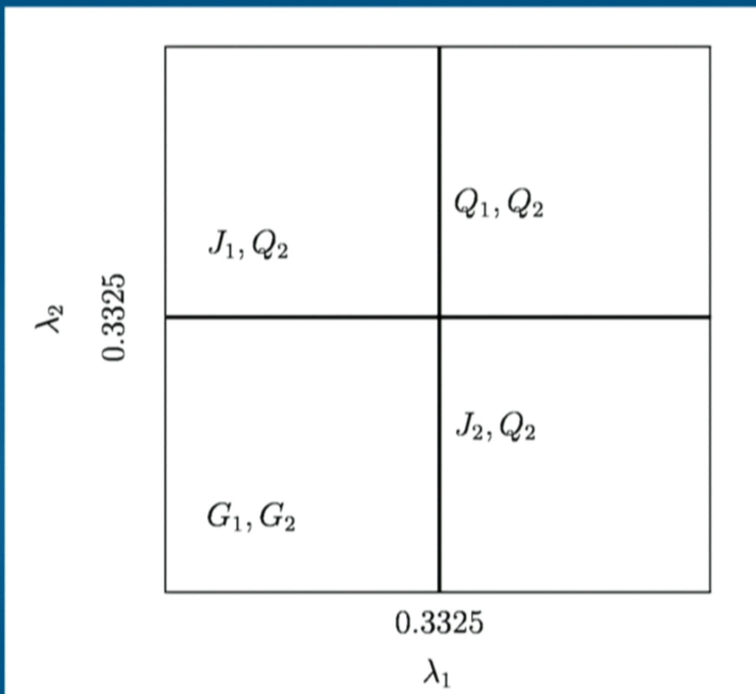
- Action gaps G when λ small
- Third term is 'mutual statistical interaction'
- For d=1 third term doesn't contribute to partition sum
- Action can be efficiently studied numerically

