Title: Information Theoretic Benefit of Entanglement in Classical Communication Settings

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Abstract: Expressions of several information theoretic quantities involve an optimization over auxiliary quantum registers. Entanglement-assisted version of some classical communication problems provides examples of such expressions. Evaluating these expressions requires bounds on the dimension of these auxiliary registers. In the classical case such a bound can usually be obtained based on the
Caratheodory theorem, but we know almost no method to bound the dimension of auxiliary quantum registers. In this talk to compare the classical and quantum sides of the problem the notion of â€œquantum convexificationâ€• will be defined. It will be shown that quantum convexification is strictly richer that the usual classical convexification. Moreover some techniques will be discussed which might be useful for bounding the dimension of quantum auxiliary registers.

Information Theoretic Benefit of Entanglement in Classical Communication Settings

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May, 2013

Joint work with

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Optimizations over auxiliar quantum systems

- [Christandl, Winter '04] Squashed entanglement: given ρ_{AB}

$$E_{\mathrm{sq}}(\rho_{AB}) = \frac{1}{2} \inf_{\rho_{ABC}} I(A;B|C)$$

$H(\rho) = -\mathrm{tr}(\rho \log \rho)$: von Neumann entropy,
$H(A|B) = H(AB) - H(B)$
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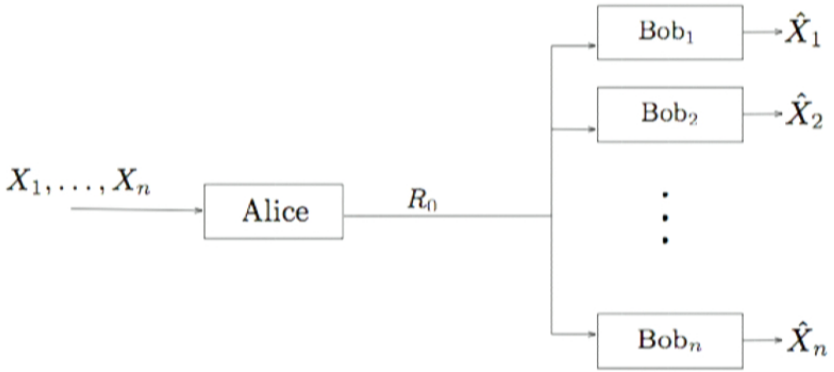
- Entanglement-assisted Gray-Wynner problem

The Gray-Wyner problem



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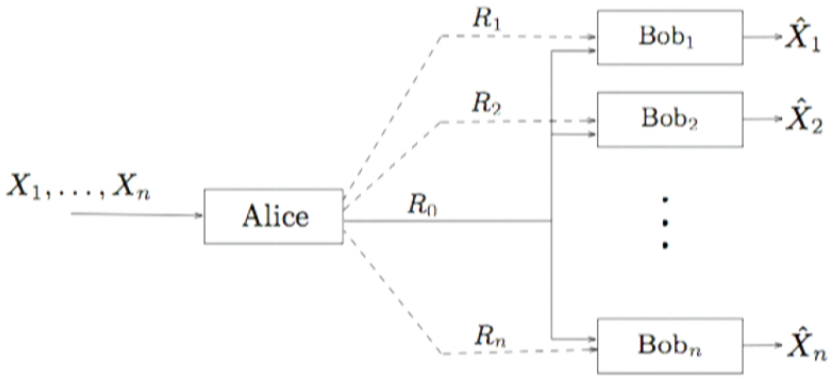

- Public message at rate R_0 to all Bobs
- Private message at rate R_j to Bob$_j$
- The goal of Bob$_j$ is to recover X_j

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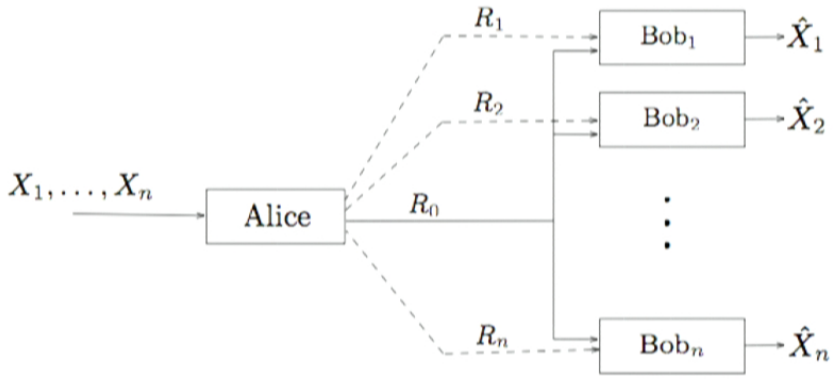

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Theorem [Gray, Wyner '74]: $(R_0, R_1, \ldots, R_n)$ is achievable iff $\exists\ C$ s.t.

$$R_0 \geq I(X_1 \ldots X_n; C) \qquad \& \qquad R_j \geq H(X_j|C)$$

The Gray-Wyner problem

Question: What if the sender and receivers share entanglement?

