Title: Emergence and Entanglement in Matrix Product States

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Abstract:

Emergence and entanglement

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Overview

- Gapped quantum phases

- Area laws
  - Entanglement rates
  - Quasi-adiabatic evolution
  - Stability of Area law

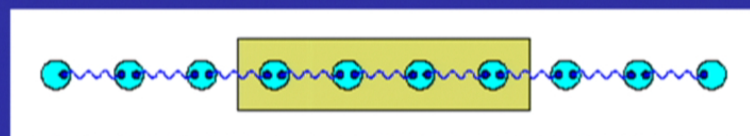
- Post MPS/PEPS methods
  - Tangent plane of MPS
  - structure of elementary excitations

Gapped quantum spin systems

- The library of exotic phases will be filled with a wealth of strongly correlated quantum systems within the near future
  - Cfr. Previous talks on gapped spin liquids, topogical quantum order, …
  - We would like to understand a few basic features of such systems:
    - Area law for the entanglement entropy?
    - Locality of elementary excitations?
    - Finite temperature regime?

Known facts about area laws

- Gapped quantum spin system in one dimension:
  - Obeys area law (Hastings)



    - Entropy is scaling like $\mathcal{O}\left(\frac{log^3 d}{\gamma}\right)$ (Arad, Kitaev, Landau, Vazirani)

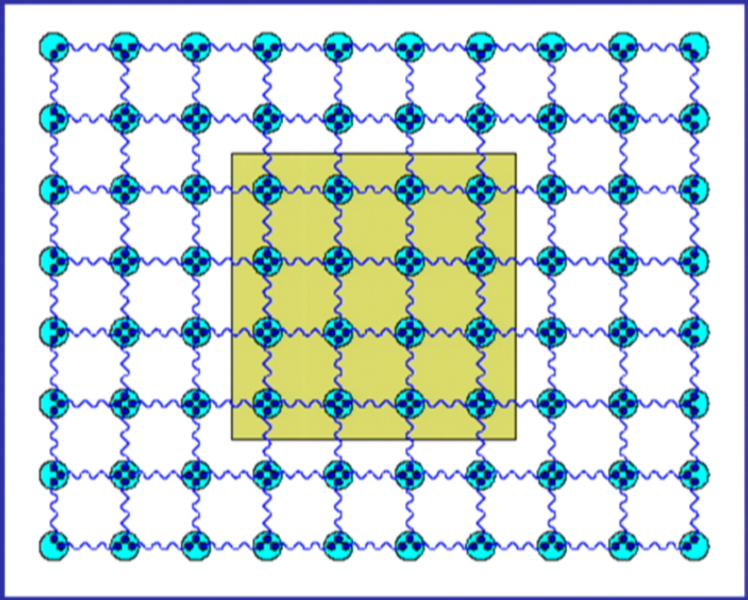
  - A consequence is the fact that ground states of gapped systems can effectively be parameterized and classified by matrix product states (MPS)

Area laws in higher dimensions?

- Existence of are law for entanglement entropy in the ground state of gapped quantum spin systems is very plausible but has not been proven

  - OK under certain conditions: density of states, ….

  - For which states/phases can one get an efficient PEPS description?



- In this talk: much more modest goals than proving area laws; instead, we will prove that a ground state satisfies an area law if and only if any other state in the same phase also exhibits an area law

Proving "stability" of area law in a quantum phase

- We define 2 ground states to be in the same phase iff there exists a 1-parameter family of local gapped Hamiltonians H(p) such that the starting and end point of this path correspond to the 2 ground states and H(p) remains gapped everywhere

- Basic strategy for proving stability result:
  - Given the adiabatic path, use Lieb-Robinson bounds to make an exact quasi-local evolution from it (quasi-adiabatic evolution)
    - Adiabatic evolution is not good enough: Fannes inequality!

$$|S(\rho) - S(\sigma)| \leq \|\rho - \sigma\|_1 \log(d-1)$$

  - For every term in the quasi-adiabatic Hamiltonian, bound the amount of entanglement that can be created
    - What is the optimal entanglement rate when acting on a subsystem?

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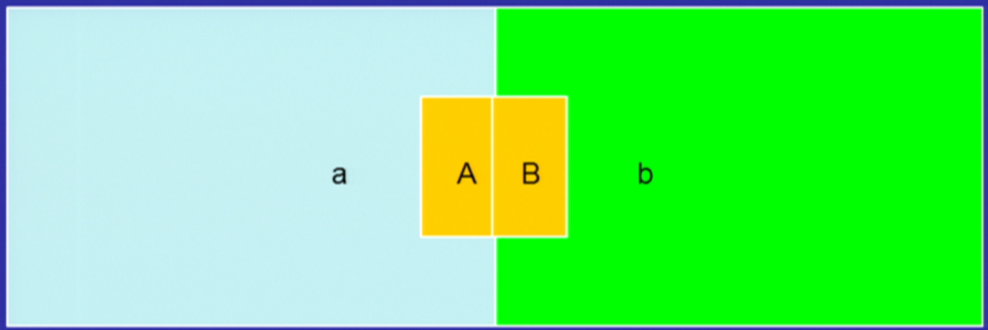
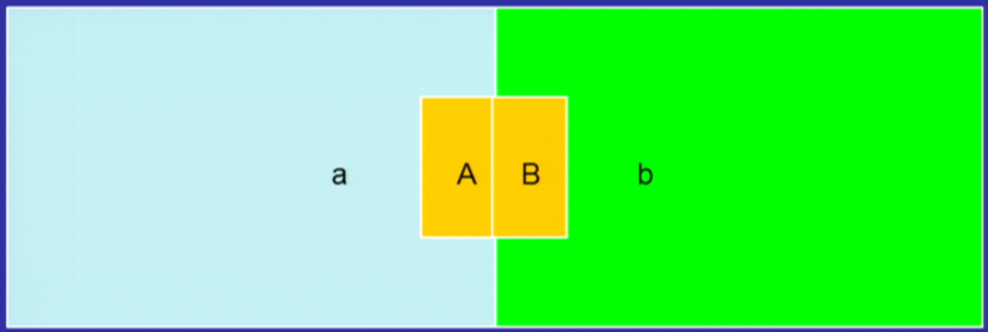
Entanglement rates



- Given a Hamiltonian acting nontrivially on A and B; how much entanglement between (Aa) and (Bb) can be created per unit time?

