

Title: Entanglement at strongly-interacting quantum critical points in 2+1D

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Abstract: <span>In two or more spatial dimensions, leading-order contributions to the scaling of entanglement entropy typically follow the "area" or boundary law.&nbsp; Although this leading-order scaling is non-universal, at a quantum critical point (QCP), the sub-leading behavior does contain universal physics.&nbsp; Different universal functions can be access through entangling regions of different geometries.&nbsp; For example, for polygonal shaped regions, quantum field theories have demonstrated that the subleading scaling is logarithmic, with a universal coefficient dependent on the number of vertices in the polygon.&nbsp; Although such universal quantities are routinely studied in non-interacting field theories, it often requires numerical simulation to access them in interacting theories.&nbsp; In this talk, I discuss quantum Monte Carlo (QMC) and numerical Linked-Cluster Expansion (NLCE) calculations of the Renyi entropies at the transverse-field Ising model QCP on the 2D square lattice.&nbsp; We calculate the universal coefficient of the vertex-induced logarithmic scaling term, and compare to non-interacting field theory calculations by Casini and Huerta. Also, we examine the shape dependence of the Renyi entropy for finite-size toroidal lattices with smooth boundaries. Such geometries provide a sensitive probe of the gapless wave function in the thermodynamic limit, and give new universal quantities that could be examined by future field-theoretical studies in 2+1D.</span>

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Roger Melko

*Entanglement scaling in two-dimensional gapless systems*, **Hyejin Ju, Ann Kallin**, Paul Fendley, Matthew B Hastings, RGM, Phys. Rev. B 85, 165121 (2012)

*Thermodynamic singularities in the entanglement entropy at a two-dimensional quantum critical point*, Rajiv Singh, RGM, Jaan Oitmaa, Phys. Rev. B 86, 075106 (2012)

*Entanglement at a Two-Dimensional Quantum Critical Point: A Numerical Linked-Cluster Expansion Study*, **Ann Kallin, Katharine Hyatt**, Rajiv Singh, RGM, Phys. Rev. Lett. 110, 135702 (2013)

*Entanglement in gapless resonating-valence-bond states*, **Jean-Marie Stéphan, Hyejin Ju**, Paul Fendley, RGM, New J. Phys. 15, 015004 (2013)

*Entanglement at a Two-Dimensional Quantum Critical Point: a T=0 projector QMC study*, **Stephen Inglis** and RGM, arXiv:1305.1069

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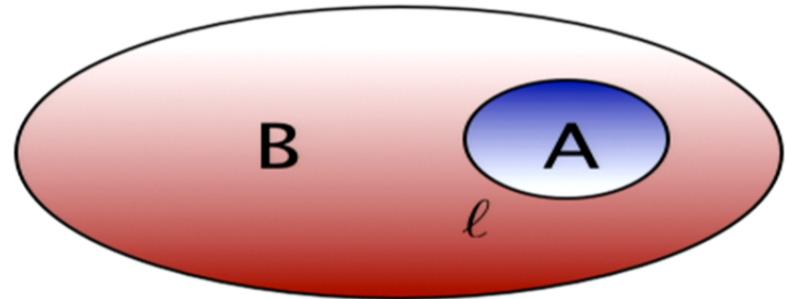
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## Renyi entropies

$$S_n(\rho_A) = \frac{1}{1-n} \ln [\text{Tr}(\rho_A^n)]$$



- We have been very successful in characterizing (via theory/numerics) exotic gapped systems – e.g. Topological entanglement entropy:

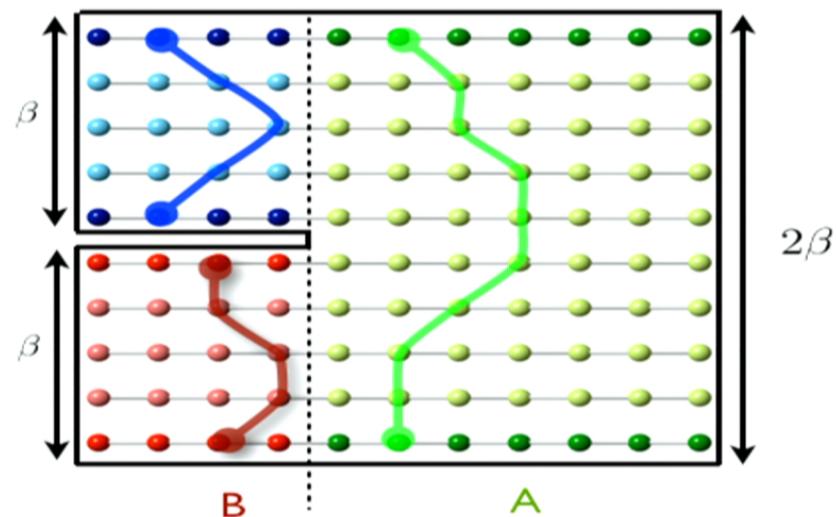
$$S_n = A\ell + \gamma$$

- Similarly, will it be possible to classify gapless systems (spin liquids, quantum critical points) by subleading scaling terms?
- For example, subleading terms are predicted to be a resource in distinguishing conventional from fractional critical points

$$\gamma_{XY*} = \gamma_{XY} + \gamma_{Z_2}$$

Swingle, Senthil, Phys. Rev. B 86, 155131 (2012)

- In gapless systems in  $D > 1$ , we are still learning how  $\gamma$  depends on subregion shape and topology
- Ultimately, want to compare field theory and numerics. However, some entangled shapes amenable to continuum theory don't work with lattice simulations, and vice versa
- Here, we will examine the shape dependence of the subleading  $\gamma$  using numerics (mostly QMC) at the simplest interacting QCP in 2D



