Title: Geometry and the entanglement spectrum in the fractional quantum Hall effect.

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Abstract: Fractional quantum hall states with nu = p/q have a characteristic geometry defined by the electric quadrupole moment of the neutral composite boson that is formed by "flux attachment" of q "flux quanta" (guiding-center orbitals) to p charged particles.   This characterizes the "Hall viscosity".   For FQHE states described by a conformal field theory with a Euclidean metric g_ab, the quadrupole moment is proportional to the "guiding-center spin" of the composite boson and the inverse metric.   The geometry gives rise to dipole moments at external edges or internal "orbital entanglement cuts", and can be seen in the entanglement spectrum.

Perimeter Institute, Waterloo ON, May 10, 2013

Geometry and Entanglement in the FQHE

F. Duncan M. Haldane
Princeton University

- hidden geometry of the Laughlin state
- geometrodynamics of the FQHE
- geometry and entanglement

Laughlin state

- originally introduced as a "lowest Landau level wavefunction"

(I will explain why this is a misleading characterization)

$$\Psi_L^q(\{\boldsymbol{r}_i\}) = \prod_{i<j}(z_i - z_j)^q \prod_i e^{-\frac{1}{2}z_i^* z_i}$$

- usual interpretation of z is

$$z = \frac{x+iy}{\sqrt{2\ell_B^2}}$$

magnetic area: $2\pi\ell_B^2$

(contains one flux quantum h/e)

The most striking feature for theorists is that this is holomorphic!

- Laughlin explained that his wavefunction had a holomorphic factor because it was a lowest-Landau level wavefunction.
- I will explain why the holomorphic character has a quite different origin!
- This will explain why the Laughlin state can be found in systems unrelated to lowest Landau level systems
- It will also reveal the fundamental geometric degree of freedom of the FQHE state.

standard derivation

- non-relativistic Galileian-invariant Landau levels

$$H = \frac{|\vec{p} - e\vec{A}(\boldsymbol{r})|^2}{2m} = \tfrac{1}{2}\hbar\omega_c(a^\dagger a + a a^\dagger)$$

(Note isotropic effective mass)

- Landau level ladder operators (in the "symmetric gauge"):

$$a = \tfrac{1}{2}z + \frac{\partial}{\partial z^*} \qquad a^\dagger = \tfrac{1}{2}z^* - \frac{\partial}{\partial z} \qquad [a, a^\dagger] = 1$$

lowest Landau level wavefunctions

$$a\psi(\boldsymbol{r}) = 0 \quad \longrightarrow \quad \psi(\boldsymbol{r}) = f(z)e^{-\frac{1}{2}z^* z}$$

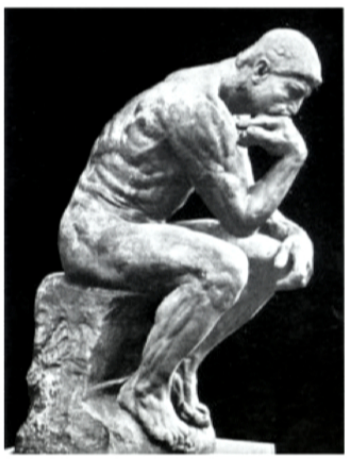
holomorphic function $\times$ Gaussian

$$\Psi_L^q(\{\boldsymbol{r}_i\}) = \prod_{i<j}(z_i - z_j)^q \prod_i e^{-\frac{1}{2}z_i^* z_i}$$

- The q = 3 Laughlin state was confirmed (by numerical exact diagonalization studies) to be the essential description of the $1/3$ FQHE
- The holomorphic factor is incidentally noticed to be a cft correlator (conformal block) of the free boson cft with boson radius $R = \sqrt{(2/q)}$. (why?)

- So it is known to work, but **why?** (In my opinion, this question was never satisfactorily answered)

a common rationalization:



"Laughlin's wavefunction cleverly lowers the Coulomb correlation energy by placing its zeroes at the locations of the particles"

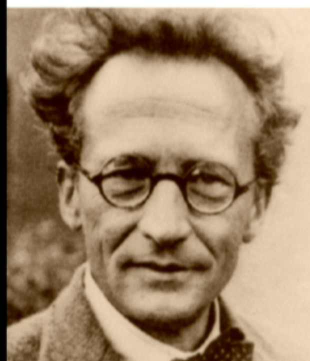
we will see that this is an empty statement

problems with this

- The "explanation" of why the Laughlin state is correct are vague rationalizations, without quantitative content.
- The relation to cft is an empirical observation, and remains unexplained
- The $1/3$ FQHE state also occurs in the second Landau level and is described by the same Laughlin **state** (but not the same "wavefunction")
- It is recently also found on Chern-insulator lattice systems (by numerical diagonalization)

The physics of the FQHE in Landau levels is the physics of non-commuting "guiding centers" (quantum geometry) which cannot be described in terms of Schrodinger wavefunctions

Schrödinger vs Heisenberg



$\Psi(\boldsymbol{r})$

wavefunction
in real space
(classical geometry)

$|\Psi\rangle$

state in
in Hilbert space

- resolution of conflict: the two formulations of QM are equivalent:

$$\Psi(\boldsymbol{r}) = \langle \boldsymbol{r}|\Psi\rangle$$

iff $\exists\ |\boldsymbol{r}\rangle$ s.t.

$$\langle \boldsymbol{r}|\boldsymbol{r}'\rangle = 0, \quad \boldsymbol{r} \neq \boldsymbol{r}'$$

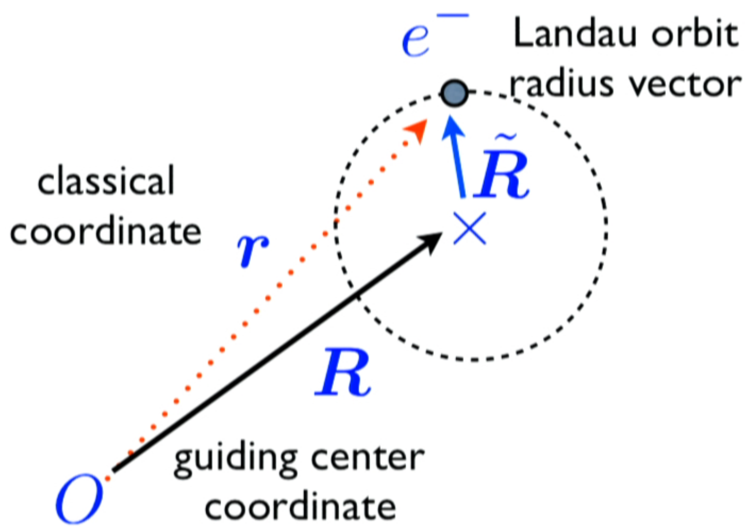
requires an orthonormal basis in real space **obeying classical locality**