$$\Gamma = \frac{\partial}{\partial t} S\left(\rho_{Aa}\right)$$

Entanglement rates



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$$\Gamma = \frac{\partial}{\partial t} S\left(\rho_{Aa}\right)$$

$$\max_{||H||=1} |\Lambda(p)| = 2 \max_{P} | Tr\,(P[X, \log Y]) |$$

$$||H|| = 1, \quad TrX = p, \quad TrY = 1 \quad 0 \leq X \leq Y$$

$$X = Y^{1/2} Z Y^{1/2}$$

$$0 \leq Z \leq I$$

- W.l.o.g. we assume Y to be diagonal with eigenvalues y_i

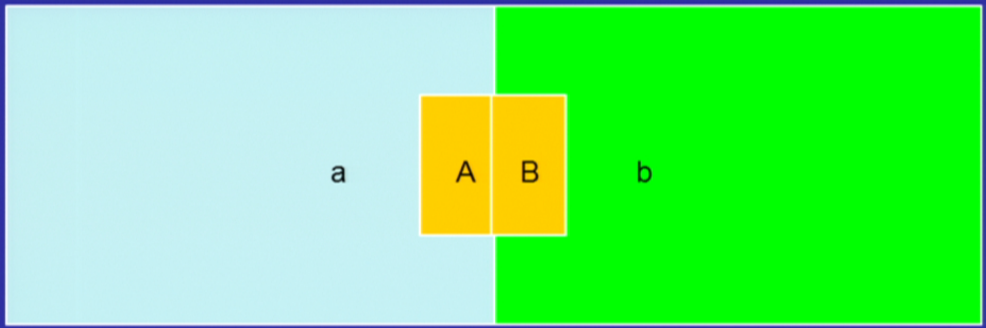
$$2|Tr(P[X, \log Y])| = 2|\sum_{i<j} \log \frac{y_i}{y_j} y_i^{1/2} y_j^{1/2} (Z_{ij} P_{ji} - Z_{ji} P_{ij})|$$

- Two different regimes: when

$$\frac{y_i}{y_j} \leq p \qquad \frac{y_i}{y_j} > p$$

- Hence:

$$\Gamma = \frac{1}{p}\Lambda(p) \leq \beta \| H \| \log \min \left(d_A, d_B \right)$$



Quasi-adiabatic evolution:

- Exact ground state evolution on an adiabatic path (Hastings '04)

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \mathcal{P}_{|\psi(t)\rangle} \frac{I}{H(t)-E_0(t)} \left(\frac{\partial}{\partial t} H(t)\right) |\psi(t)\rangle$$
$$= \int_{-\infty}^{\infty} dz F(z) e^{iHz} \left(\frac{\partial}{\partial t} H(t)\right) e^{-iHz} |\psi(t)\rangle$$

$$F(-z) = -F(z)$$

$$\forall |\omega| \geq \min_t \gamma(H(t)) : \qquad \tilde{F}(\omega) = \frac{1}{\omega}$$

$$\tilde{F}(0) = 0$$

- Every choice of F(z) satisfying the conditions leads to the exact evolution, hence defining an effective Hamiltonian. Lieb Robinson bounds allow to bound the norm of all terms involved

$$K(s) = \sum_{\mathbf{i}\in\lambda} k_{\mathbf{i}}(s)$$

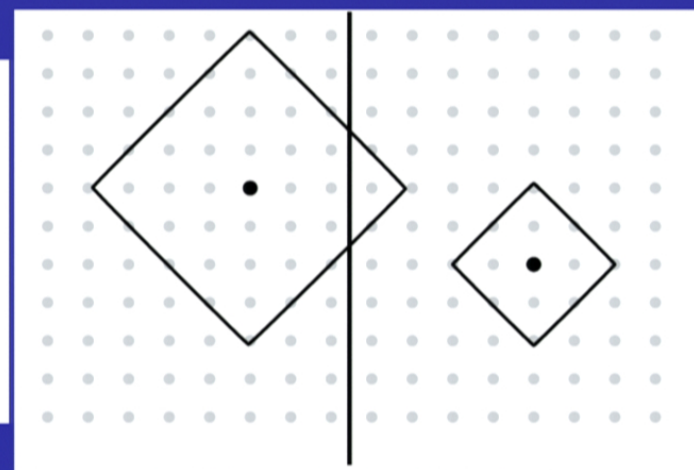
a

$$k_{\mathbf{i}}(s) = \sum_{r=0} k_{\mathbf{i}}(s,r)$$

$$||k_0(r,s)|| \lesssim r^{D-1}\frac{||h||||h'||\gamma}{\kappa^3}e^{-rv/2} + \frac{||h'||}{\gamma}\left(\frac{\xi}{r}\right)^n$$

$$\xi = v_{LR}/\gamma$$

$$\begin{aligned}
\frac{dS_L(s)}{ds} &= i\,Tr(K(s)[\,|\Psi(s)\rangle\langle\Psi(s)|\,, \log\rho_L\otimes I_R]) \\
&= i\,Tr(\sum_{\mathbf{i},r} k_{\mathbf{i}}(s,r)[\,|\Psi(s)\rangle\langle\Psi(s)|\,, \log\rho_L\otimes I_R]) \\
&\lesssim \sum_{(\mathbf{i}_P,i_O)}\sum_{r\geq i_O} \log d_l\, r^D ||k_{\mathbf{i}_P,i_O}(s,r)|| \qquad (18) \\
&\lesssim \log d_l\, A\,(\sum_r r^{D+1}||k_0(s,r)||)\,.
\end{aligned}$$



$$\frac{dS_L(s)}{ds} \lesssim A\frac{||h'(s)||}{\gamma(s)}\xi(s)^{D+2}\log d_l$$

Example:

- Suppose that we start from a product state and corresponding Hamiltonian H(0), and want evolve to the ground state of H(1). Suppose the gap along the path is always larger than γ. Then the entanglement of the ground state of H(1) is bounded above by

$$S_1 \leq A \log(d) \left(\int_0^1 dt \| \frac{\partial}{\partial t} h(t) \| \right) \frac{v_{LR}^{D+2}}{\gamma^{D+3}}$$

with A the area of the region of interest, D the spatial dimension, d the dimension of the local Hilbert space, and v_{LR} the Lieb-Robinson velocity

Feynman's idea of simulating quantum many-body systems

"It's really quite insane actually: we are trying to find the energy by taking the expectation of an operator which is located here and we present ourselves with a functional which is dependent on everything all over the map. That's something wrong. Maybe there is some way to surround the object, or the region where we want to calculate things, by a surface and describe what things are coming in across the surface. It tells us everything that's going on outside. I'm talking about a new kind of idea but that's the kind of stuff we shouldn't talk about at a talk, that's the kind of stuff you should actually do!" (1988)

This clearly hints at some holographic principle: store relevant degrees in the boundary

DMRG/MPS/PEPS are exactly implementing this idea

Area laws provide a new guiding principle for interacting systems on the lattice

- A large part of the corner in Hilbert space containing low-energy states can be parameterized by tensor network states : the entanglement is distributed in a *local* way such as to capture the area law in a way that does not lead to an exponential number of parameters
    - Matrix product states (MPS) in 1D (as also used in DMRG)
    - projected entangled pair states (PEPS) in higher dimensions
    - multiscale entanglement renormalization ansatz for critical systems

