Title: Negativity , Contextuality, Magic and the Power of Quantum Computation

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Abstract:

Negative Quasi-Probability, Contextuality, Quantum Magic and the Power of Quantum Computation

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Joint work with: V. Veitch, M. Howard,
C. Ferrie, D. Gross, T. Bohdanowicz

Quantum Landscapes, Perimeter Institute, 2013

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Motivation

Big Picture

- There are a variety of *conceptual* features of quantum mechanics which are *presumed* to be "non-classical":
  - very old ideas: superposition, entanglement, collapse of the wavefunction, etc
  - less old ideas: non-locality, contextuality, negative quasi-probability, etc
- For some, these features are confounding to the point of denying their validity!
  - many-worlds theorists deny collapse
  - dynamical collapsicans deny macro superposition
  - ...and Bill Unruh denied non-locality just two days ago!

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- Meanwhile, with the advent of quantum information, there are clearly defined *operational* advantages to quantum theory:
    - exponential speed-up with quantum computation, secure quantum communication and improved success probability at CHSH games, etc
- What are the necessary and sufficient resources required for these *operational* advantages of quantum information?
    - It is already clear that *non-locality* is a key resource for quantum communication (given LOCC paradigm), but the resource requirements for quantum speed-up are not well understood (eg, MBQC, standard circuit model, DQC1 model)
    - Also a practical issue: identifying essential quantum resources could simplify the design requirements for prototype quantum computers

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Motivation

Here are the questions I will address today

- Which aspects of quantum theory are required for the power of quantum computation?
    - Can we identify whether any of the older *conceptual* notions of non-classicality correspond to the resources required for the *operational* advantages of quantum computation?
- Can finding the conditions for these operational advantages clarify which *conceptual* features are truly non-classical and which are not?
- Can this approach clarify whether quantum states are "physical states" or "states of incomplete knowledge" or both?

A broader hope:

- Help clarify conceptual role of quantum physics to guide the construction of a quantum theory of gravity

Outline of Results

I will describe a paradigm (which is *oddly* limited to systems of odd-prime dimensional qudits) for which:

- Negativity of the DWF occurs if and only if the corresponding quantum state exhibits contextuality (using the graph-theoretic formulation of Cabello, Severini and Winter)
- Negativity/Contextuality of a distinguished discrete Wigner function (DWF) is operationally relevant: it defines a necessary (and possibly sufficient) condition for universal quantum computation

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Outline of Results

For our quantum landscapes theme, there are some useful insights:

- The non-negative states, transformations and measurements of the DWF define a *classical probabilistic model* for a large and convex subtheory of quantum theory
- The non-negative subtheory is the "maximally classical" quantum subtheory that includes the stabilizer subtheory = stabilizer operations + bound magic states
- The classical probabilistic model for this quantum subtheory:
  - includes entanglement - and offers a geometric explanation of it as a generic consequence of interpreting quantum states as states of incomplete knowledge
  - includes macro superposition, the *collapse of the wavefunction*, quantum teleportation, the no cloning principle, and other "confounding" quantum features (as in Spekkens toy theory, the ERL subtheory, etc) . . . is there #$%@ on the road?

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Resources for Quantum Computation?

Some Candidates

- Entanglement? ... Provably necessary in circuit model, but absent in DQC1.
- Purity/Coherence/Superposition? ... Unclear.
- Discord? ... Ok, probably not discord.
- Negative Wigner function and contextuality? ... Yes!

Quantum Resources

Resources arise naturally under operational restrictions, e.g., fundamental or practical restrictions on the quantum formalism.

Quantum Resources from operational restrictions

Limitations of fault-tolerant stabilizer computation give a set of resource-constraints for quantum computation!

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Resources for Fault Tolerance

Eastin-Knill, 2009

A transversal (and hence fault-tolerant) encoded gate set can not be universal.

Fault Tolerance with Stabilizer Operations

- Stabilizer operations are a typical choice of for fault tolerant gates - they form a subgroup of the unitary group.
- Stabilizer operations are not universal - this set is efficiently simulatable by the Gottesman-Knill theorem.
- This defines a natural restriction on the set of quantum operations.
- Thus an additional resource is needed for universal quantum computation - consumption of resource states.

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Magic State Computing (Bravyi, Kitaev 2005)

Magic State Model

- Operational restriction: only stabilizer operations (states, gates and projective measurement) can be realized
- Additional resource: preparation of non-stabilizer "magic" state ρ_R

Magic State Distillation

- Convert several noisy magic states ρ_R to produce a few very pure magic states $\tilde{\rho}_R$
- Consume pure magic states $\tilde{\rho}_R$ to perform non-stabilizer unitary gates (using only fault tolerant stabilizer operations)

An Open Question

Are all non-stabilizer states a resource that promotes stabilizer computation to quantum computation?

