Title: Thermodynamical Property of Entanglement Entropy for Excited States

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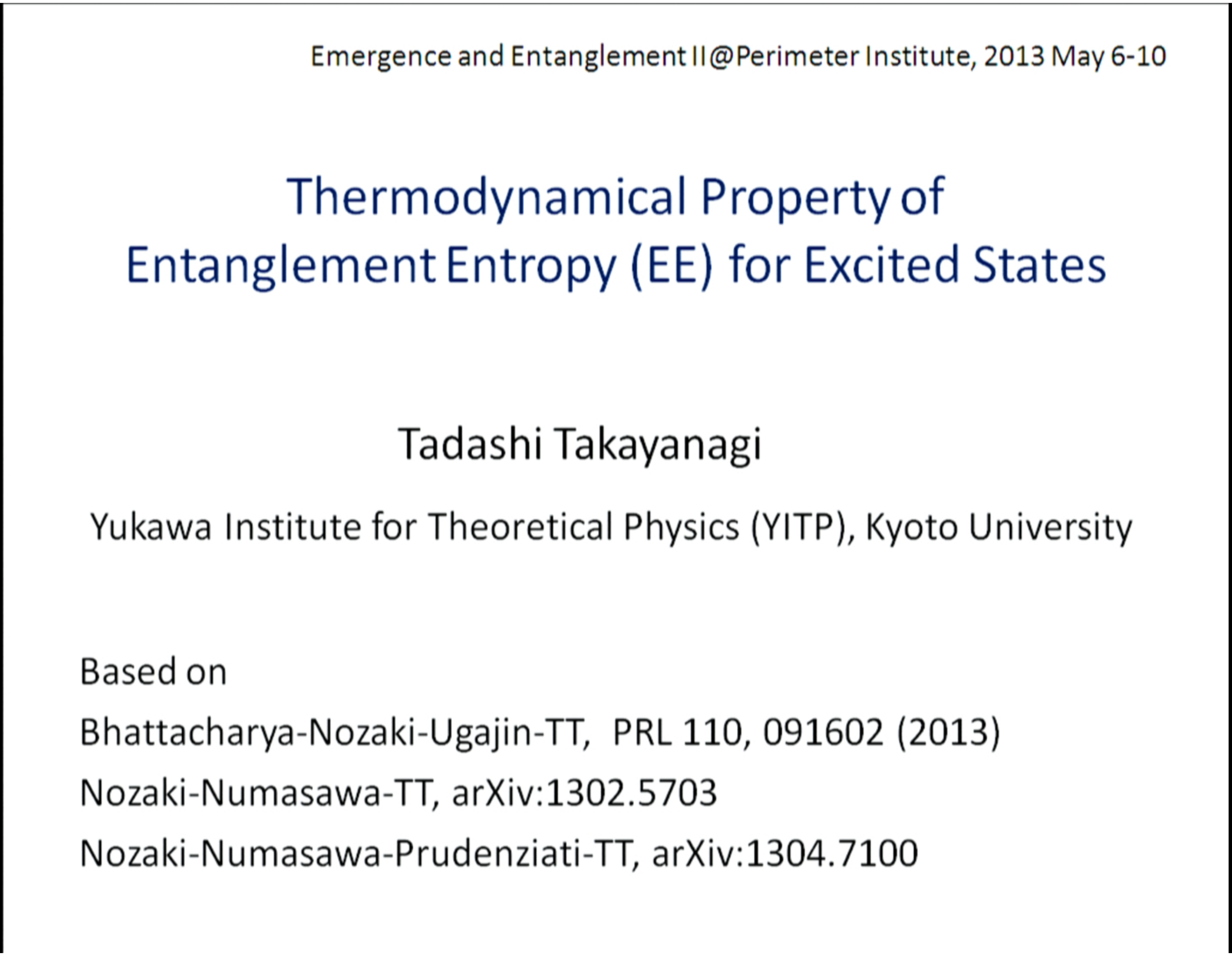
Abstract: We will point out that there is a universal thermodynamical property of entanglement entropy for excited states. We will derive this by using the AdS/CFT correspondence in any dimension. We will also directly confirm this property from direct field theoretic calculations in two dimensions. We will define a new quantity called entanglement density by taking derivatives of entanglement entropy with respect to the shape of subsystem. We will show that this quantity coincides with the energy density by taking the small subsystem limit and show that this is another equivalent statement of the thermodynamical property.