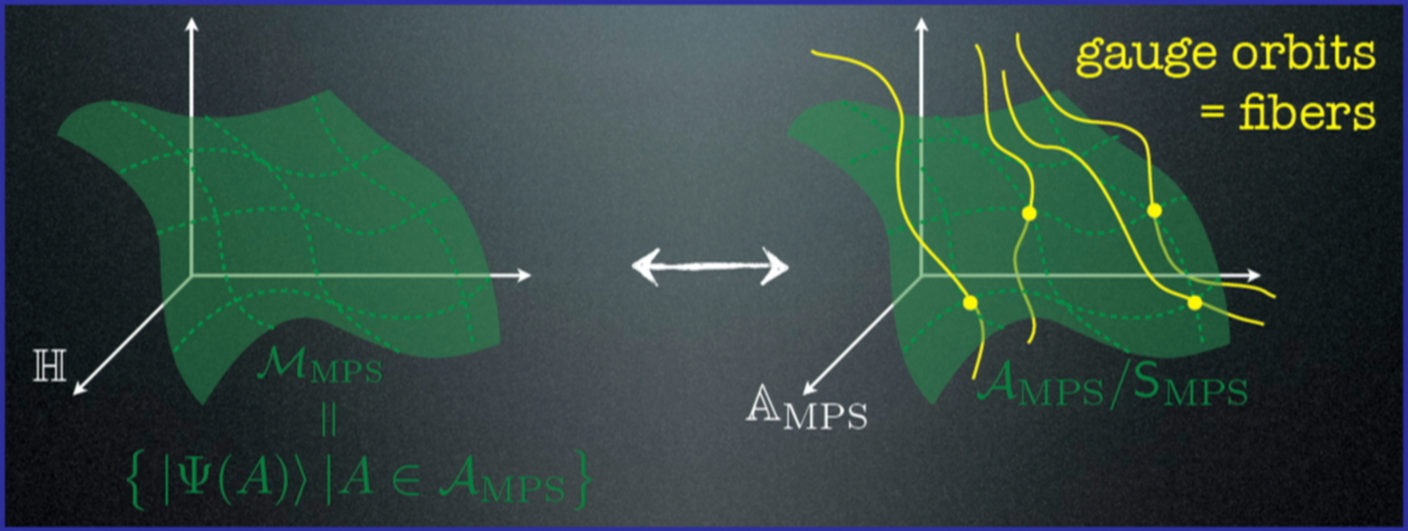
Emergence

- Vacuum (ground state) is absolute emptiness; we can only observe excitations on top of it
- Instead of focus on ground states:
  - What about physics on top of that ground state; is it possible to understand the low-energy physics (e.g. elementary excitations, low T physics) ?
    - Construction of wavefunctions corresponding to exact eigenstates, (not quasi-particles: infinite lifetime)
  - Post MPS methods: build a Hilbert/Fock space with those excitations on top of MPS/PEPS

(see Jutho Haegeman et al. arXiv:1305.1894)

Manifold of MPS: fibre bundles

Haegeman, Marien, Osborne, FV '12

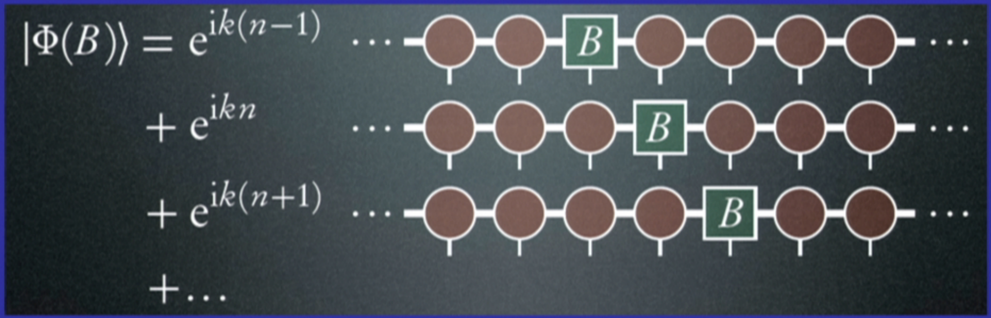


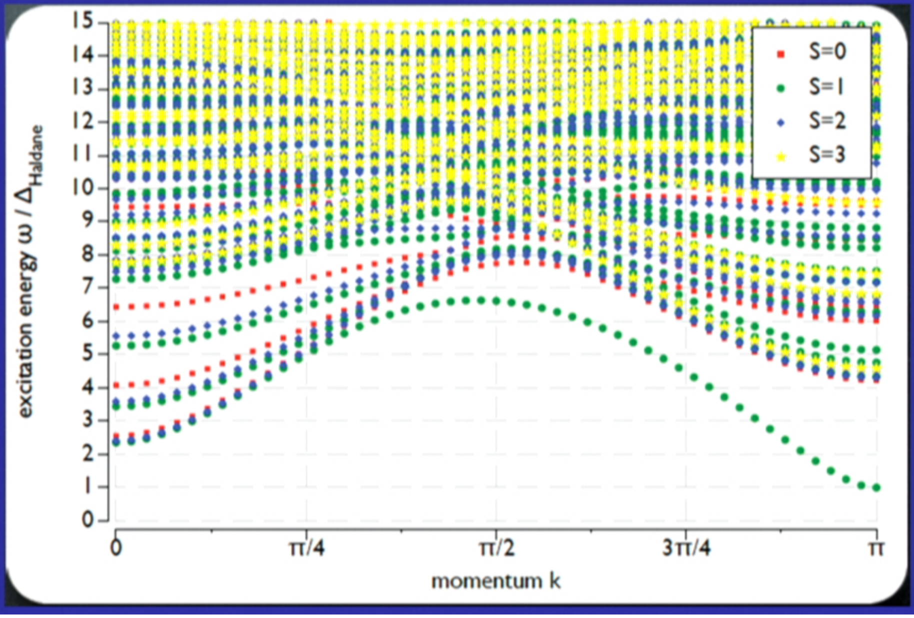
- MPS description is not unique: there are gauge degrees of freedom

$$A^i \leftrightarrow X A^i X^{-1}$$

Those gauge transformation play crucial role in classifying all phases of matter

Excitations in the tangent plane



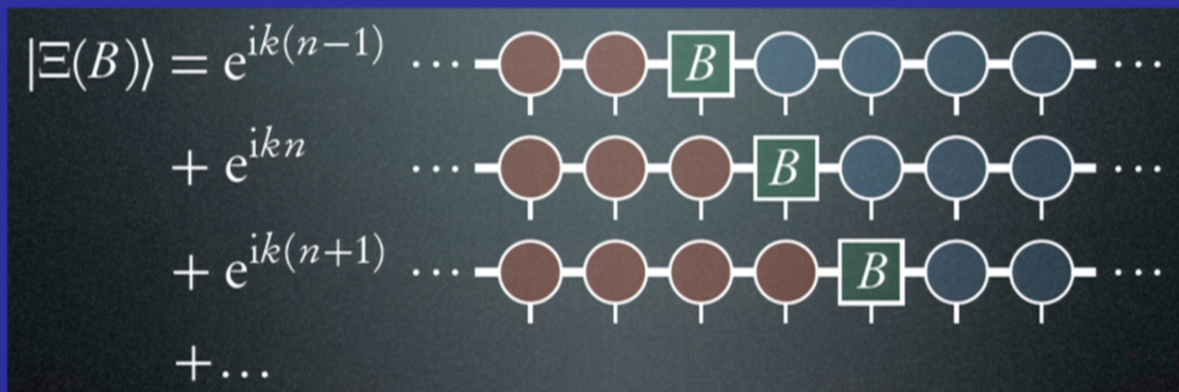


Spin 1 Heisenberg model

$$\Delta^{(\infty)}_{\text{Haldane}} = 0.410479248463^{+6\times10^{-12}}_{-3\times10^{-12}}.$$