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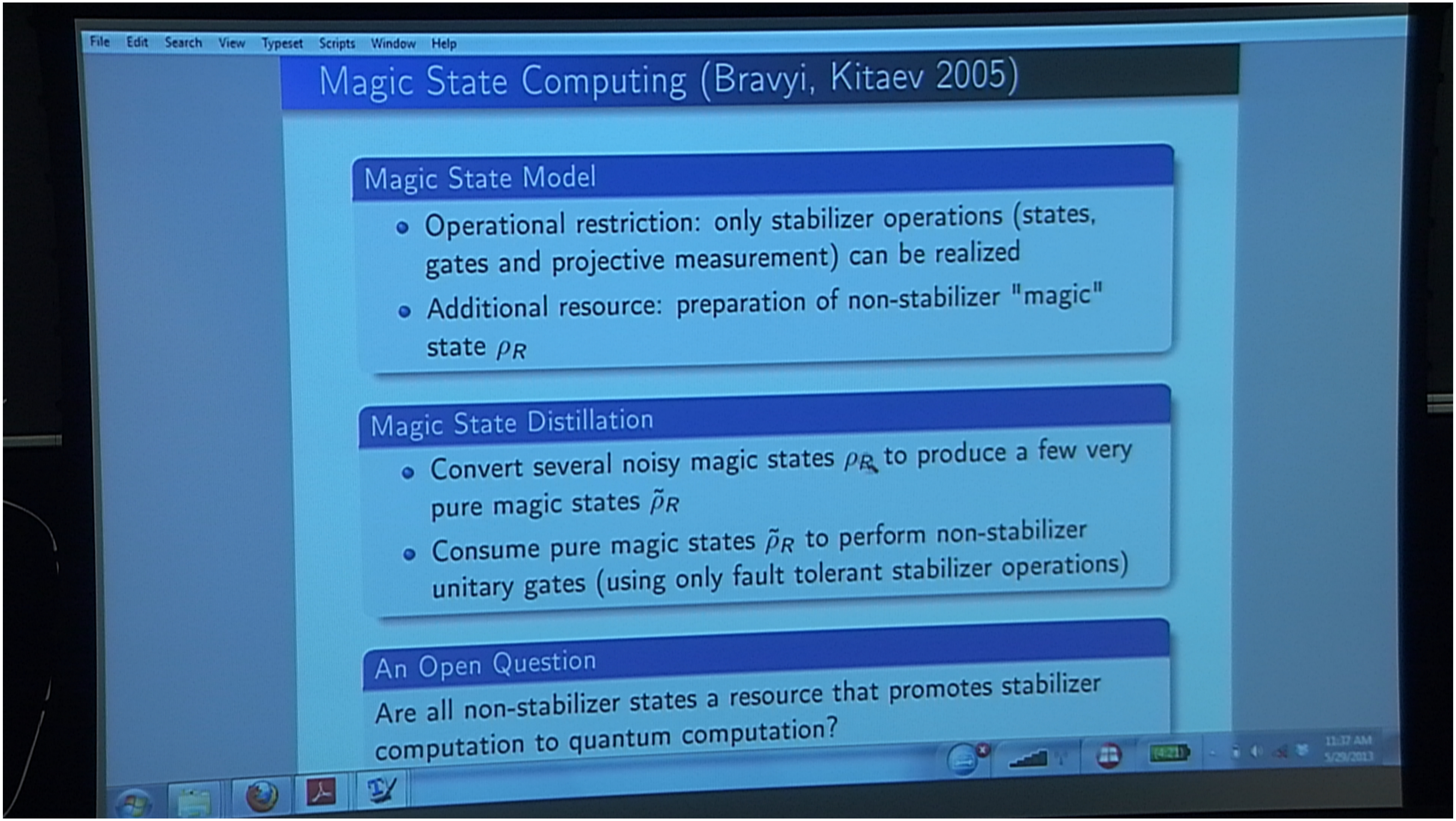
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Quasi-Probability Representations

Probably the most well-known quasi-probability representations for quantum theory is the Wigner function:

$$\mu_\rho^{\mathrm{Wigner}}(q,p) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} d\xi d\eta \ \mathrm{Tr}\left[\rho e^{i\xi(Q-q)+i\eta(P-p)}\right]$$

- This function on the classical phase space (eg, $\mathbb{R}^2$ for 1 particle in 1d) is called a quasi-probability representation because it is known that the function $\mu_\rho^{\mathrm{Wigner}}(q,p)$ takes on negative values for a broad class of quantum states.
- This function gives an *equivalent formulation* of quantum mechanics in the sense that one can reproduce all the quantum predictions using only these real-valued phase-space functions:

$$Pr(q \in \Delta) = \int_\Delta dq \int dp \mu_\rho^{\mathrm{Wigner}}(q,p)$$

Non-uniqueness

The Wigner function is highly non-unique!

- (i) The choice $\Lambda = \mathbb{R}^2$ is non-unique.
- (ii) The choice of map taking quantum states to real-valued functions is non-unique.
- (iii) The (often implicit) choice of map taking measurements to conditional probabilities is generally non-unique.

Indeed there are many, many examples of quasi-probability representation that have been defined in the last 70 years: the Husimi function, the P-representation, the Q-representation, and more recently a variey of representations for finite dimensional quantum systems, eg Wootters representation.

