Title: Generalized Relative Entropies, Entanglement Monotones and One-Shot Information Theory

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Abstract: We introduce two relative entropy quantities called the min- and max-relative
entropies and discuss their properties and operational meanings.

These relative entropies act as parent quantities for tasks such as data compression, information
transmission and entanglement manipulation in one-shot information theory. Moreover, they lead us to define entanglement monotones which have interesting operational interpretations.

Relative Entropies, Entanglement Monotones & One-shot Information Theory

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Quantum Relative Entropy

- A fundamental quantity in Quantum Mechanics & Quantum Information Theory is the Quantum Relative Entropy of ρ w.r.t. σ,

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Quantum Relative Entropy

- A fundamental quantity in Quantum Mechanics & Quantum Information Theory is the Quantum Relative Entropy

  of ρ w.r.t. σ, $\rho \geq 0, \ \mathrm{Tr}\,\rho = 1, \ \sigma \geq 0$:

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Quantum relative entropy as a *"distance"* measure between states

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Quantum relative entropy as a *"distance"* measure between states

"distance"

~~*symmetric*~~
~~*triangle inequality*~~

$$D(\rho \| \sigma) \geq 0 \quad \text{if} \quad \rho, \sigma \ \text{states}$$

$$= 0 \text{ if \& only if } \rho = \sigma$$

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Quantum relative entropy as a "distance" measure between states

"distance"

~~*symmetric*~~
~~*triangle inequality*~~

$$D(\rho\,\|\,\sigma) \geq 0 \quad \text{if} \quad \rho, \sigma \quad \text{states}$$
$$= 0 \text{ if \& only if } \rho = \sigma$$

..........(1)

- Monotonicity of Quantum Relative Entropy under a linear completely positive trace-preserving (CPTP) map Λ:

$$D(\Lambda(\rho)\,\|\,\Lambda(\sigma)) \leq D(\rho\,\|\,\sigma) \qquad \text{..........(2)}$$

Many properties of other entropies can be proved using (1) & (2)

e.g. (2) ⟶ *Strong subadditivity of the von Neumann entropy*

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Outline

- *Consider 2 generalized relative entropy quantities*
- *Discuss their properties and operational significance*
- *Give a motivation for defining them*
- *Define 2 entanglement monotones*

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- *Definition 1:* The max- relative entropy of a state ρ & a positive operator σ is

$$D_{\max}(\rho \| \sigma) := \inf\left\{\gamma : \rho \leq 2^{\gamma}\sigma\right\}$$

$$\text{supp}\,\rho \subseteq \text{supp}\,\sigma$$

$$(2^{\gamma}\sigma - \rho) \geq 0$$

$$D_{\max}(\rho \| \sigma) = \log\left(\lambda_{\max}(\sigma^{-1/2}\rho\sigma^{-1/2})\right)$$

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pseudoinverse

- *asymmetric measure*

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Why are $D_{\min}(\rho\|\sigma)$ & $D_{\max}(\rho\|\sigma)$ relative entropies?

- Like $D(\rho\|\sigma)$ we have

$$D_*(\rho\|\sigma) \geq 0$$

for $* = \max, \min$

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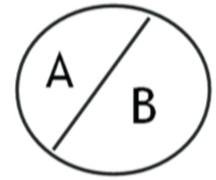
For a bipartite state ρ_{AB} :



- ***Conditional min-entropy*** *[Renner]*

$$H_{\min}(A|B)_\rho := \max_{\sigma_B}\{-D_{\max}(\rho_{AB}\,||\,I_A\otimes\sigma_B)\}$$

just as: *Quantum conditional entropy*

$$S(A|B) = -D(\rho_{AB}\,||\,I_A\otimes\rho_B) = \max_{\sigma_B}\{-D(\rho_{AB}\,||\,I_A\otimes\sigma_B)\}$$

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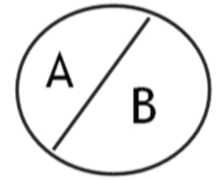
For a bipartite state ρ_{AB} :



- *Max-information* [Berta, Christandl, Renner]

$$I_{\max}(A:B)_{\rho} := \min_{\sigma_B} D_{\max}(\rho_{AB} \,\|\, \rho_A \otimes \sigma_B)$$

just as: *Quantum mutual information*

$$I(A:B) = D(\rho_{AB} \,\|\, \rho_A \otimes \rho_B)$$
$$= \min_{\sigma_B} D(\rho_{AB} \,\|\, \rho_A \otimes \sigma_B)$$

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Operational significance of $D_{\min}(\rho \| \sigma)$

- ***State Discrimination:*** Bob receives a state

  ρ or σ

- He does a measurement to infer which state it is

  POVM $\Pi \; [\rho]$ & $(I - \Pi) \; [\sigma]$

Possible errors	*inference*	*actual state*
Type I	σ	ρ
Type II	ρ	σ

  hypothesis testing

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- *Suppose* $\Pi = \pi_{\rho}$ (POVM element)

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- *Suppose* $\Pi = \pi_\rho$ (POVM element)