Emergence and Entanglement II@Perimeter Institute, 2013 May 6-10

Thermodynamical Property of Entanglement Entropy (EE) for Excited States

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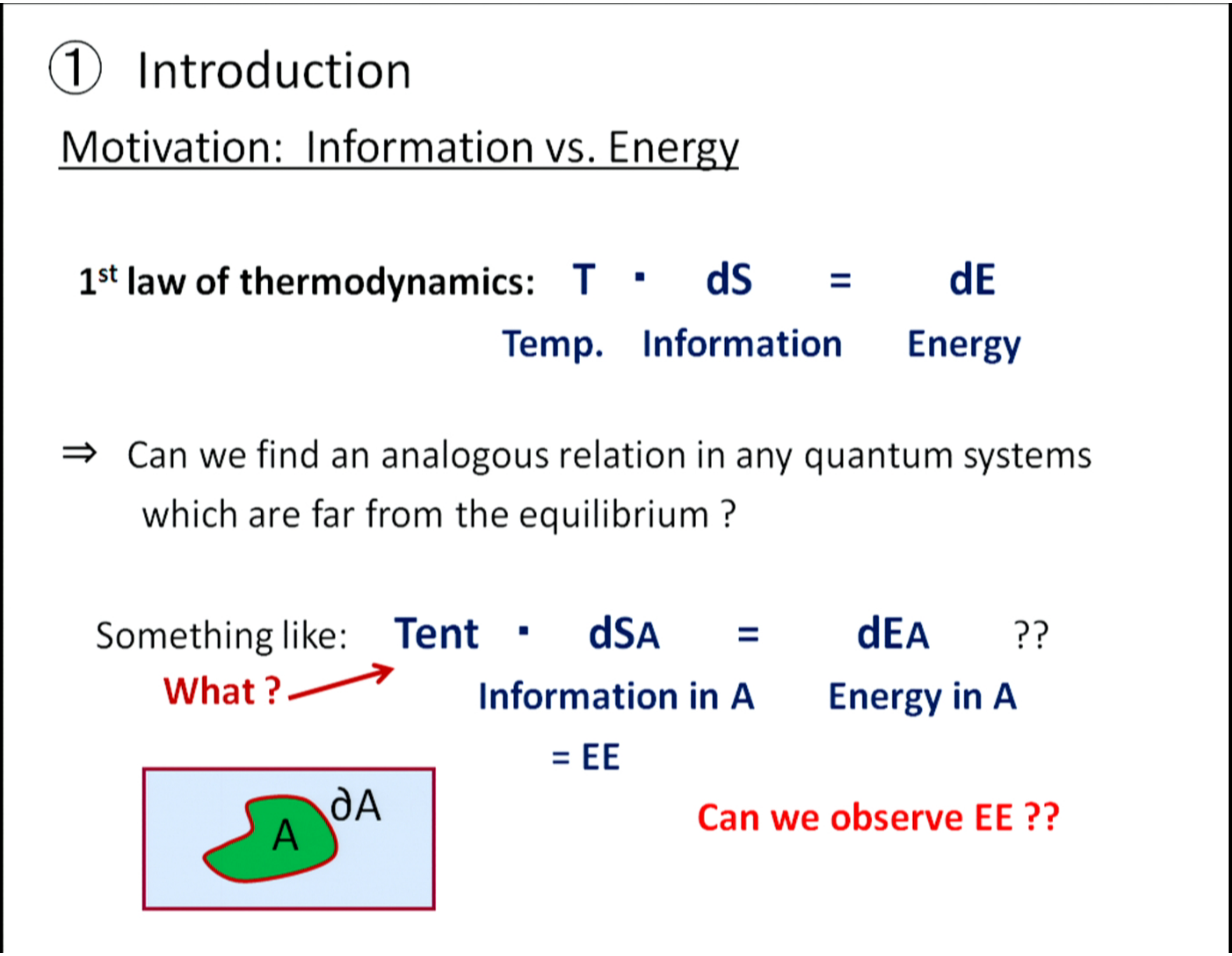
Based on

Bhattacharya-Nozaki-Ugajin-TT, PRL 110, 091602 (2013)

Nozaki-Numasawa-TT, arXiv:1302.5703

Nozaki-Numasawa-Prudenziati-TT, arXiv:1304.7100



① Introduction

Motivation: Information vs. Energy

1st law of thermodynamics: $T \cdot dS = dE$

Temp. Information Energy

⇒ Can we find an analogous relation in any quantum systems which are far from the equilibrium ?

Something like: $\mathrm{Tent} \cdot dS_A = dE_A$??

What ? →

Information in A Energy in A

= EE

Can we observe EE ??

