Title: Initial-boundary value problems for Einstein's field equations

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Abstract: We discuss well-posed initial-boundary value formulations
in general relativity. These formulations allow us to construct solutions of
Einstein's field equations inside a cylindrical region, given suitable initial
and boundary data. We analyze the restrictions on the boundary data that result
from the requirement of constraint propagation and the minimization of spurious
reflections, and choosing harmonic coordinates we show how to cast the problem
into well-posed form. Then, we consider the particular case where the boundary
represents null infinity of an asymptotically flat spacetime. Here, the rÃ´le of
the boundary conditions is to provide adequate regularity and gauge conditions
at infinity.

As an application of our setup we mention ongoing work on
the computation of quasi-stationary scalar field configurations on a
non-rotating supermassive black hole background.

Initial-boundary value problems for Einstein's field equations

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May 23, 2013

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Outline

- Problem and motivation
- Towards absorbing boundary conditions
- The harmonic formulation
- Wave systems
- Conclusions

Living Reviews in Relativity **15**: 9, 2012
OS and Manuel Tiglio

The problem

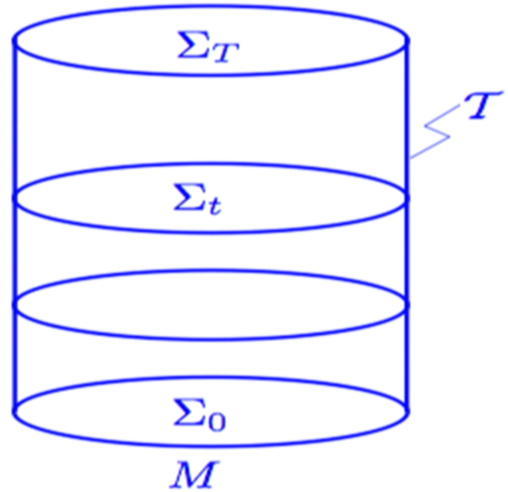
Solve Einstein's field equations

$$G_{\mu\nu} = 0$$

on a spacetime manifold of the form $M = [0, T] \times \Sigma$ (Σ: a 3-dimensional manifold with boundary $\partial\Sigma$) with given

- initial data on $\Sigma_0 := \{0\} \times \Sigma$
- boundary data on $\mathcal{T} := [0, T] \times \partial\Sigma$

such that $\mathcal{T}$ is time-like and each Σ_t is space-like with respect to (M, g).



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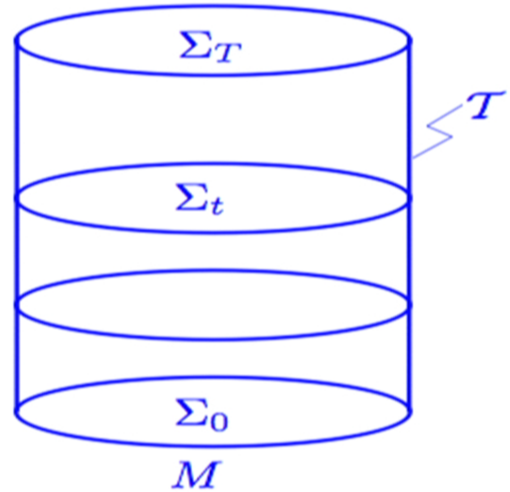

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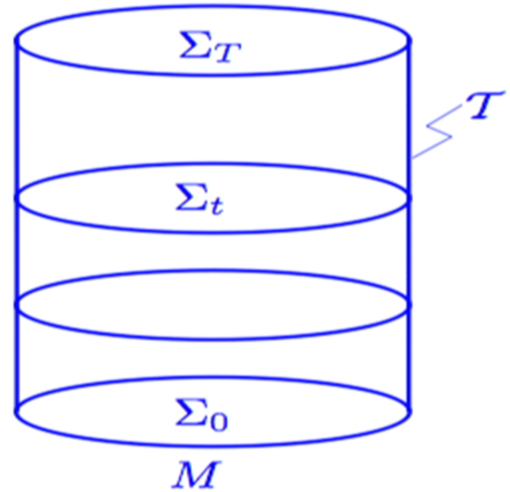