- classical locality (and Schrödinger-Heisenberg equivalence) fails after Landau quantization!

$$\boldsymbol{r} = \boldsymbol{R} + \tilde{\boldsymbol{R}}$$

$$\boldsymbol{r} = r^a \boldsymbol{e}_a$$

$$[r^a, r^b] = 0$$



non-commutative algebra

$$[\tilde{R}^a, \tilde{R}^b] = i\ell_B^2 \epsilon^{ab}$$

$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}$$

$$[R^a, \tilde{R}^b] = 0$$

$$p_a - eA_a(\boldsymbol{r}) \equiv \epsilon_{ab} \hbar \tilde{R}^a / \ell_B^2$$

$$\ell_B^2 = \frac{\hbar}{eB} > 0$$

$$\boldsymbol{r} = \boldsymbol{R} + \tilde{\boldsymbol{R}}$$

eliminated by Landau quantization

- residual guiding center degrees of freedom are non-commutative

$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}$$

- isomorphic to phase space, obeys uncertainty principle

guiding centers cannot be localized within an area less than $2\pi\ell_B^2$

- The Hamiltonian governing the residual guiding-center degrees of freedom:

$$H = \int \frac{d^2\boldsymbol{q}\ell_B^2}{2\pi} U(\boldsymbol{q}) \sum_{i<j} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_i - \boldsymbol{R}_j)}$$

$$U(\boldsymbol{q}) = \frac{\tilde{V}(\boldsymbol{q}) f(\boldsymbol{q}) f(-\boldsymbol{q})}{2\pi\ell_B^2}$$

$$\tilde{V}(q) = \int d^2\boldsymbol{r}_{ij}\; V(\boldsymbol{r}_{ij}) e^{i\boldsymbol{q}\cdot\boldsymbol{r}_{ij}}$$

Fourier transformed Coulomb interaction

$$f_n(q) = \langle \psi_n | e^{i\boldsymbol{q}\cdot\tilde{\boldsymbol{R}}} | \psi_n \rangle = L_n(u) e^{\frac{1}{2}u}$$

$$u = \tfrac{1}{2}|\boldsymbol{q}|^2 \ell_B^2$$

Landau level form factor
(n = landau level index)

(depends on Landau orbit)

- in this limit, the state is an unentangled product of a non-trivial state of the guiding centers with a trivial state of the Landau orbits

$$|\Psi\rangle = |\Psi_{\boldsymbol{R}}\rangle \otimes |\Psi_{\tilde{\boldsymbol{R}}}\rangle$$

full state (which has a Schrödinger representation)

FQHE is here !

depends only on $U(\boldsymbol{q})$

Trivial state of Landau orbits

$$\prod_{\otimes} |\psi_n(\tilde{\boldsymbol{R}}_i)\rangle$$

characterized by form factor $f_n(\boldsymbol{q})$

- In what follows, I will regard the essential FQHE state as the purely-guiding center state defined by

$$H = \int \frac{d^2\boldsymbol{q}\ell_B^2}{2\pi} U(\boldsymbol{q}) \sum_{i<j} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_i - \boldsymbol{R}_j)}$$

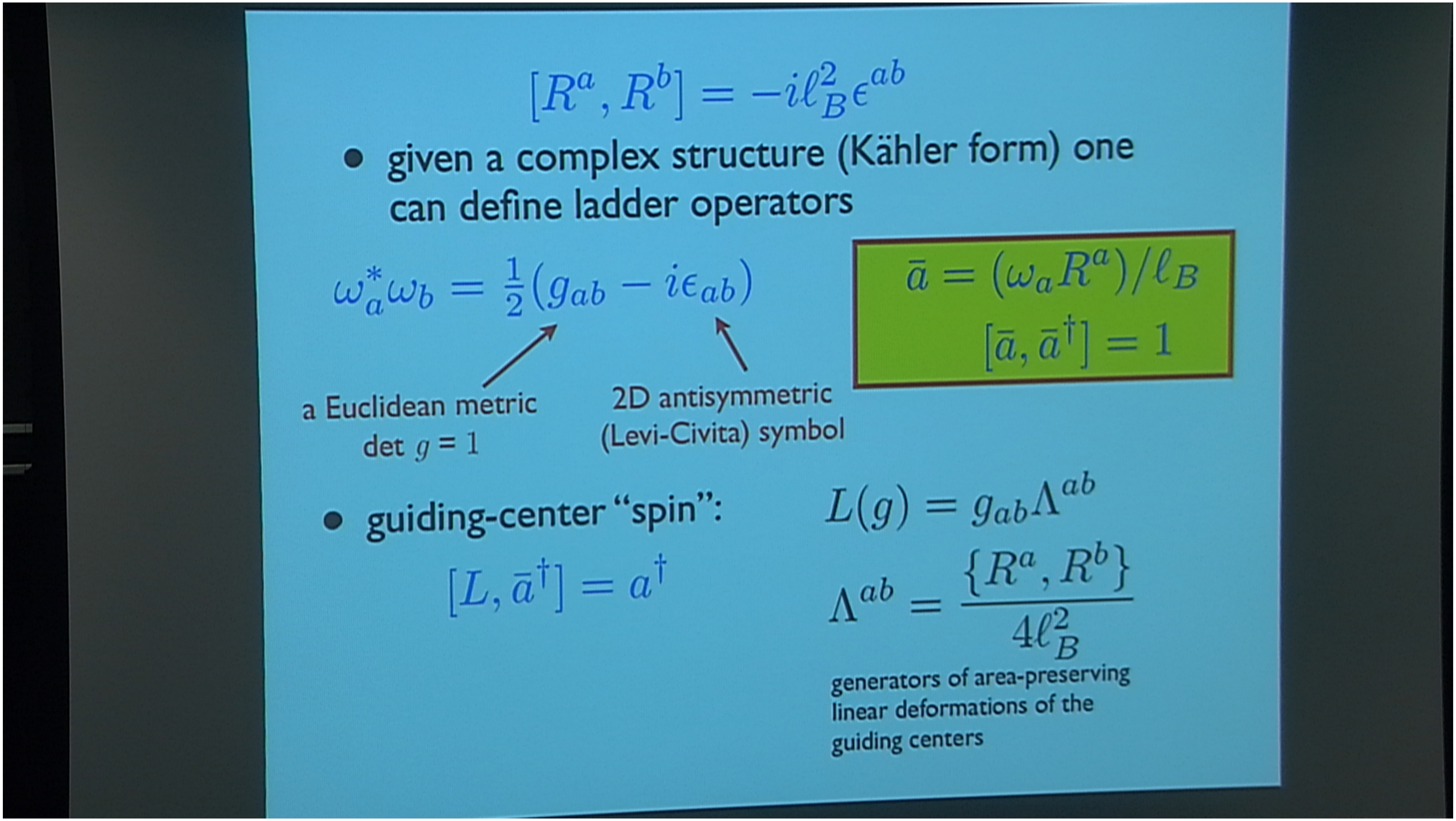
$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}$$