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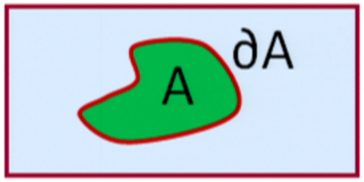

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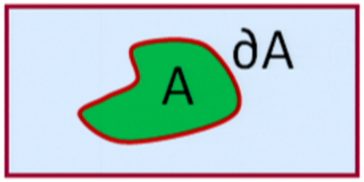
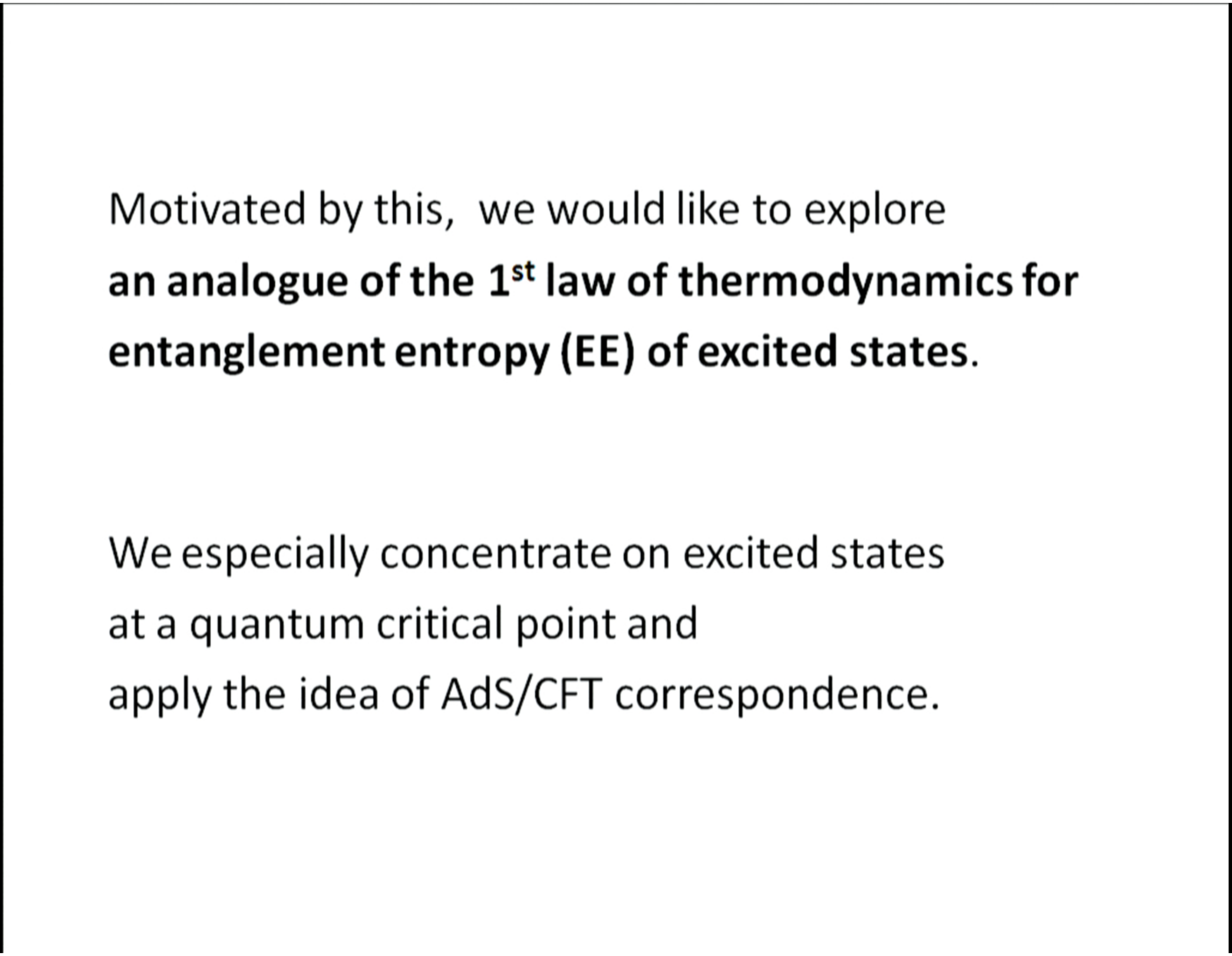


Can we observe EE ??



Motivated by this, we would like to explore **an analogue of the 1st law of thermodynamics for entanglement entropy (EE) of excited states**.

We especially concentrate on excited states at a quantum critical point and apply the idea of AdS/CFT correspondence.