General Class of Quasi-probability Representations

We can define the *general class of quasi-probability representation* of QM as any pair of affine maps:

$$\mu_\rho : \rho \to \mu_\rho$$

$$\xi_k : E_k \to \xi_k$$

with $\mu_\rho : \Lambda \to \mathbb{R}$ and $\xi_k : \Lambda x \mathbb{K} \to \mathbb{R}$ (with $k \in \mathbb{K}$ an index labeling measurement outcomes), that satisfy the *law of total probability*

$$\Pr(k) = \int_\Lambda d\lambda \xi_k(\lambda)\mu_\rho(\lambda) = \mathrm{Tr}(E_k \rho)$$

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Frames and Quasi-probability representations

The freedom in choosing (i) an ontic space Λ and (ii) a pair of affine maps satisfying the law of total probability is equivalent to selecting a frame of operators and a dual frame.

- In finite-dimensional Hilbert space a frame of operators $\{F(\lambda)\}$ is just a spanning set, viz. an overcomplete basis, indexed by $\lambda \in \Lambda$.
- Taking $\{F(\lambda)\}$ to be a frame of Hermitian operators and $\{F^*(\lambda)\}$ a Hermitian frame dual to $\{F(\lambda)\}$ gives a quasi-probability representation where

$$\mu_\rho(\lambda) = \mathrm{Tr}(F(\lambda)\rho)$$

$$\xi_k(\lambda) = \mathrm{Tr}(F^*(\lambda)\rho)$$

- Note: For any operator A, a dual frame satisfies

$$A = \int d\lambda F^*(\lambda)\mathrm{Tr}(F(\lambda)A)$$

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No go theorem

No-Go Theorem for a Fully Non-Negative Quasi-Probability Representation:

- It is impossible to construct a quasi-probability representation for QM for which all states and all measurements are represented by non-negative functions, ie, for which all of quantum theory is represented as a classical probability theory.
- Can prove using the theory of frames: a frame of non-negative operators can not have a dual frame consisting of non-negative operators.

Refs: Ferrie and Emerson (J. Phys. A, 2008), Spekkens (PRL, 2008).

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Choice of Quasi-Probability Representation

- For different choice of quasi-probability representation, different sets of states and/or measurements will be non-negative (viz classical)
- Can align the choice of frame and dual frame to capture operationally important restrictions with subsets of non-negatively represented states and measurements
- This is the approach taken by David Gross (2006) to represent the Clifford subtheory non-negatively (for odd-dimensional qudits)
- The Clifford/stabilizer subtheory is central to quantum error correction and fault-tolerance; the stabilizer subtheory admits an efficient classical simulation scheme (Gottesman-Knill theorem) and therefore offers no quantum speed-up.

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Slice of the Quantum State Space and Stabilizer Polytope

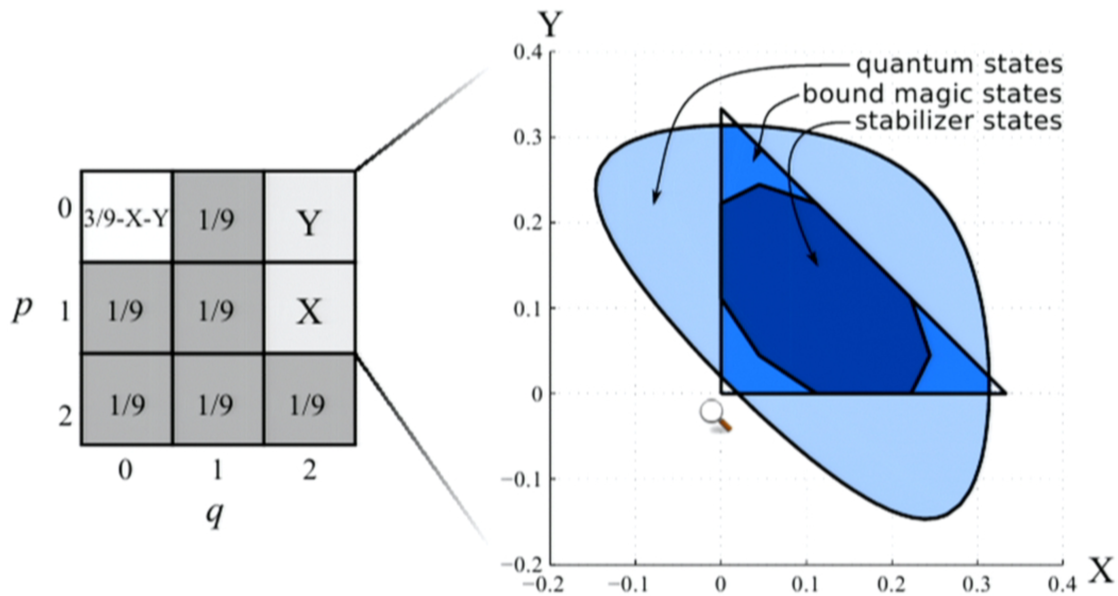

Figure: Slice defined by fixing six entries of the Wigner function and varying the remaining through their possible values to create the plot.