Haegeman, FV '11

- In the case of symmetry breaking, elementary excitations are typically domain walls between the two phases: topological nontrivial excitations (cfr. Mandelstam ansatz)



Spin 1 XXZ

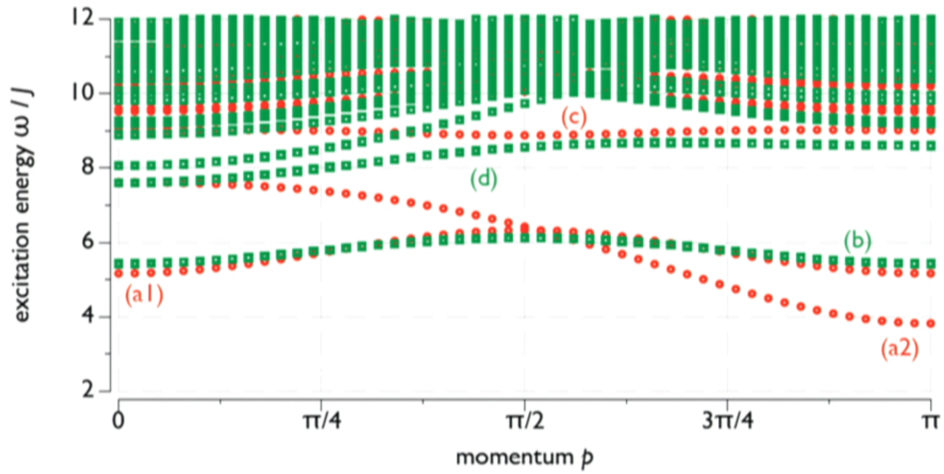


Figure 3.20: Spectrum of the lowest lying excitations of the $S = 1$ XXZ antiferromagnet with anisotropy parameter $\Delta = 3$ at $D = 32$. Red circles indicate topologically non-trivial excitations whereas green squares indicate topologically trivial excitations. We refer to the text for more information about the indicated branches.

(a) kinks with $\omega = 2\Delta J$: $|\cdots \uparrow\downarrow\uparrow\uparrow\downarrow\uparrow \cdots\rangle$ (a1) and $|\cdots \uparrow\downarrow\uparrow 0 \downarrow\uparrow \cdots\rangle$ (a2)

(b) spin deviation state with $\omega = 2\Delta J$: $|\cdots \uparrow\downarrow\uparrow 0 \uparrow\downarrow \cdots\rangle$

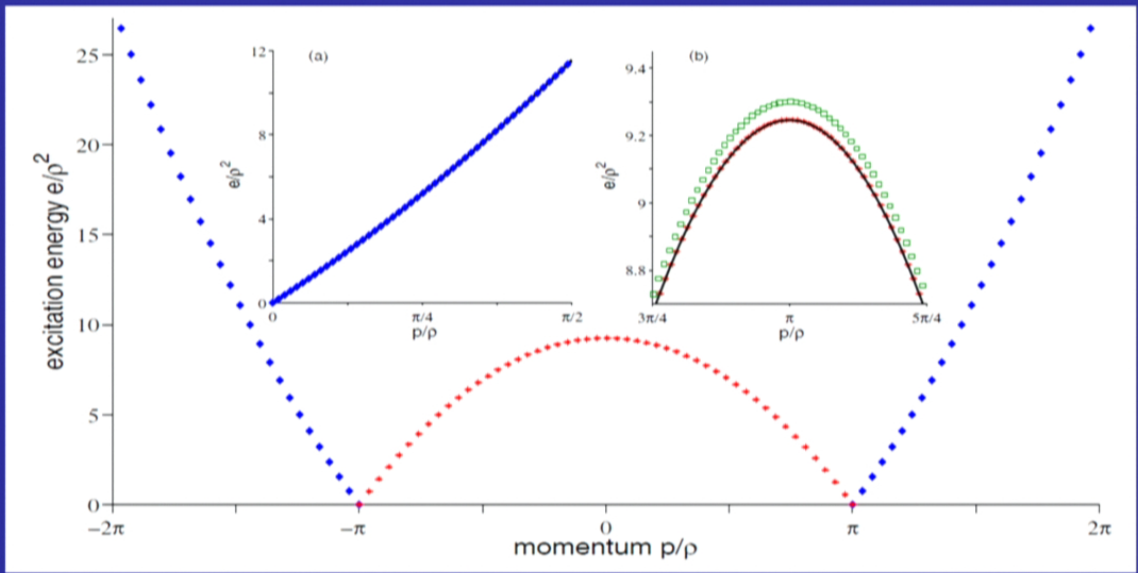
(c) topologically non-trivial bound state with $\omega = 3\Delta J$: $|\cdots \uparrow\downarrow 00 \downarrow\uparrow \cdots\rangle$

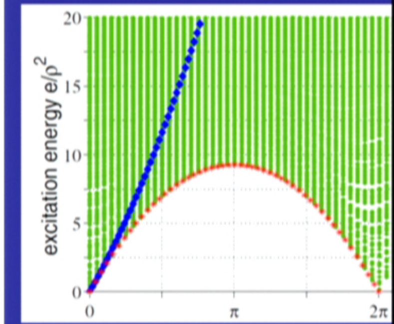
(d) topologically trivial bound state with $\omega = 3\Delta J$: $|\cdots \uparrow\downarrow 00 \uparrow\downarrow \cdots\rangle$

Haegeman et al. '11

Lieb Liniger and cMPS ansatz

$$\mathcal{H} = \int_{-\infty}^{+\infty} dx \left[\frac{d\hat{\psi}^\dagger(x)}{dx} \frac{d\hat{\psi}(x)}{dx} + c\hat{\psi}^\dagger(x)\hat{\psi}^\dagger(x)\hat{\psi}(x)\hat{\psi}(x) \right]$$