Applications of EE

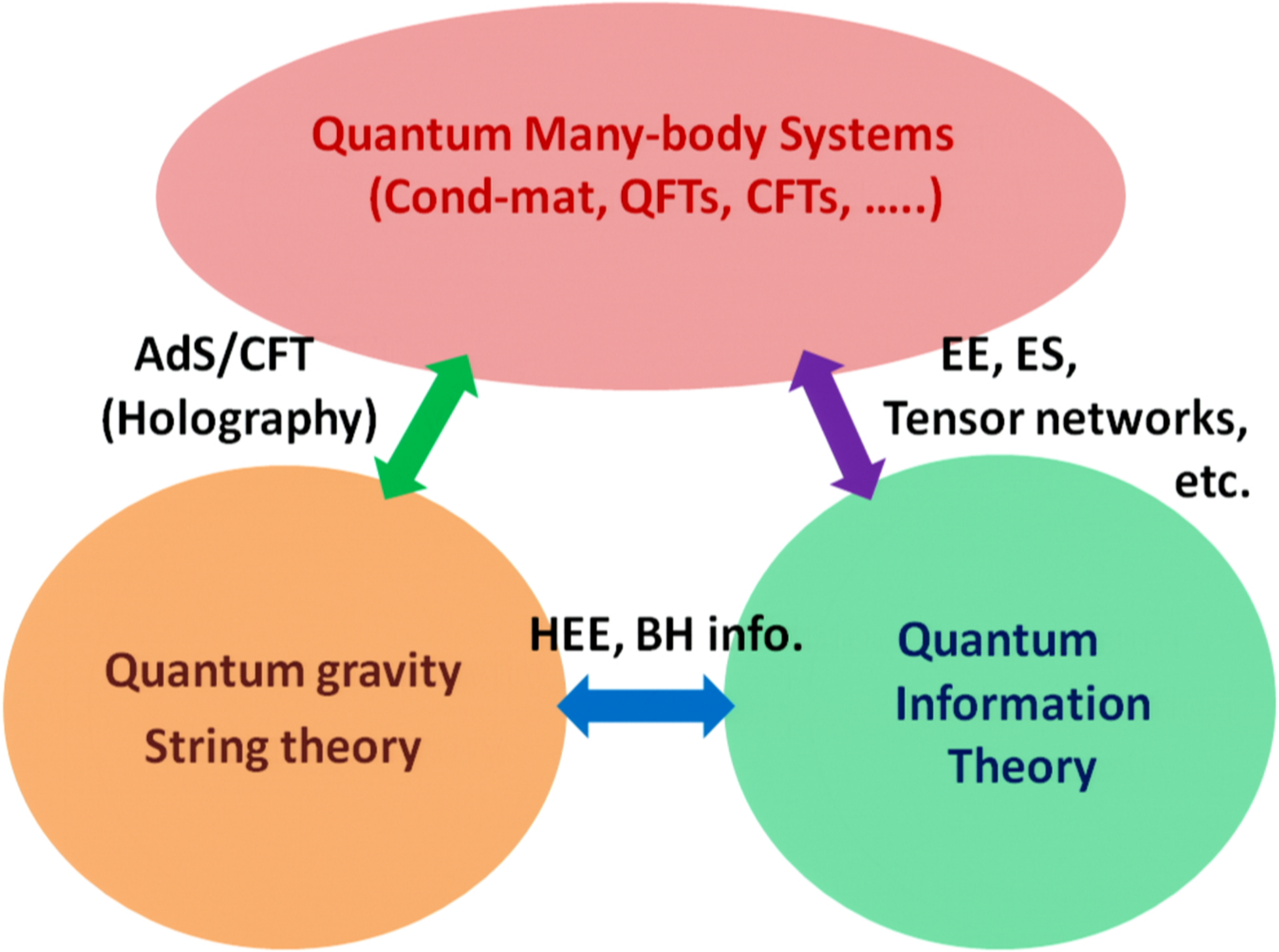
- EE = A quantum order parameter

⇒ Classify quantum phases.
Applicable in non-equilibrium systems.

- The entanglement entropy (EE) is a useful bridge between gravity (string theory) and cond-mat physics.

Gravity ⟷ Entanglement ⟷ Cond-mat.

$g_{\mu\nu}$ AdS/CFT (Holography) $S_A \approx \mathrm{Area}$ systems $|\Psi\rangle$



Quantum Many-body Systems
(Cond-mat, QFTs, CFTs, …..)