$$H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$$

$$(h/J)_c = 3.044$$

**n-sheeted Riemann surface:**  
sometimes integer  
 $n > 2$  is all we have

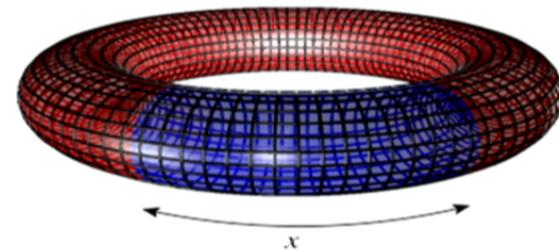
# Outline

- Review of entanglement entropy scaling in 1D

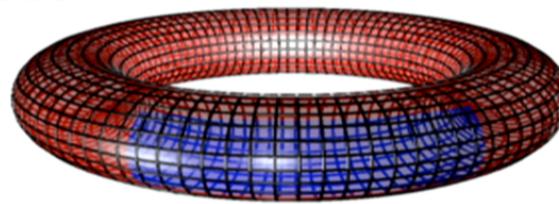


- 2D: area law and beyond

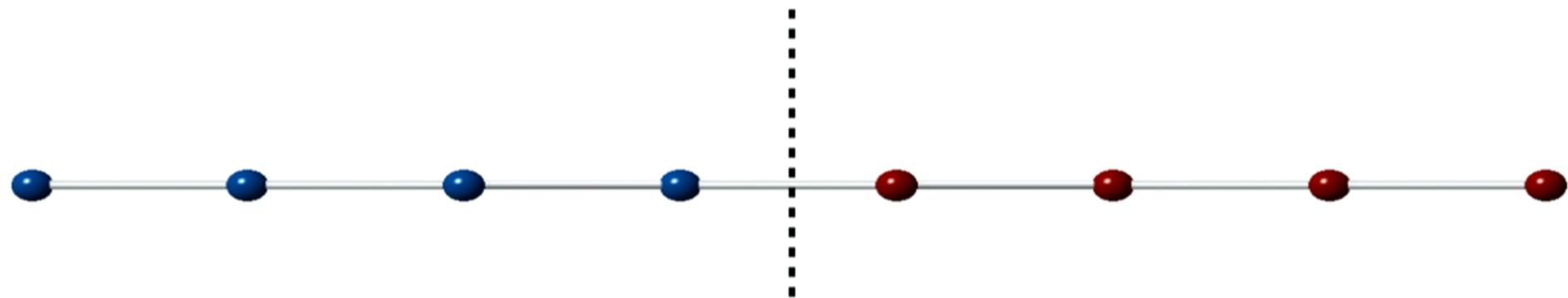
- Two-cylinder entropies



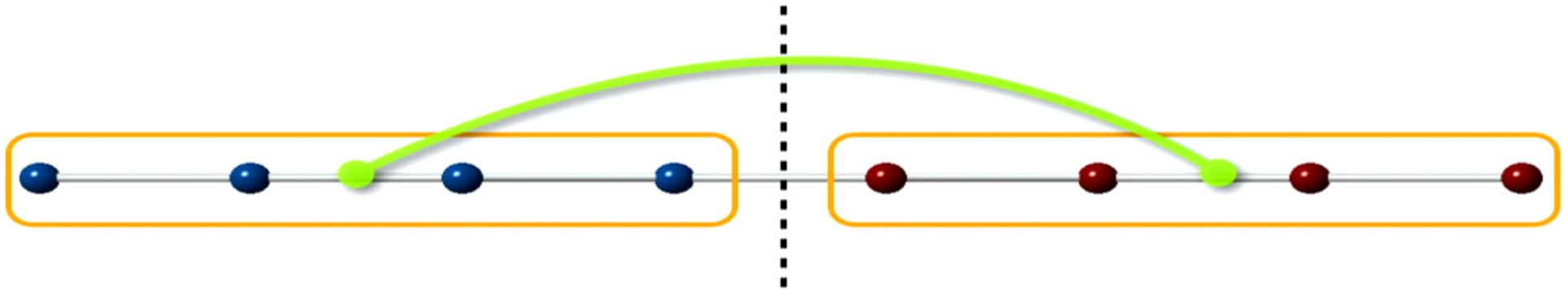
- Universal contributions due to vertices



- Critical systems in 1D... Scale invariance: assume  $\mathcal{O}(1)$  unit of entanglement entropy at **each** length scale



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$$\mathcal{O}(1) \times (1+1+1+\dots) = \log(L)$$

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C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. B424, 443 (1994)  
G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003)  
Calabrese and Cardy, J. Stat. Mech: Theory Exp. P06002 (2004)

- Proper way: conformal mapping from a cylinder to a plane:

$$S_n = \frac{c}{3\eta} \left(1 + \frac{1}{n}\right) \log \left[ \frac{\eta L}{\pi a} \sin \frac{\pi x}{L} \right] + \dots$$

C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. B424, 443 (1994)  
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- Finite-size scaling form allows a detailed comparison with numerics