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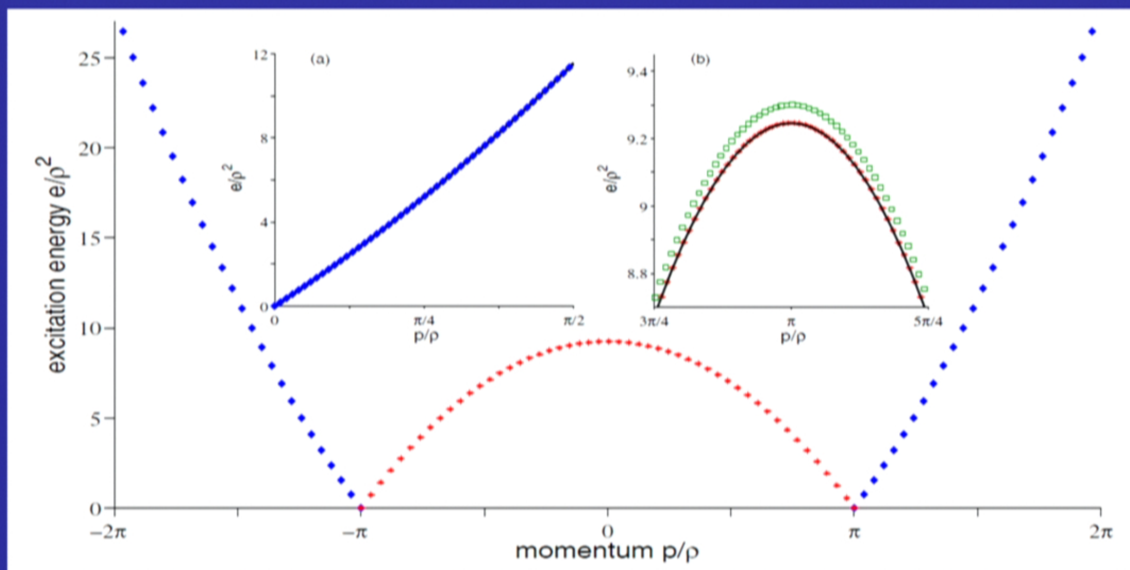

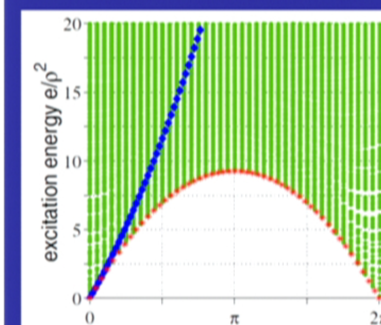

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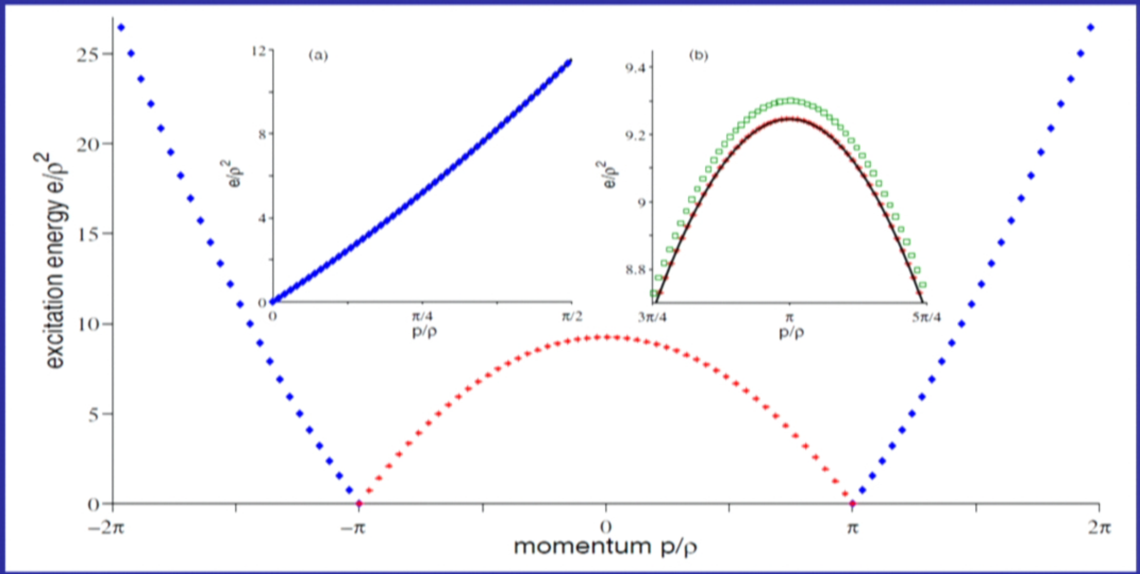