"quantum geometry*"

$$\rho(\boldsymbol{q}) = \sum_i e^{i\boldsymbol{q}\cdot\boldsymbol{R}}$$

$$[\rho(\boldsymbol{q}), \rho(\boldsymbol{q}')] = 2i\sin\left(\tfrac{1}{2}\epsilon^{ab} q_a q_b \ell_B^2\right)\rho(\boldsymbol{q} + \boldsymbol{q}')$$ GMP 1985

* "triple" {algebra,representation,Hamiltonian} satisfies Connes' definition


$[R^a, R^b] = -i\ell_B^2\epsilon^{ab}$
given a complex structure (Kähler form) one can define ladder operators
$\omega_a^*\omega_b = \frac{1}{2}(g_{ab} - i\epsilon_{ab})$
a Euclidean metric
det $g = 1$
2D antisymmetric (Levi-Civita) symbol
$\bar{a} = (\omega_a R^a)/\ell_B$
$[\bar{a}, \bar{a}^\dagger] = 1$
guiding-center "spin":
$L(g) = g_{ab}\Lambda^{ab}$
$[L, \bar{a}^\dagger] = a^\dagger$
$\Lambda^{ab} = \frac{\{R^a, R^b\}}{4\ell_B^2}$
generators of area-preserving linear deformations of the guiding centers

- New insight: the choice of the Euclidean metric g_{ab} is (so far) **arbitrary** (previous work always chose it as $\mathrm{diag}(1,1)$ to be congruent to the shape of the Landau orbits)

- The metric is a (hidden) variational parameter of the Laughlin **state,** and is the **fundamental physical degree of freedom** of FQHE states.

(the metric is fixed as $\mathrm{diag}(1,1)$ in the "Laughlin wavefunction")

- one can now write the Heisenberg form of the Laughlin state, liberated from any dependence on the Landau orbit geometry

$$|\Psi_L^q(g)\rangle = \prod_{i<j} \left(\omega_a^*(R_i^a - R_j^a)\right)^q |\Psi_0(g)\rangle$$

$$\omega_a R_i^a |\Psi_0(g)\rangle = 0 \qquad \omega_a^* \omega_b = \tfrac{1}{2}(g_{ab} - i\epsilon_{ab})$$

- It is the exact zero-energy ground state of the "pseudopoential" model with

$$U(\boldsymbol{q}; g) = \sum_{m<q} V_m L_m(q_g^2 \ell_B^2) e^{-\frac{1}{2} q_g^2 \ell_B^2}$$

$$V_m > 0 \qquad q_g^2 \equiv g^{ab} q_a q_b$$

- coherent state basis

$$\bar{a}|\bar{z}\rangle = \bar{z}|\bar{z}\rangle \quad |\bar{z}\rangle = e^{\bar{z}\bar{a}^\dagger - \bar{z}^*\bar{a}}|0\rangle$$

$$S(\bar{z}, \bar{z}^*; \bar{z}', \bar{z}^*) = \langle \bar{z}|\bar{z}'\rangle = e^{\bar{z}^*\bar{z}' - \frac{1}{2}(\bar{z}'^*\bar{z}' + \bar{z}^*\bar{z})}$$

- non-null eigenstates of the overlap define an orthonormal basis

$$\int \frac{d\bar{z}' d\bar{z}'^*}{2\pi} S(\bar{z}, \bar{z}^*; \bar{z}', \bar{z}'^*)\psi(\bar{z}', \bar{z}'^*) = \lambda\psi(\bar{z}, \bar{z}^*)$$

- non-null eigenstates are degenerate with $\lambda = 1$

$$\psi(\bar{z}, \bar{z}^*) = f(\bar{z}^*)e^{-\frac{1}{2}\bar{z}^*\bar{z}}$$

holomorphic!

"accidentally" coincide with lowest-Landau level wavefunctions if $\bar{z} = z^*$!!!

- This is the true origin of holomorphic functions in the theory of the FQHE
- NOTHING to do with lowest Landau level states, derives from overlaps between states in a non-orthogonal overcomplete basis!
- Has obvious parallels in theory of flat-band Chern insulators, where the projected lattice-site basis is non-orthogonal and overcomplete

$$|\Psi_L^q\rangle = \prod_i \int \frac{d\bar{z}_i^* d\bar{z}_i}{2\pi} \prod_{i<j} (\bar{z}_i^* - \bar{z}_j^*)^q \prod_i e^{-\frac{1}{2}\bar{z}_i^* \bar{z}_i} |\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N\rangle$$

"Laughlin wavefunction"

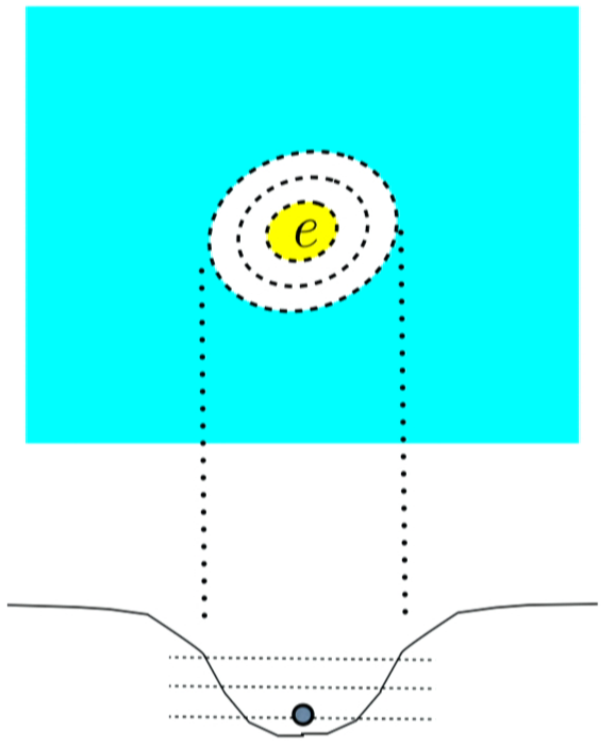
many-particle coherent state

$$\bar{a}_i|\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N\rangle = \bar{z}_i|\bar{z}_1, \bar{z}_2, l \ldots, \bar{z}_N\rangle$$

- The metric is a physical degree of freedom that characterizes the **shape** of the correlation hole surrounding a particle in the Laughlin state
- The $1/q$ Laughlin state can be characterized as describing a "condensate" of "composite bosons" formed by "attaching" q "flux quanta" (orbitals) to the particles.
- more generally, the composite boson is formed by attaching q "flux quanta" to p particles.

