Title: Hexagon functions and six-gluon scattering in planar N=4 super-Yang-Mills

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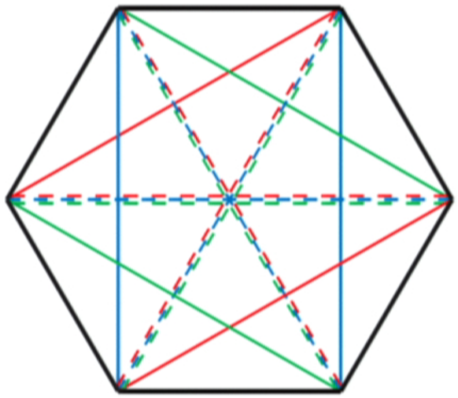
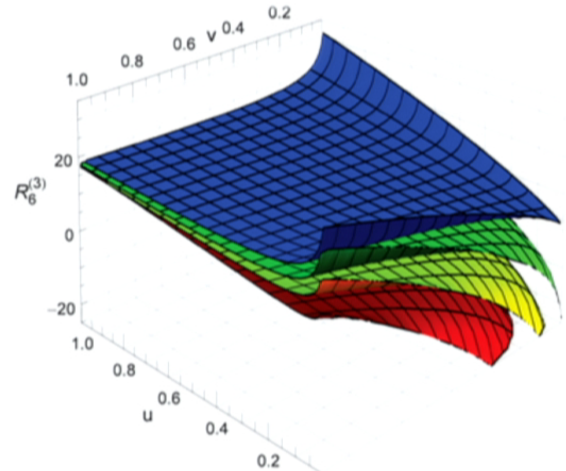
Abstract: Hexagon functions are a class of iterated integrals, depending on three variables (dual conformal cross ratios) which have the correct branch cut structure and other properties to describe the scattering of six gluons in planar N=4 super-Yang-Mills theory. We classify all hexagon
functions through transcendental weight five, using the coproduct for their Hopf algebra iteratively, which amounts to a set of first-order differential equations. As an example, the three-loop remainder function is a particular weight-six hexagon function, whose symbol was determined
previously.

The differential equations can be integrated numerically for generic values of the cross ratios, or analytically in certain kinematics limits, including the near-collinear and multi-Regge
limits. These limits allow us to impose constraints from the operator product expansion and multi-Regge factorization directly at the function level, and thereby to fix uniquely a set of Riemann-Zeta-valued constants that could not be fixed at the level of the symbol. The near-collinear limits agree precisely with recent predictions by Basso, Sever and Vieira based on integrability. The multi-Regge limits agree with a factorization formula of Fadin and Lipatov, and determine three constants entering the impact factor at this order. We plot the three-loop remainder function for various slices of the Euclidean region of positive cross ratios, and compare it to the two-loop one. For large ranges of the cross ratios, the ratio of the three-loop to the two-loop remainder function is relatively constant, and close to -7.

Hexagon Functions and Six-Gluon Scattering in Planar N=4 SYM





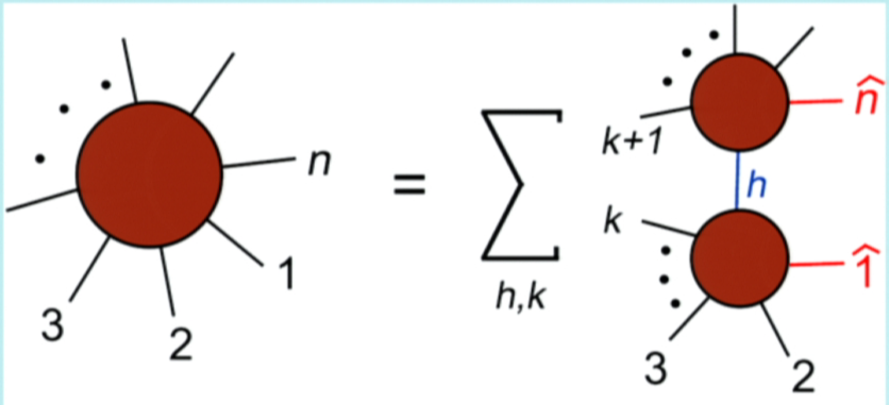
L. Dixon, J. Drummond, M. von Hippel
and J. Pennington

1306.nnnn

Perimeter Institute June 12, 2013

Bootstrapping Amplitudes in D=4

- Many (perturbative) bootstraps for integrands:
- BCFW (2004,2005) for trees (bootstrap in n)



- Trees can be fed into loops via unitarity

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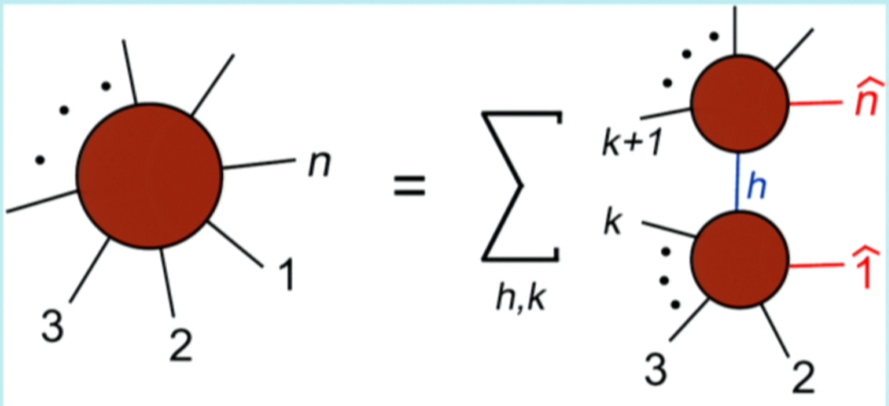

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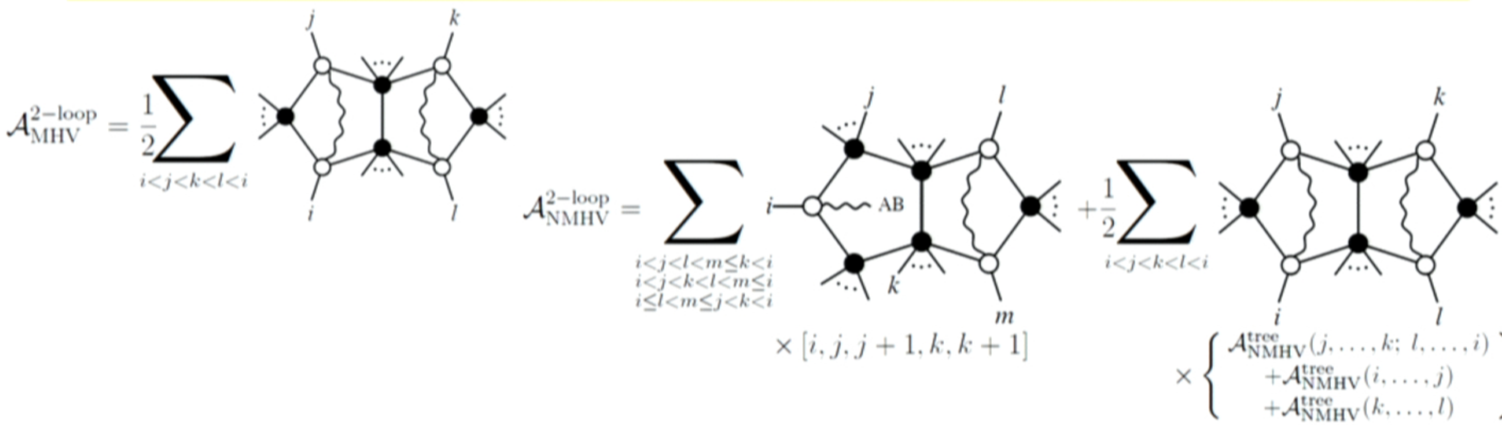
All planar N=4 SYM integrands

Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka, 1008.2958, 1012.6032

- All-loop BCFW recursion relation for integrand ☺
- Or new approach Arkani-Hamed et al. 1212.5605
- Manifest Yangian invariance ☺
- Multi-loop integrands in terms of "momentum-twistors" ☺
- Still have to do integrals over the loop momentum ☹

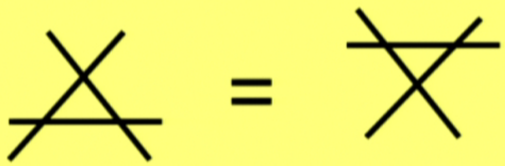


Integrable scattering in D=2

- Infinitely many conserved charges

→ Factorizable S-matrices.

2→2 *S* matrix must satisfy Yang-Baxter equations



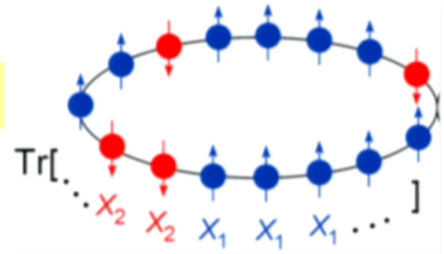
- Many-body S matrix a simple product of 2→2 S matrices.

Integrability and planar N=4 SYM

- Single-trace operators ←→ 1-d spin systems

Tr[$\ldots X_2 X_2 X_1 X_1 X_1 \ldots$] → $SU(2)$ (spin 1/2)

Tr[$\ldots \mathcal{D}^+ \mathcal{D}^+ X_1 X_1 X_1 \ldots$] → $SL(2)$ (noncompact)



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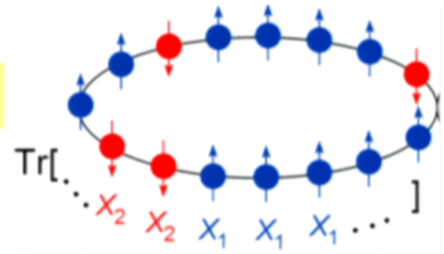
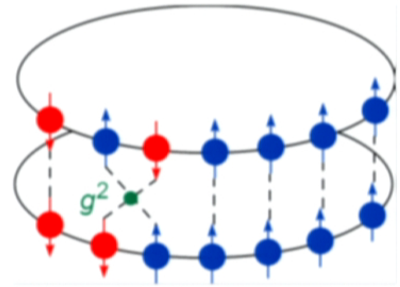


• Anomalous dimensions from spin-chain Hamiltonian. Local in planar limit, though range increases with number of loops



• N=4 SYM Hamiltonian **integrable:**
 – infinitely many conserved charges
 – scattering of quasi-particles (magnons) via 2 → 2 S matrix obeying YBE

Lipatov (1993); Minahan, Zarembo (2002); Beisert, Kristjansen, Staudacher (2003); …

• Also: integrability of $AdS_5 \times S^5$ σ-model Bena, Polchinski, Roiban (2003)

Bootstrapping multi-loop amplitudes

- Use “boundary value data” (like near collinear limit) to assist an iterative bootstrap for multi-loop amplitudes in planar N=4 SYM

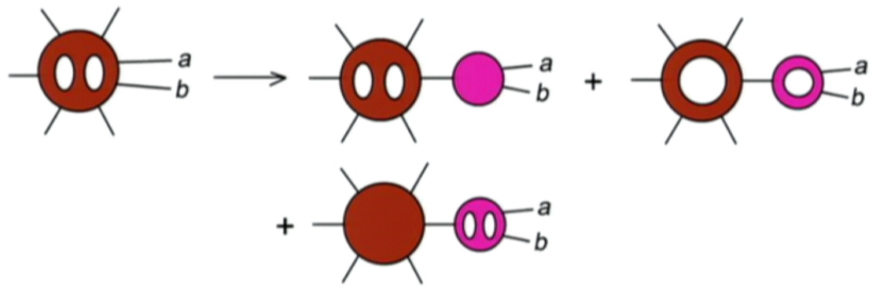
Formula for $R_6^{(2)}(u_1,u_2,u_3)$

• First worked out analytically from Wilson loop integrals
Del Duca, Duhr, Smirnov, 0911.5332, 1003.1702
17 pages of Goncharov polylogarithms.

Wilson loop OPEs

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; GMSV, 1010.5009, 1102.0062

• Remarkably, $R_6^{(2)}(u_1, u_2, u_3)$ can be recovered directly from analytic properties, using "near collinear limits"



• Wilson-loop equivalence → this limit is controlled by an operator product expansion (OPE)

• **Possible to go to 3 loops, by combining OPE expansion with symbol** LD, Drummond, Henn, 1108.4461

Here, promote symbol to unique function $R_6^{(3)}(u_1,u_2,u_3)$

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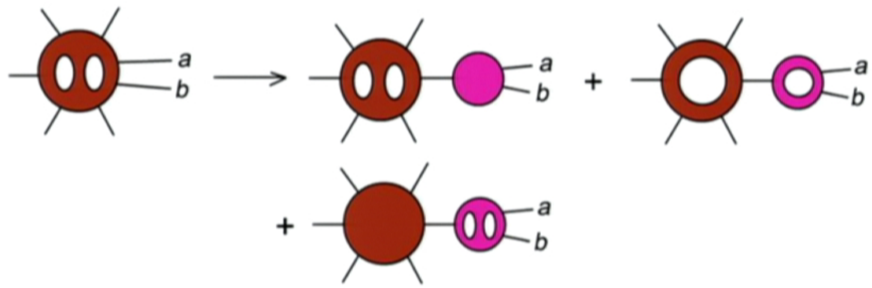