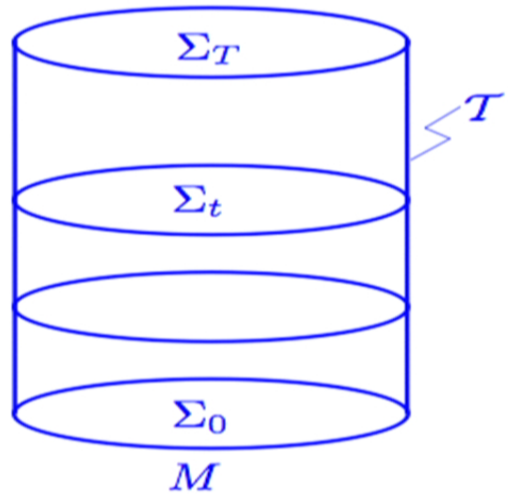
Initial data consist of the first and second fundamental form of Σ_0, and they must satisfy the constraints.
Contrary to the initial conditions, the boundary conditions are not always unique, for they determine how much radiation escapes M.

The problem

We require the problem to be *well-posed*:

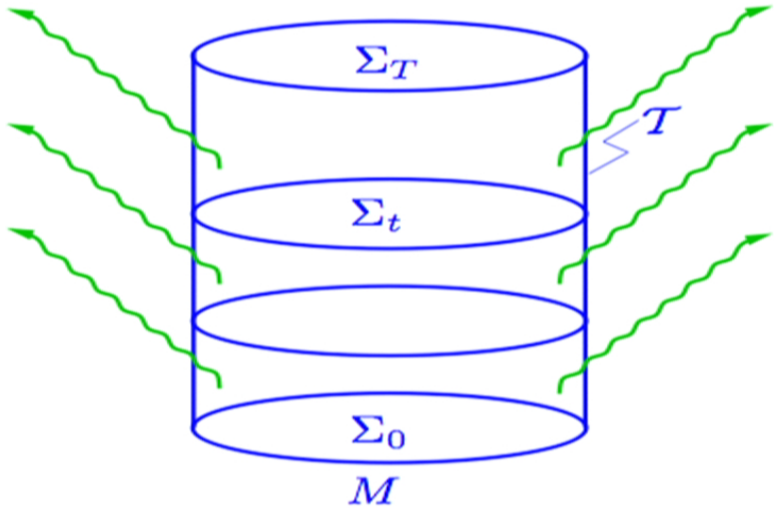
(i) Solution exists and is unique, up to diffeomorphisms which leave M, Σ_0 and $\mathcal{T}$ invariant.

(ii) Stability of the solution with respect to small fluctuations of the initial and boundary data.



Motivation

- Numerical modeling of isolated systems, like compact binaries. $\mathcal{T}$: absorbing boundary (should be transparent, i.e. minimize spurious reflections of gravitational waves)



Friedrich-Nagy '99, Kreiss-Winicour '06, Kreiss-Reula-S-Winicour '09

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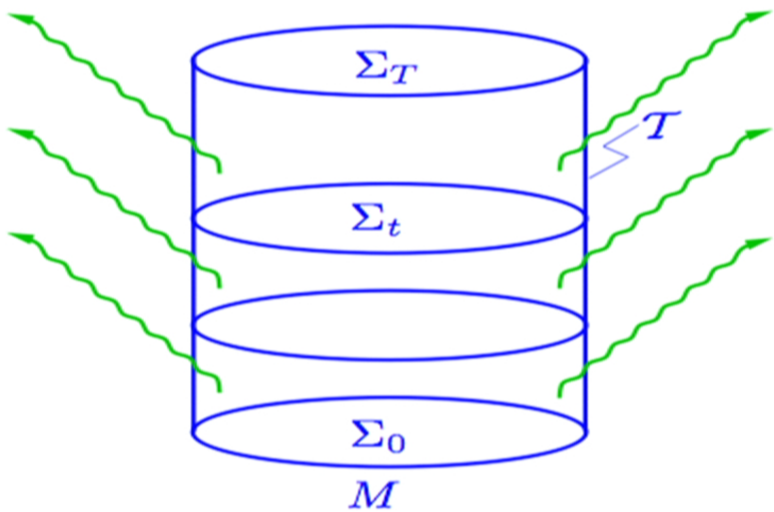