The metric describes the shape of the composite boson

$1/3$ Laughlin state



If the central orbital is filled, the next two are empty

The composite boson has inversion symmetry about its center

It has a "spin"

$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$		
1	0	0	.....	$L = \frac{1}{2}$
$-$ $\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	.....	$- L = \frac{3}{2}$

$$s = -1$$

the electron excludes other particles from a region containing 3 flux quanta, creating a potential well in which it is bound

- The composite boson behaves as a neutral particle because the Berry phase (from the disturbance of the the other particles as its "exclusion zone" moves with it) cancels the Bohm-Aharonov phase

- It behaves as a boson provided its statistical spin cancels the particle exchange factor when two composite bosons are exchanged

p particles	$(-1)^{pq} = (-1)^p$	fermions
q orbitals	$(-1)^{pq} = 1$	bosons

- The shape of the composite boson is determined by minimizing the sum of the correlation energy and the background potential energy.
- If there is no background potential, the metric is flat and the charge density is uniform
- If there is a background potential $g_{ab}(\boldsymbol{r})$ varies with position to give a charge density fluctuation

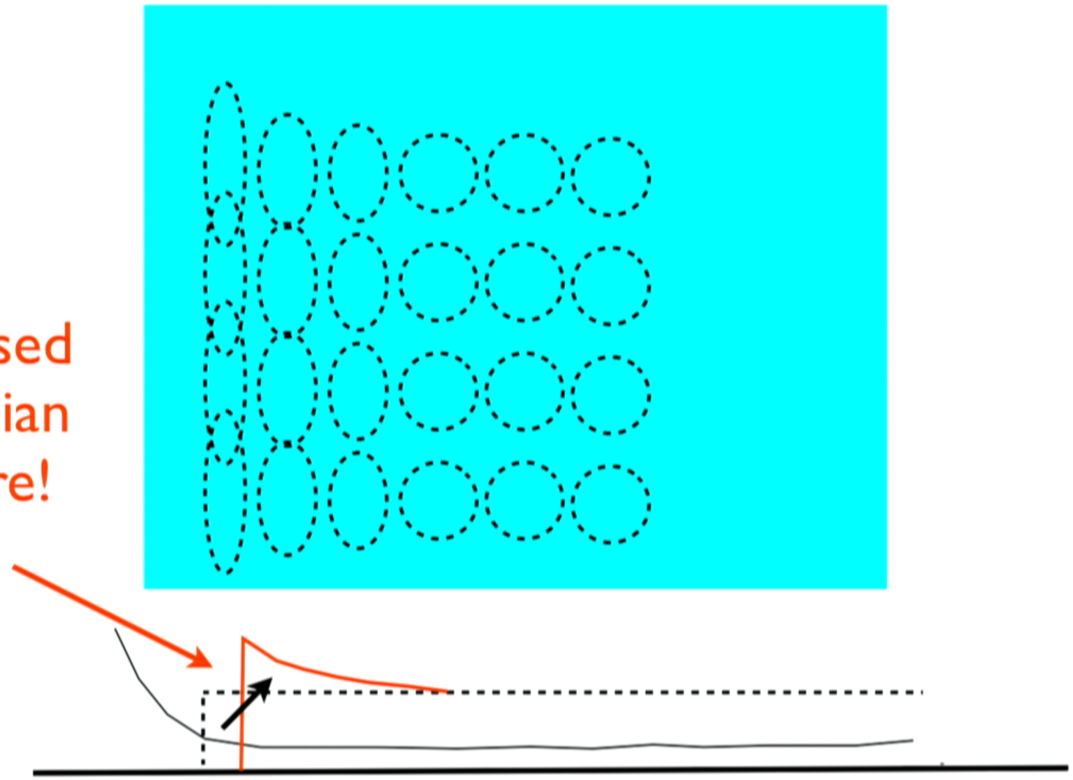
$$\delta\rho(\boldsymbol{r}) = esK(\boldsymbol{r})$$

"spin"

Gaussian curvature of metric

$$K(r) = \underbrace{\tfrac{1}{2}\partial_a\partial_b g^{ab}}_{\text{from variation of second moment of charge distribution}} + \underbrace{\tfrac{1}{8}g_{ab}\epsilon_{cd}\epsilon^{ef}\partial_e g^{ac}\partial_f g^{bd}}_{\text{from Berry phase associated with shape change}}$$

- metric deforms (preserving $\det\ g = 1$)in presence of non-uniform electric field

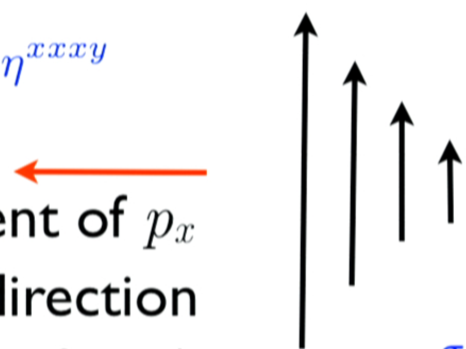


produces a dipole momemt

- Hall viscosity $\eta^{abcd} = \frac{eBs}{4\pi q}\frac{1}{2}\left(g^{ac}\epsilon^{bd} + g^{bd}\epsilon^{ac} + a \leftrightarrow b\right)$

(plus a similar term from the Landau orbit degrees of freedom (Avron et al))