$$\eta = 1$$

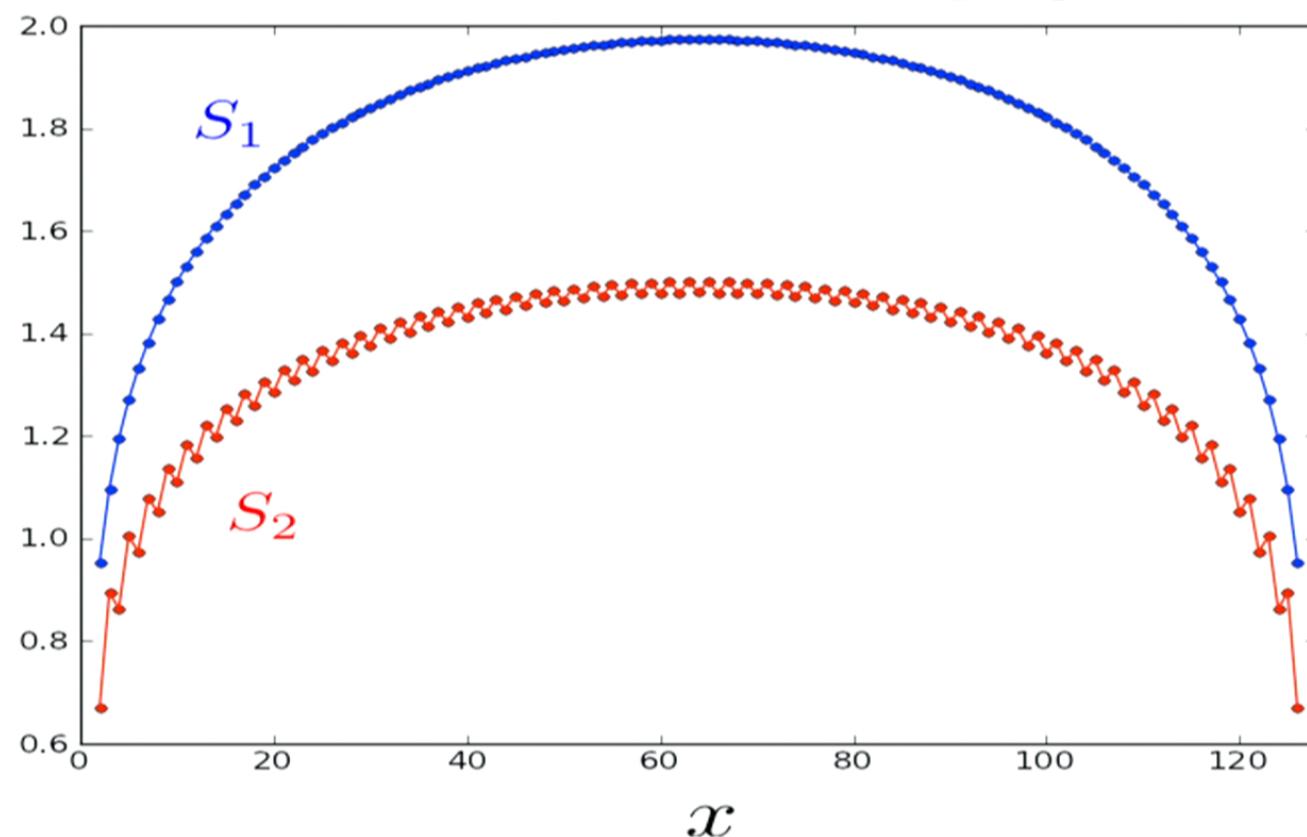


$$\eta = 2$$

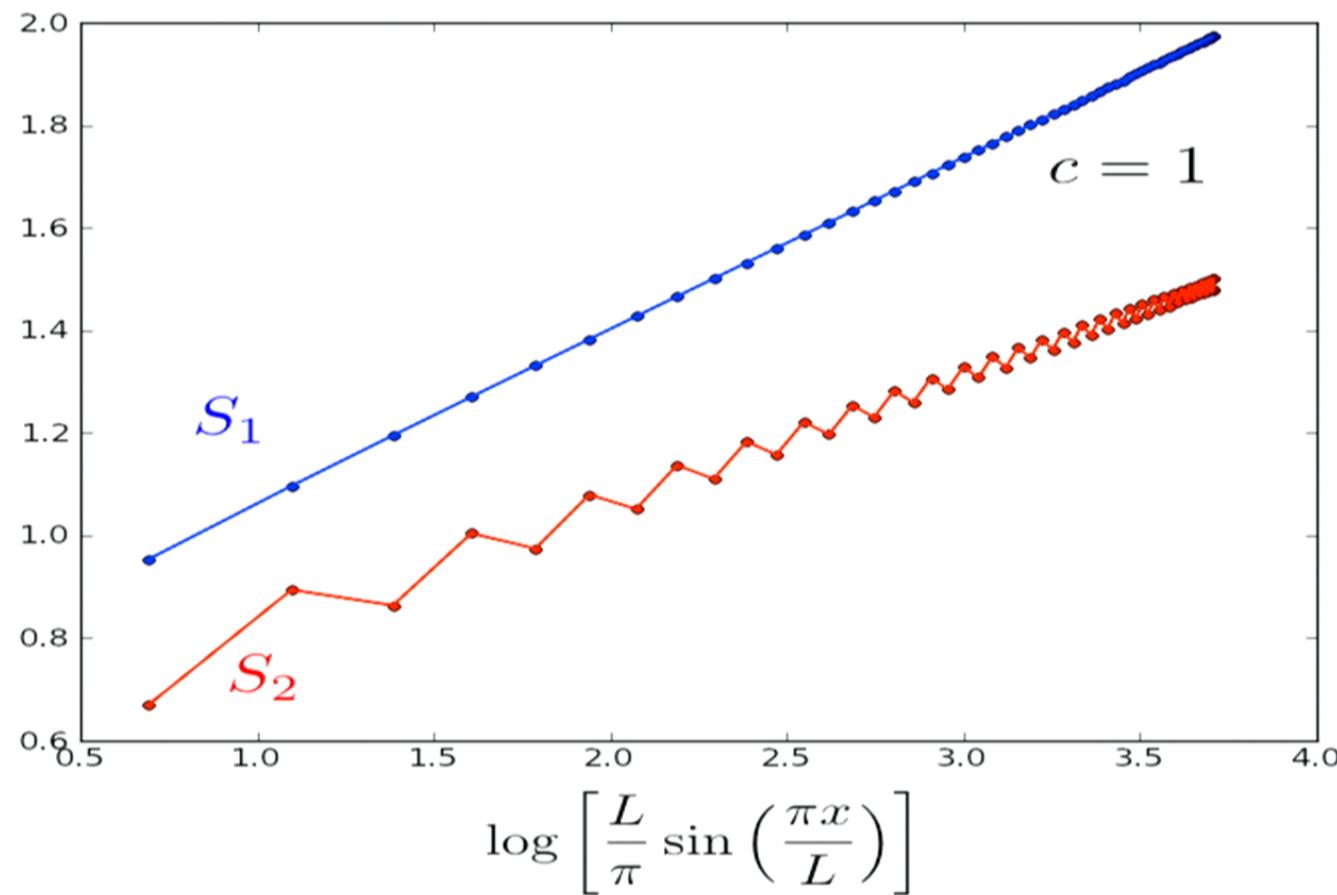


- There is a lot of synergy between theory and DMRG in 1+1...

$$H = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$



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- In a critical system in 2D, the leading-order scaling is area-law:

$$S_n = A\ell + \dots$$

this is confirmed numerically on a variety of boson models

- General expectation: geometry-dependent subleading term

S. Ryu and T. Takayanagi

Y. Zhang, T. Grover, Vishwanath

Casini, Huerta, Myers

$$S_n = A\ell + \gamma_n$$

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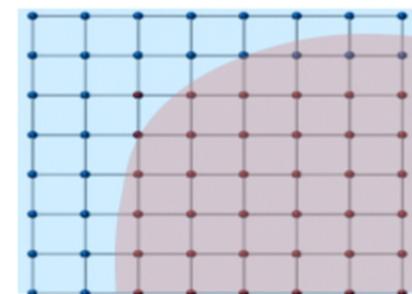
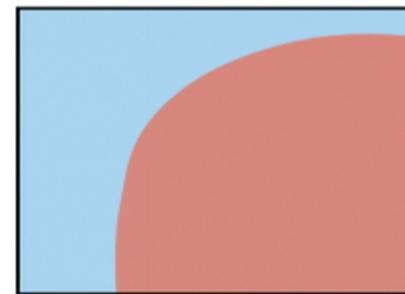
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$$S_n = A\ell + \gamma_n$$

smooth, curved surfaces have  
defects on the lattice scale



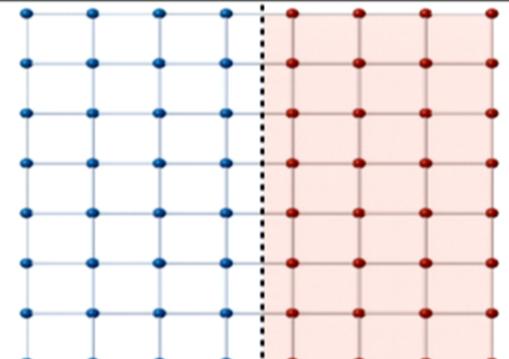
- Want to avoid studying pixelization effects at this stage...