Prob(Type I error)

$$\alpha = \mathrm{Tr}((I - \Pi)\rho)$$
$$= 0$$

Bob never infers the state to be σ *when it is* ρ

Prob(Type II error)

$$\beta = \mathrm{Tr}(\Pi\sigma)$$
$$= \mathrm{Tr}(\pi_\rho \sigma)$$

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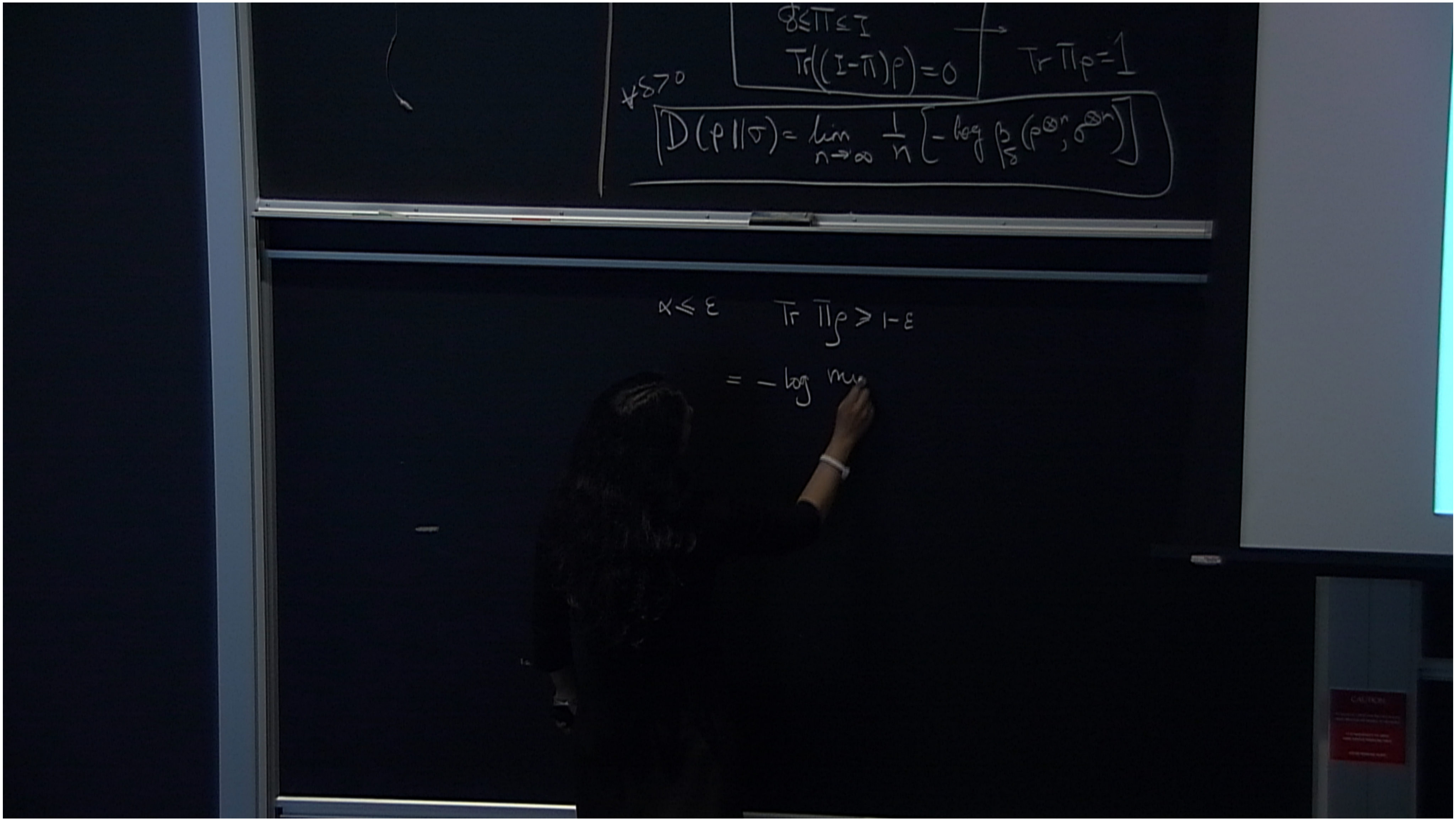
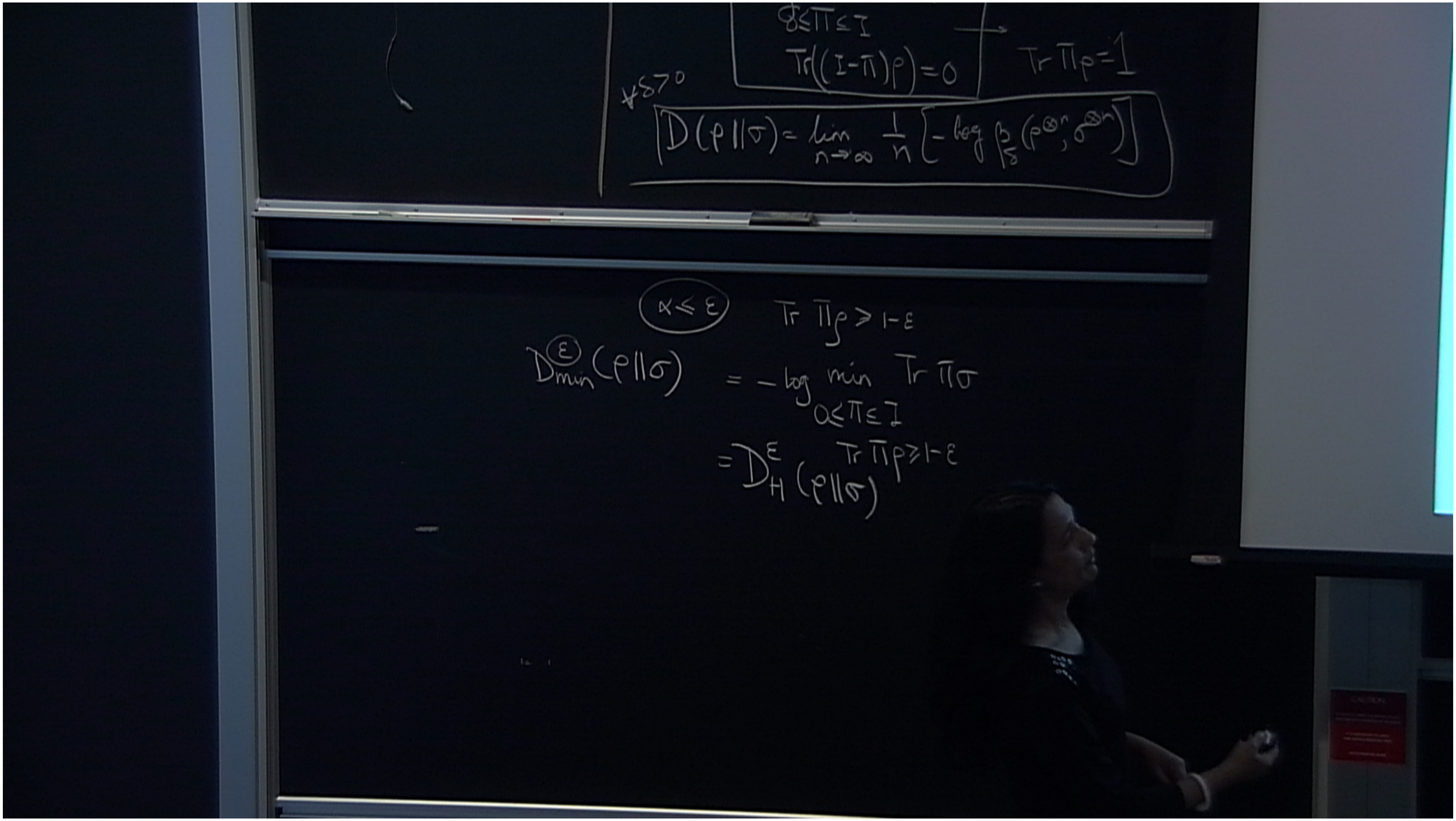
Operational interpretations of the max-relative entropy (i)