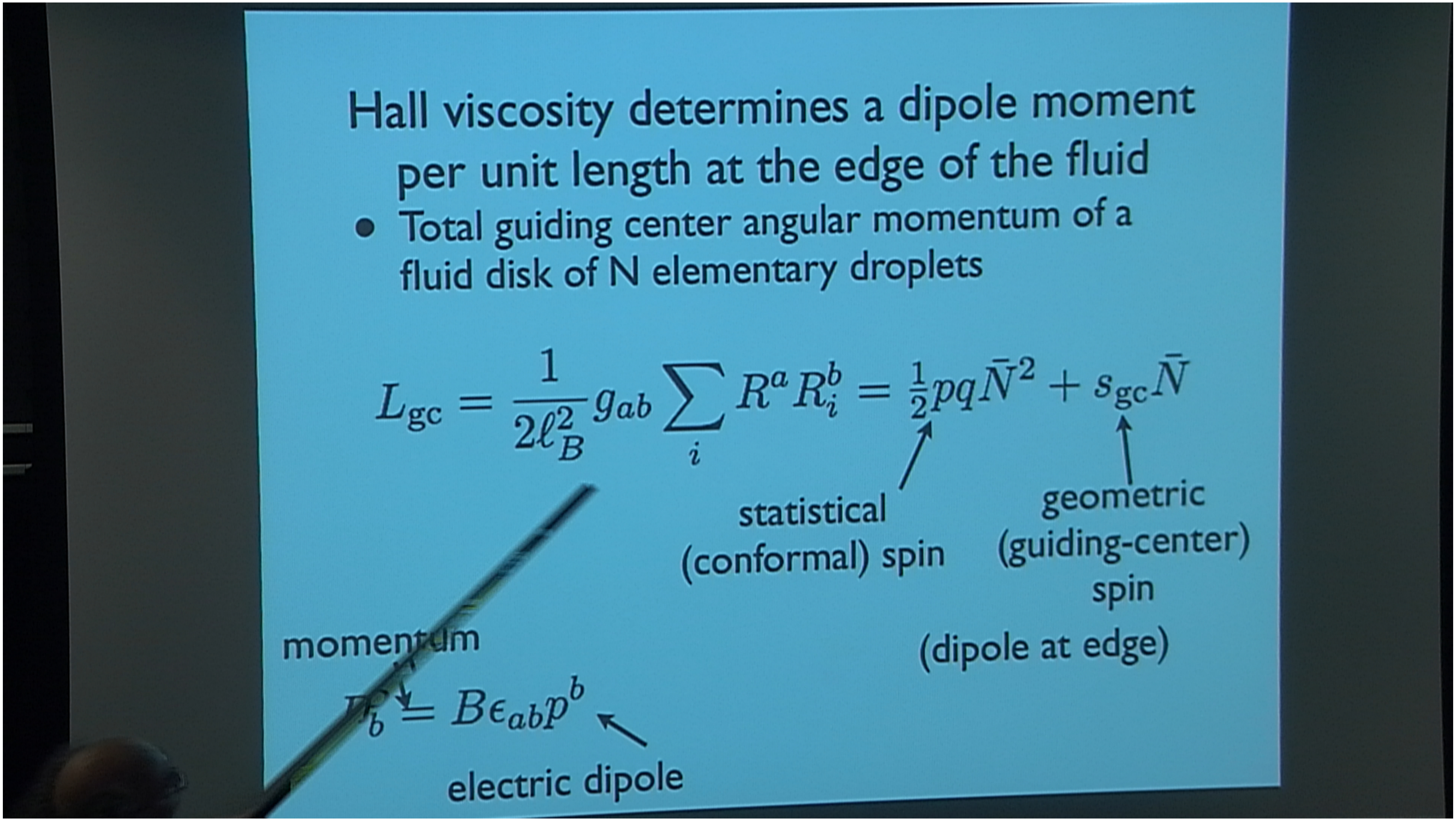
η^{xxxy}

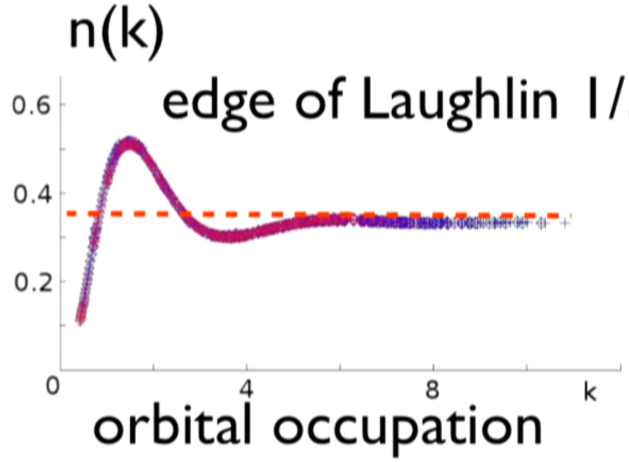


current of p_x
in x-direction
(stress force)

$\frac{dv^y}{dx}$

$\sigma^a_b = \epsilon_{be}\eta_H^{aecf}\epsilon_{cf}\partial_c v^d$


Hall viscosity determines a dipole moment per unit length at the edge of the fluid
• Total guiding center angular momentum of a fluid disk of N elementary droplets
$L_{\rm gc} = \frac{1}{2\ell_B^2} g_{ab} \sum_i R^a R_i^b = \frac{1}{2}pq\bar{N}^2 + s_{\rm gc}\bar{N}$
statistical (conformal) spin
geometric (guiding-center) spin
(dipole at edge)
momentum
$= B\epsilon_{ab}p^b$
electric dipole

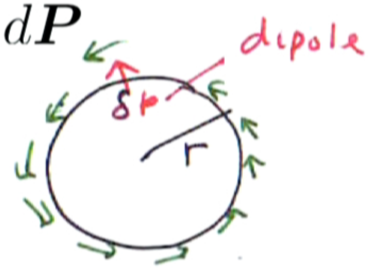


The dipole at a segment of the edge has a momentum

$$dP_a = \frac{\hbar}{e\ell_B^2}\epsilon_{ab}dp^b$$

momentum dipole

momentum $d\boldsymbol{P}$



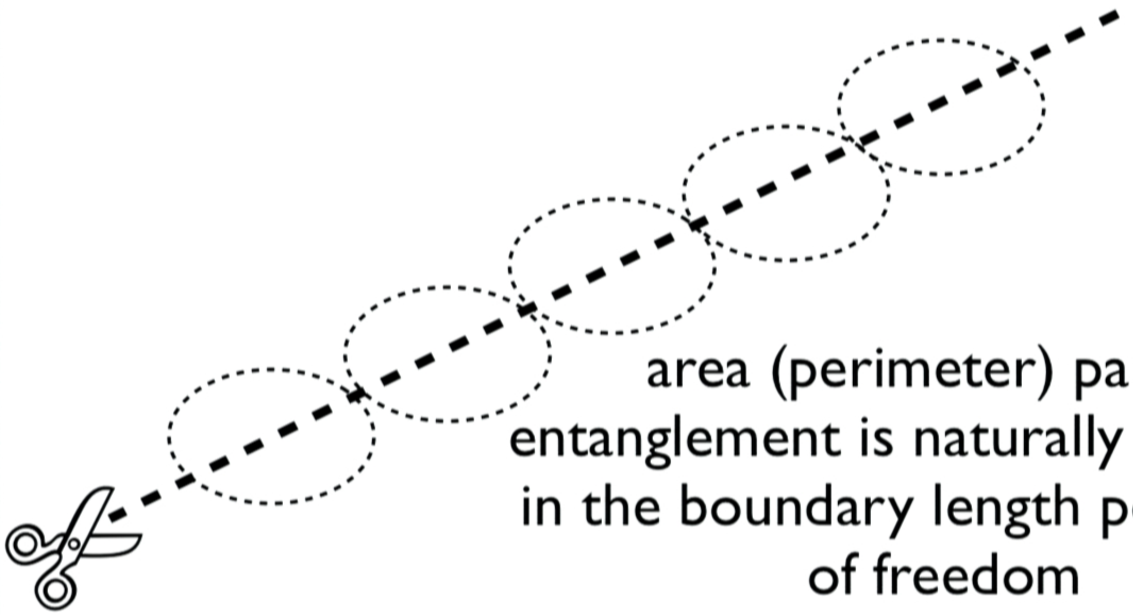
circular droplet

doesn't contribute to total momentum:

$$\oint dP_a = 0$$

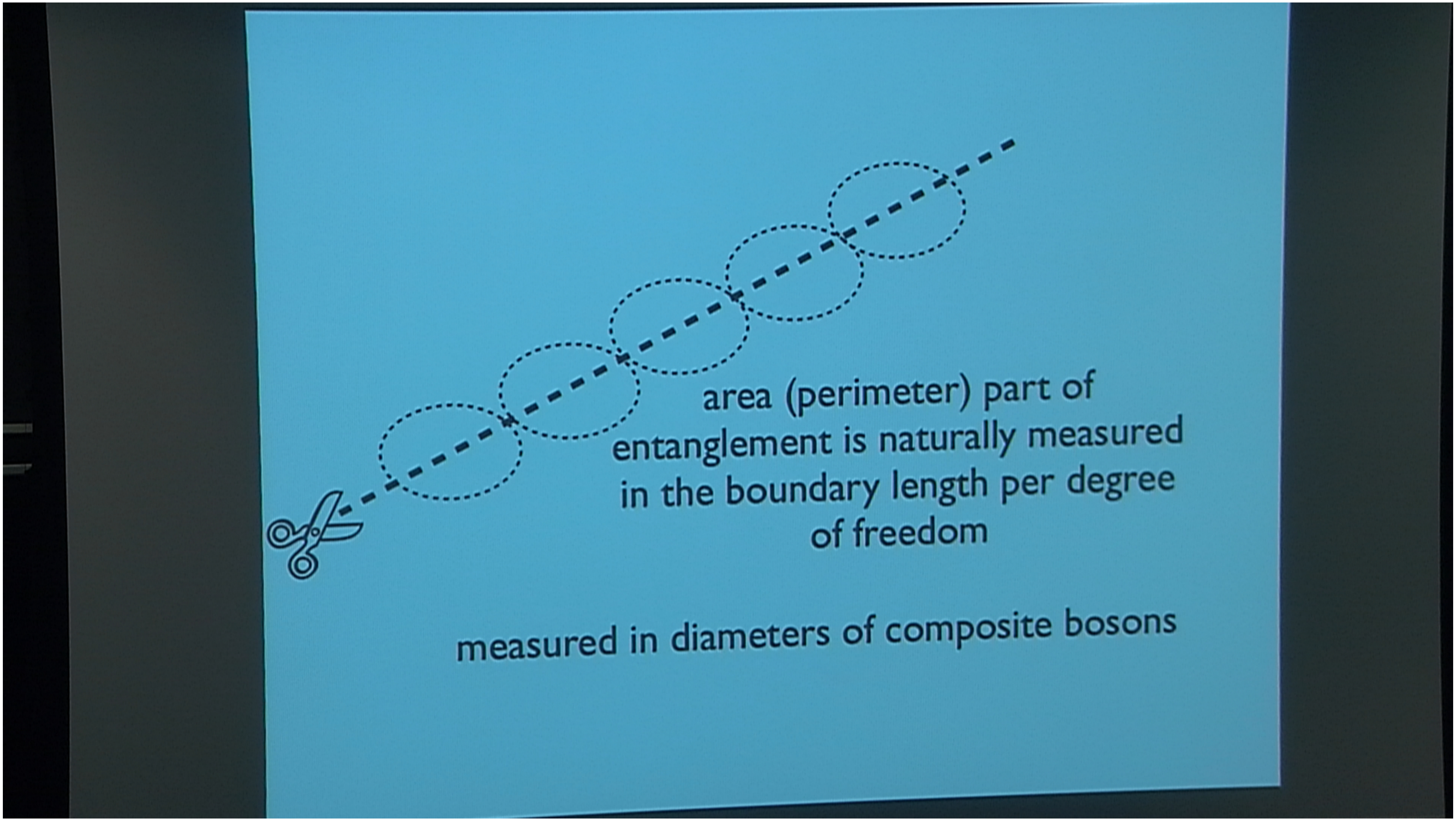
it does contribute an extra term to total angular momentum:

$$\Delta L^z(\boldsymbol{g}) = \hbar \oint \epsilon^{ab} g_{bc} r^c dP_a \neq 0$$



area (perimeter) part of
entanglement is naturally measured
in the boundary length per degree
of freedom

measured in diameters of composite bosons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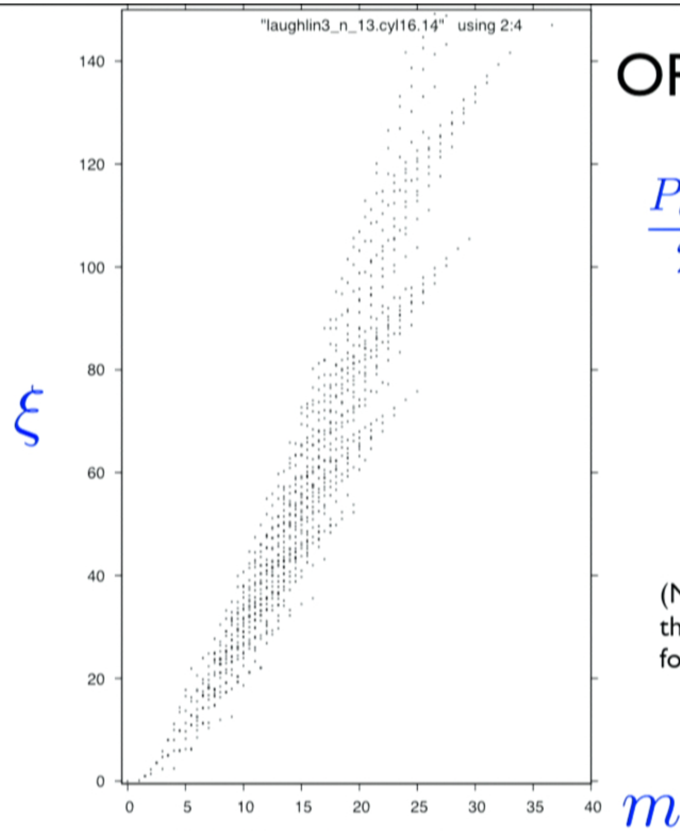