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Clifford/Stabilizer Subtheory

Let p be a prime number and define the boost and shift operators:

$$X\,|j\rangle = |\,j+1 \mod p\rangle$$
$$Z\,|j\rangle = \omega^{j}\,|j\rangle\,,\ \omega = \exp\left(\frac{2\pi i}{p}\right)$$

The generalized Pauli (Heisenberg-Weyl) operators in prime dimension:

$$T_{(a_1,a_2)} = \begin{cases} i^{a_1 a_2} Z^{a_1} X^{a_2} & (a_1, a_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \omega^{-\frac{a_1 a_2}{2}} Z^{a_1} X^{a_2} & (a_1, a_2) \in \mathbb{Z}_p \times \mathbb{Z}_p,\ p \neq 2 \end{cases}$$

where $\mathbb{Z}_p$ are the integers modulo p.
For Hilbert space $H_a \otimes H_b \otimes \cdots \otimes H_u$ we have:

$$T_{(a_1,a_2)\oplus(b_1,b_2)\cdots\oplus(u_1,u_2)} \equiv T_{(a_1,a_2)} \otimes T_{(b_1,b_2)} \cdots \otimes T_{(u_1,u_2)}.$$

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Discrete Wigner Representation for Odd Dimension

Consider a frame of *phase space point operators*

$$A_{\mathbf{0}} = \frac{1}{d}\sum_{\boldsymbol{u}} T_{\boldsymbol{u}},\; A_{\boldsymbol{u}} = T_{\boldsymbol{u}} A_{\mathbf{0}} T_{\boldsymbol{u}}^{\dagger}.$$

The Gross-Wootters discrete Wigner function (DWF) of a state $\rho \in L(\mathbb{C}^{p^n})$, with $d = p^n$ and p odd, is a quasi-probability distribution over $\Lambda = \mathbb{Z}_p^n \times \mathbb{Z}_p^n$, i.e., a set of $d \times d$ points, where

$$W_\rho(\boldsymbol{u}) = \frac{1}{d}\mathrm{Tr}(A_{\boldsymbol{u}}\rho),$$

The DWF for a quantum measurement operator E_k is then the conditional (quasi-)probability function over Λ,

$$W_{E_k}(\boldsymbol{u}) = \mathrm{Tr}(A_{\boldsymbol{u}} E_k).$$

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Discrete Wigner Representation for Odd Dimension

- The phase space point operators in dimension p^n are tensor products of n copies of the p dimensional system phase space point operators, eg. $A_{(0,0)\oplus(0,0)} = A_{(0,0)} \otimes A_{(0,0)}$.
- The phase space point operators $A_{\boldsymbol{u}}$ are Hermitian so the discrete Wigner representation is real-valued.
- There are d^2 such operators for d-dimensional Hilbert space, corresponding to the d^2 phase space points $\boldsymbol{u} \in \Lambda$.
- Of course, the Born rule is reproduced by the law of total probability

$$\Pr(k) = \sum_{\boldsymbol{u}} W_\rho(\boldsymbol{u}) W_{E_k}(\boldsymbol{u}) = \mathrm{Tr}(\rho E_k)$$

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Example of Discrete Wigner Representation for Qutrits

1/3	1/3	1/3
0	0	0
0	0	0

Figure: Wigner representation of qutrit $|0\rangle$ state

1/6	1/6	1/6
1/6	-1/3	1/6
1/6	1/6	1/6

Figure: Wigner representation of qutrit $|0\rangle - |1\rangle$ state

Discrete Wigner Representation for Odd Dimension

- A state ρ has positive representation if $W_\rho(\boldsymbol{u}) \geq 0 \; \forall \boldsymbol{u} \in \mathbb{Z}_p^n \times \mathbb{Z}_p^n$ and negative representation otherwise.
- When $W_\rho(\boldsymbol{u}) \geq 0 \; \forall \boldsymbol{u} \in \mathbb{Z}_p^n \times \mathbb{Z}_p^n$ this function can be interpreted as a probability distribution over Λ.
- A measurement with POVM $M = \{E_k\}$ has positive representation if $W_{E_k}(\boldsymbol{u}) \geq 0 \; \forall \boldsymbol{u} \in \mathbb{Z}_p^n \times \mathbb{Z}_p^n$, $\forall E_k \in M$ and negative representation otherwise.
- When $W_{E_k}(\boldsymbol{u}) \geq 0 \; \forall \boldsymbol{u}$ this function can be interpreted as the (conditional) probability of getting outcome k given that the system is actually at point $\boldsymbol{u}$,
  $W_{E_k}(\boldsymbol{u}) = \Pr(\text{outcome } k | \text{location } \boldsymbol{u})$.