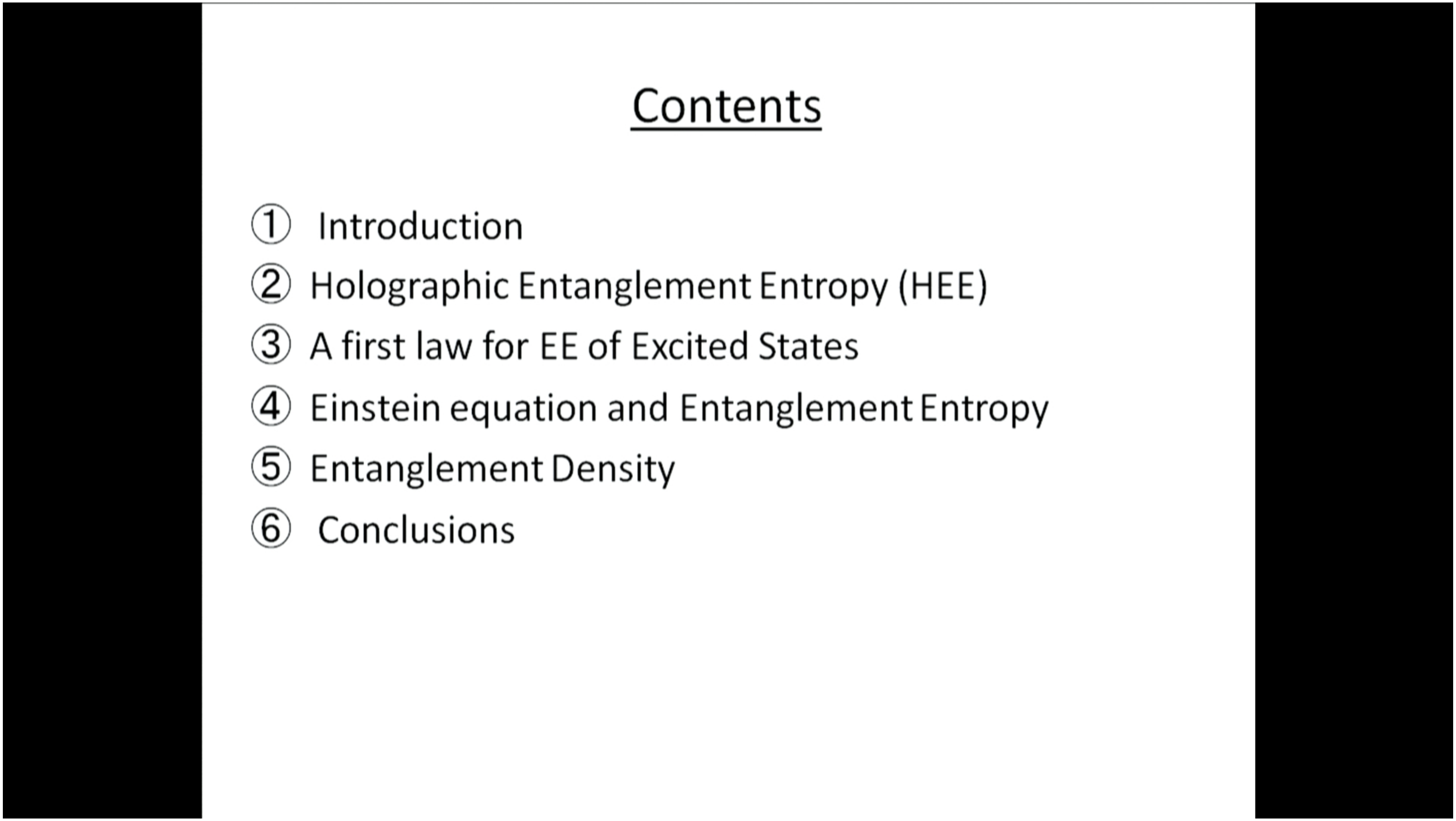
AdS/CFT
(Holography)

EE, ES,
Tensor networks,
etc.

Quantum gravity
String theory

HEE, BH info.

Quantum
Information
Theory



Contents

② Holographic Entanglement Entropy (HEE)

(2-1) AdS/CFT (the best example of holography)

AdS/CFT

Quantum Gravity (String theory) on d+2 dim. AdS spacetime (anti de-Sitter space) = **Conformal Field Theory (CFT) on d+1 dim. Minkowski spacetime**

⬇ **Classical limit**

General relativity with Λ＜0 (Geometrical)

⬇ **Large N limit**
Strong coupling limit

Strongly interacting quantum critical system

Basic Principle
（Bulk-Boundary relation）: $\boxed{Z_{Gravity} = Z_{CFT}}$

② Holographic Entanglement Entropy (HEE)

(2-1) AdS/CFT (the best example of holography)

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General relativity with Λ < 0 (Geometrical)