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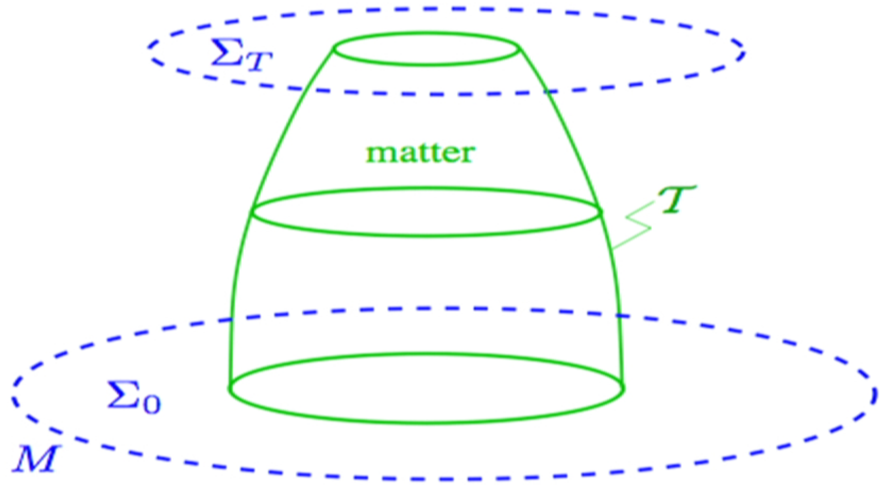
Motivation

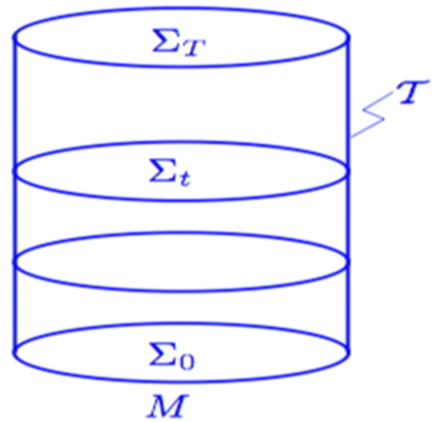
- $\mathcal{T}$: worldtube formed by the surface of a star



Free boundary value problem, difficult!

Motivation

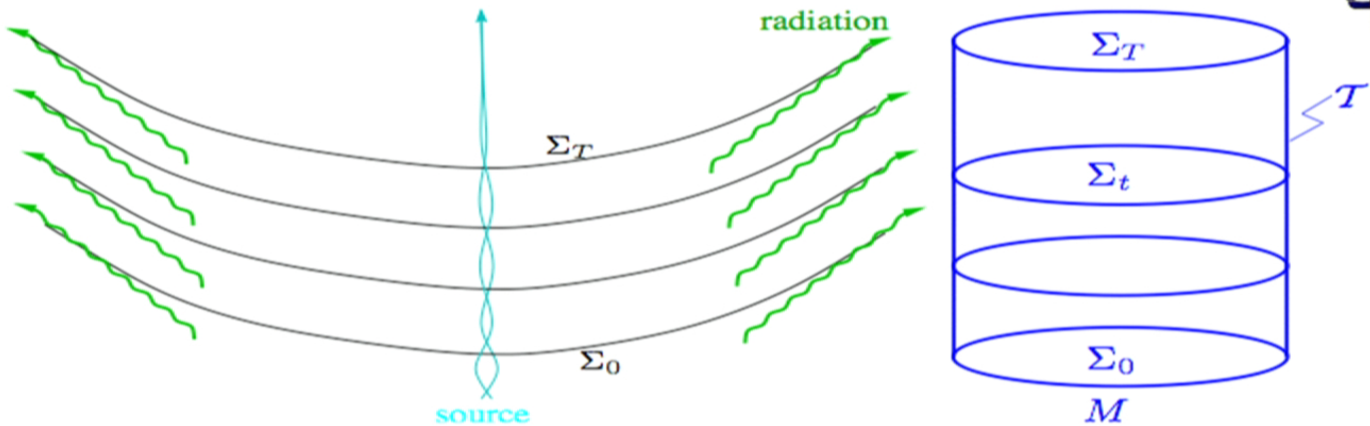
- $\mathcal{T}$: conformal boundary of Anti-de-Sitter spacetime



Boundary data: A smooth Lorentzian conformal structure on $\mathcal{T}$, satisfying appropriate compatibility (corner) conditions at $\partial\Sigma_0$.

Friedrich '95: For small enough $T > 0$ one has existence and uniqueness of the solutions (with negative cosmological constant).