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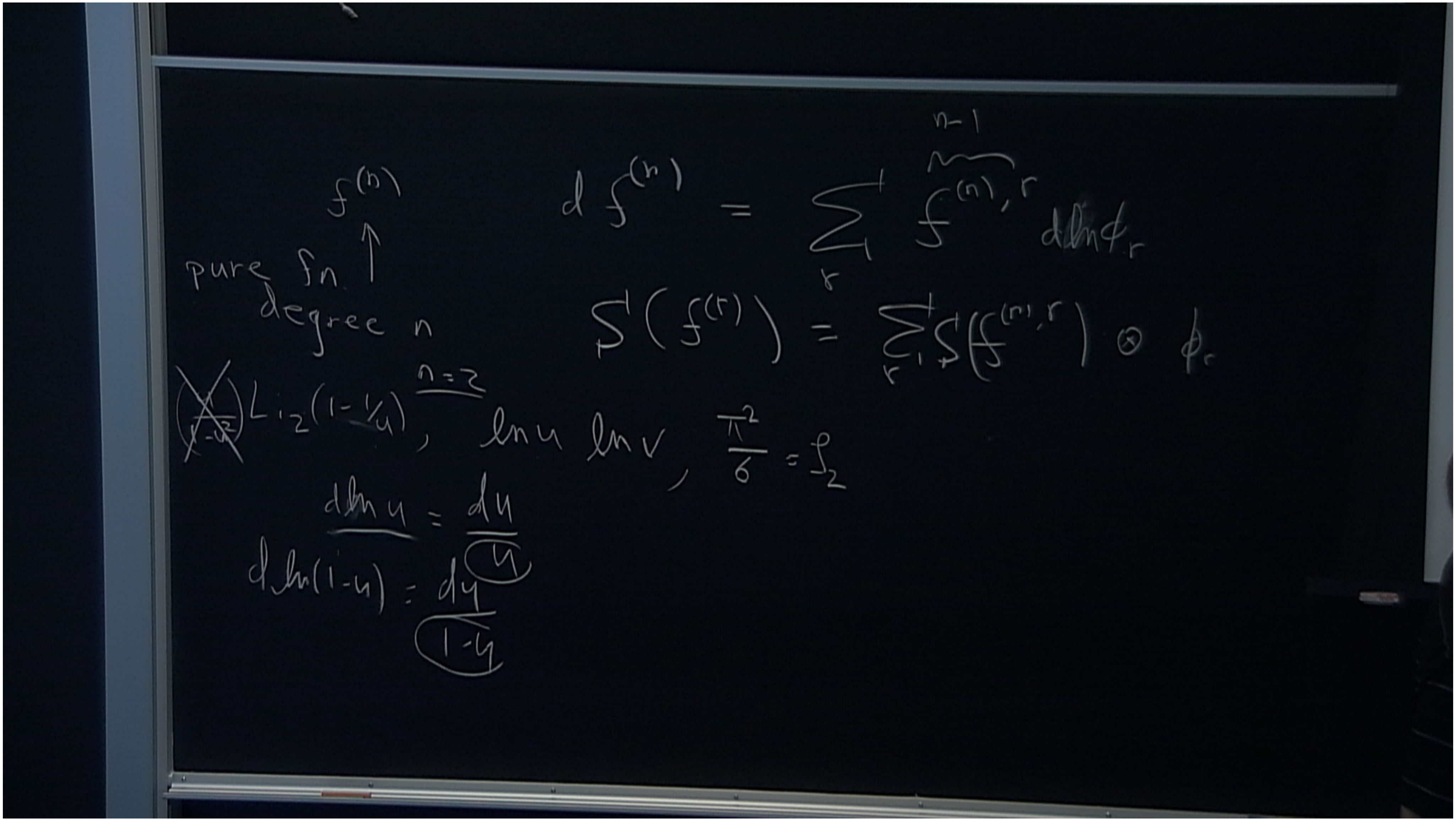
f^(n)
pure fn
degree n
d f^(n) = Σ' f^(n),r dlnφ_r
S'(f^(n)) = Σ' S'(f^(n),r) ⊗ φ_r
n=2
Li_2(1-1/u), ln u ln v, π²/6 = ζ_2
dln u = du/u
dln(1-u) = du/(1-u)

$S[\ R_6^{(2)}(u,v,w)\ ]$ in these variables

GSVV, 1006.5703

$$
\begin{aligned}
-8\,S[R_6^{(2)}] \;=\;& u \otimes (1-u) \otimes \frac{u}{(1-u)^2} \otimes \frac{u}{1-u} \\
&+ 2\,(u \otimes v + v \otimes u) \otimes \frac{w}{1-v} \otimes \frac{u}{1-u} \\
&+ 2\,v \otimes \frac{w}{1-v} \otimes u \otimes \frac{u}{1-u} \\
&+ u \otimes (1-u) \otimes y_u y_v y_w \otimes y_u y_v y_w \\
&- 2\,u \otimes v \otimes y_w \otimes y_u y_v y_w
\end{aligned}
$$

$$+\ 5 \text{ permutations of } (u,v,w)$$

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& + 2\,v \otimes \frac{w}{1-v} \otimes u \otimes \frac{u}{1-u} \\
& + u \otimes (1-u) \otimes y_u y_v y_w \otimes y_u y_v y_w \\
& - 2\,u \otimes v \otimes y_w \otimes y_u y_v y_w \\
& \\
& + 5 \text{ permutations of } (u,v,w)
\end{aligned}
$$

First entry

- Always drawn from $\{u, v, w\}$ GMSV, 1102.0062
- Because first entry controls branch-cut location
- Only massless particles

→ all cuts start at origin in $s_{i,i+1}$, $s_{i,i+1,i+2}$

→ Branch cuts all start from 0 or ∞ in

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}^2 s_{45}^2}{s_{123}^2 s_{345}^2}$$

Final entry

- Always drawn from

$$\left\{\frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w\right\}$$

- Seen in structure of various Feynman integrals
[e.g. Arkani-Hamed et al., 1108.2958]
related to amplitudes
Drummond, Henn, Trnka 1010.3679;
LD, Drummond, Henn, 1104.2787, V. Del Duca et al., 1105.2011,...
- Same condition also from Wilson super-loop approach Caron-Huot, 1105.5606

Generic Constraints

- **Integrability** (must be symbol of some function)
- S_3 permutation **symmetry** in $\{u, v, w\}$
- Even under "**parity**":
every term must have an even number of y_i – 0, 2 or 4

$$i\sqrt{\Delta} \leftrightarrow -i\sqrt{\Delta}$$
$$z_+ \leftrightarrow z_-$$
$$y_i \leftrightarrow 1/y_i$$

- Vanishing in **collinear** limit

$$v \to 0 \qquad u + w \to 1$$

OPE Constraints

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; GMSV, 1010.5009; 1102.0062
Basso, Sever, Vieira [BSV], 1303.1396; 1306.2058

- $R_6^{(L)}(u,v,w)$ vanishes in the collinear limit,

$$v = 1/\cosh^2\tau \to 0 \qquad\qquad \tau \to \infty$$

In **near**-collinear limit, described by an Operator Product Expansion, with generic form