Discrete Wigner Representation for Odd Dimension

1. Discrete Hudson's theorem (Gross, 2006): a pure state $|S\rangle$ has positive representation if and only if it is a stabilizer state. Hence for any state in STAB we know $\mathrm{Tr}(A_{\boldsymbol{u}}S) \geq 0 \ \forall \boldsymbol{u}$.
2. Clifford unitaries act as permutations of phase space. This means that if U is a Clifford then,

$$W_{U\rho U^{\dagger}}(\boldsymbol{v}) = W_{\rho}(\boldsymbol{v}'),$$

   for each point $\boldsymbol{v}$.
3. Hence Clifford operations preserve non-negativity.
4. Note: only a small subset of the possible permutations of phase space correspond to Clifford operations.

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Stabilizer Operations Preserve Positive Representation

Observation

Negative Wigner representation is a resource that can not be created by stabilizer operations.

Proof

Let $\rho \in L(\mathbb{C}_{d^n})$ be an n qudit quantum state with positive Wigner representation. Observe the following:

1. $U\rho U^\dagger$ is positively represented for any Clifford (stabilizer) unitary U.
2. $\rho \otimes S$ is positively represented for any stabilizer state S.
3. state-update, $M\rho M^\dagger / \mathrm{Tr}\left(M\rho M^\dagger\right)$, is positively represented for any stabilizer projector M.

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A question

Positive Representation ≡ Stabilizer State?

Do all non-stabilizer states have negative Wigner representation?

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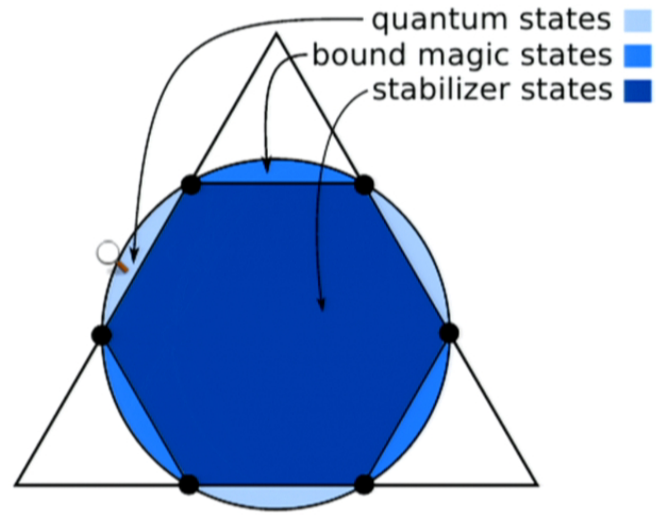
Stabilizer Polytope

Stabilizer Polytope

- Convex polytope with stabilizer states as vertices
- Can be defined from set of "facets"

Wigner Facets

The Wigner simplex has d^2 facets = small subset of stabilizer polytope facets



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Slice of the Quantum State Space and Stabilizer Polytope

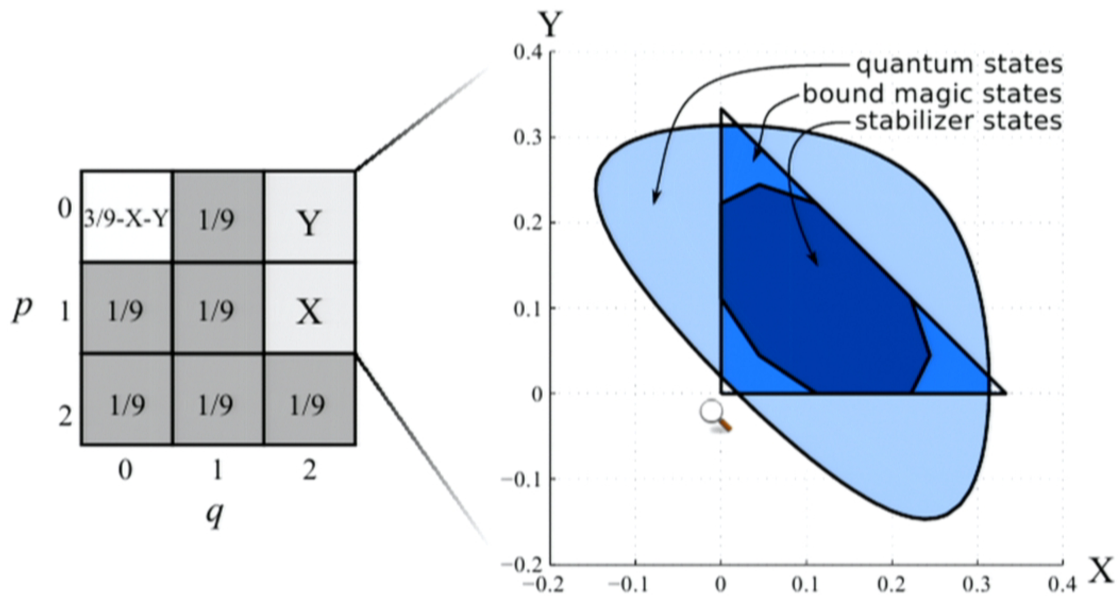

Figure: Slice defined by fixing six entries of the Wigner function and varying the remaining through their possible values to create the plot.