0 ≤ Π ≤ I
Tr((I−Π)ρ)=0
Tr Πρ=1
∀δ>0
D(ρ||σ) = lim n→∞ 1/n [−log β_δ(ρ^⊗n, σ^⊗n)]
α ≤ ε
Tr Πρ ≥ 1−ε


0≤Π≤I
Tr((I−Π)ρ)=0
Tr Πρ=1
∀δ>0
D(ρ||σ) = lim n→∞ 1/n [−log β_δ(ρ^⊗n; σ^⊗n)]
α≤ε
Tr Πρ ≥ 1−ε
D_min^ε(ρ||σ) = −log min Tr Πσ
0≤Π≤I
Tr Πρ≥1−ε
= D_H^ε(ρ||σ)

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$$\sigma = \lambda \rho_s + (1 - \lambda)\omega$$

- ***Theorem 2*** *[ND,T.Rudolph]:*

The separability of the state σ of a bipartite system is given by:

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Min- & Max relative entropies: $D_{\min}(\rho \| \sigma),\ D_{\max}(\rho \| \sigma)$

act as **parent quantities** *for* **one-shot rates** *of protocols*

just as

Quantum relative entropy: $D(\rho \| \sigma)$

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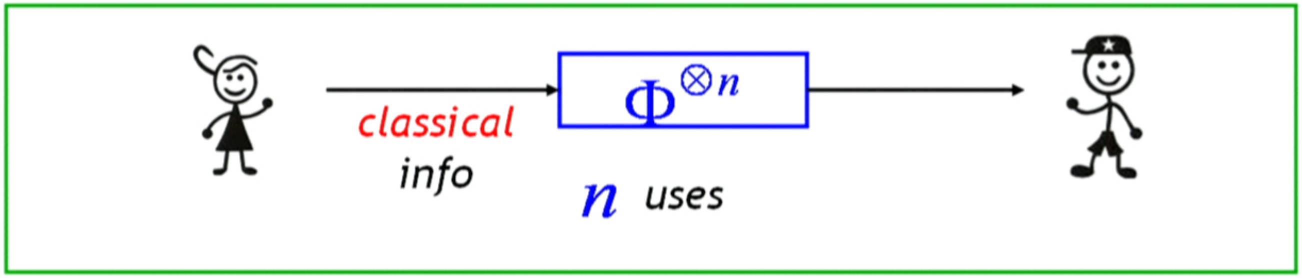
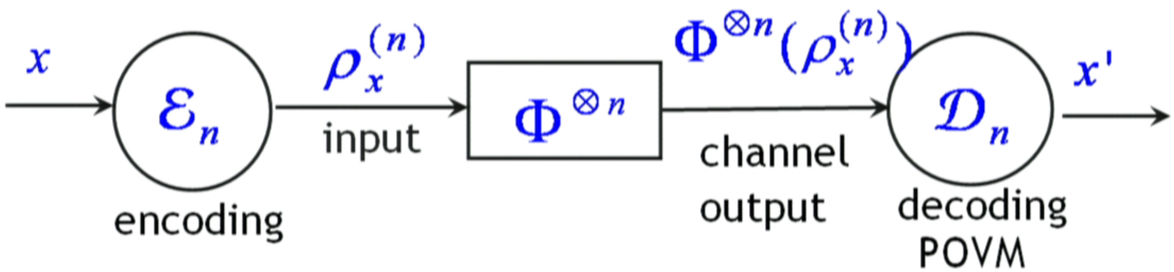
“asymptotic, memoryless setting”