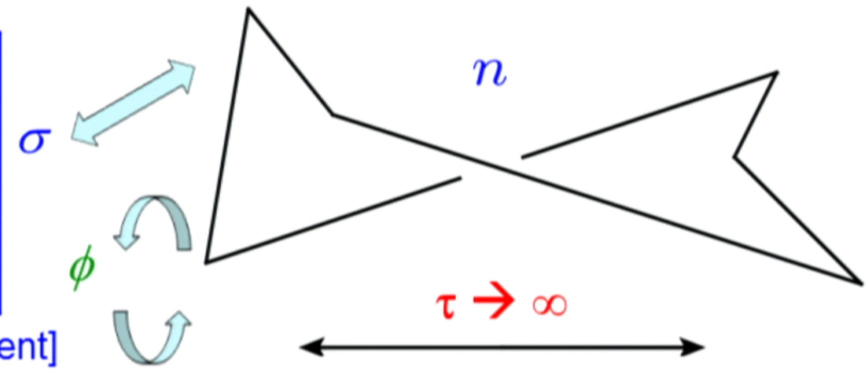
$$R_6^{(L)}(u,v,w) = R_6^{(L)}(\tau,\sigma,\phi) \sim \int dn\ C_n(g)\ \exp[-E_n(g)\tau]$$

$$u = \frac{e^\sigma \sinh\tau \tanh\tau}{2(\cosh\sigma \cosh\tau + \cos\phi)}$$

$$v = \frac{1}{\cosh^2\tau}$$

$$w = u\, e^{-2\sigma}$$

[BSV parametrization a little different]



OPE Constraints (cont.)

- Using conformal invariance, send one long line to ∞, put other one along x^-

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- Using conformal invariance, send one long line to ∞, put other one along x^-
- Dilatations, boosts, azimuthal rotations preserve configuration.
- σ, ϕ conjugate to twist p, spin m of conformal primary fields (flux tube excitations)
- Expand anomalous dimensions in coupling g^2:

$$E_n(g) = E_n^{(0)} + g^2\, E_n^{(1)} + g^4\, E_n^{(2)} + \ldots$$

$$\exp[-E_n(g)\tau]$$
$$= \exp[-E_n^{(0)}\tau] \times \left[1 - g^2\,\tau\, E_n^{(1)} + g^4(\frac{1}{2}\tau^2\,[E_n^{(1)}]^2 - \tau\, E_n^{(2)}) + \ldots\right]$$

OPE Constraints (cont.)

- As $\tau \to \infty$, $v = 1/\cosh^2\tau$ → $\tau^{L\text{-}1} \sim [\ln v]^{L\text{-}1}$
- Extract this piece from symbol by only keeping terms with L-1 leading v entries

$$\underbrace{v \otimes \ldots \otimes v}_{\text{clip } L-1 \text{ entries}} \otimes \underbrace{\ldots}_{\text{keep } L+1 \text{ entries}}$$

- Powerful constraint: fixes 3 loop symbol up to 2 parameters. But not powerful enough for $L > 3$
- New results of BSV give

$$v^{1/2}\, e^{\pm i\phi}\, [\ln v]^{k}, \quad k = 0,1,2,\ldots L\text{-}1$$

and even

$$v^{1}\, e^{\pm 2i\phi}\, [\ln v]^{k}, \quad k = 0,1,2,\ldots L\text{-}1$$

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Constrained Symbol

- Leading discontinuity constraints reduced symbol ansatz to just 2 parameters: DDH, 1108.4461

$$\mathcal{S}[R_6^{(3)}] = \mathcal{S}[X] + \alpha_1 \mathcal{S}[f_1] + \alpha_2 \mathcal{S}[f_2]$$

Reconstructing the function

- One can build up a complete description of the pure functions $F(u,v,w)$ with correct branch cuts iteratively in the weight n, using the $(n\text{-}1,1)$ element of the co-product $\Delta_{n\text{-}1,1}(F)$

Duhr, Gangl, Rhodes, 1110.0458

$$\Delta_{n-1,1}(F) \equiv \sum_{i=1}^{3} F^{u_i} \otimes \ln u_i + F^{1-u_i} \otimes \ln(1-u_i) + F^{y_i} \otimes \ln y_i$$

which specifies all first derivatives of F:

$$\left.\frac{\partial F}{\partial u}\right|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}} F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}} F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}} F^{y_w}$$

$$\sqrt{\Delta}\, y_u \left.\frac{\partial F}{\partial y_u}\right|_{y_v,y_w} = (1-u)(1-v-w)F^u - u(1-v)F^v - u(1-w)F^w - u(1-v-w)F^{1-u} + uv\, F^{1-v} + uw\, F^{1-w} + \sqrt{\Delta}\, F^{y_u}\,.$$

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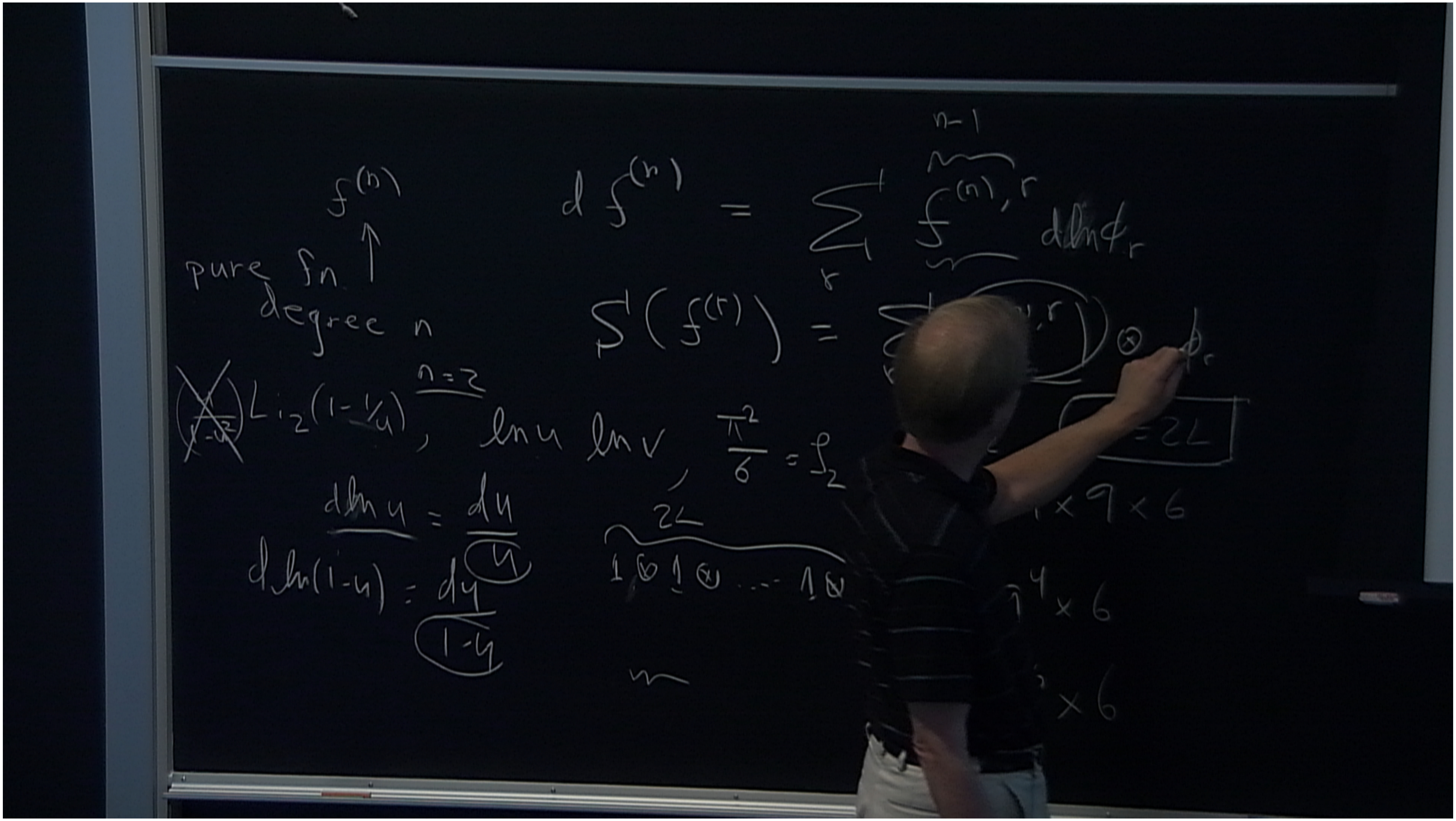
pure fn
degree n
n=2
ln u ln v
d ln u = du/u
d ln(1-u) = du/(1-u)