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Magic States and Negative Quasi-Probability

Distillable Magic States for Odd Dimensional Qudits

- There is a large class of non-stabilizer quantum states (*bound magic states*) that are not useful for magic state distillation.
- Hence negative quasi-probability is necessary condition for a state to be distillable
- Is the boundary for negativity also a boundary for contextuality?

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State-dependent contextuality

Consider the graph-based contextuality formalism introduced by Cabello, Severini and Winter

- Consider a set of binary yes-no tests, which we quantum mechanically represent by a set of rank-one projectors, Π, with eigenvalues $\lambda(\Pi) \in \{1, 0\}$.
- Compatible tests are those whose representative projectors commute, and a context is a set of mutually compatible tests.
- Commuting rank-1 projectors cannot both take on the value $+1$ i.e., the respective propositions are mutually exclusive and cannot both be answered in the affirmative.
- These (mutual orthogonality) relations can be represented by a graph Γ where connected vertices correspond to compatible and exclusive tests.

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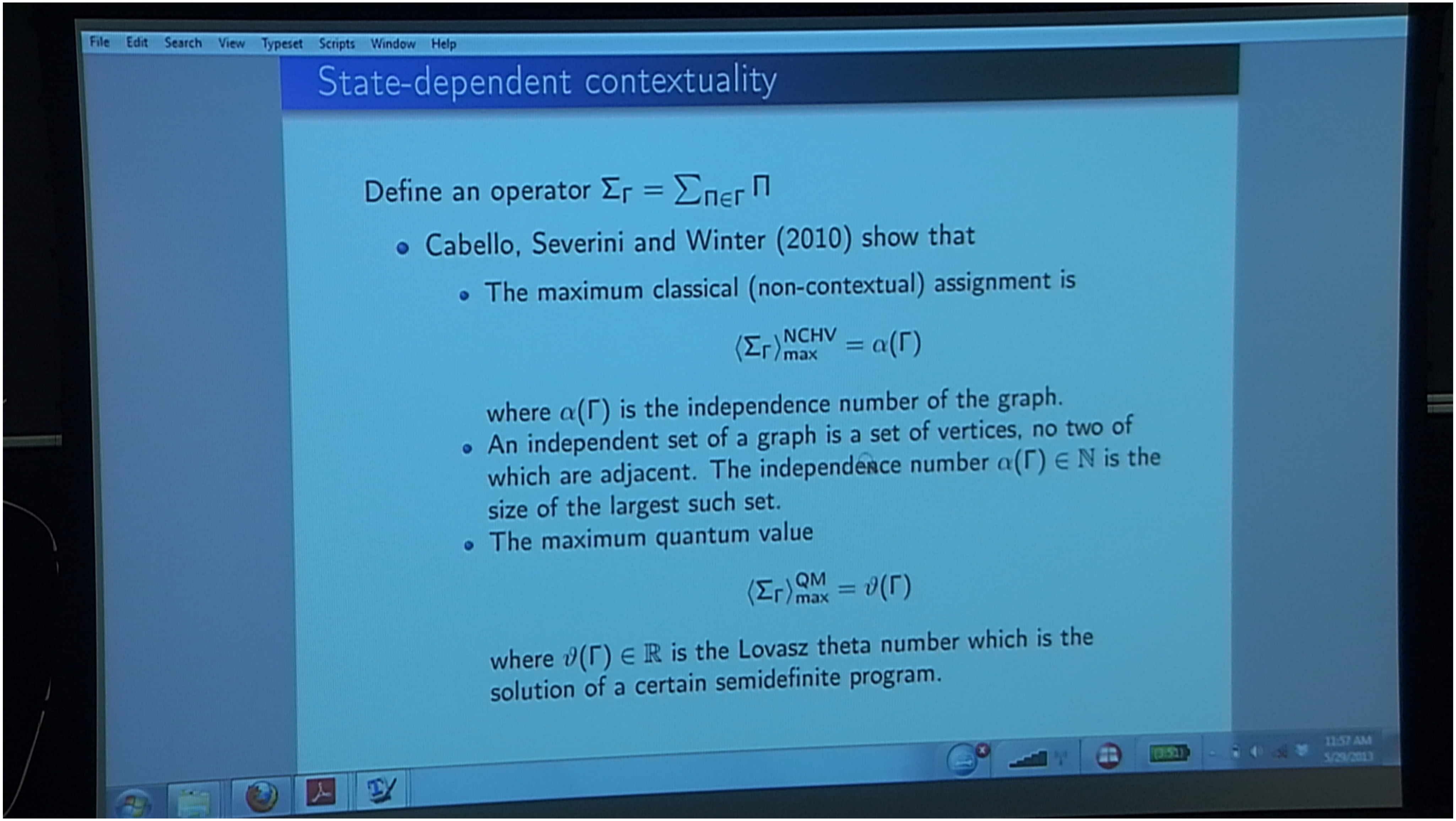
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State-dependent contextuality
Define an operator $\Sigma_\Gamma = \sum_{\Pi \in \Gamma} \Pi$
Cabello, Severini and Winter (2010) show that
The maximum classical (non-contextual) assignment is
$\langle \Sigma_\Gamma \rangle_{\max}^{\mathrm{NCHV}} = \alpha(\Gamma)$
where $\alpha(\Gamma)$ is the independence number of the graph.
An independent set of a graph is a set of vertices, no two of which are adjacent. The independence number $\alpha(\Gamma) \in \mathbb{N}$ is the size of the largest such set.
The maximum quantum value
$\langle \Sigma_\Gamma \rangle_{\max}^{\mathrm{QM}} = \vartheta(\Gamma)$
where $\vartheta(\Gamma) \in \mathbb{R}$ is the Lovasz theta number which is the solution of a certain semidefinite program.