- One requires : **prob. of error** $p_e^{(n)} \to 0$ as $n \to \infty$

- If inputs are product states: $\rho_x^{(n)} = \rho_{x_1} \otimes \rho_{x_2} \otimes \ldots\ldots \otimes \rho_{x_n}$

- Capacity : Product state capacity

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[HSW Theorem]

$$C_p(\Phi) = \chi^*(\Phi) = \max_{\{p_i,\rho_i\}} \min_{\sigma_B} D\left(\rho_{XB} \,\|\, \rho_X \otimes \sigma_B\right)$$

Holevo-capacity

$$\rho_{XB} = \sum_x p_x |x\rangle\langle x| \otimes \Phi(\rho_x);$$

$\forall\, 0<\varepsilon<1$

[ND, Mosonyi, Hsieh, Brandao]

$$C_\varepsilon^{(1)}(\Phi) \approx \chi^*_{\max,\varepsilon'}(\Phi) = \max_{\{p_i,\rho_i\}} \min_{\sigma_B} D_{\max}^{\varepsilon'}\left(\rho_{XB} \,\|\, \rho_X \otimes \sigma_B\right)$$

smooth max-Holevo capacity

$\varepsilon' = f(\varepsilon)$

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Relation to other entanglement monotones

$$E_{\max}(\rho) = LR_g(\rho)$$ = *log robustness* of ρ

- $$LR_g(\rho) := \log\left(1 + R_g(\rho)\right)$$

$R_g(\rho)$ = *global robustness of* ρ [Harrow & Nielsen]

= a measure of the extent to which another state can be mixed with ρ until the mixture becomes separable.