- One may induce a universal subleading term from a smooth boundary on the infinite plane:

**finite correlation length**

Metlitski, Fuertes, Sachdev, PRB 80, 115122 (2009)

$$S_n = A\ell + r_n \frac{L^{D-1}}{\xi^{D-1}},$$



$\epsilon$ , large -  $N$  :  $r_n$  changes sign (!) near  $n \approx 1$  at the W-F fixed point

$$r_n \propto \frac{(n-1)}{\epsilon^2}$$

violates some entanglement monotonicity theorem?

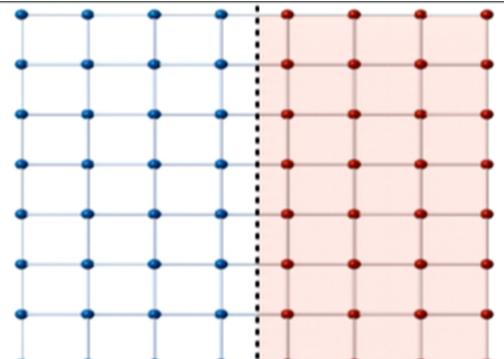
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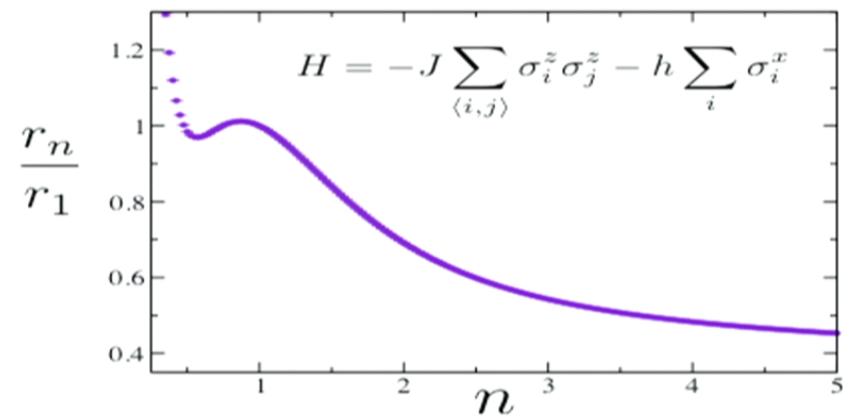
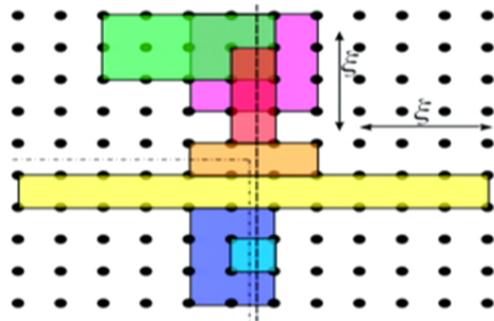
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H. Casini and M. Huerta, Phys. Rev. D 85, 125016 (2012)

- Numerical Linked Cluster Expansion (all Renyi entropies)

Kallin, Hyatt, Singh, RGM, Phys. Rev. Lett. 110, 135702 (2013)

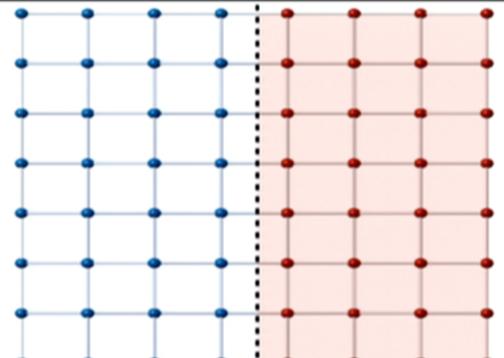


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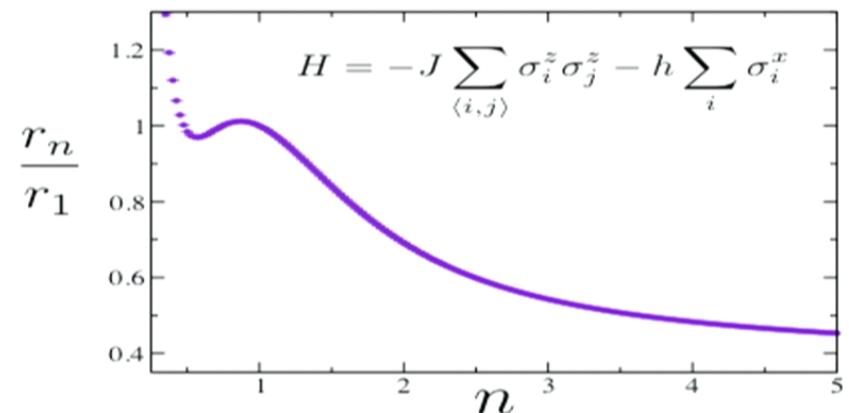
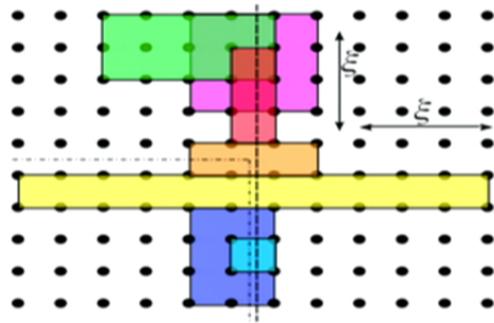
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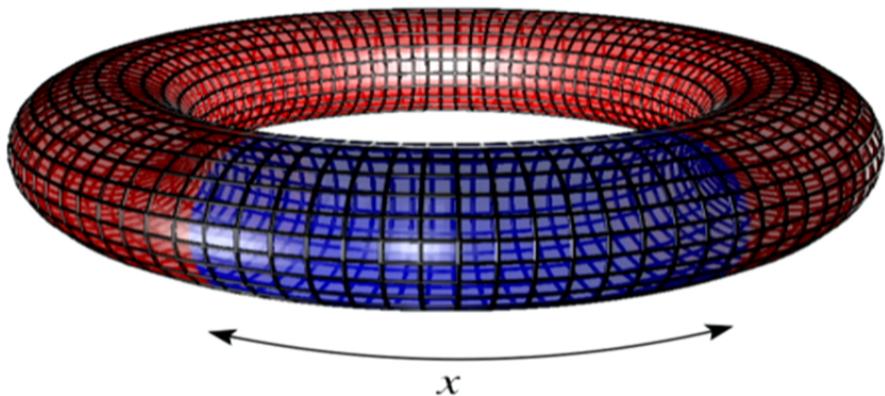
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QMC geometries – 2D torus, two smooth boundaries