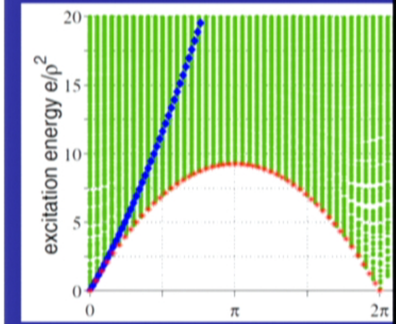

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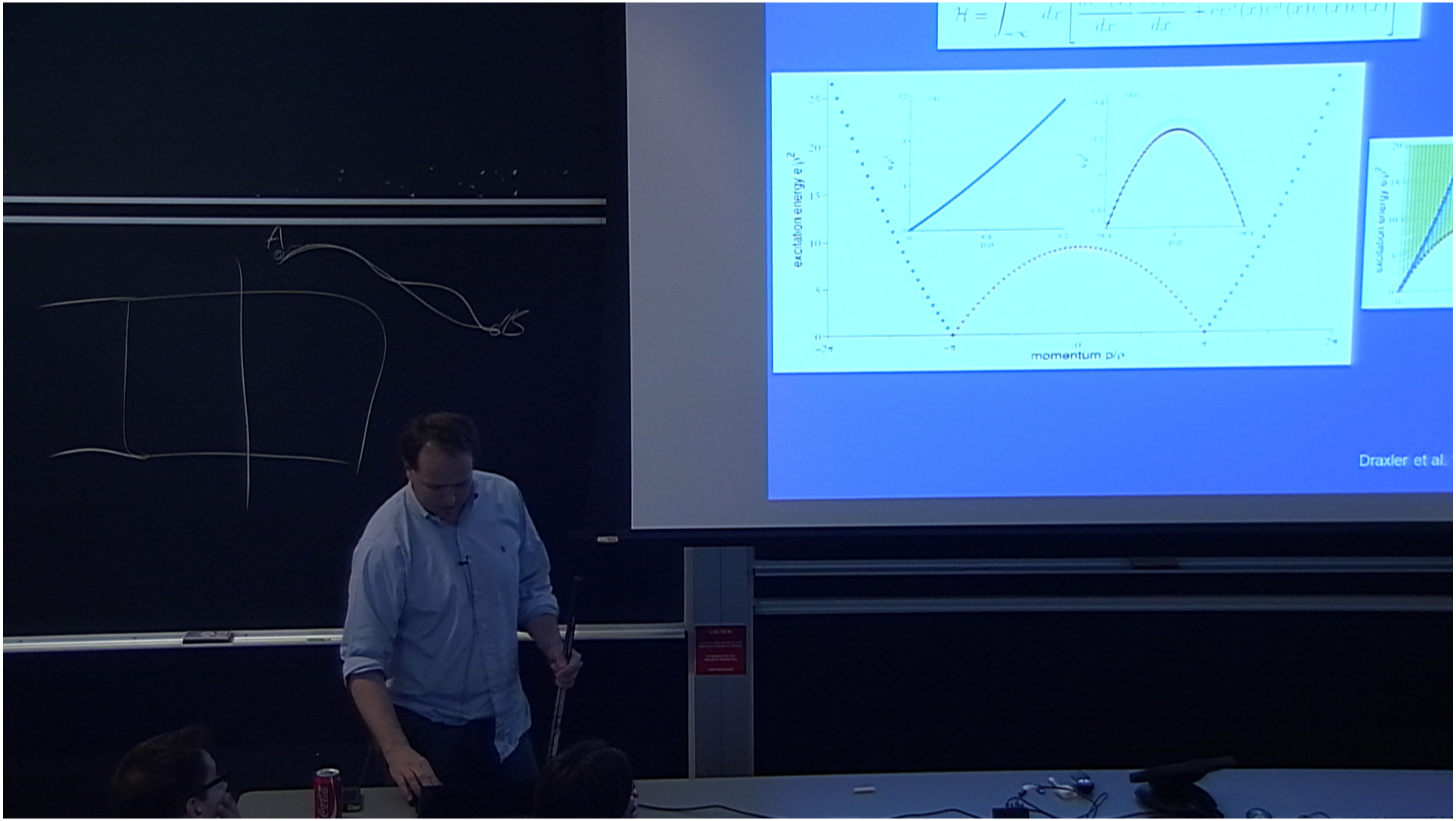
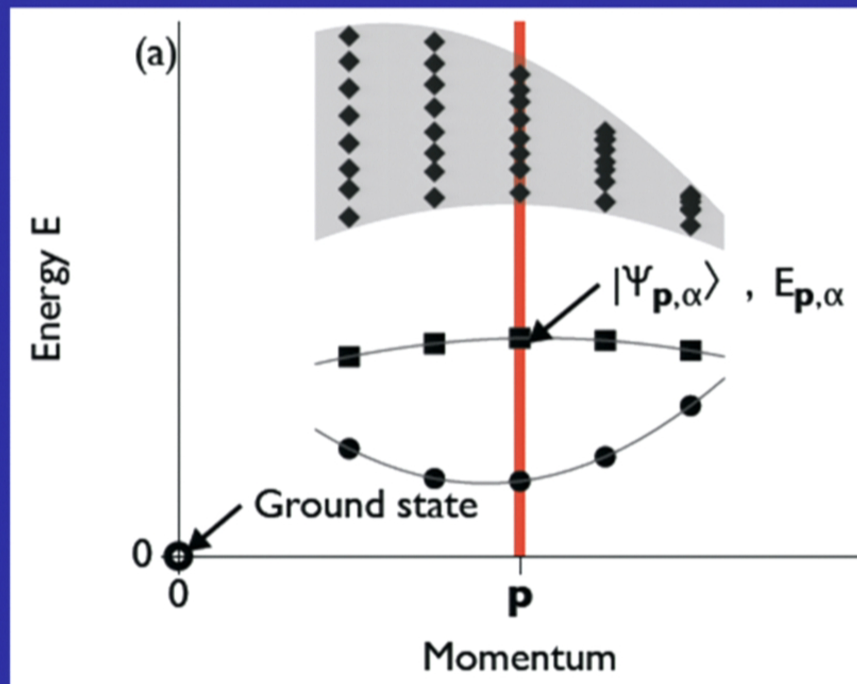
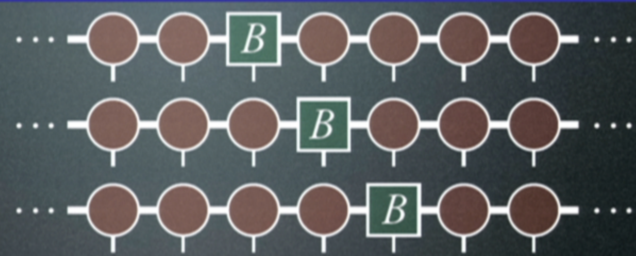

excitation energy e/ρ^2
momentum p/ρ
Draxler et al.

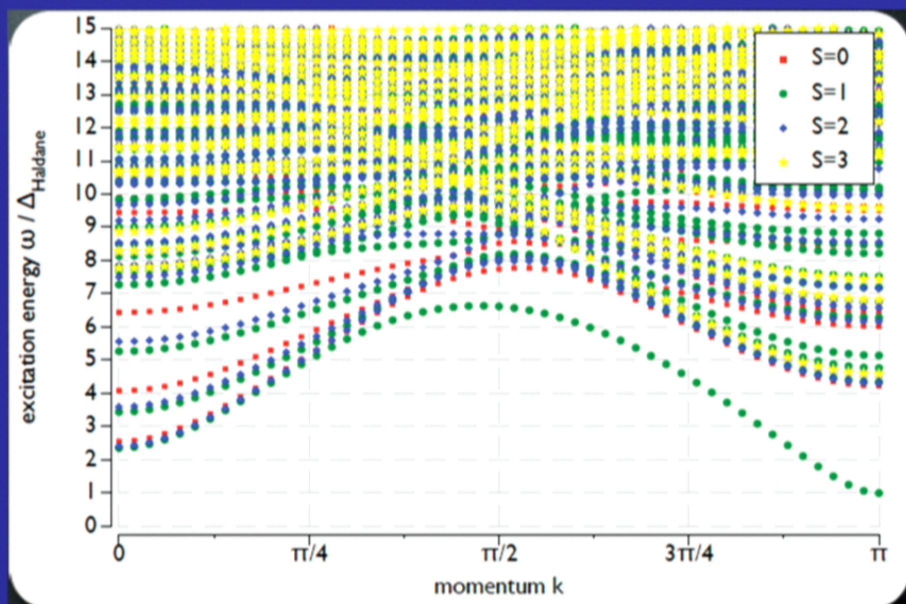
Dispersion relations for gapped quantum spin chain



Excitations in the tangent plane

$$|\Phi(B)\rangle = e^{ik(n-1)} \cdots + e^{ikn} \cdots + e^{ik(n+1)} \cdots + \ldots$$