How many hexagon functions?

First entry $\{u, v, w\}$; non-product

Weight	y^0	y^1	y^2	y^3	y^4
1	3 HPLs	-	-	-	-
2	3 HPLs	-	-	-	-
3	6 HPLs	$\tilde{\Phi}_6$	-	-	-
4	9 HPLs	$3 \times F_1$	$3 \times \Omega^{(2)}$	-	-
5	18 HPLs	$G, 3 \times K_1$	$5 \times M_1, N, O, 6 \times Q_{\rm ep}$	$3 \times H_1, 3 \times J_1$	-
6	27 HPLs	4	27	29	$3 \times R_{\rm ep} + 15$

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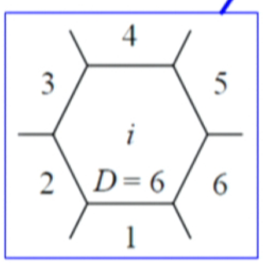
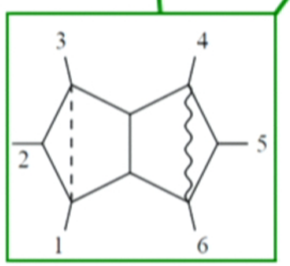
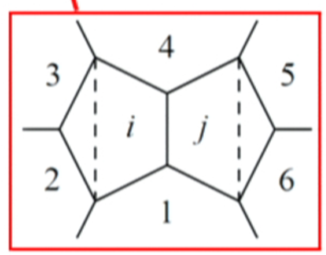






$R_6^{(3)}(u,v,w)$

$$R_6^{(3)}(u,v,w) = R_{\rm ep}(u,v,w) + R_{\rm ep}(v,w,u) + R_{\rm ep}(w,u,v) + P_6(u,v,w) + c_1\zeta_6 + c_2(\zeta_3)^2$$

$$P_6 = -\frac{1}{4}(\Omega^{(2)}(u,v,w)\mathrm{Li}_2(1-1/w) + \mathrm{cyc}) - \frac{1}{16}(\tilde{\Phi}_6)^2 + \frac{1}{4}\mathrm{Li}_2(1-1/u)\mathrm{Li}_2(1-1/v)\mathrm{Li}_2(1-1/w)\,.$$

Many relations among coproduct coefficients for $R_{\rm ep}$:

$$R_{\rm ep}^{v} = -R_{\rm ep}^{1-v} = -R_{\rm ep}^{1-u}(u\leftrightarrow v) = R_{\rm ep}^{u}(u\leftrightarrow v)\,, \qquad R_{\rm ep}^{y_v} = R_{\rm ep}^{y_u}\,,$$

$$R_{\rm ep}^{w} = R_{\rm ep}^{1-w} = R_{\rm ep}^{y_w} = 0$$

Only 2 indep. $R_{\rm ep}$ coproduct coefficients

$$R_{\rm ep}^{y_u} = -\frac{1}{32}H_1(u,v,w) - \frac{3}{32}H_1(v,w,u) - \frac{1}{32}H_1(w,u,v) + \frac{3}{128}J_1(u,v,w) + \frac{3}{128}J_1(v,w,u)$$
$$+ \frac{3}{128}J_1(w,u,v) - \frac{1}{8}H_2^u\,\tilde\Phi_6 - \frac{1}{8}H_2^v\,\tilde\Phi_6 - \frac{1}{32}\ln^2 u\,\tilde\Phi_6 + \frac{1}{16}\ln u\,\ln v\,\tilde\Phi_6$$
$$- \frac{1}{16}\ln u\,\ln w\,\tilde\Phi_6 - \frac{1}{32}\ln^2 v\,\tilde\Phi_6 - \frac{1}{16}\ln v\,\ln w\,\tilde\Phi_6 + \frac{1}{32}\ln^2 w\,\tilde\Phi_6 + \frac{11}{16}\zeta_2\,\tilde\Phi_6\,,$$

$$R_{\rm ep}^{u} = -\frac{2}{3}Q_{\rm ep}^u(u,v,w) + \frac{2}{3}Q_{\rm ep}^u(u,w,v) - \frac{2}{3}Q_{\rm ep}^u(v,w,u) - \frac{1}{3}Q_{\rm ep}^u(v,u,w) + Q_{\rm ep}^u(w,v,u)$$
$$+ \frac{1}{32}M_1(u,v,w) - \frac{1}{32}M_1(v,u,w) + \frac{5}{32}\ln u\,\Omega^{(2)}(u,v,w) - \frac{3}{32}\ln u\,\Omega^{(2)}(v,w,u)$$
$$- \frac{1}{32}\ln u\,\Omega^{(2)}(w,u,v) - \frac{5}{32}\ln v\,\Omega^{(2)}(u,v,w) - \frac{1}{32}\ln v\,\Omega^{(2)}(v,w,u) - \frac{3}{32}\ln v\,\Omega^{(2)}(w,u,v)$$
$$+ \frac{1}{8}\ln w\,\Omega^{(2)}(u,v,w) + \frac{1}{16}\ln w\,\Omega^{(2)}(v,w,u) + \frac{1}{8}\ln w\,\Omega^{(2)}(w,u,v) + R_{\rm ep,\,rat}^u\,,$$