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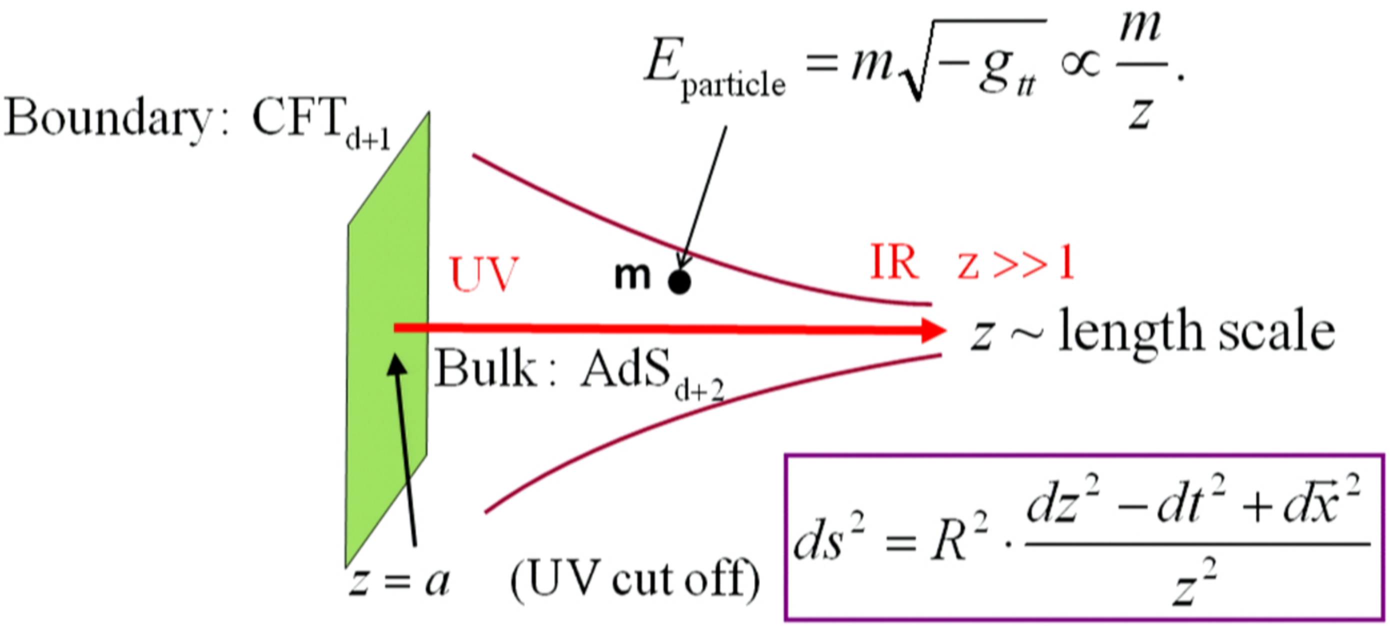
$$\boxed{Z_{Gravity} = Z_{CFT}}$$

The meaning of the extra dimension



$$E_{\text{particle}} = m\sqrt{-g_{tt}} \propto \frac{m}{z}.$$

$\text{Boundary}: \ \text{CFT}_{d+1}$

UV

m

IR z >>1

$z \sim$ length scale

$\text{Bulk}: \ \text{AdS}_{d+2}$

$z = a$ (UV cut off)

$$\boxed{ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}}$$

The radial direction z corresponds to the length scale

in CFT under the RG flow. (1/z ~ Energy Scale)

(2-2) Holographic Entanglement Entropy Formula

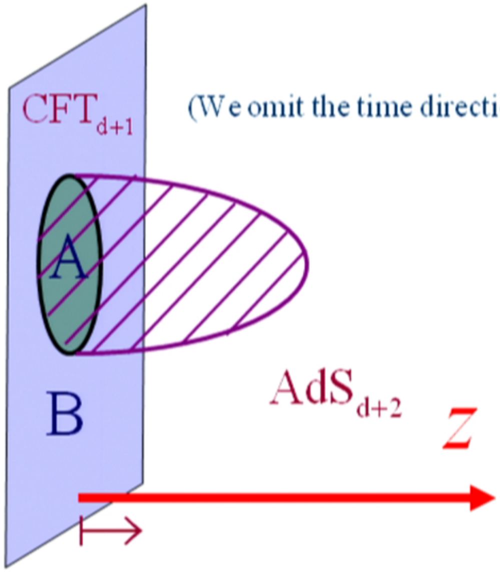
[Ryu-TT 2006]

$$\boxed{S_A = \underset{\substack{\partial\gamma_A = \partial A \\ \gamma_A \approx A}}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\gamma_A)}{4G_N} \right]}$$