Theorem [Winter '12]: $(R_0, R_1, \ldots, R_n)$ is achievable iff $\exists\, Q_1, \ldots, Q_n$ s.t.

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- Does entanglement help?
- Entanglement does not increase the capacity of point-to-point classical channels.
- Given $X_1, \ldots, X_n$ and auxiliary quantum Q does there exists auxiliary classical C s.t.

$$\big(I(X_1 \ldots X_n; Q), H(X_1|Q), \ldots, H(X_n|Q)\big)$$

$$= \big(I(X_1 \ldots X_n; C), H(X_1|C), \ldots, H(X_n|C)\big)$$

Another example

- Let X, Y, Z be (classical) with a given distribution.
- Compute

$$\sup_{C-X-YZ} I(C;Y) - I(C;Z)$$

$E - F - G$ means $I(E;G|F) = 0$

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Claim: WLOG we may assume $|\mathcal{C}|$ is bounded

Proof: $I(C;Y) - I(C;Z) = H(Y) - H(Z) + H(Z|C) - H(Y|C)$

$H(Y|C) = \sum_{c \in \mathcal{C}} p(c) H(Y|C=c)$

Use Caratheodory theorem: every point in ConvHull$(\mathcal{S})$ where $\mathcal{S} \subseteq \mathbb{R}^n$ can be written as a convex combination of at most $n+1$ points of $\mathcal{S}$.

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Question: What happens if we take C to be quantum?

What we like...

- Compute regions defined by conditioning on a quantum auxiliary register
- Conditioning on quantum auxiliary registers vs auxiliary random variables
- Is entanglement ever helpful in a classical communication setting?

Convexification

- Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be finite sets. Consider the set of distributions $p(x_1, \ldots, x_n)$ on $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$
- Consider the maps $p(x_1, \ldots, x_n) \mapsto (H(X_1), \ldots, H(X_n))$.

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- ConvHull$(\mathcal{G})$ is computable (have a bound on $|\mathcal{C}|$)
- Define QConvHull$(\mathcal{G})$ to be the set of points

$$\big(p(x_1, \ldots, x_n), H(X_1|Q), \ldots, H(X_n|Q)\big), \qquad \forall p(x_1, \ldots, x_n), \rho^{Q}_{x_1 \ldots x_n}$$

Question: Is the inclusion ConvHull$(\mathcal{G}) \subseteq$ QConvHull$(\mathcal{G})$ strict or not?

Main Result (1)

Theorem: The followings are equivalent

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Proof: (1) $\Rightarrow$ (2) is immediate.

(3) $\Rightarrow$ (1): Let

$$\left(p(x_1,\ldots,x_n), H(X_1|\mathbf{Q}),\ldots,H(X_n|\mathbf{Q})\right) \in \mathrm{QConvHull}(\mathcal{G})$$

Define $X = X_1 \ldots X_n$, consider $X \to \mathbf{Q}$ as a channel, and then use (3).

Proof of (2) $\Rightarrow$ (3)

Fix a channel $X \to \mathbf{Q}$, and a distribution $p(x)$ on X.

$$I(\mathbf{Q}; Y) - I(\mathbf{Q}; Z) \leq \max_{p(c|x)} I(C; Y) - I(C; Z), \quad \forall p(y|x), p(z|x).$$