2 pages of 1-d HPLs

Similar (but shorter) expressions for lower degree functions

Numerical integration contours

$$\frac{\partial\tilde{\Phi}_6}{\partial\ln y_v} = -\Omega^{(1)}(w,u,v) \quad \Longrightarrow \quad \tilde{\Phi}_6 = \sqrt{\Delta(u,v,w)} \int_1^u \frac{du_t \Omega^{(1)}(w_t,u_t,v_t)}{v_t[u(1-w)+(w-u)u_t]}$$

base point $(u,v,w) = (1,1,1)$

$$v_t = 1 - \frac{(1-v)\,u_t\,(1-u_t)}{u\,(1-w)+(w-u)\,u_t}$$

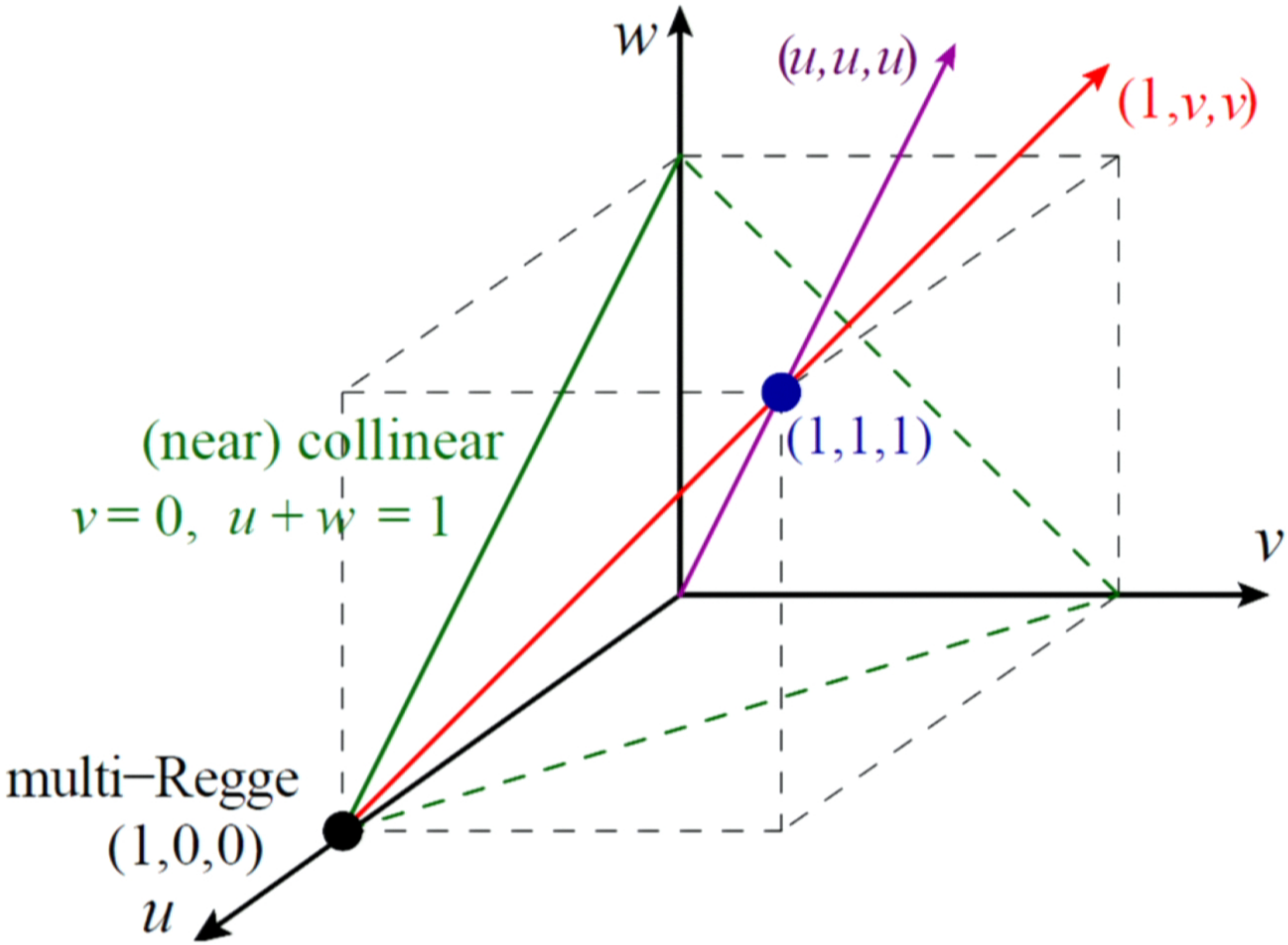
$$w_t = \frac{(1-u)\,w\,u_t}{u\,(1-w)+(w-u)\,u_t}\,.$$

$$\frac{\partial\tilde{\Phi}_6}{\partial\ln(y_u/y_w)} = \ln(u/w)\ln v \quad \Longrightarrow \quad \tilde{\Phi}_6 = \sqrt{\Delta(u,v,w)} \int_0^u \frac{du_t \ln(u_t/w_t)\ln v_t}{(1-v_t)[uw+(1-u-w)u_t]}$$

base point $(u,v,w) = (0,0,1)$

$$v_t = \frac{vu_t(1-u_t)}{uw+(1-u-w)u_t}$$

$$w_t = \frac{uw(1-u_t)}{uw+(1-u-w)u_t}$$


w
(u,u,u)
(1,v,v)
(1,1,1)
(near) collinear
v = 0, u + w = 1
v
multi−Regge
(1,0,0)
u