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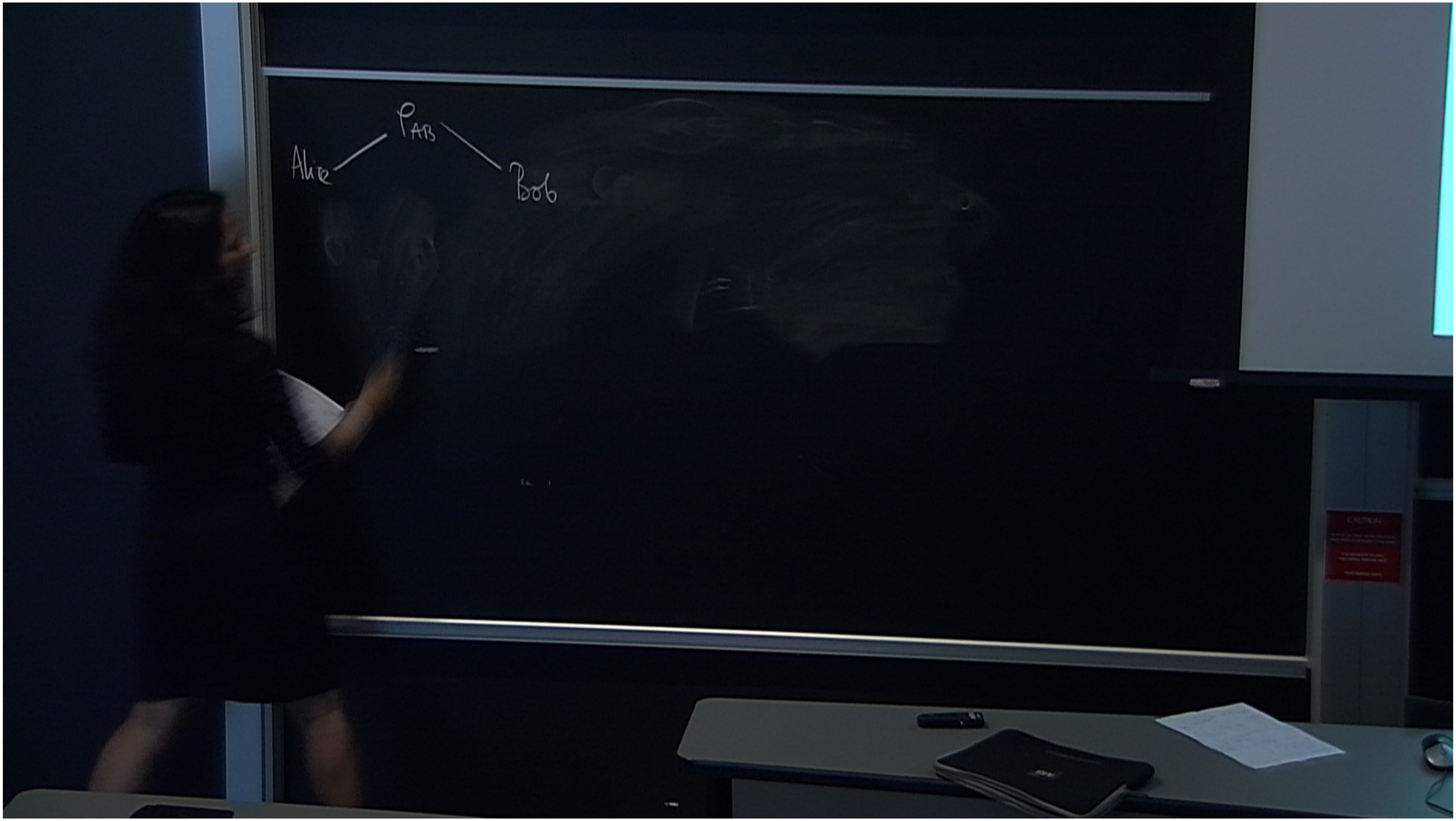
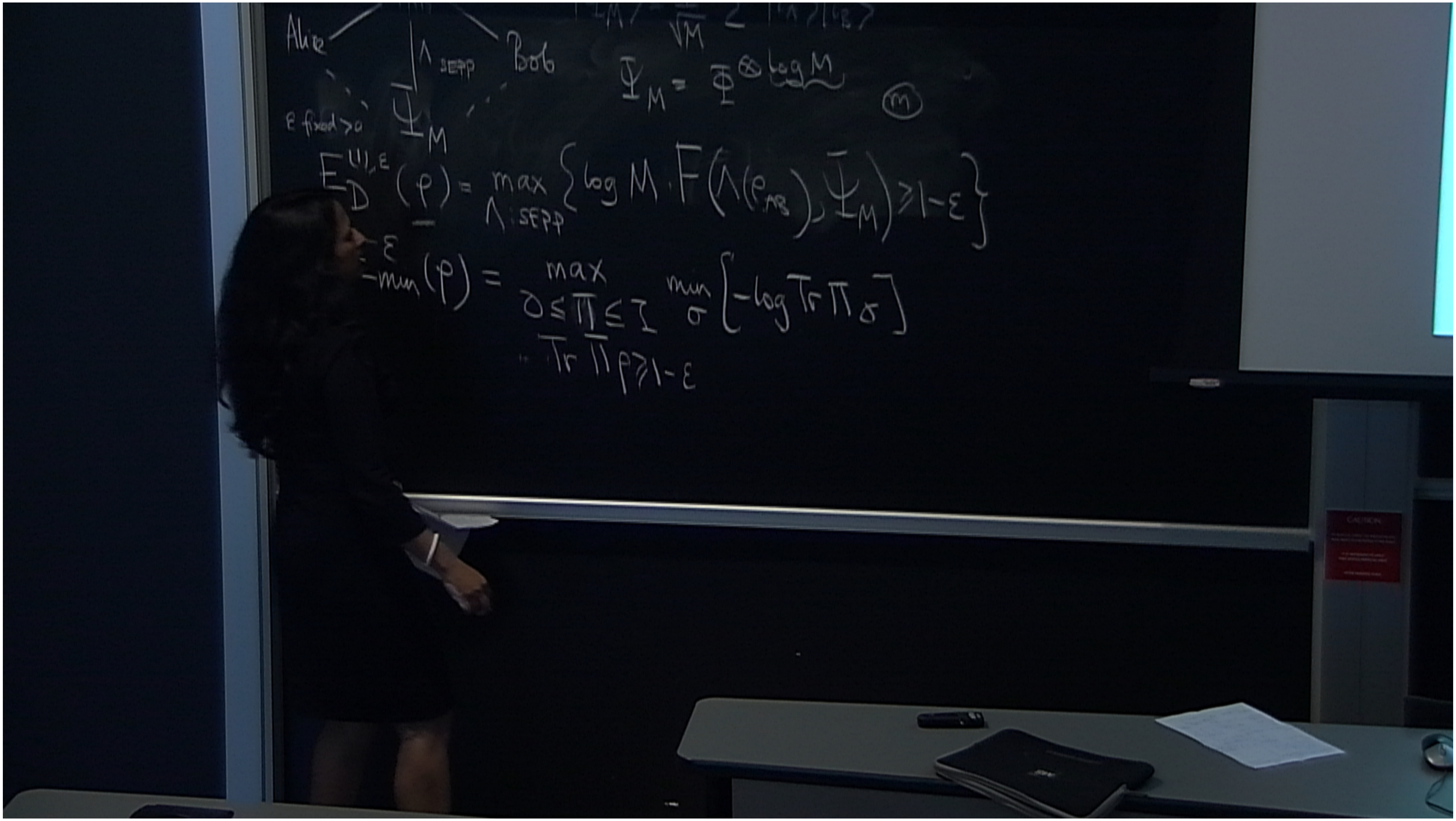
$$E_{\max}(\rho) \quad \text{and} \quad E_{\min}(\rho)$$