γ_A is the minimal area surface (codim.=2) such that

$$\partial A = \partial\gamma_A \text{ and } A \sim \gamma_A \ .$$

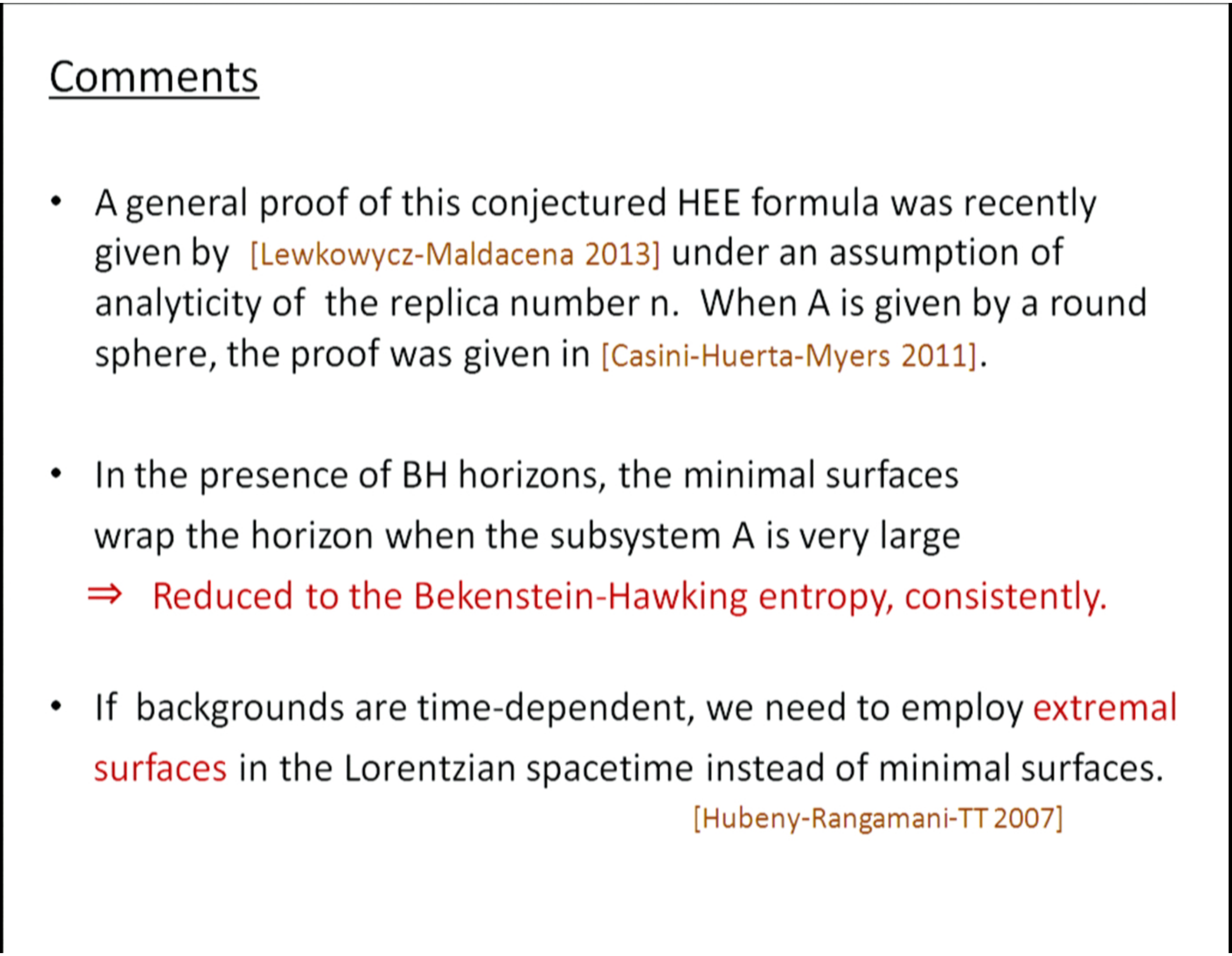
homologous



(We omit the time direction.)

$z > a$ (UV cut off)

$$ds^2_{AdS} = R^2_{AdS} \frac{-dt^2 + \sum_{i=1}^{d-1} dx_i^2 + dz^2}{z^2}$$



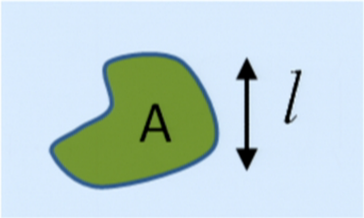
Comments

- A general proof of this conjectured HEE formula was recently given by [Lewkowycz-Maldacena 2013] under an assumption of analyticity of the replica number n. When A is given by a round sphere, the proof was given in [Casini-Huerta-Myers 2011].
- In the presence of BH horizons, the minimal surfaces wrap the horizon when the subsystem A is very large

  ⇒ Reduced to the Bekenstein-Hawking entropy, consistently.
- If backgrounds are time-dependent, we need to employ extremal surfaces in the Lorentzian spacetime instead of minimal surfaces.

  [Hubeny-Rangamani-TT 2007]

General Behavior of HEE



$$S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)}\Gamma(d/2)}\left[\underline{p_1\left(\frac{l}{a}\right)^{d-1}} + p_3\left(\frac{l}{a}\right)^{d-3} + \cdots + \left\{\begin{array}{ll} p_{d-1}\left(\frac{l}{a}\right) + \underline{p_d} & (\text{if } d = \text{even}) \\ p_{d-2}\left(\frac{l}{a}\right)^2 + \underline{q}\log\left(\frac{l}{a}\right) & (\text{if } d = \text{odd}) \end{array}\right.\right],$$

Area law divergence

where $p_1 = (d-1)^{-1},\ p_3 = -(d-2)/[2(d-3)], \ldots.$

$\ldots\ldots\quad q = (-1)^{(d-1)/2}(d-2)!!/(d-1)!!\quad .$

A universal quantity which characterizes odd dim. CFT. (so called `F')

Agrees with conformal anomaly (central charge)

③ A first law for the EE of Excited States

[Bhattacharya-Nozaki-Ugajin-TT 2012]

(3-1) Setup

Since the EE in a QFT is UV divergent, we would like to focus on the difference between the EE for an excited state and that for the ground state at a quantum critical point.