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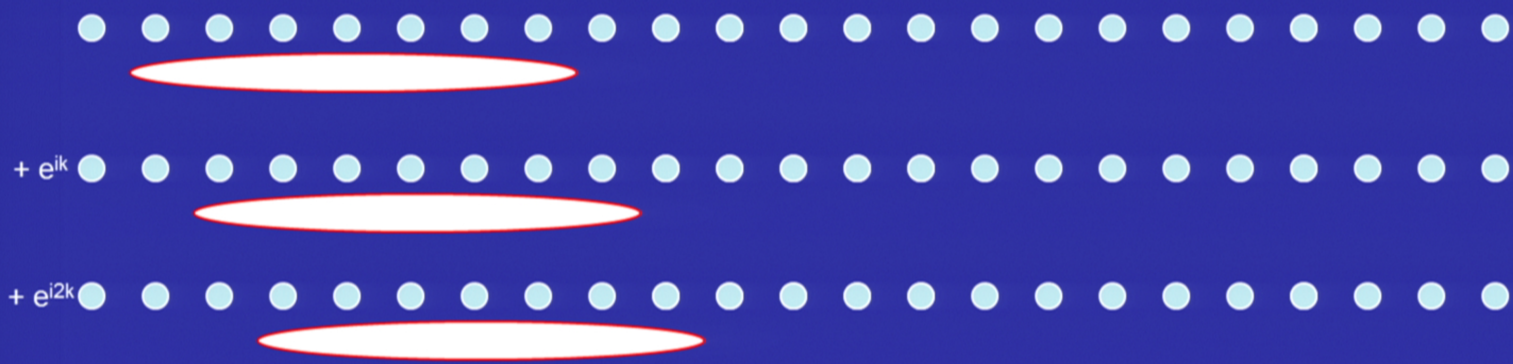
Haegeman, FV '11

Excitations

- Why does this Feynman-Bijl type ansatz for excitations work so well?
  - Consequence of Lieb-Robinson bounds: non-relativistic analogue of the particle-like excitation in QFT



- The length on which those blocks have to act is proportional to the gap between the 1-particle band and the continuum band above it

- the information of the low-lying excited states and the dispersion relation is hence encoded in the ground state;
  - Also: the entanglement entropy of the elementary excited state is equal to the one of the ground state + 1

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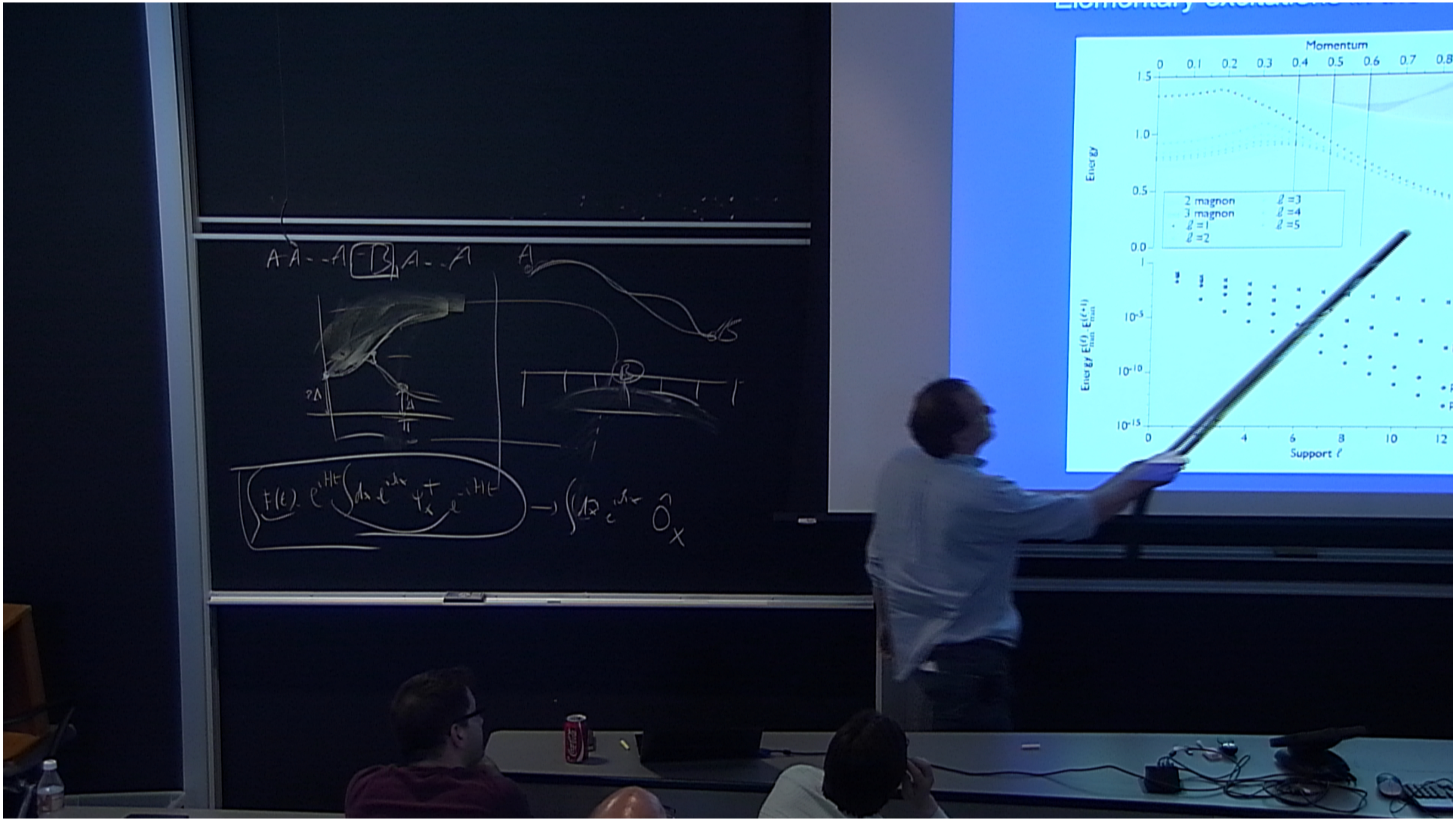
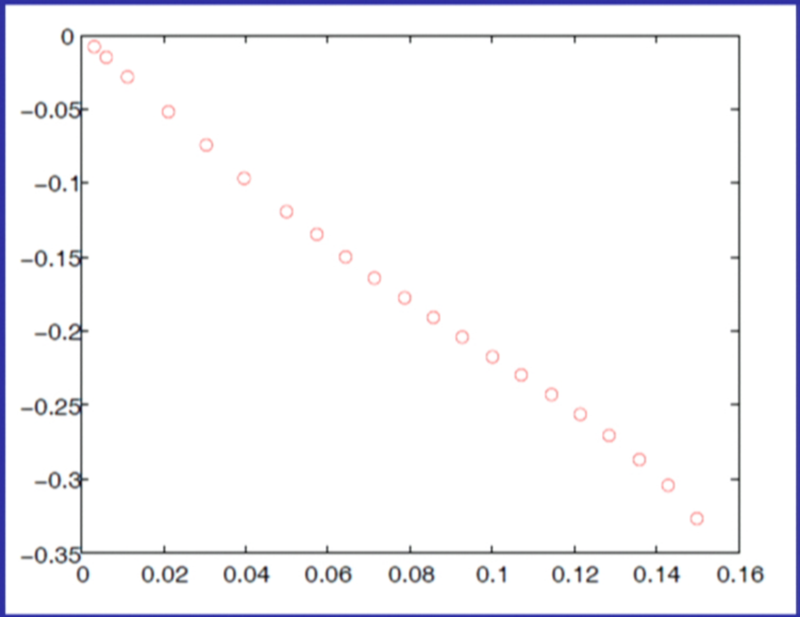