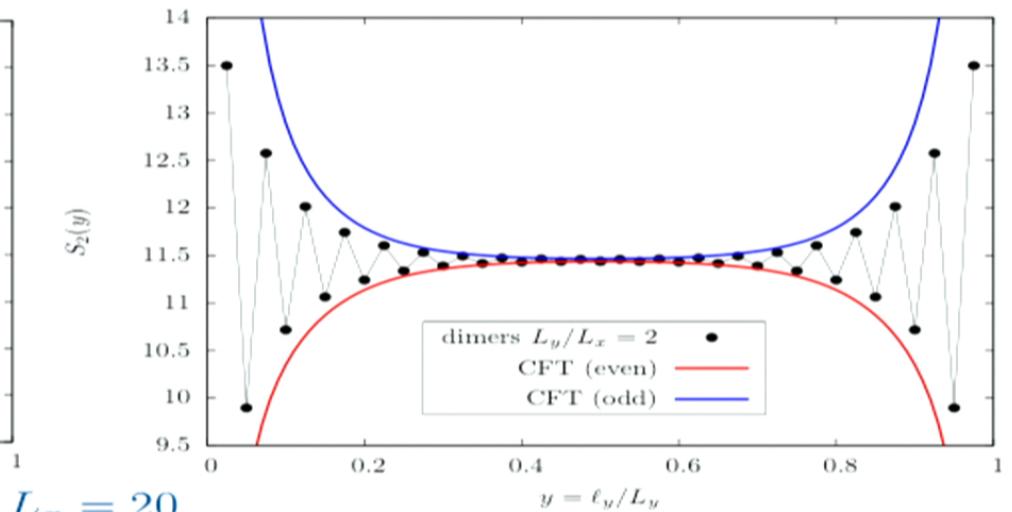
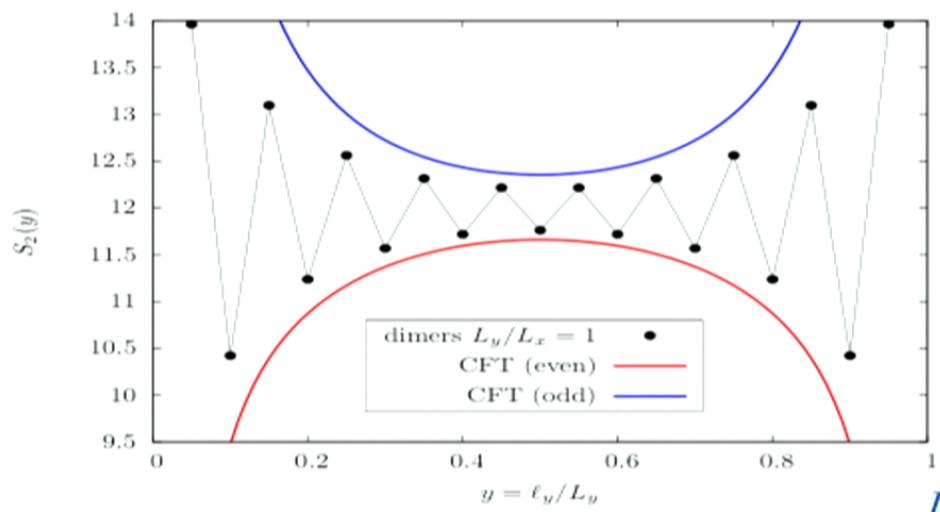
$$S_n = A\ell + \gamma(x/L_x, L_x/L_y)$$

- Jean-Marie Stéphan: “RVB–shape function”, derived in a quantum Lifshitz free scalar field theory

$$J_n(y) = \frac{n}{1-n} \log \left[ \frac{\eta(\tau)^2}{\theta_3(2\tau)\theta_3(\tau/2)} \frac{\theta_3(2y\tau)\theta_3(2(1-y)\tau)}{\eta(2y\tau)\eta(2(1-y)\tau)} \right]$$

$$y = x/L$$

$$\tau = iL_x/L_y$$

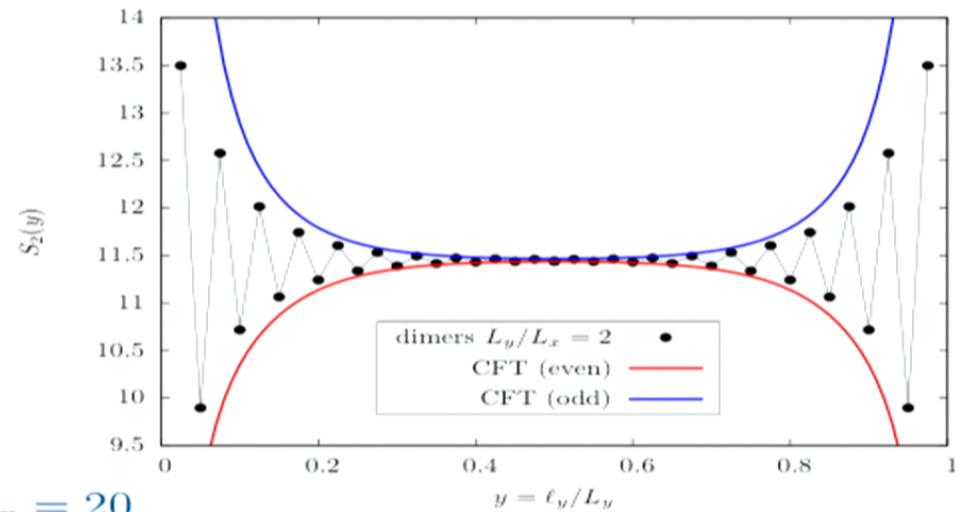
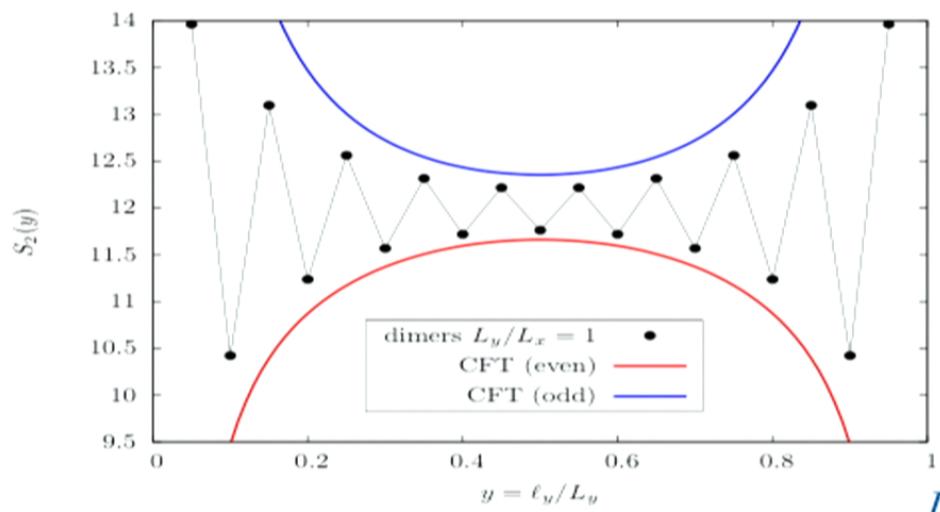