Pushing the boundaries out

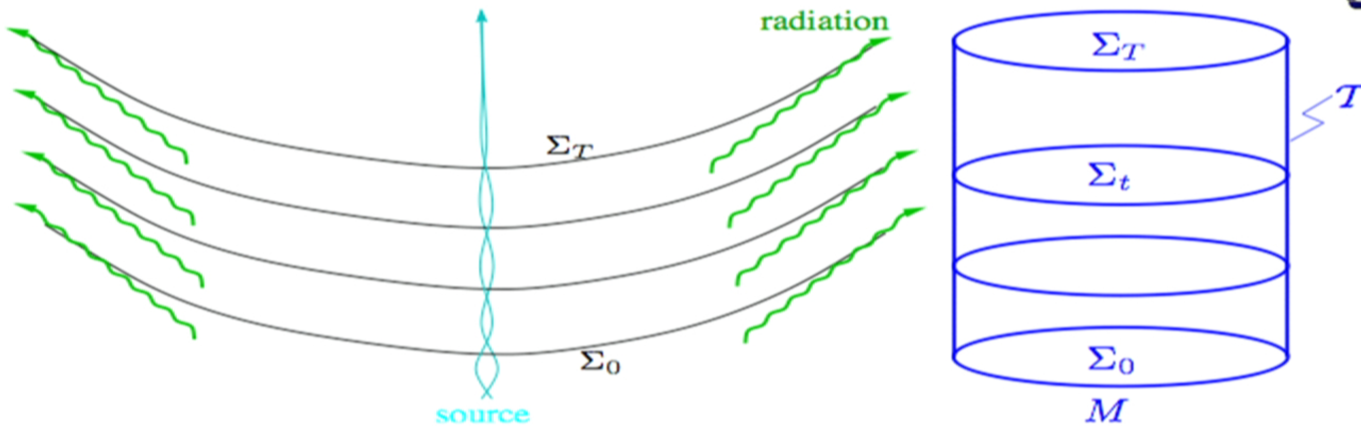


Use hyperboloidal time slices and compactify null infinity ($\mathcal{T}$=scri).
Advantages:

- No artificial boundaries and conditions ($\mathcal{T}$ is outflow)
- Unambiguous way of defining the gravitational wave amplitude

However, the compactification introduces singular terms in the equations, so one needs appropriate regularity conditions.

Pushing the boundaries out



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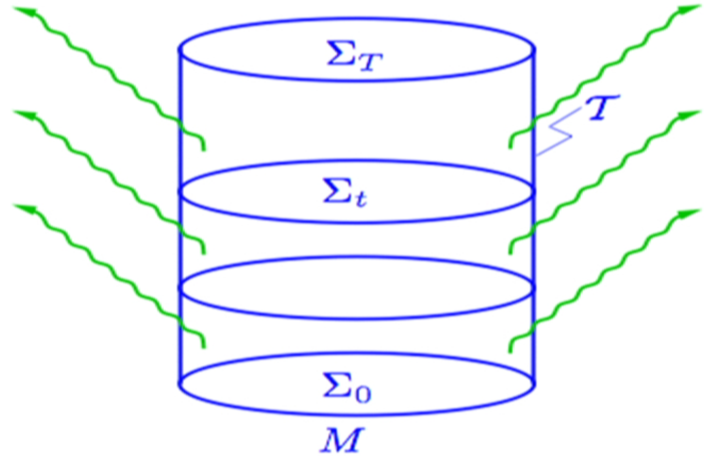
Pushing the boundaries out

Existing approaches for the full nonlinear Einstein equations:

- Friedrich '83 (conformal field equations)
- Zenginoğlu '08 (metric formulation)
- Moncrief, Rinne '09
  (CMC slices, hyperbolic-elliptic system based on metric fields)
- Bardeen, Buchman, S '11
  (CMC slices, hyperbolic-elliptic system based on triad fields)

Absorbing boundaries

Consider an asymptotically flat spacetime of total (ADM) mass m which is truncated at $\mathcal{T}$, and try to specify boundary conditions which make the artificial boundary $\mathcal{T}$ as transparent as possible. (L. Buchman, OS, '06,'07)



Assumptions:

(i) The outer boundary lies in the weak field region, $R \gg m$.

(ii) $\partial\Sigma_t$ are approximate metric 2-spheres with constant area $4\pi R^2$.