$$e^{2\pi i \frac{b}{d}}$$

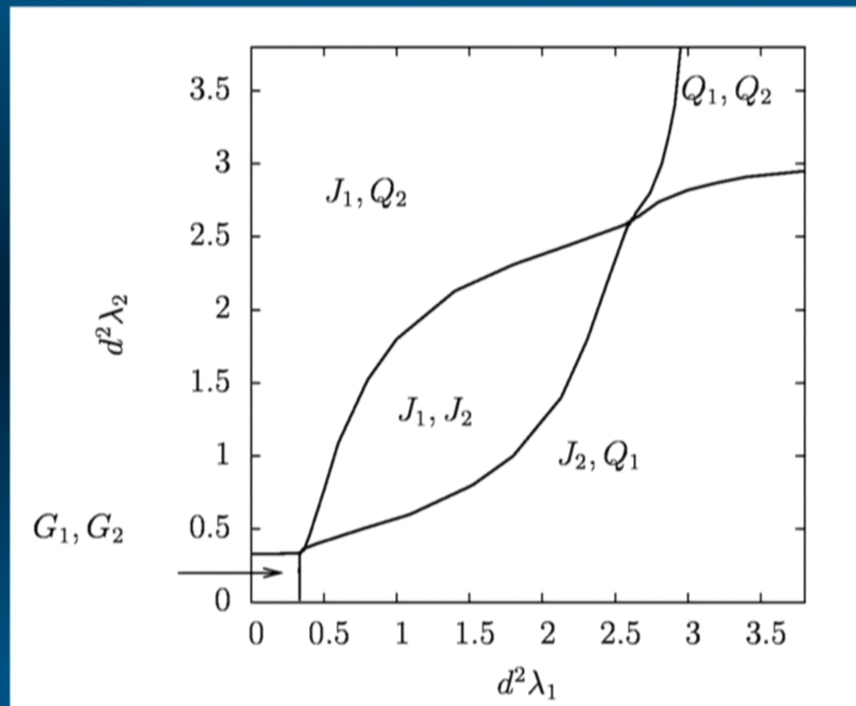


# Phase Diagrams

d=1




d=3




Labels are what variables are gapped in that phase

F. Alet and E. Sorensen, PRE **67** 01570 (2003)1

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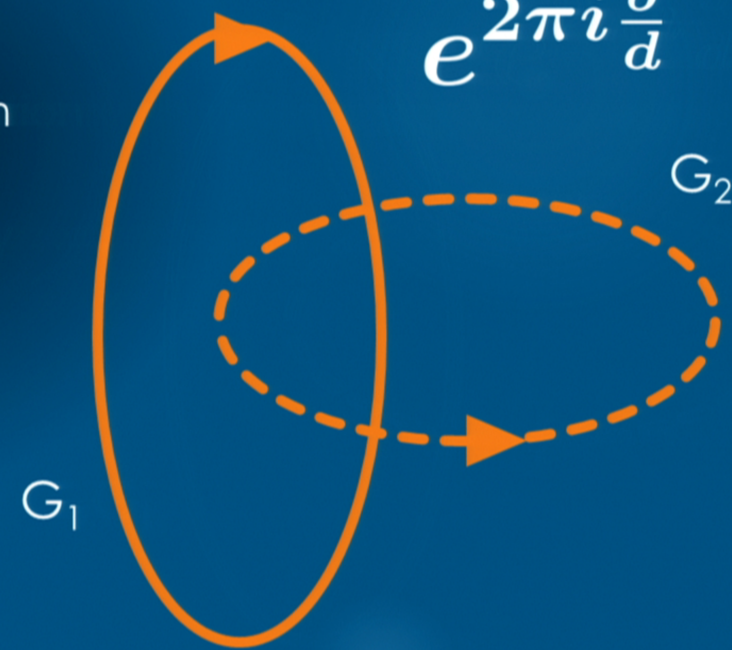

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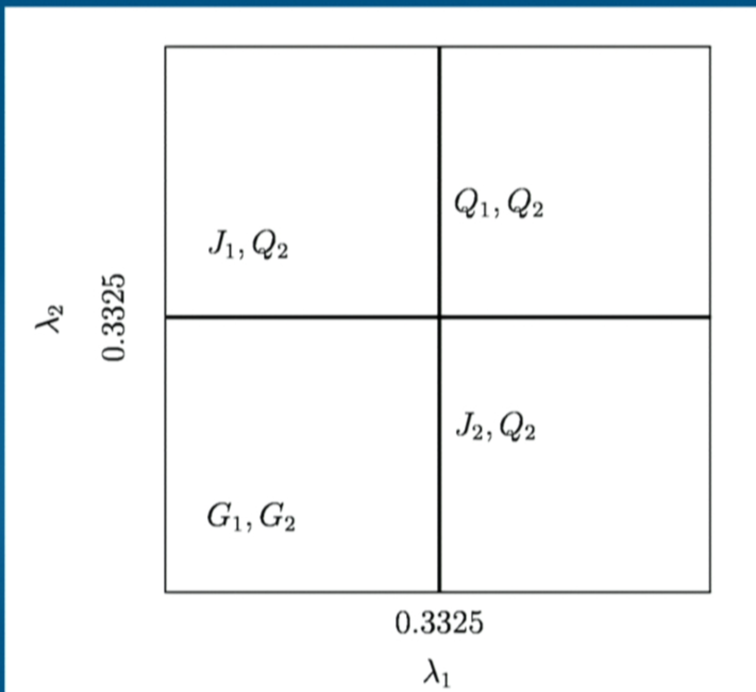


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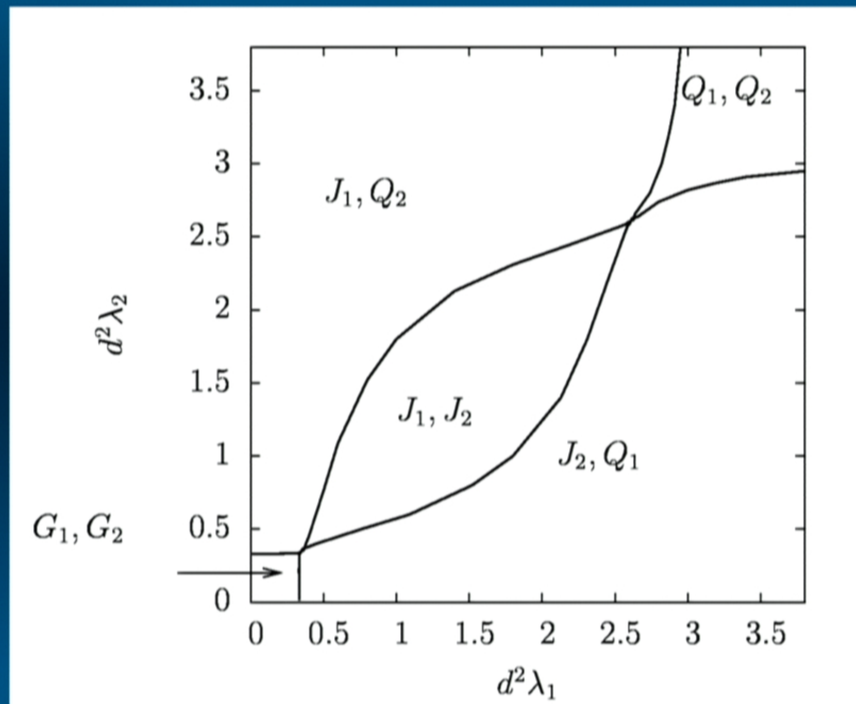


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# Is it topological?
# Part 1: Topological Invariant