Momentum
Energy
2 magnon
3 magnon
Support ℓ

- This is a starting point for building a Fock/Hilbert space on top of ground state in a systematic way

- First step: determine S-matrix of 2-particle excitation
  - Example: scattering state of 2 particles with momentum $\pi \pm x$ in spin 1 Heisenberg model, and determination of scattering length



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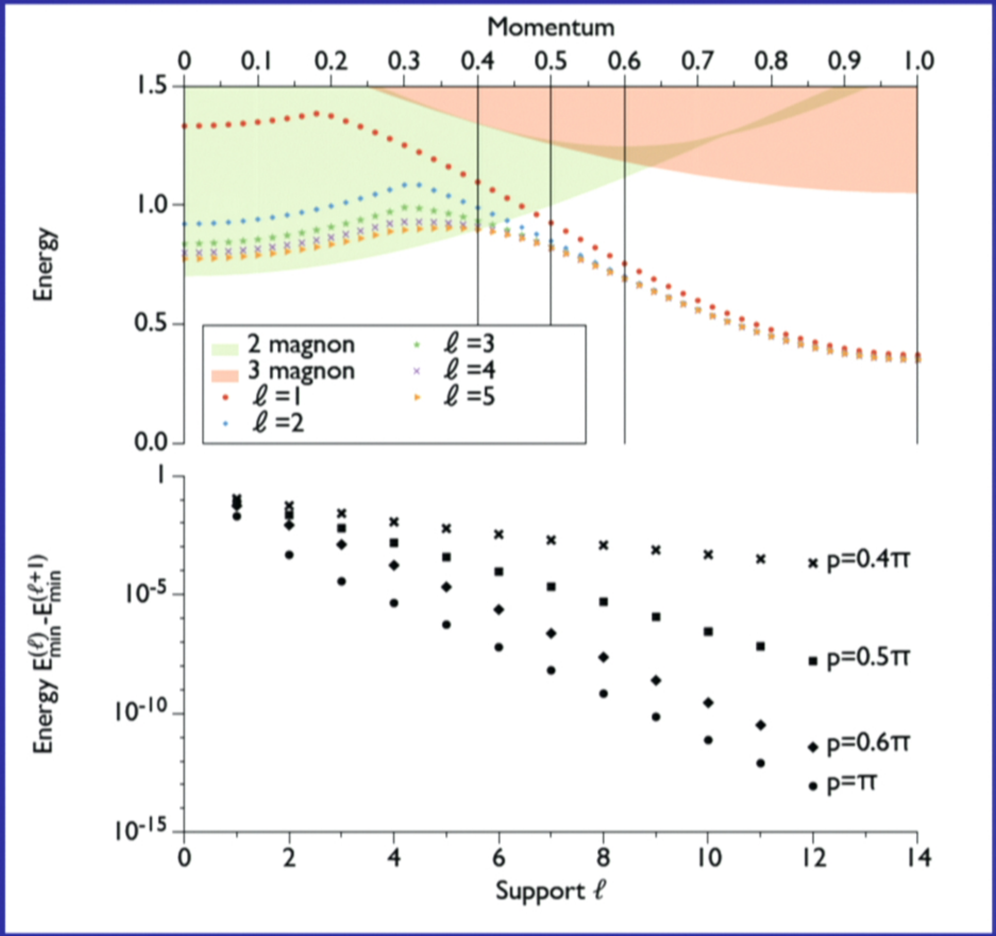
Elementary excitations in the AKLT model



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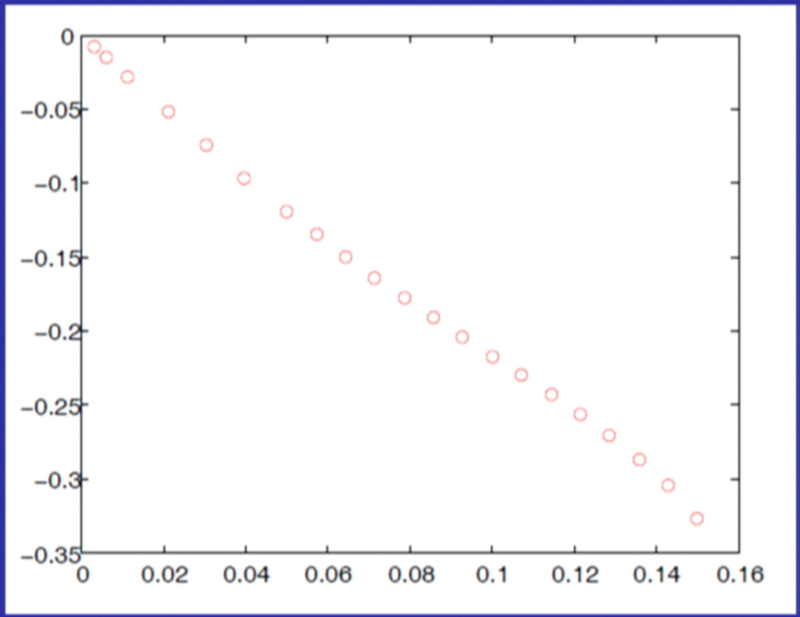

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Conclusion

- Area laws are key to understand strongly correlated quantum spin systems
- Area law is stable within a phase for quantum spin systems
- MPS/PEPS/MERA are the embodiment of area laws
- Post MPS/PEPS methods hold lots of promise, both from the physics and computational physics point of view

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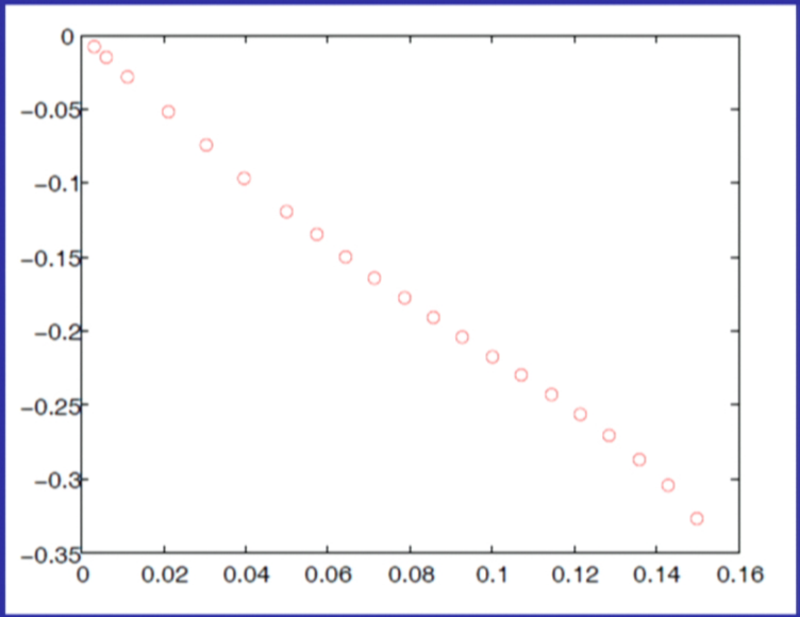

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