These assumptions imply that we may linearize the field equations about an exterior Schwarzschild spacetime of mass m near the outer boundary. The boundary corresponds to the surface of areal radius equal to R.

Absorbing boundaries

Linearized gravitational waves on a Schwarzschild background are described by the Regge-Wheeler-Zerilli equations:

$$\Box_{\tilde{g}}\phi + V_{even,odd}(r)\phi = 0, \qquad V_{odd}(r) = \frac{\ell(\ell+1)}{r^2} - \frac{6m}{r^3}.$$

where here ℓ is the angular momentum number, r the areal radius and $\tilde{g} = -dt^2 + dr^2 + \frac{2m}{r}(dt - dr)^2$ (outgoing Eddington-Finkelstein coords.), and ϕ is a gauge-invariant linear combination of the metric perturbations.

Solutions as series expansions in m/R. (Bardeen-Press '73)
For example, the odd-parity quadrupolar outgoing solution is

$$\phi = U''(r-t) - \frac{3}{r}\left(1 - \frac{2m}{r}\right)U'(r-t) + \frac{3}{r^2}U(r-t) + \frac{m}{2}\int_{r-t}^{\infty} K_2(t,r,s)U(s)ds$$

plus quadratic corrections in m/R. The integral describes the *scattering* off the curvature (backscatter).

Absorbing boundaries

Zeroth order in m/R:

- Free outgoing wave solutions:

$$\phi_{\nearrow\ell}(t,r) = a_\ell^\dagger a_{\ell-1}^\dagger \cdots a_1^\dagger U(r-t), \qquad a_\ell^\dagger := -\partial_r + \ell/r.$$

- They satisfy

$$(b_-)^{\ell+1}\phi_{\nearrow\ell}(t,r) = 0, \qquad b_- := r^2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r}\right).$$

- Therefore, given fixed $L = 1, 2, 3, \ldots$, the boundary condition

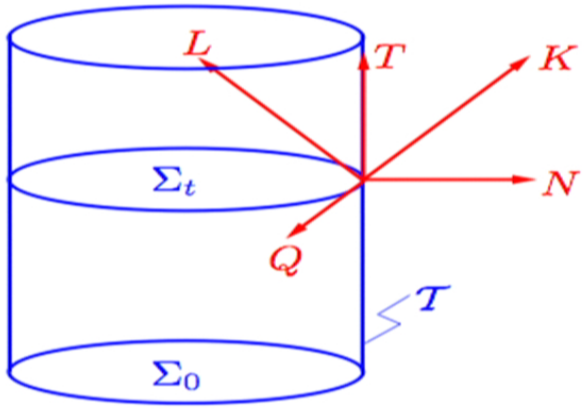
$$\mathcal{B}_L : \quad (b_-)^{L+1}\phi(t,R) = 0, \qquad b_- = r^2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r}\right)$$

is perfectly absorbing to zeroth order in m/R for all linearized gravitational waves with $\ell \leq L$. (Bayliss-Turkel '80)

Absorbing boundaries

More geometrically written:

- T: future-directed unit vector on $\mathcal{T} \perp \partial\Sigma_t$.
- N: outward unit normal to $\mathcal{T}$
- $K = T + N$: outgoing null vector
- $L = T - N$: ingoing null vector
- Q: complex unit null vector $\perp$ to T, N.
- Weyl scalar $\Psi_0 := C_{\alpha\beta\gamma\delta} K^\alpha Q^\beta K^\gamma Q^\delta$.



In the linearized case, the boundary conditions $\mathcal{B}_L$ are equivalent to

$$\mathcal{B}_L: \quad (b_-)^{L-1}(r^5\Psi_0)\big|_{\mathcal{T}} = 0, \qquad b_- := r^2 K^\mu \frac{\partial}{\partial x^\mu}$$

In this geometric form, the boundary conditions can also be applied to the nonlinear problem! (note $\Psi_0 \sim 1/r^5$ according to the Peeling theorem)