Fixing all the constants

- 11 bts constants (plus $\alpha_{1,2}$) before analyzing limits

Fixing all the constants

- 11 bts constants (plus $\alpha_{1,2}$) before analyzing limits

Fixing all the constants

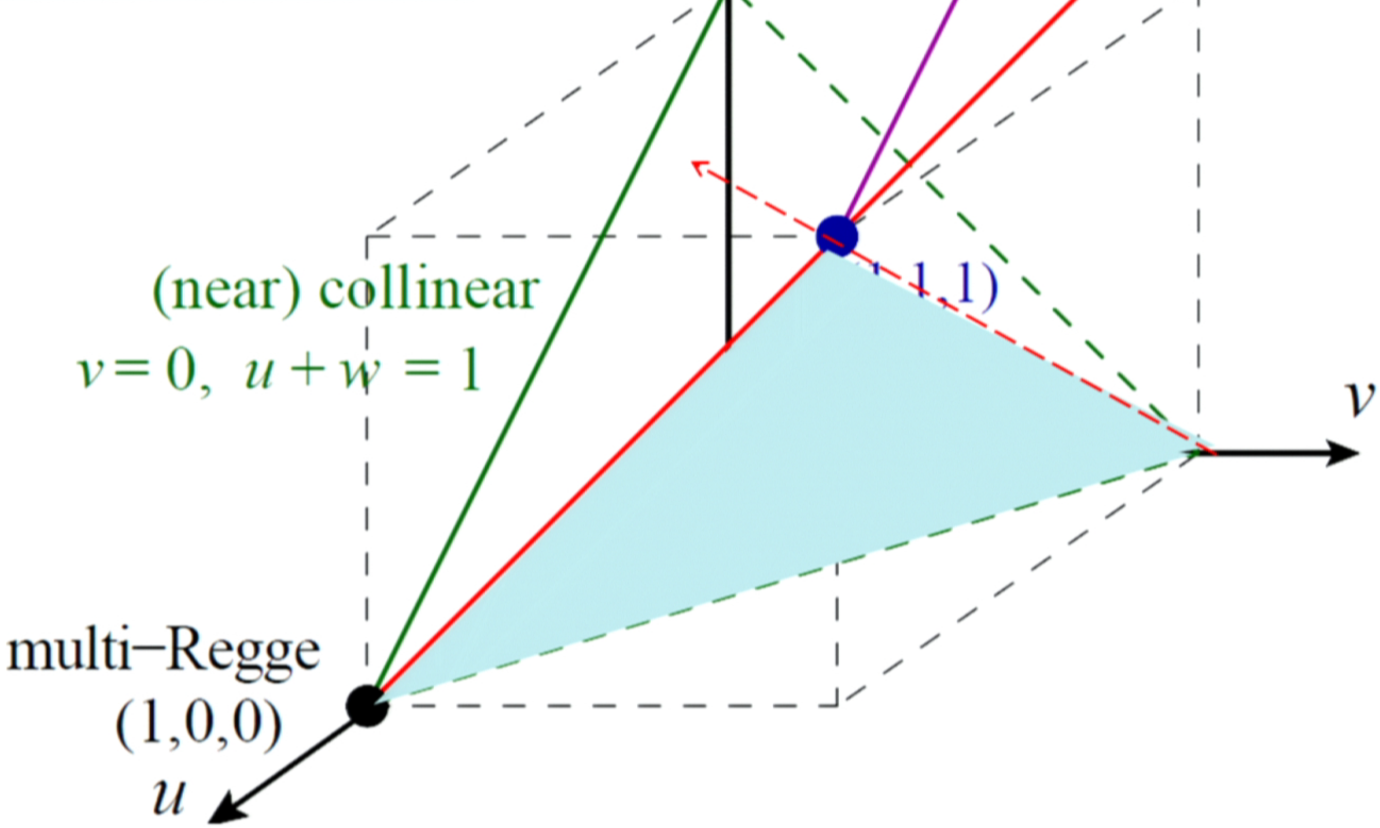
- 11 bts constants (plus $\alpha_{1,2}$) before analyzing limits
- Vanishing of collinear limit $v \to 0$ fixes everything, except α_2 and 1 bts constant
- Near-collinear limit,

$$v^{1/2}\, e^{\pm i\phi}\, [\ln v]^{k}, \; k = 0,1$$

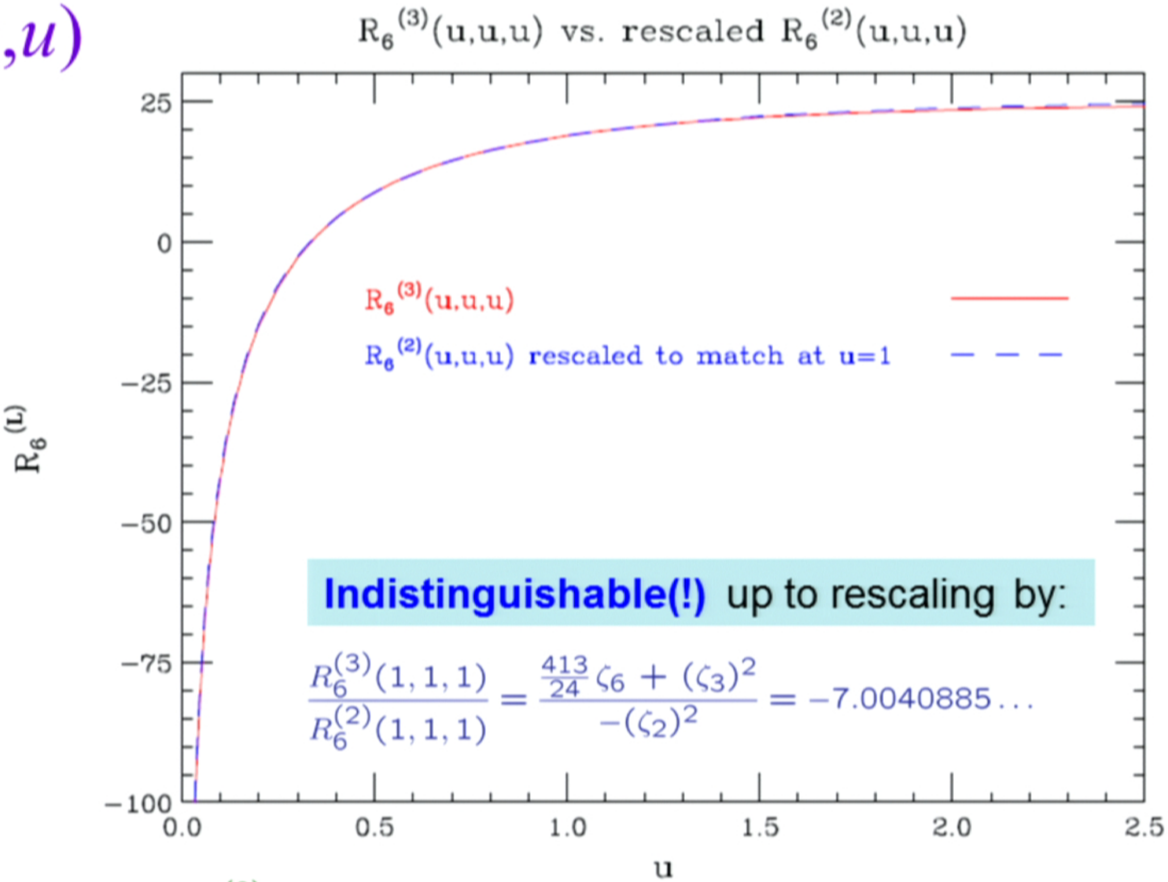
fixes last 2 constants

(α_2 agrees with Caron-Huot+He and BSV)