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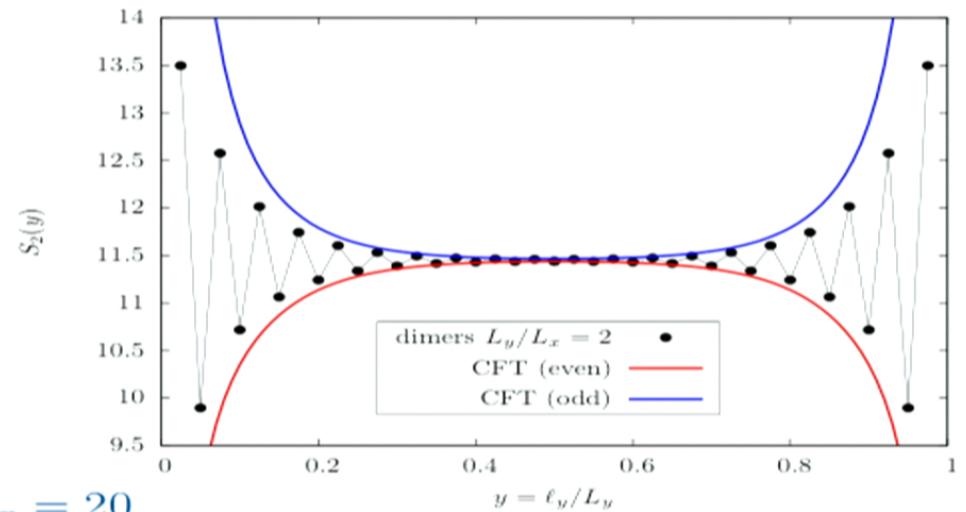
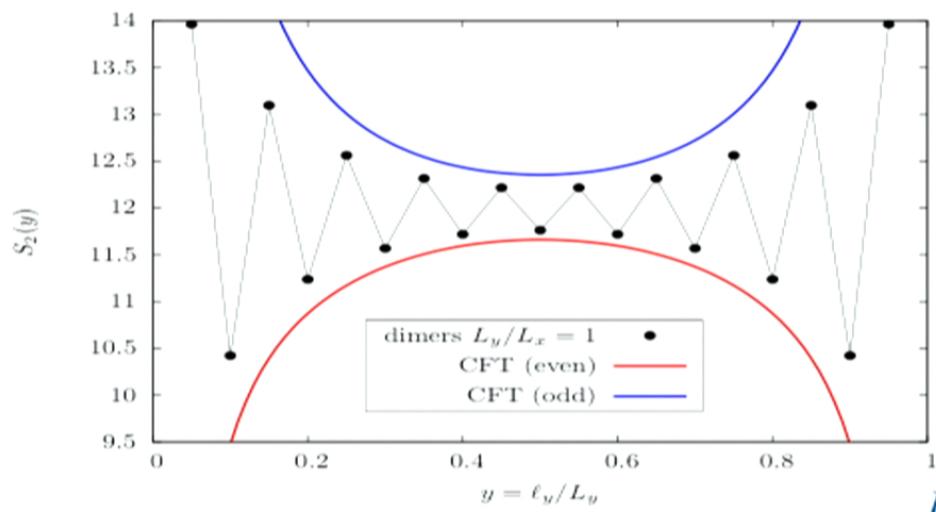


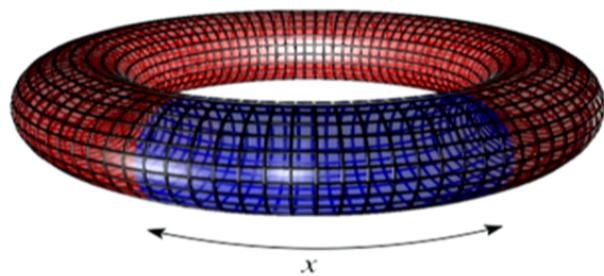
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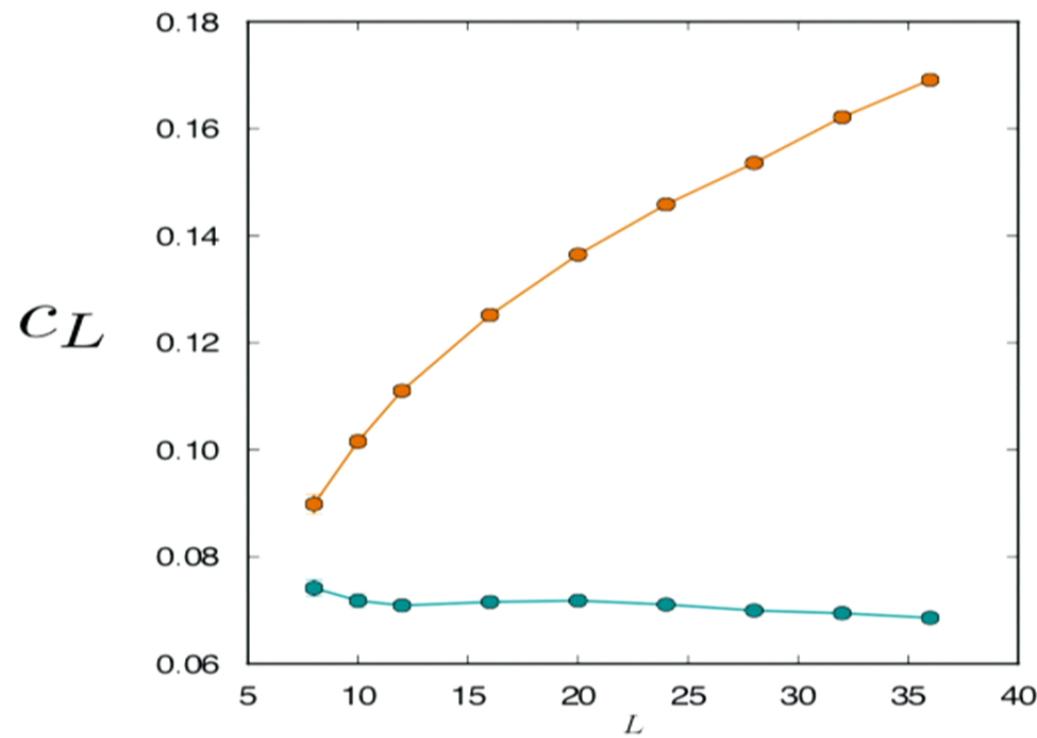
$$\tau = iL_x/L_y$$