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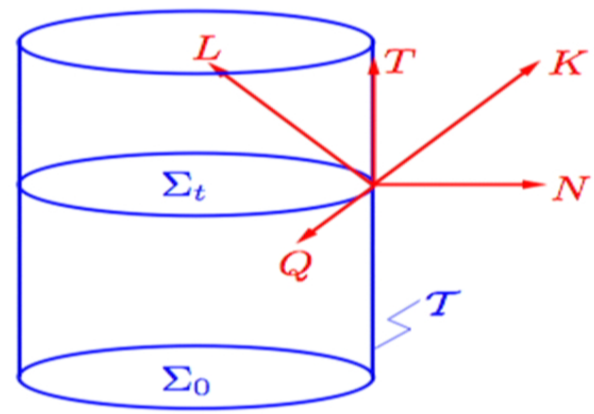

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Absorbing boundaries

In particular, for $L = 1$, the condition $\mathcal{B}_1$ implies

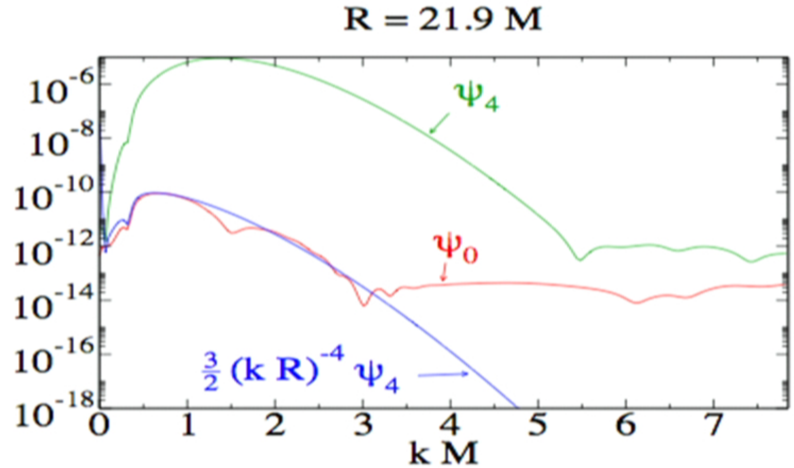
$$\partial_t \Psi_0|_{\mathcal{T}} = 0,$$

which is being used in numerical calculations of compact binaries.

The reflection coefficient for monochromatic quadrupolar waves with wave number k can be estimated as

$$\gamma(kR) \simeq \frac{3}{2}(kR)^{-4}$$

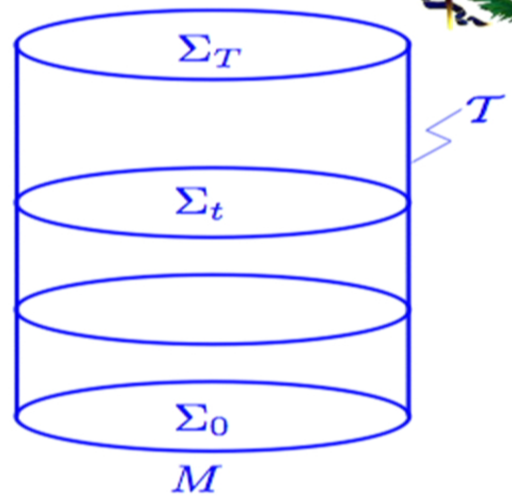
in the wave zone $kR \gg 1$.



Rinne, Lindblom, Scheel
Class. Quantum Grav. **24** 4053 (2007).

The harmonic formulation

- Choose a fixed background metric $\mathring{g}$ on M, such that $\mathcal{T}$ time-like and Σ_t space-like.
- Define $h_{ab} := g_{ab} - \mathring{g}_{ab}$, the difference between the dynamical and the background metric.
- Define $C^d{}_{ab} := \Gamma^d{}_{ab} - \mathring{\Gamma}^d{}_{ab}$ (a tensor field of type (1,2) on M).
- Harmonic gauge: $g^{ab} C^d{}_{ab} = 0$ ($\mathrm{Id} : (M, g) \to (M, \mathring{g})$ is a wave map)



Einstein's vacuum field equations are

$$g^{ab} \mathring{\nabla}_a \mathring{\nabla}_b h_{cd} = Q_{cd}(h; C, C) \qquad \text{(quasilinear system of wave equations)}$$

$$C_d := g^{ab}(\mathring{\nabla}_a h_{bd} - \frac{1}{2} \mathring{\nabla}_d h_{ab}) = 0 \qquad \text{(constraints).}$$