ORBITAL CUT

$$\frac{P_a L^a}{2\pi} = \frac{\sum_\alpha m_\alpha e^{-\xi_\alpha}}{\sum_\alpha e^{-\xi_\alpha}} = \eta_H^{cd}\epsilon_{ac}\epsilon_{bd}\frac{L^a L^b}{2\pi\ell_B^2}$$

$$+\frac{1}{24}(c-\nu) - h$$

conformal anomaly

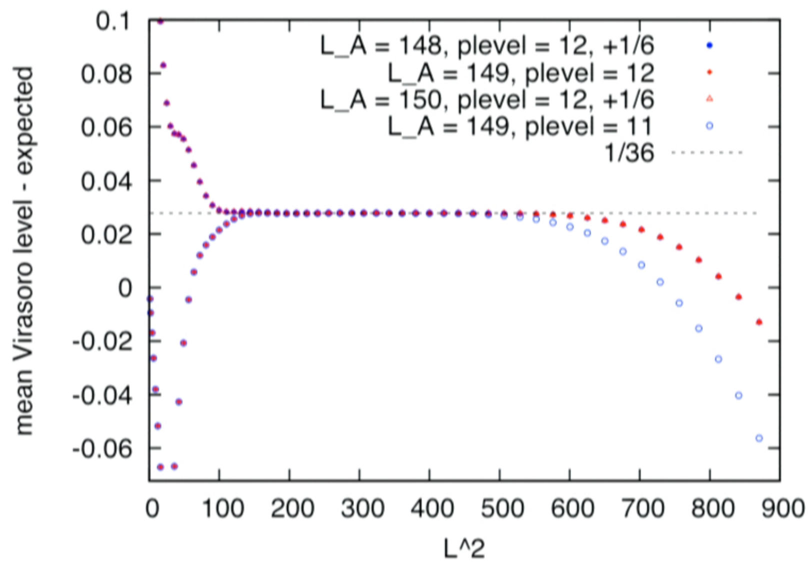
chiral anomaly

virasoro level of sector

(NOT "real-space cut" which requires the Landau orbit degrees of freedom and their form factor to be included

- Hall viscosity gives "thermally excited" momentum density on entanglement cut, relative to "vacuum", at von Neumann temperature T = 1

Yeje Park, Z Papic, N Regnault



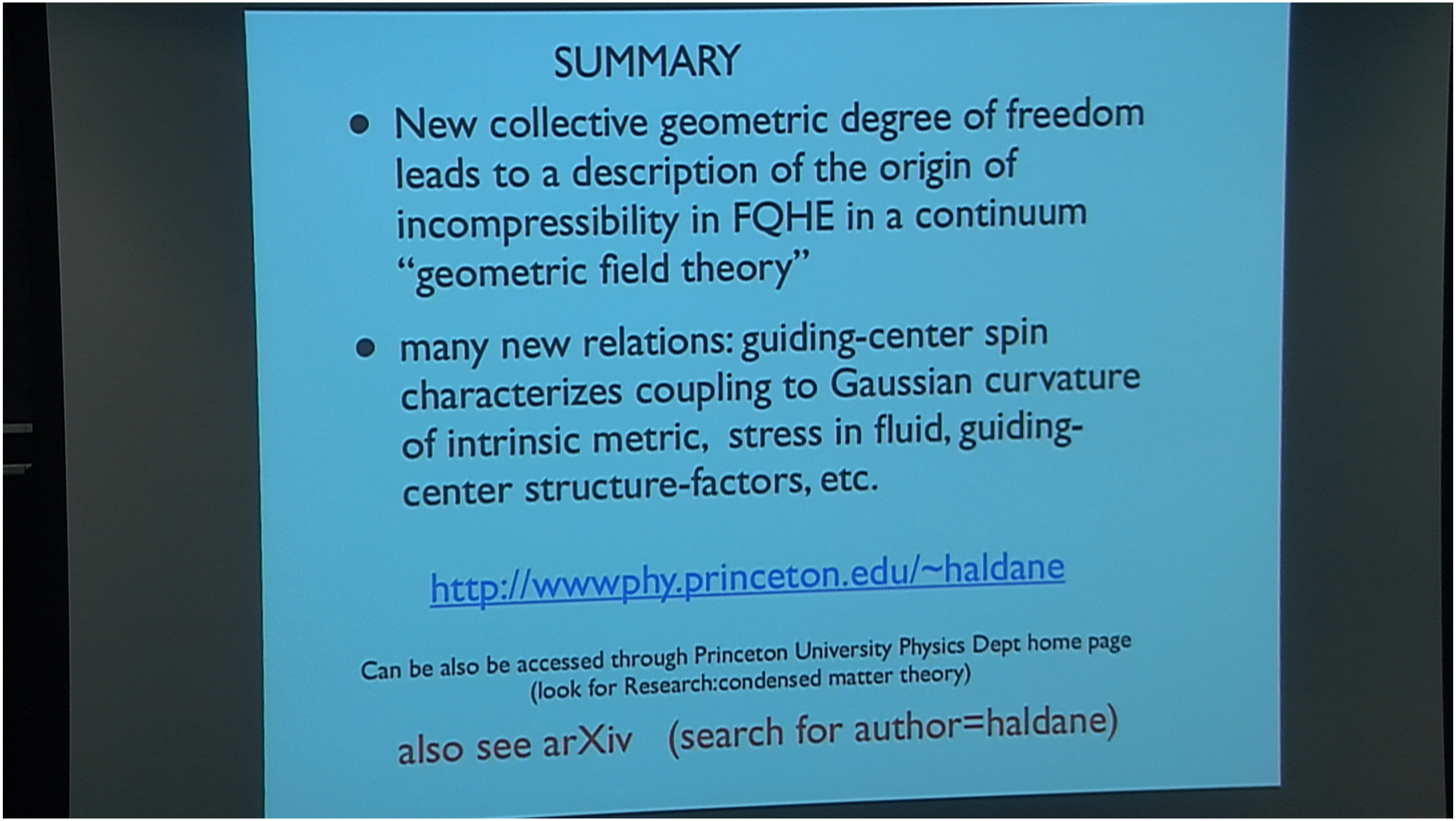
SUMMARY

- New collective geometric degree of freedom leads to a description of the origin of incompressibility in FQHE in a continuum "geometric field theory"
- many new relations: guiding-center spin characterizes coupling to Gaussian curvature of intrinsic metric, stress in fluid, guiding-center structure-factors, etc.