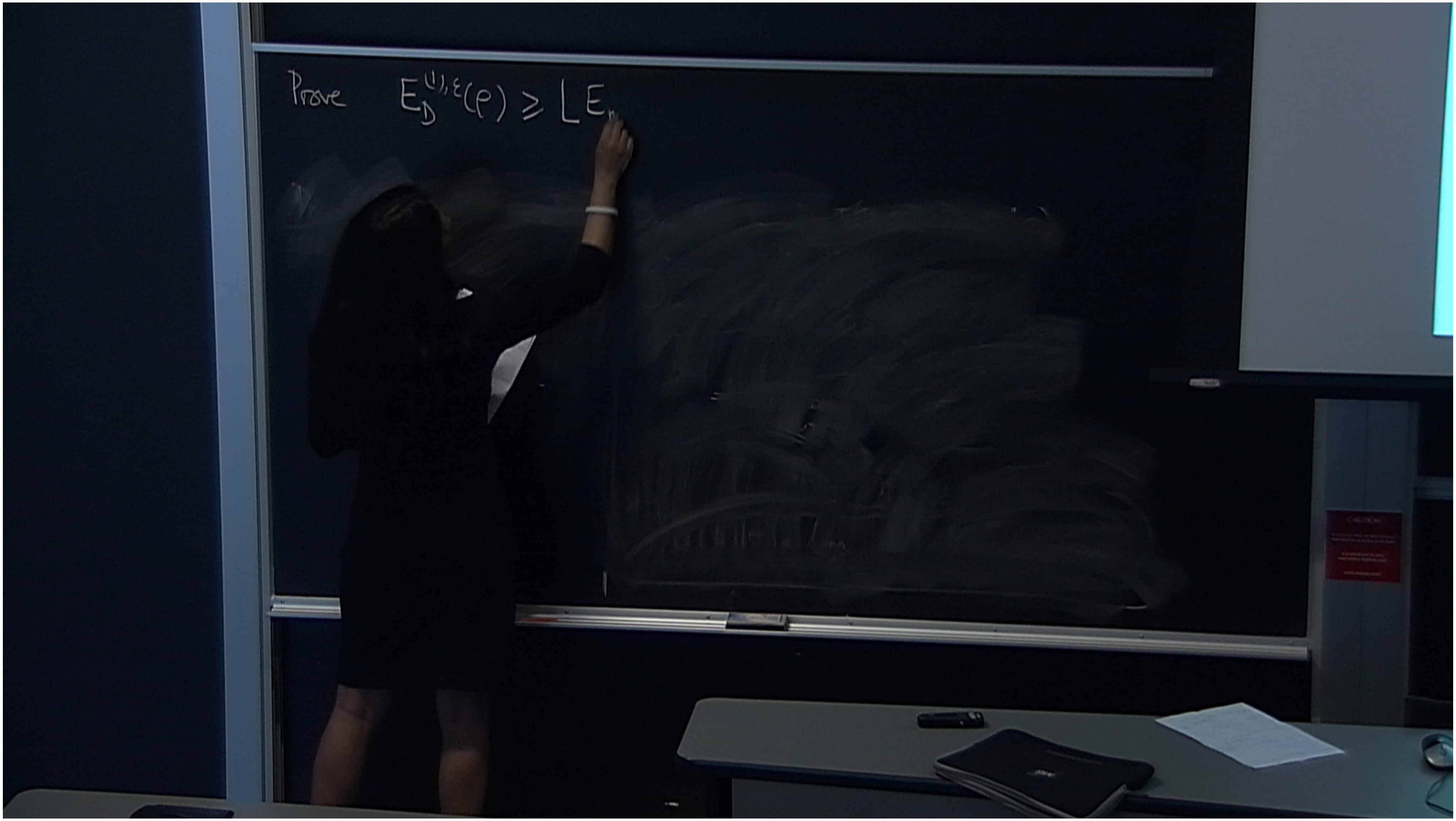
have interesting operational significances in entanglement manipulation