$$S_{\log} = a\ell + c_L \log(\sin(\pi y)) + d$$

$$S_{\text{RVB}} = a\ell + c_L J(y) + d$$

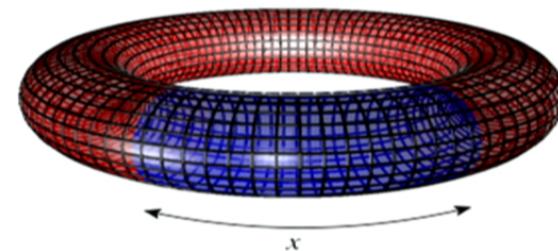


clear  $L$ -dependence remaining

consistent with  $L$ -independence

$$c \approx 0.07$$

## Two-cylinder Renyi entropies



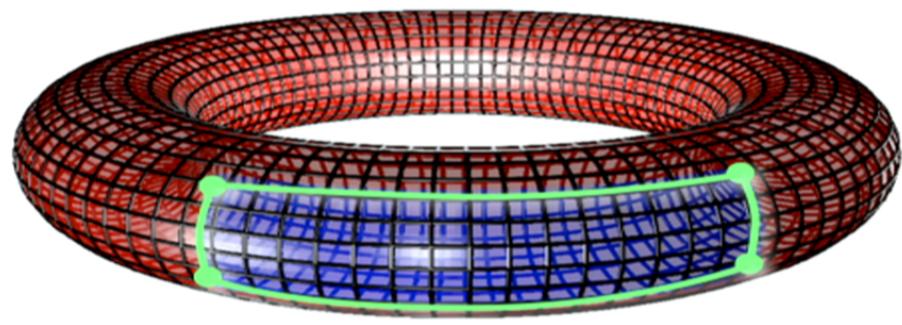
- Numerical results at quantum critical point of TFIM are consistent with NO additive logarithm

- also true in RVBs, pi-flux fermions
  - in clear contrast to case with broken continuous symmetry:

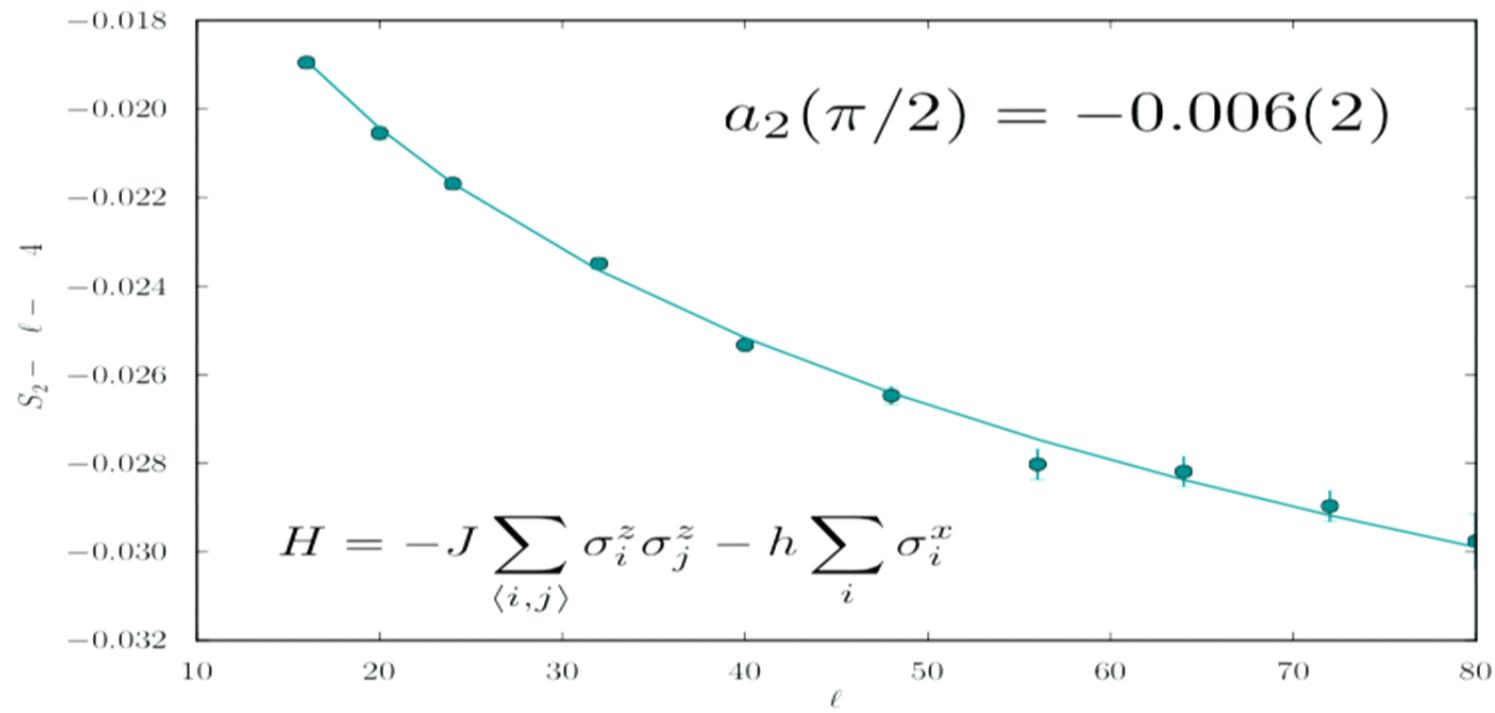
$$\frac{N_G(D-1)}{2} \log(\ell)$$

Kallin, Hastings, RGM, Singh, PRB 84, 165134 (2011)  
Metlitski, Grover arXiv:1112.5166 (2011)

- Fun to speculate regarding the universality of  $J(y)$  and its coefficient
- Need systematic QMC calculations at other fixed points