- Since we have U(1) symmetry in 2D, natural to look at Hall conductivity $\boldsymbol{\sigma}^{xy}$

Add source terms:

$$\delta S[J_1, J_2] = i\sum_R \vec{J}_1(R)\cdot\vec{A}^{\text{ext}}(R) + i\sum_r \vec{J}_2(r)\cdot\vec{A}^{\text{ext}}(r)$$

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Carry source terms through change of variables:

$$\begin{aligned}\delta S[G_1, G_2] &= i2\left(\frac{2\pi c}{d}\right)\sum_r(\vec{\nabla}\times\vec{A}^{\text{ext}})(r)\cdot\vec{A}^{\text{ext}}(r)\\ &+ i\frac{1}{d}\sum_R \vec{G}_1(R)\cdot\vec{A}^{\text{ext}}(R) + i\frac{1}{d}\sum_r \vec{G}_2(r)\cdot\vec{A}^{\text{ext}}(r)\end{aligned}$$

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Carry source terms through change of variables:

Chern-Simons term

$$\begin{aligned} \delta S[G_1, G_2] &= i2\left(\frac{2\pi c}{d}\right) \sum_r (\vec{\nabla} \times \vec{A}^{\text{ext}})(r) \cdot \vec{A}^{\text{ext}}(r) \\ &+ i\frac{1}{d} \sum_R \vec{G}_1(R) \cdot \vec{A}^{\text{ext}}(R) + i\frac{1}{d} \sum_r \vec{G}_2(r) \cdot \vec{A}^{\text{ext}}(r) \end{aligned}$$

$$\sigma^{xy} = 2\frac{c}{d}!$$

# Is it topological?
# Part 1: Topological Invariant

- Quantized Hall Conductivity!
- Binding *d* vortices to *c* particles
- d=1:
  - Short-ranged entanglement (SPT)
  - Integer number of phases agrees with classification[1]
  - Integer Quantum Hall Effect for Bosons[2,3]
- d>1:
  - Long-ranged entanglement
  - Fractional charges and mutual statistics
  - Fractional Quantum Hall Effect for Bosons

[1] Chen, Gu, Liu, Wen, Science **338** 1604 (2012)
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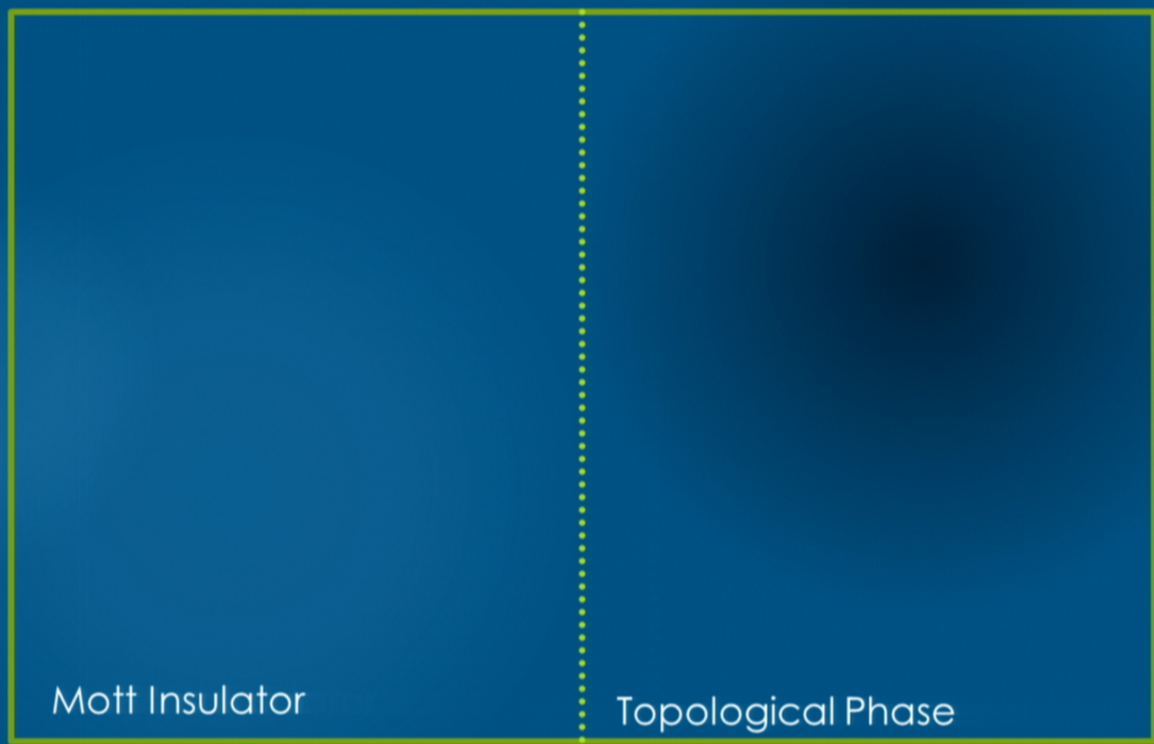
# Is it topological?
# Part 2: Edge States

$$S[\vec{Q}_1, \vec{J}_2] = \sum_{r,r'} \frac{1}{\lambda} |\eta(r)\vec{Q}_1(r) - \vec{J}_2(r)|^2$$