Plot $R_6^{(3)}(u,v,w)$ on some slices

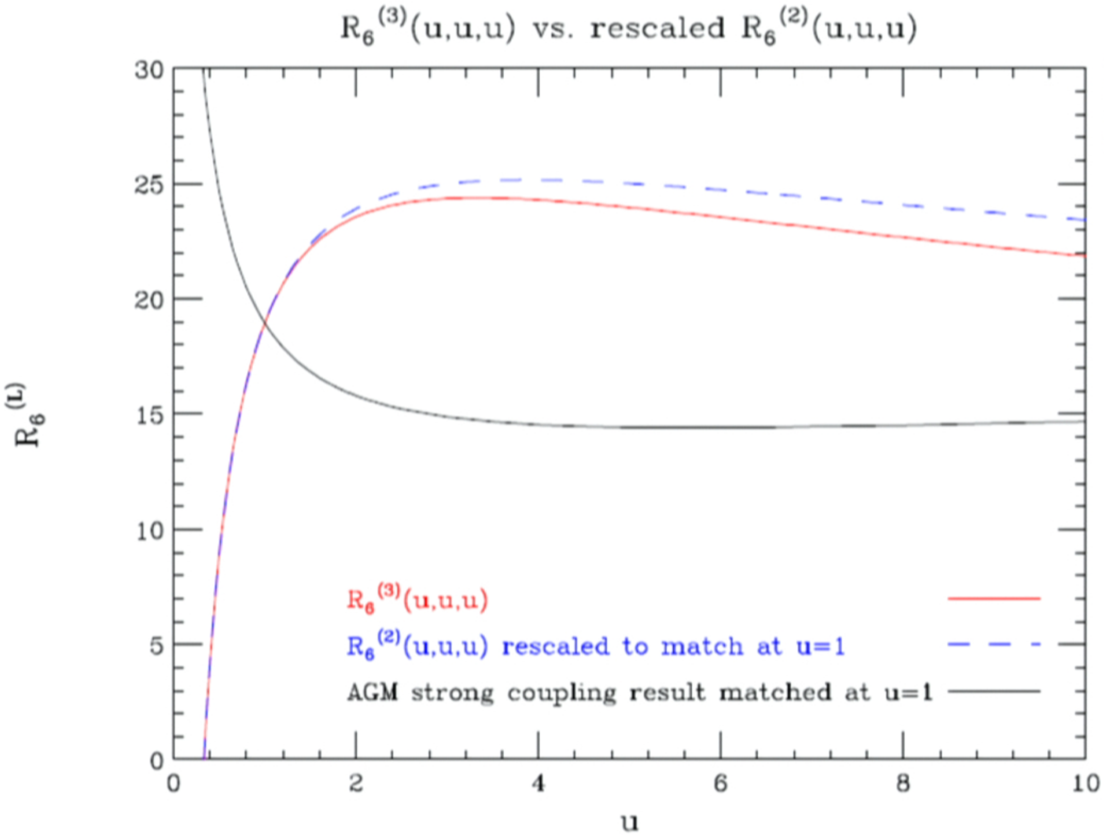


(u,u,u)



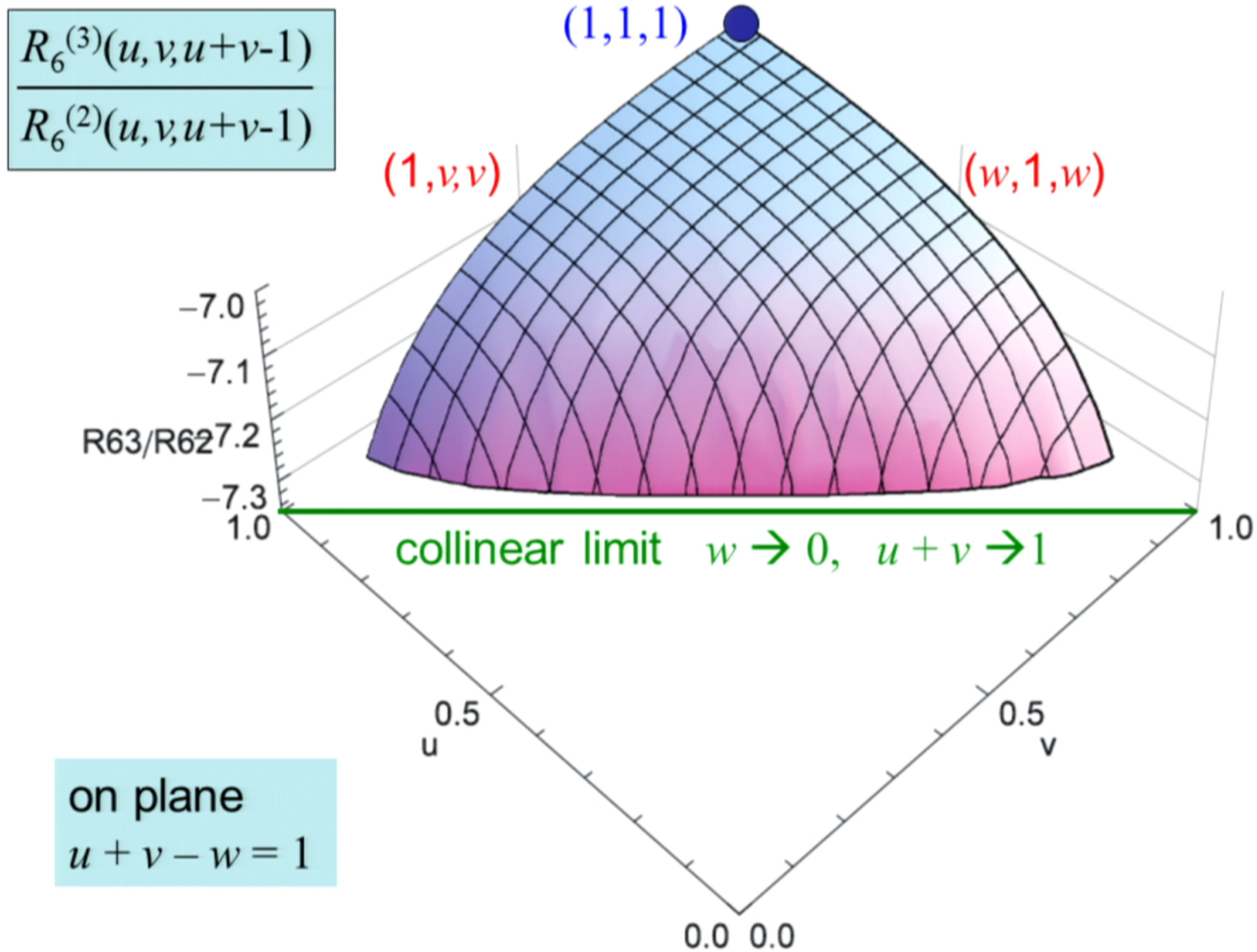
cf. cusp ratio: $\frac{\gamma_K^{(3)}}{\gamma_K^{(2)}} = \frac{22\,\zeta_4}{-4\,\zeta_2} = -3.61885\ldots$

Proportionality ceases at large u



ratio ~ -1.23

0911.4708



$\frac{R_6^{(3)}(u,v,u+v-1)}{R_6^{(2)}(u,v,u+v-1)}$
(1,1,1)
(1,v,v)
(w,1,w)
–7.0
–7.1
R63/R62
–7.2
–7.3
1.0
1.0
collinear limit $w \to 0, \quad u+v \to 1$
0.5
u
0.5
v
0.0 0.0
on plane
$u+v-w=1$

On to 4 loops

LD, Duhr, Pennington, in progress

- In the course of 1207.0186, we "determined" the 4 loop remainder-function symbol.
- However, still 113 undetermined constants ☹
- Consistency with LLA and NLLA multi-Regge limits → 81 constants 😐
- Consistency with BSV's $\nu^{1/2}\, e^{\pm i\phi}$ → 4 constants ☺
- Adding BSV's $\nu^{1}\, e^{\pm 2i\phi}$ → 0 constants!! ☺☺

[Thanks to BSV for supplying this info!]

Conclusions

- Bootstraps are wonderful things
- Applied successfully to D=2 integrable models
- To perturbative amplitudes & integrands
- To anomalous dimensions in planar N=4 SYM
- Now, nonperturbatively to whole D=2 scattering problem on OPE/near-collinear boundary of phase-space for scattering amplitudes
- With knowledge of function space and this boundary data, can determine perturbative N=4 amplitudes over full phase space, without need to know any integrands at all