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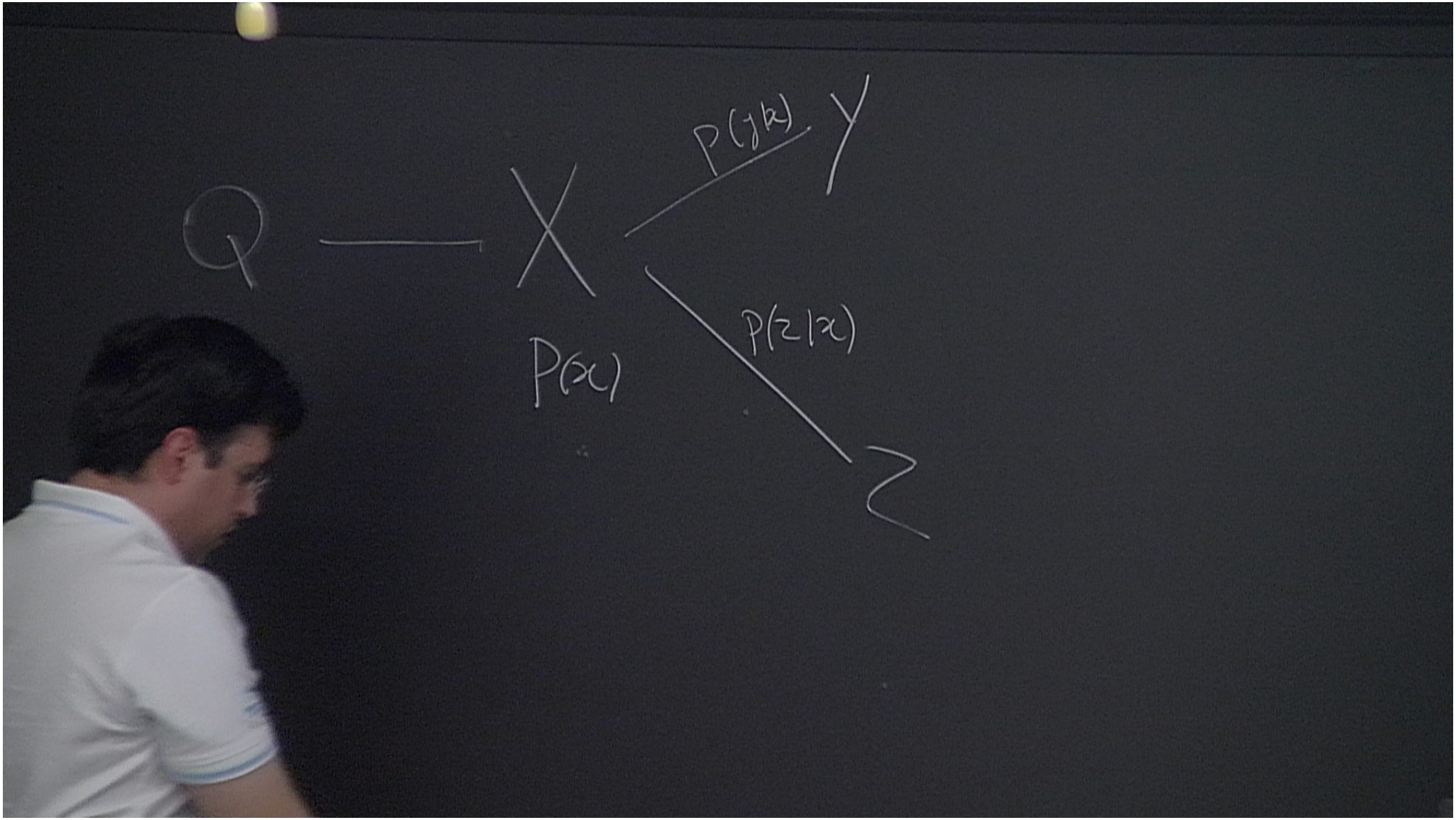
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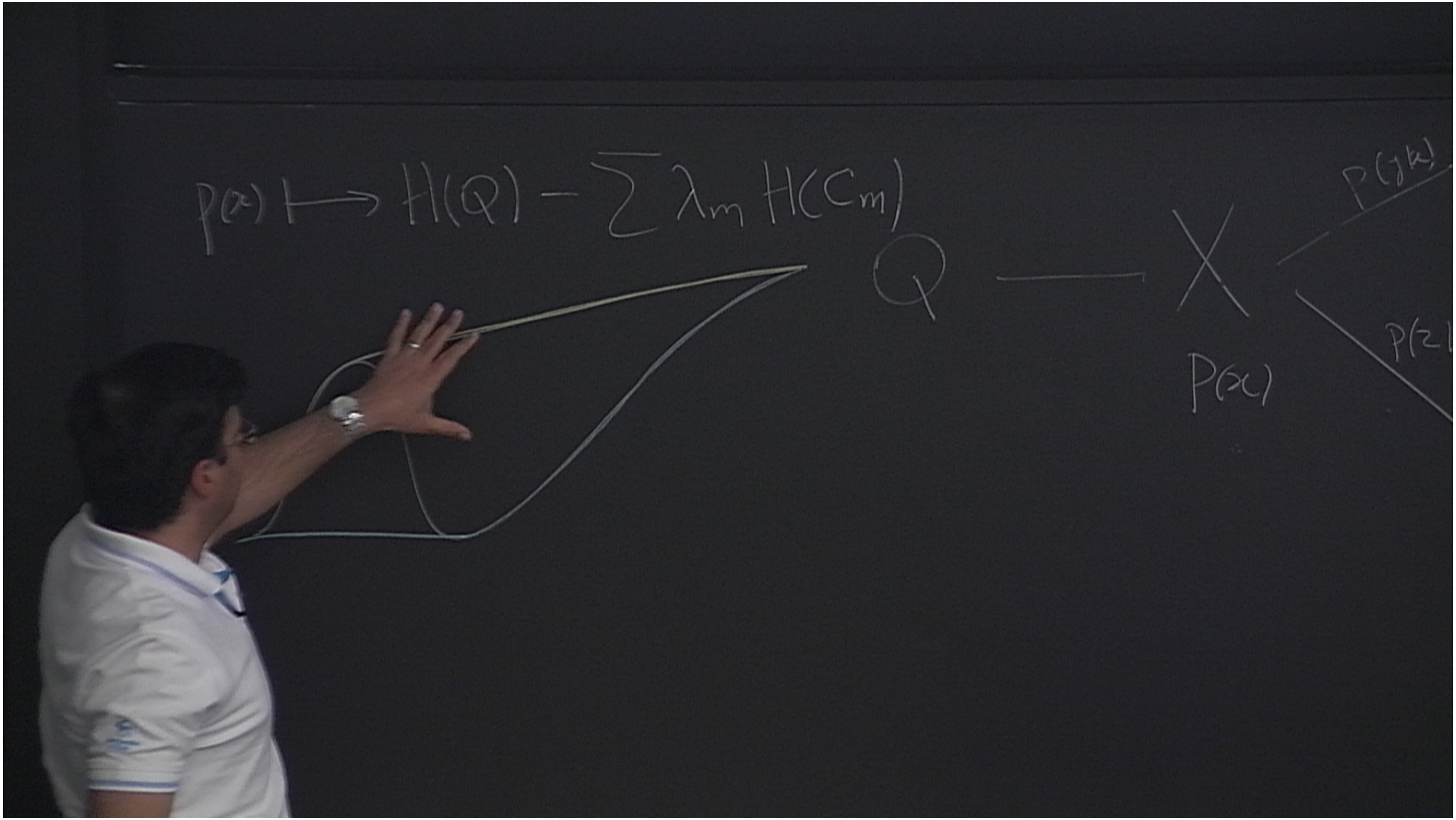
- Therefore, the function $p(x) \mapsto H(\mathbf{Q}) - \sum_m \lambda_m H(C_m)$ is almost linear.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- $I(\mathbf{Q}; X) - I(D; X)$ is zero at extreme points.
- Thus $I(\mathbf{Q}; X) - I(D; X)$ is almost zero everywhere!