http://wwwphy.princeton.edu/~haldane

Can be also be accessed through Princeton University Physics Dept home page
(look for Research:condensed matter theory)

also see arXiv (search for author=haldane)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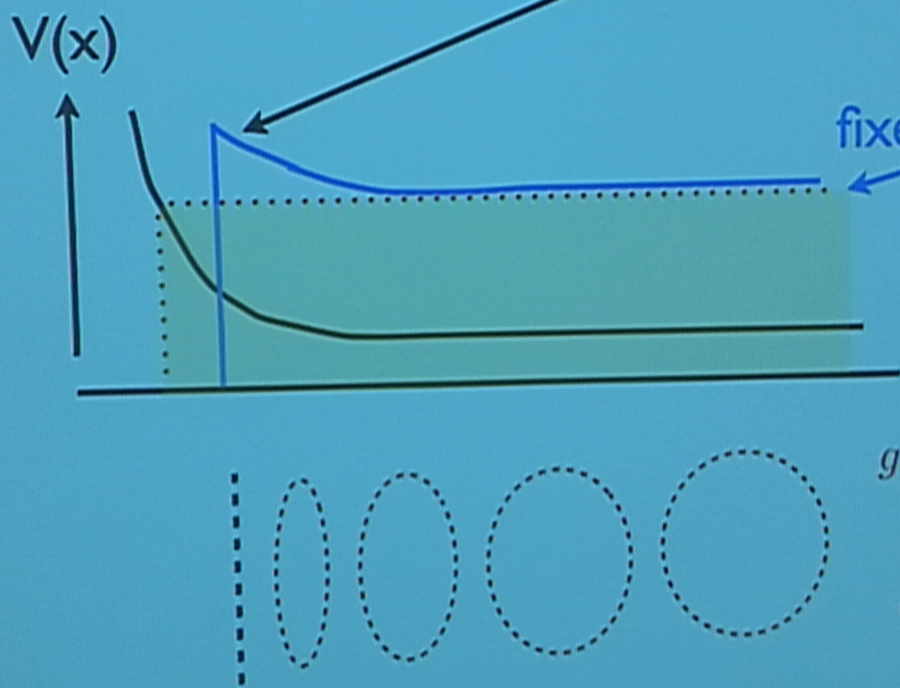
- near edges:

fluid is compressed at edges by creating Gaussian curvature

$$\delta J^0_e = \frac{e^* s}{2\pi} J^0_g$$

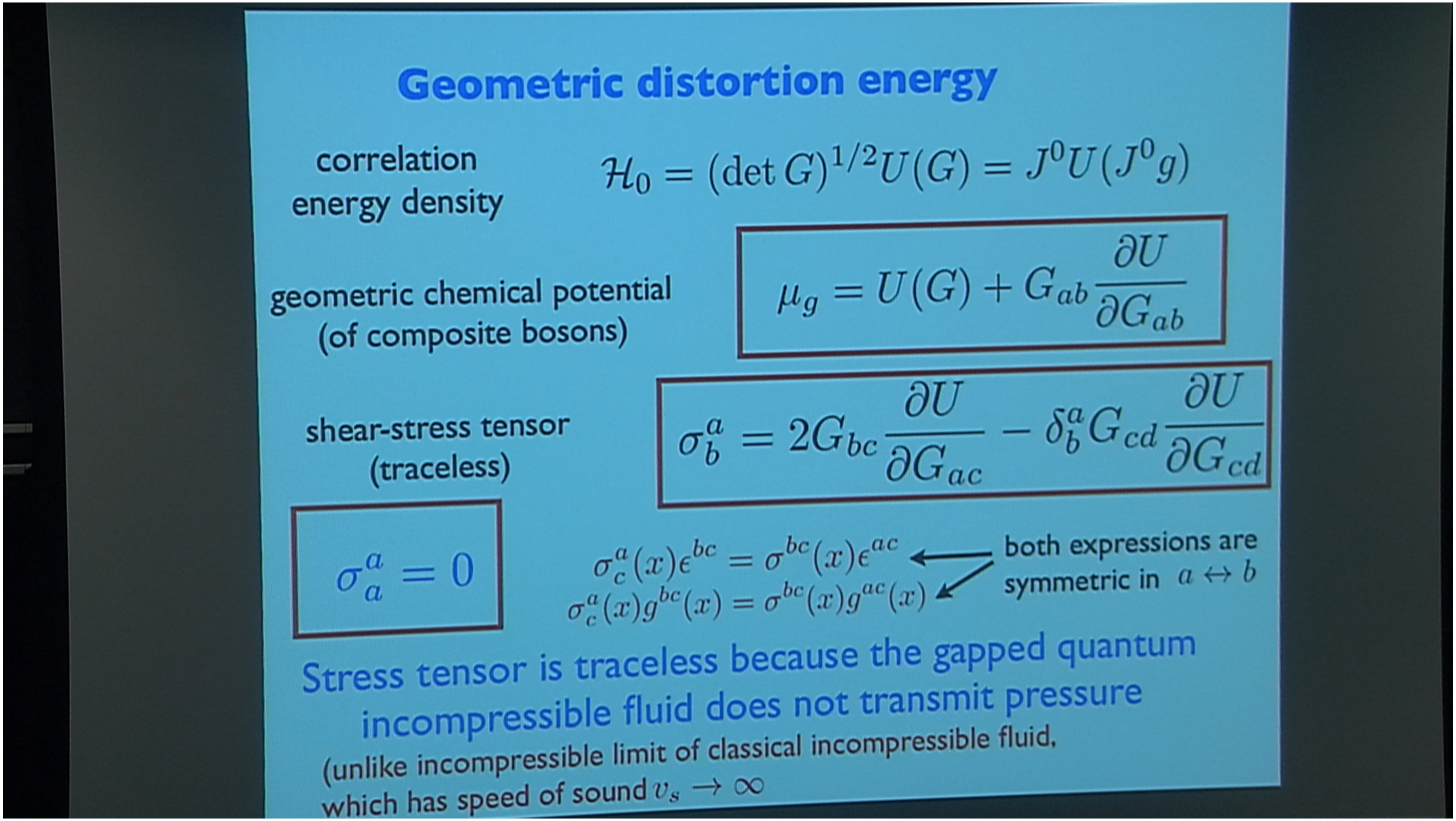
fluid density fixed by flux density



$$g = \begin{pmatrix} \alpha(x) & 0 \\ 0 & \frac{1}{\alpha(x)} \end{pmatrix}$$

$$J^0_g = -\frac{1}{2}\frac{d^2}{dx^2}\frac{1}{\alpha(x)}$$

For larger s, fluid becomes more compressible (less distortion needed for a given density change)


Geometric distortion energy
correlation energy density
$\mathcal{H}_0 = (\det G)^{1/2} U(G) = J^0 U(J^0 g)$
geometric chemical potential (of composite bosons)
$\mu_g = U(G) + G_{ab} \frac{\partial U}{\partial G_{ab}}$
shear-stress tensor (traceless)
$\sigma^a_b = 2 G_{bc} \frac{\partial U}{\partial G_{ac}} - \delta^a_b G_{cd} \frac{\partial U}{\partial G_{cd}}$
$\sigma^a_a = 0$
$\sigma^a_c(x)\epsilon^{bc} = \sigma^{bc}(x)\epsilon^{ac}$
$\sigma^a_c(x) g^{bc}(x) = \sigma^{bc}(x) g^{ac}(x)$
both expressions are symmetric in $a \leftrightarrow b$
Stress tensor is traceless because the gapped quantum incompressible fluid does not transmit pressure
(unlike incompressible limit of classical incompressible fluid, which has speed of sound $v_s \to \infty$