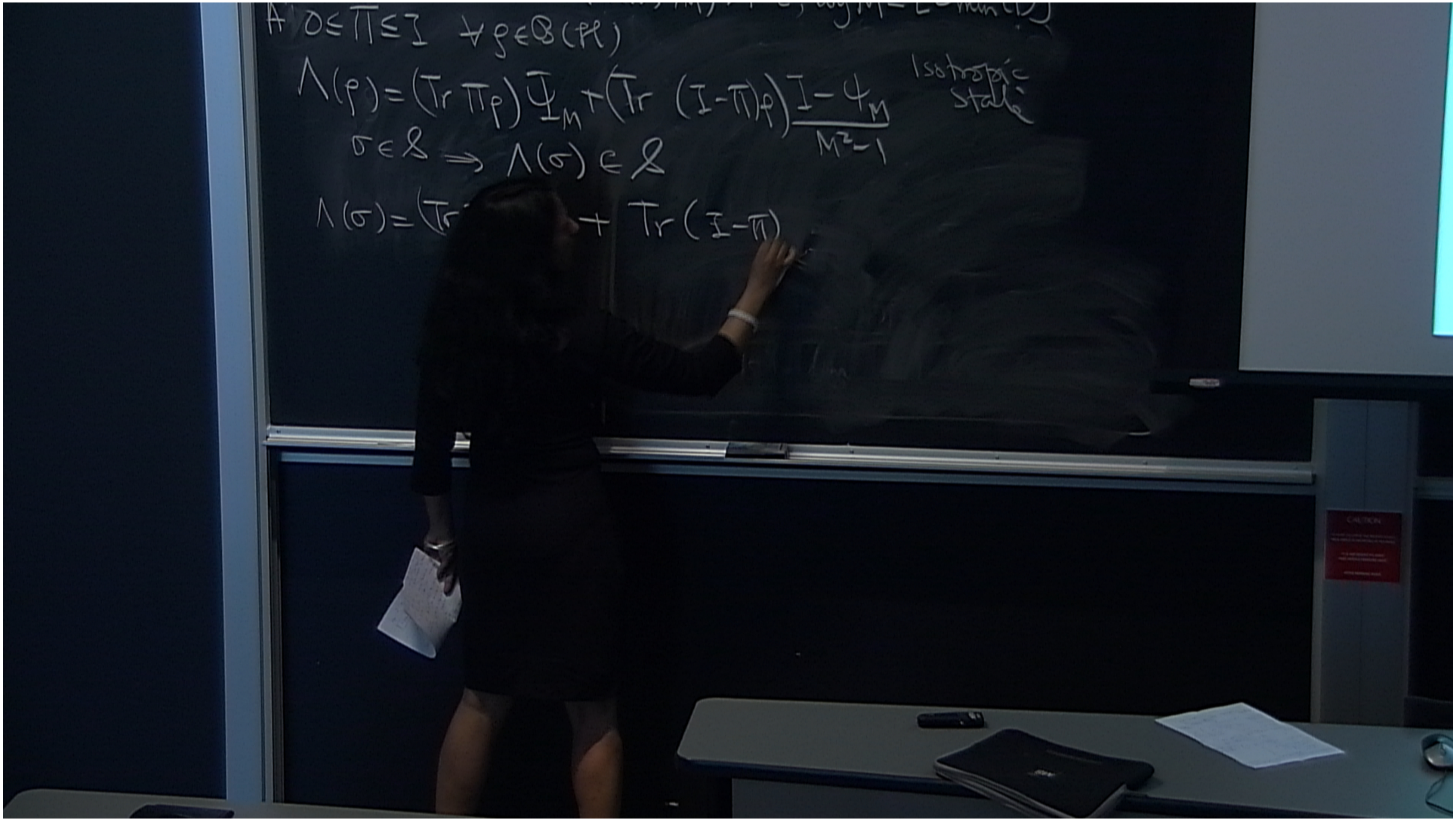
- *What is entanglement manipulation ?*

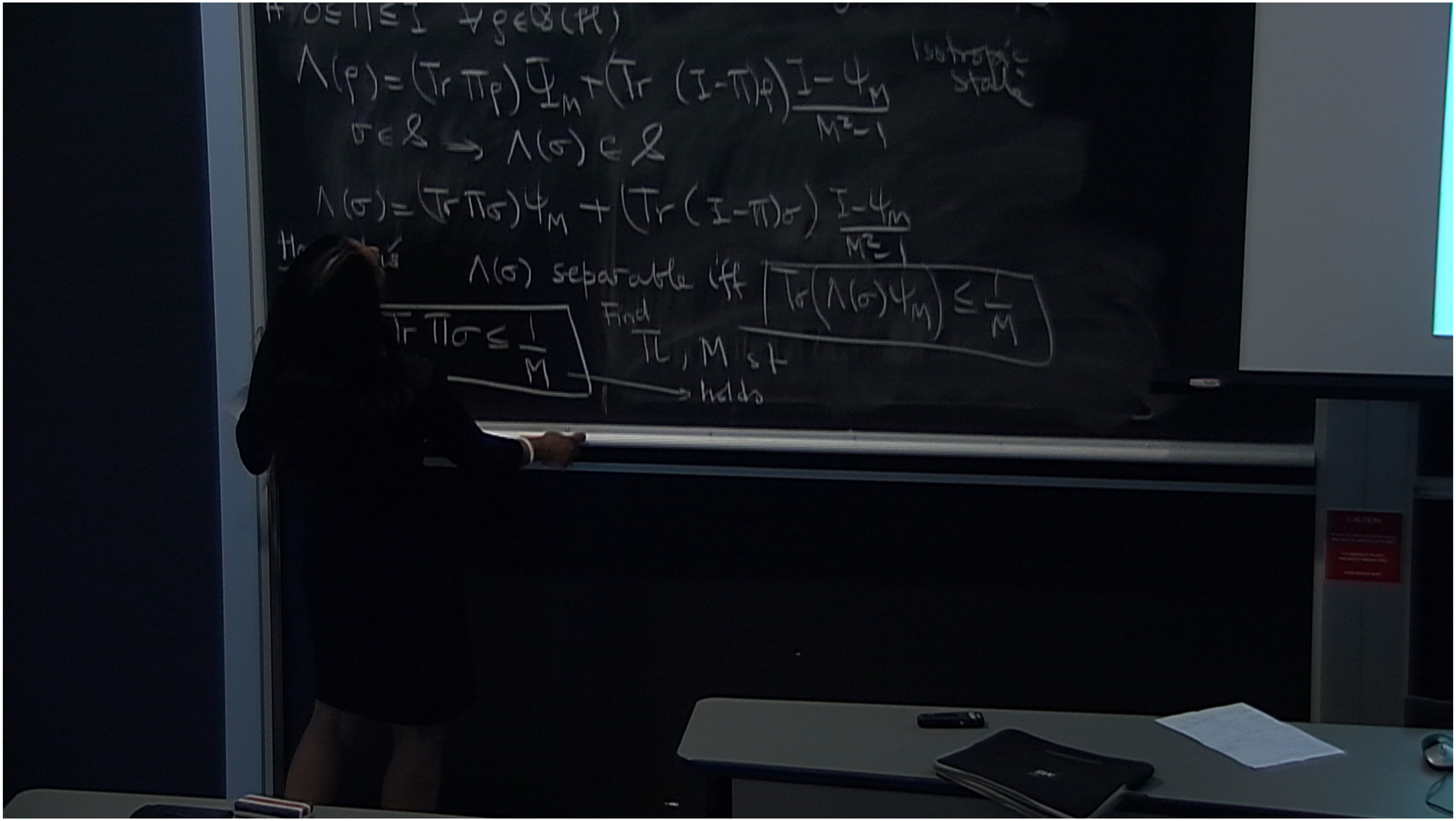
= Transformation of entanglement from one form to another