Main Result (2)

Theorem: The followings are equivalent

(1) $\mathrm{QConvHull}(\mathcal{G}) = \mathrm{ConvHull}(\mathcal{G})$.

(2) For any $p(x, y, z)$ consider the optimization problem

$$\sup_{\mathbf{Q}-X-YZ} I(\mathbf{Q}; Y) - I(\mathbf{Q}; Z).$$

(3) For a c-q channel $X \to \mathbf{Q}$, consider the function $p(x) \mapsto I(X; \mathbf{Q})$. Then for every $\epsilon > 0$ there exists a c-c channel $X \to C$ such that $\left| I(X; \mathbf{Q}) - I(X; C) \right| \leq \epsilon$ for all $p(x)$. Then the supremum is attained at a classical $\mathbf{Q}$.

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Theorem: There is a counterexample for part (1).

Counterexample Based on Zero-Error Capacity

- Let $X \to Y$ be a channel
- Fix a distribution on X
- Suppose $\exists M, C$ such that

$$I(C;M) = 0, \quad H(X|CM) = 0, \quad MC - X - Y, \quad H(M|CY) = 0$$

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- [Leung et al '10] Entanglement may increase the one-shot zero-error capacity of classical channels.
- The above equations can be written in terms of the *conditioning* problem $\to$ counterexample

Main Result (3)

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Theorem: Part (3) holds if $|X| = \dim\mathbf{Q} = 2$.