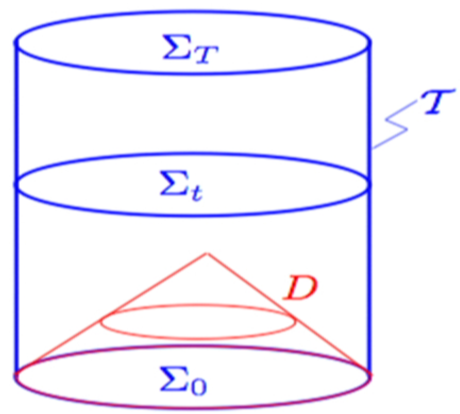
Constraint propagation

The field equations and the definition of the constraints imply that the constraint fields C_d obey the *constraint propagation system*

$$g^{ab}\nabla_a\nabla_b C_d + R_d{}^c C_d = 0,$$

which is also a system of wave equations.

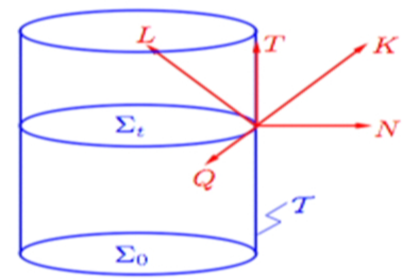
- The constraint propagation system guarantees that $C_d = 0$ in the domain of dependence D of the initial surface, provided the initial data is such that $C_d|_{\Sigma_0} = 0, \quad \partial_t C_d|_{\Sigma_0} = 0.$
- However, in order to guarantee constraint propagation beyond D one needs to impose appropriate boundary conditions.
- One possibility is to set $C_d|_{\mathcal{T}} = 0$.



Boundary conditions

We have 10 wave equations, so we need 10 boundary conditions.
Consider then the following boundary conditions:

$$\mathring{\nabla}_K h_{KK} = 0 \qquad \text{(gauge)}$$
$$\mathring{\nabla}_K h_{KQ} = 0 \qquad \text{(gauge)}$$
$$\mathring{\nabla}_K h_{KL} = 0 \qquad \text{(gauge)}$$
$$\mathring{\nabla}_K^2 h_{QQ} = \mathring{\nabla}_Q (2\mathring{\nabla}_K h_{KQ} - \mathring{\nabla}_Q h_{KK}) \qquad (\Psi_0 = 0),$$
$$\mathring{\nabla}_K h_{Q\bar{Q}} = -\mathring{\nabla}_L h_{KK} + \mathring{\nabla}_Q h_{K\bar{Q}} + \mathring{\nabla}_{\bar{Q}} h_{KQ} \qquad \text{(constraint } C_K = 0)$$
$$\mathring{\nabla}_K h_{LQ} = -\mathring{\nabla}_L h_{KQ} + \mathring{\nabla}_Q h_{KL} - \mathring{\nabla}_{\bar{Q}} h_{QQ} \qquad \text{(constraint } C_Q = 0)$$
$$\mathring{\nabla}_K h_{LL} = -\mathring{\nabla}_L h_{Q\bar{Q}} + \mathring{\nabla}_Q h_{\bar{Q}L} + \mathring{\nabla}_{\bar{Q}} h_{QL} \qquad \text{(constraint } C_L = 0)$$



Generalized Sommerfeld boundary conditions for the fields h_{ab}.
Note the hierarchical order on the left-hand side of the equations!

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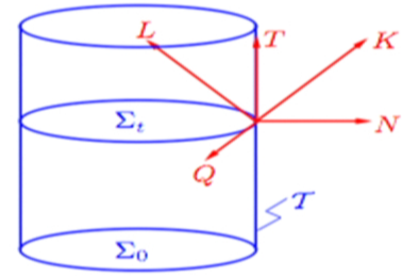

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Boundary conditions

Simple analogy from electrodynamics:
$\Box A^\mu = 0$, $C := \nabla_\mu A^\mu$ with boundary conditions

$$\nabla_K A_K = 0 \qquad \text{(gauge)}$$
$$\nabla_K A_Q = \nabla_Q A_K \qquad \text{(radiation)}$$
$$\nabla_K A_L = -\nabla_L A_K + \nabla_Q A_{\bar{Q}} + \nabla_{\bar{Q}} A_Q \qquad \text{(constraint } C = 0\text{)}$$

The radiation condition is gauge-invariant; it is transparent to outgoing electromagnetic waves in the normal direction.