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Graph of Stabilizer Projectors

We construct a set of stabilizer projectors for a system of two p-dimensional qudits such that:

$$\Sigma_{\text{tot}} = \Sigma_{\text{sep}} + \Sigma_{\text{ent}} = p^3 \mathbb{I}_{p^2} - \left(A_{(0,0)} \otimes \mathbb{I}_p\right)$$

- Then for any state $\sigma \in \mathcal{H}_p$ we have

$$\text{Tr}\left[\Sigma_{\text{tot}}\left(\rho \otimes \sigma\right)\right] > p^3 \iff \text{Tr}\left[A_{(0,0)}\rho\right] < 0.$$

- First via numerical search for $p = 3$ and $p = 5$ and then via general proof we show that

$$\alpha(\Gamma_{\text{tot}}) = p^3 \Rightarrow \langle \Sigma_{\text{tot}} \rangle^{\text{NCHV}}_{\text{max}} = p^3$$

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Graph of Stabilizer Projectors

Choosing $|\nu\rangle = \frac{|1\rangle - |p-1\rangle}{\sqrt{2}}$ we get

$$\mathrm{Tr}\left[A_{(0,0)}|\nu\rangle\langle\nu|\right] = -1,$$

and hence

$$\mathrm{Tr}\left[\Sigma_{\mathrm{tot}}|\psi^{\Gamma}_{\mathrm{max}}\rangle\langle\psi^{\Gamma}_{\mathrm{max}}|\right] = \langle\Sigma_{\mathrm{tot}}\rangle^{\mathrm{QM}}_{\mathrm{max}} = p^3 + 1,$$

for a maximal violation.

From the above it follows that:

- (i) a state is non-contextual if and only if it is positively represented in the discrete Wigner function,
- (ii) maximally negative states exhibit the maximum possible amount of contextuality

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Entanglement from Epistemic Restriction

Entanglement without non-locality:

- The two qutrit Bell state

$$|B\rangle = \frac{|00\rangle + |11\rangle + |22\rangle}{\sqrt{3}}$$

  is an entangled stabilizer state
- Its density operator does *not* admit a convex decomposition into factored qutrit states
- But under stabilizer measurements it can not exhibit any form of contextuality
- Morever, its discrete Wigner function *must* admit the decomposition