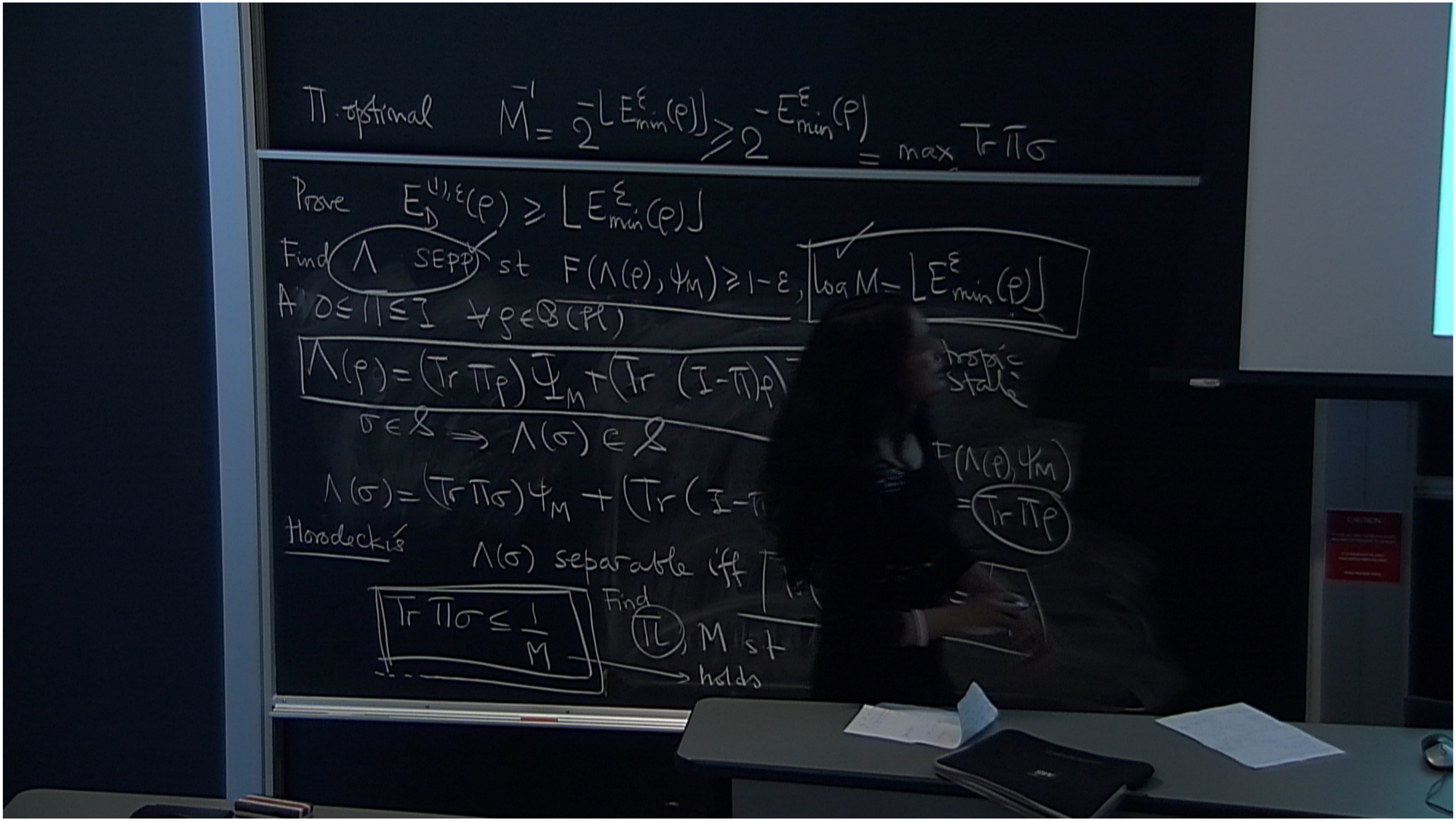

Alice
ρ_{AB}
Bob


Alice
Bob
ε fixed > 0
max
min


Prove $E_D^{(1),\varepsilon}(\rho) \geq LE$


0 ≤ Π ≤ I ∀ ρ ∈ S(H)
Λ(ρ) = (Tr Πρ) Φ_M + (Tr (I − Π)ρ) (I − Φ_M)/(M² − 1)
Isotropic state
σ ∈ S ⇒ Λ(σ) ∈ S
Λ(σ) = (Tr
+ Tr (I − Π)


isotropic state
Λ(σ) separable iff
Find
Π, M st
holds


Π optimal
Prove
Find
SEPP
st
Horodecki's
Λ(σ) separable iff
Find
M st
holds

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- Thanks to all my collaborators:

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