Residual gauge transformations: $A_\mu \mapsto A_\mu + \nabla_\mu \chi$:
The Lorentz gauge and the gauge boundary condition imply:

$$\Box \chi = 0, \qquad \nabla_K^2 \chi|_{\mathcal{T}} = 0,$$

which yields a well-posed problem for χ.
(Similar situation in linearized harmonic system.)

Strong well-posedness

Well-posedness?

The key is to show *strong well-posedness* (Kreiss). For the wave equation $\Box\phi = 0$ with boundary conditions $\nabla_K \phi = g$ this implies, in particular, an estimate of the form

$$\|\nabla\phi\|_{L^2(\Sigma_T)} + \|\nabla\phi\|_{L^2(\mathcal{T})} \leq C(T)\left(\|\nabla\phi\|_{L^2(\Sigma_0)} + \|g\|_{L^2(\mathcal{T})}\right),$$

where

$$\|\nabla\phi\|^2_{L^2(\Sigma)} := \int\limits_{\Sigma} (\nabla^\mu\phi)(\nabla_\mu\phi) d^3x, \qquad \|\nabla\phi\|^2_{L^2(\mathcal{T})} := \int\limits_0^T \int\limits_{\partial\Sigma} (\nabla^\mu\phi)(\nabla_\mu\phi) d^2x dt.$$

The important point is the boundary term $\|\nabla\phi\|_{L^2(\mathcal{T})}$ which allows us to estimate the gradient of ϕ at the boundary. For systems of wave equations this offers the possibility to consider the generalized Sommerfeld-boundary conditions with the hierarchical structure!

Wave systems

In order to formulate the boundary conditions we recall:

- T: future-directed unit time-like vector on $\mathcal{T}$.
- N: outward unit normal to $\mathcal{T}$
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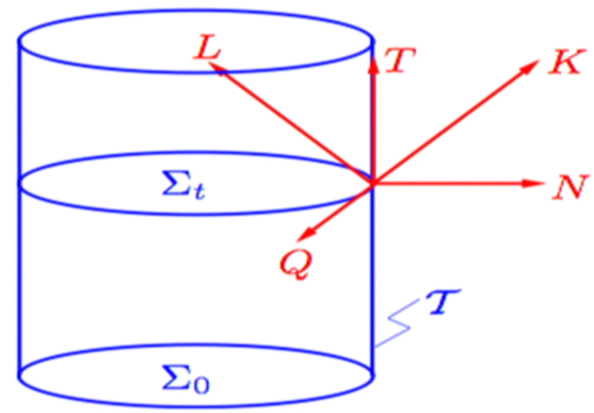

- Here, "unit", "orthogonal", "time-like" etc. refer to the dynamical metric $g_{ab}(\Phi)$, so these vector fields depend implicitly on the solution! (but algebraically)
- How to choose the time-like vector field T? See Friedrich '09 for a proposal based on the extrinsic geometry of $\mathcal{T}$.

Wave systems

Theorem (H.-O. Kreiss, O. Reula, OS, J. Winicour, '09)
Consider the IBVP

$$g^{ab}(\Phi)\mathring{\nabla}_a\mathring{\nabla}_b\Phi^A = F^A(\Phi, \mathring{\nabla}\Phi), \qquad \text{(quasilinear wave system)}$$

$$\Phi^A\big|_{\Sigma_0} = \phi_0^A, \qquad n^a\mathring{\nabla}_a\Phi^A\Big|_{\Sigma_0} = \pi_0^A, \qquad \text{(initial conditions)}$$

$$K^a\mathring{\nabla}_a\Phi^A\Big|_S = c^{aA}{}_B\,\mathring{\nabla}_a\Phi^B\Big|_S + q^A, \qquad \text{(generalized Sommerfeld conditions)}$$

where $c^{aA}{}_B = 0$ for $B \leq A$ (nilpotent).
If the data is smooth and satisfies the compatibility (corner) conditions at $\partial\Sigma_0$, then, for small enough $T > 0$, there exists a unique solution on $M = [0,T] \times \Sigma$ depending continuously on the data ϕ_0^A, π_0^A and q^A.

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Conclusions

- It is now possible to formulate a well-posed IBVP for Einstein's vacuum equations in harmonic coordinates.
- The formulation is based on a new general theorem for systems of wave equations that can also be applied to IBVPs for other physical theories (Maxwell in potential form, Yang-Mills, ...)
- Geometric uniqueness is still an open problem.
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