$$W_{|B\rangle\langle B|} = \Sigma_l p_l W_l^A \otimes W_l^B$$

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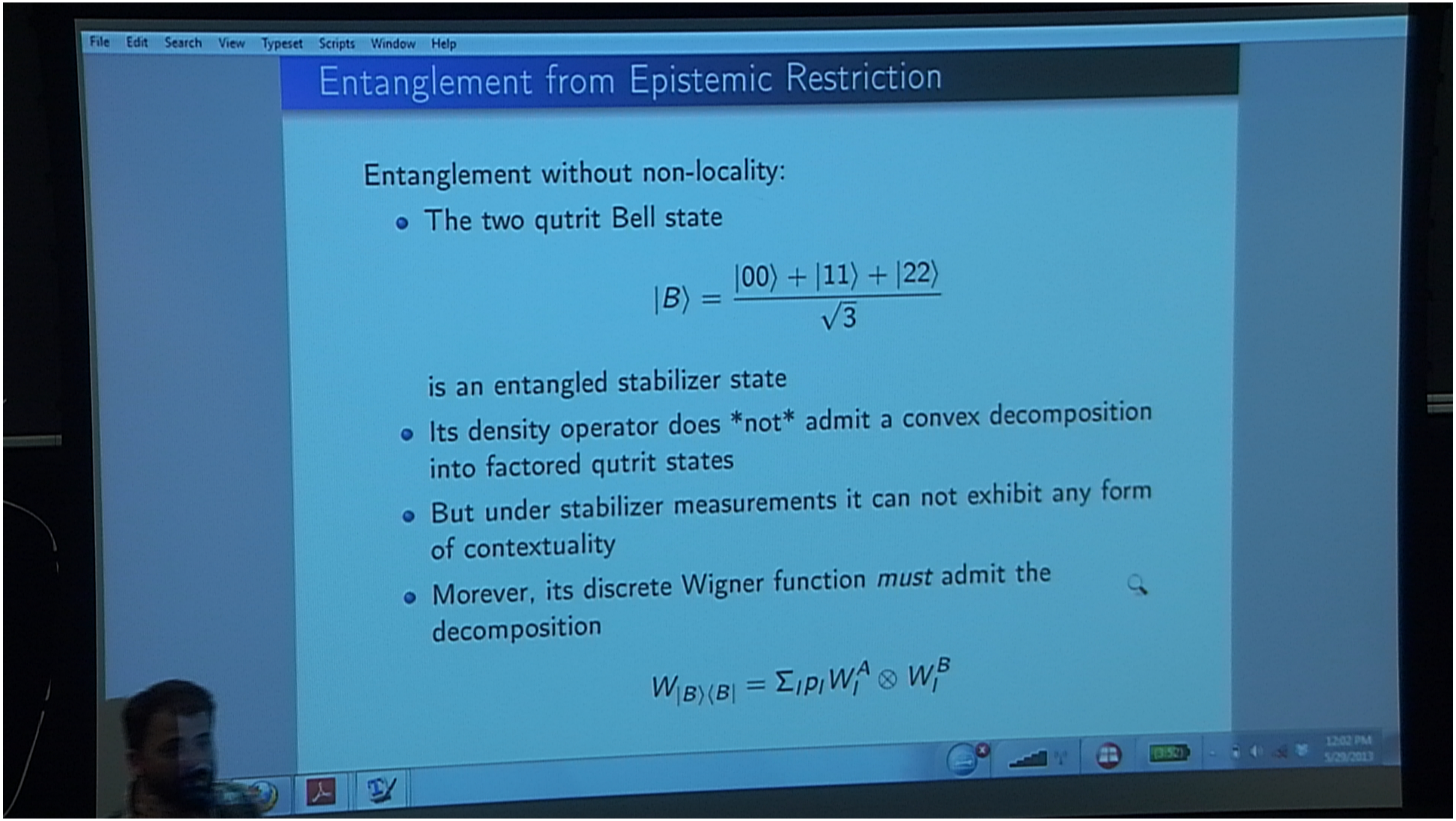
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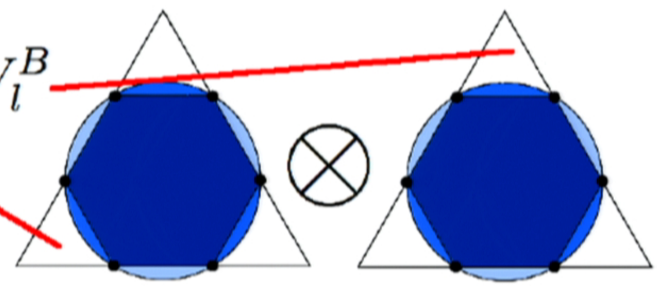
Entanglement from Epistemic Restriction

- Note that W_l^A and W_l^B come from *forbidden regions* of the single-qutrit Wigner probability simplex – that is, W_l^A and W_l^B are not valid single qutrit quantum states

$$\sum_l p_l W_l^A \otimes W_l^B$$



- Entanglement arises naturally from the epistemic restriction, i.e. from incompleteness of quantum states!

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Extended Gottesman-Knill Theorem

Scope

- Prepare ρ with positive representation
- Act on input with Clifford U_F (corresponding to linear size F)
- Perform measurement $\{E_k\}$ with positive representation

Simulation Protocol

- Sample phase space point (u, v) according to distribution $W_\rho(u, v)$
- Evolve phase space point according to $(u, v) \rightarrow \boldsymbol{F}^{-1}(u, v)$
- Sample from measurement outcome according to $\tilde{W}_{\{E_k\}}(u, v)$

See also *Positive Wigner functions render classical simulation of quantum computation efficient*, A. Mari and J. Eisert

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Linear Optics

Results

- There exist mixed states with positive Wigner representation that are not convex combinations of gaussian states (Brocker and Werner, 1995)
- Computations using linear optical transformations and measurements as well as preparations with positive Wigner function can be efficiently classically simulated.[a]

[a] Veitch, Wiebe, Ferrie and Emerson, NJP 2013)

Odd Dimension	*Infinite Dimension*
Stabilizer Operations	Linear Optics
Stabilizer States	Gaussian States
Discrete Wigner Function	Wigner Function

Table: Comparison of Odd and Infinite Dimensional Formalisms

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Summary and Open Questions

Summary

- Negative Wigner function is a resource for FT stabilizer computation
- Bound states for magic state distillation
- Extension of Gottesman-Knill
- A state has negative quasi-probability if and only if it violates a contextuality inequality

Main Refs:

Veitch et al, NJP 14, 113011

Howard et al, forthcoming

Key Conceptual Point

Large convex subtheory with superposition and collapse: dynamical collapse, pilot-waves and many worlds aren't needed, but incompleteness is!

Future Work

- Resource theory for stabilizer formalism? Already done.
- How to extend to other operational restrictions?
- Is contextuality sufficient for distillability?