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Mott Insulator

Topological Phase

$\eta = 0$

$\eta = 1$

# K-matrix interpretation

- Alternative way to implement duality:

$$Z = \sum_J e^{-S[J]} \to \int d\alpha \sum_Q e^{-S[\nabla\times\alpha]+i2\pi\alpha Q}$$

- Do change of variables as before, dualize resulting F variables

$$S[J_1, J_2] \to S[Q_1, J_2] \to S[F_1, G_2] \to S[G_1, G_2]$$

- Get Chern-Simons action in terms of $\alpha$ fields (can ignore gapped G)

$$K = \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \qquad t_1 = \begin{bmatrix} 1 \\ c \end{bmatrix} \qquad \sigma_{xy} = t_1^T K^{-1} t_1$$

- Can reproduce conductivity, fractional charges, mutual statistics

Wen, 'Quantum Field Theory for Many-Body Systems'
Polyakov, 'Gauge Fields and Strings'

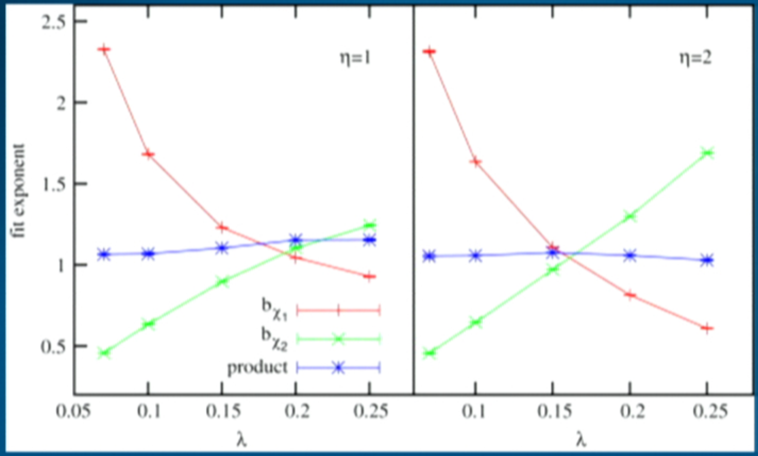
# K-matrix interpretation of edges

$$K = \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix}$$

- Can use bulk-edge correspondence to get following edge action

$$S = d \int dx d\tau \frac{i}{2\pi}(\partial_x \phi_1)(\partial_\tau \phi_2) + \text{quadratic terms}$$

- Integrate one species out
- Find correlator with respect to resulting action, get non-universal number for decay exponent
- Products of decay exponents $=d^2$



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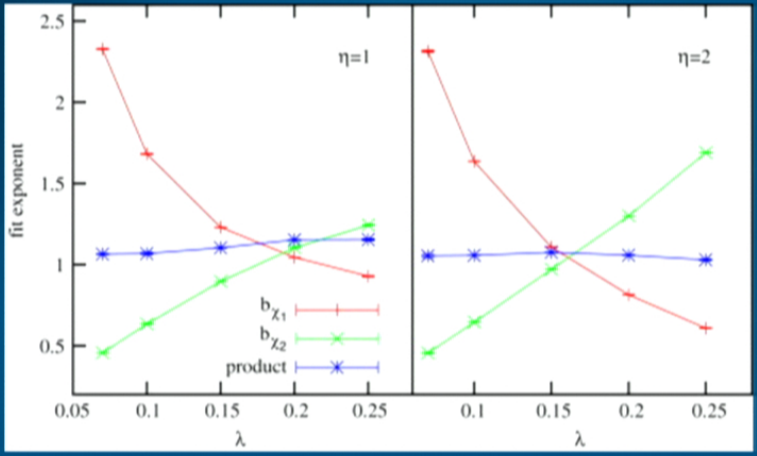

# Conclusion of U(1)xU(1) model

- Found a physical action which bound bosons to vortices, giving the bosons flux
- Showed that the resulting system was topological
- Studied the system numerically
- Possible uses of this model:
  - Phase transitions between bosonic quantum Hall plateaus
  - Effects of disorder
  - Adding defects
- Moral: Binding charges to topological defects can lead to topological behaviour
- Can we apply this to other systems? Ex:
  - (1+1)D, $Z_N \times Z_N$
  - (3+1)D, SO(3) x U(1)