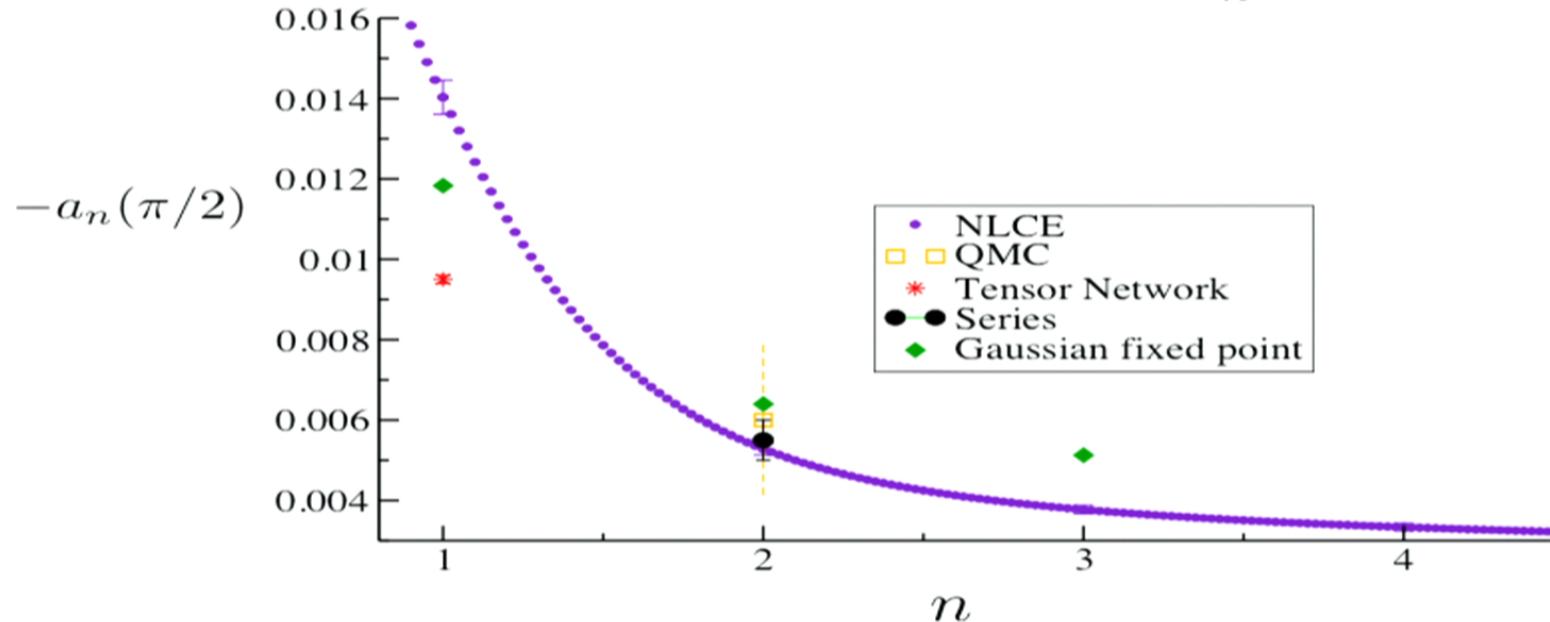


$$S_n = A\ell + 4a_n(\theta) \log(\ell)$$



## Results for 90-degree corner coefficient from the literature

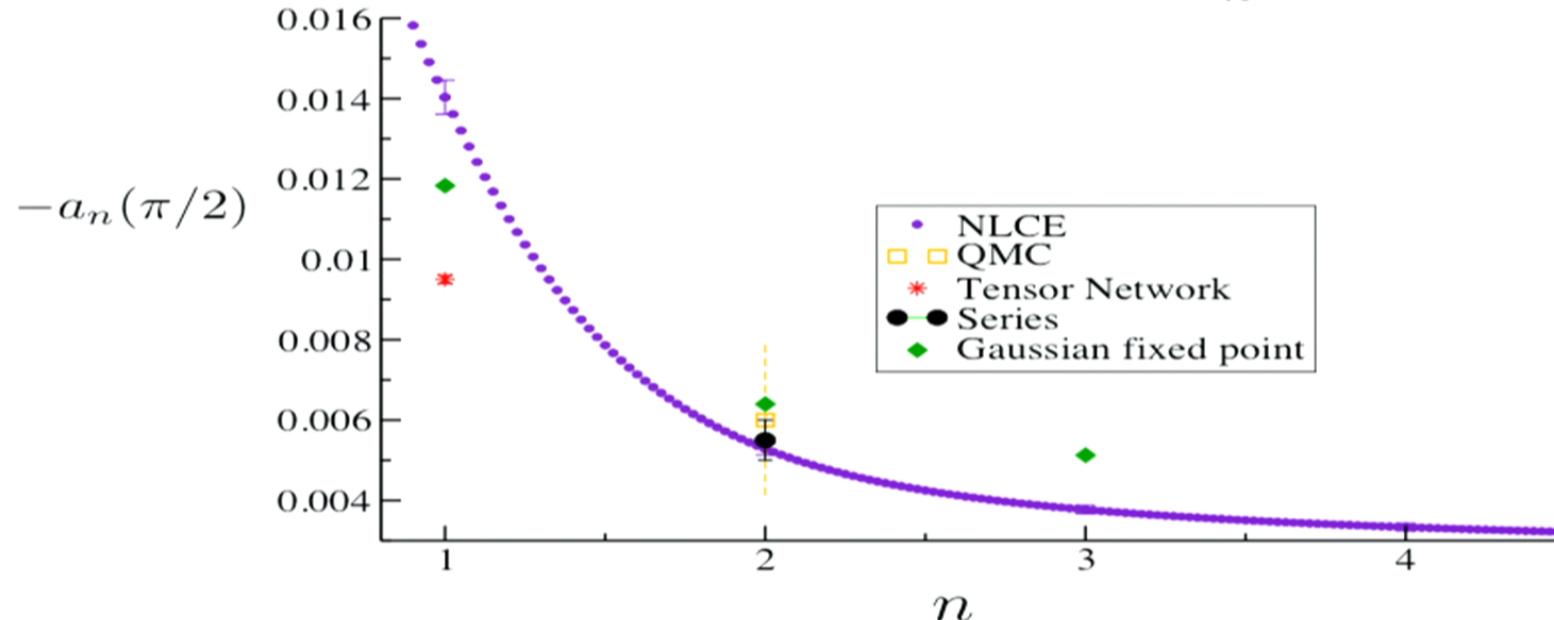
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## universal corner coefficient



- We're starting to get a handle on these numerical values for the Ising QCP
- Can we use this number to distinguish Gaussian from Ising? Which numerical method is working best?
- Wish List:
  - Coefficient of the corner log in  $\epsilon$  and/or  $1/N$  expansion at W-F fixed point
  - More tensor network/series expansion calculations
  - Next: QMC/series and field theory calculations for the O(N) transition

# Discussion

- We have the technology to take a more serious look at entanglement in gapless systems.
- Numerics/field theory/holography has potential to give c,F-theorems working relevance for condensed-matter systems
- See Tarun Grover– best talk that never happened

11/07, Tarun Grover  
1:45 KITP  
p.m.

Quantum Entanglement and Stability of Gapless Spin-liquids [Podcast] [Aud] [Cam]

[arXiv:1211.1392](#)

- Renyi entropies will be the quantities eventually measured in experiments

Cardy, PRL 106, 150404 (2011)  
Abanin, Demler, PRL 109, 020504 (2012)  
Pichler, Bonnes, Daley, Läuchli, Zoller arXiv:1302.1187