Dimension bounds on quantum auxiliary systems

- We compared $\sup_{\mathbf{Q}-X-YZ} I(\mathbf{Q};Y) - I(\mathbf{Q};Z)$ in the two cases
  - $\mathbf{Q}$ is classical
  - $\mathbf{Q}$ is quantum
- What if we compare the followings
  - $\mathbf{Q}$ is quantum
  - $\mathbf{Q}$ is quantum of dimension $\leq d$
- How the curves (surfaces) $p(x) \mapsto I(X;\mathbf{Q})$ behave when we increase dim$\mathbf{Q}$?

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- Does entanglement helps in parallel repetitions of Bell settings?

Simulation of bipartite correlations

Alice x^n — Bob y^n with distribution $\prod_j p(x_j, y_j)$

f

a^n — b^n with distribution $\tilde{p}(a^n, b^n | x^n, y^n)$

Classical case: shared randomness

Theorem [Yassaee et al '12]: A rate R is achievable iff there exists an auxiliary random variable F such that

$$R \geq I(X;F|Y)$$

and

$$\begin{aligned} &F - X - Y, \\ &A - FX - YB, \\ &B - FY - XA, \\ &|\mathcal{F}| \leq |\mathcal{X}||\mathcal{Y}||\mathcal{A}||\mathcal{B}| + 1. \end{aligned}$$

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$$A, X, Y, \mathbf{Q} \sim \{\tilde{p}(a,x,y); \rho^{\mathbf{Q}}_{a,x,y}\}$$

$$A, B, X, Y \sim \tilde{p}(a,b,x,y) = \tilde{p}(a,x,y)\tilde{p}(b|a,x,y),$$

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$$\mathbf{Q} - X - Y,$$

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$$\exists \text{ CPTP map } \Psi \text{ s.t. } \Psi(\mathbf{Q}, Y) = B.$$

Theorem [B., Gohari, '12]:

(1) Every achievable R is in $\bigcap_{\epsilon>0} \mathcal{S}_\epsilon$

(2) All $R \in \mathcal{S}_0$ are achievable

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Proof: (1) follows from similar steps as in the classical case.
(2) follows from a remote state preparation protocol with side information

Example: CHSH-type correlations

$$p_\alpha(a,b|x,y)=\begin{cases}\frac{\alpha}{2} & a\oplus b=xy\\ \frac{1-\alpha}{2} & \text{otherwise}\end{cases}$$

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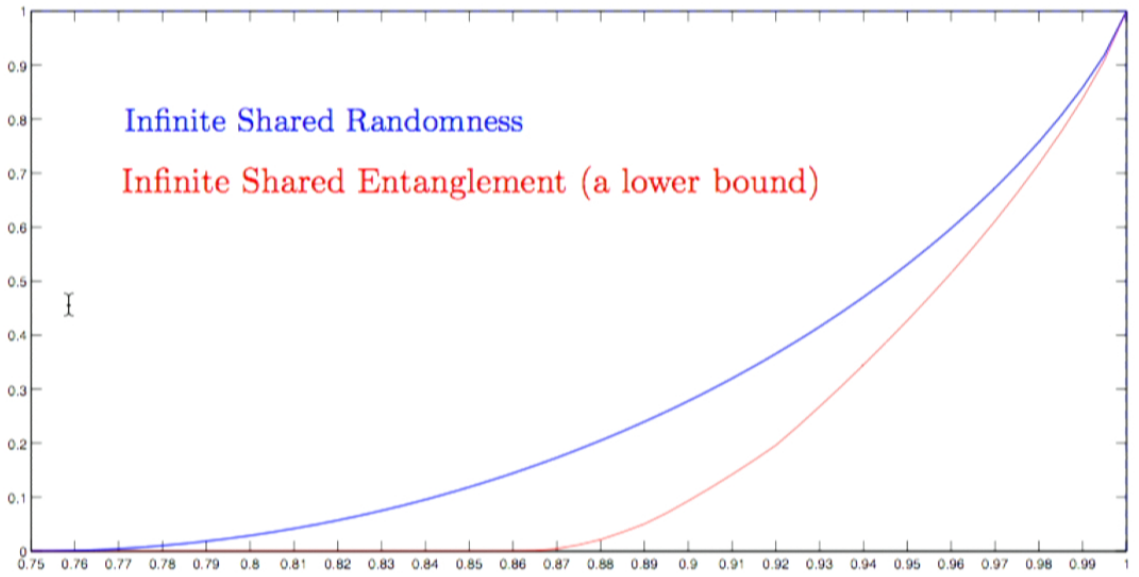

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