In other words, we will consider excited states and calculate:

$$\Delta S_A = S_A - S_A^{\text{Ground State}}.$$

This is always finite and we want to compare this entropy with the energy in A: $\Delta E_A = \int_A dx^d T_{tt}.$

(3-2) Holographic Calculation

Consider an asymptotically AdSd+2 background

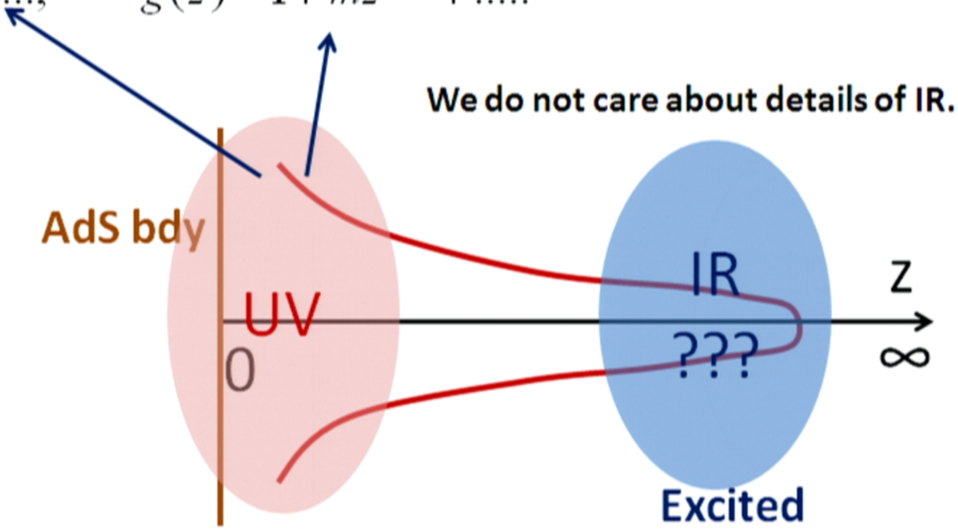
(= an excited state in CFTd+1):

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + g(z)dz^2 + \sum_{i=1}^{d} dx_i^2\right) \ ,$$

$$f(z) = 1 - mz^{d+1} + \ldots, \qquad g(z) = 1 + mz^{d+1} + \ldots..$$

$$\Rightarrow T_{tt} = \frac{dR^{d+1}m}{16\pi G_N}.$$

energy density

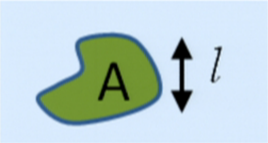


Holographic Entanglement Entropy Analysis

If we assume a small subsystem A with the size l such that

$$ml^{d+1} << 1,$$

then we can show



$$T_{ent} \cdot \Delta S_A = \Delta E_A,$$

$$\text{where} \quad \Delta S_A = S_A - S_A^{\text{PureAdS}}, \qquad \Delta E_A = \int_A dx^d T_{tt}.$$

The `entanglement temperature' is given by

$$T_{ent} = \frac{c}{l}.$$

The constant c is universal in that it only depends on the shape of the subsystem A: $e.g. \quad c = \frac{d+2}{2\pi}$ when A = a round sphere.

Example 1. Excited States in 2d CFT

$$\Delta S_A = \frac{2\pi^2}{3}(h+\bar{h})\frac{l^2}{l_{tot}^2} = \frac{\pi}{3} l \cdot \Delta E_A .$$

$(h,\bar{h}) =$ conformal dim.

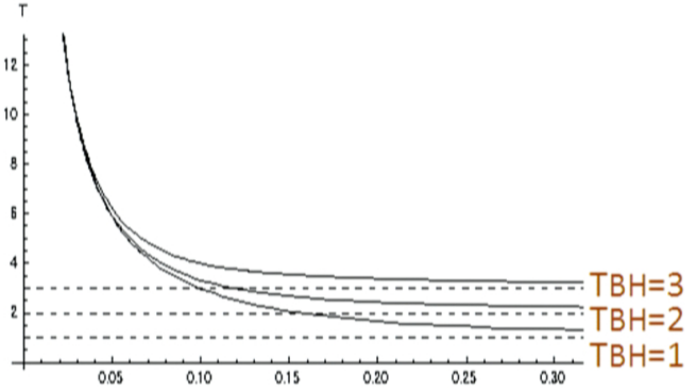
[Agrees with Alcaraz-Berganza-Sierra 11]

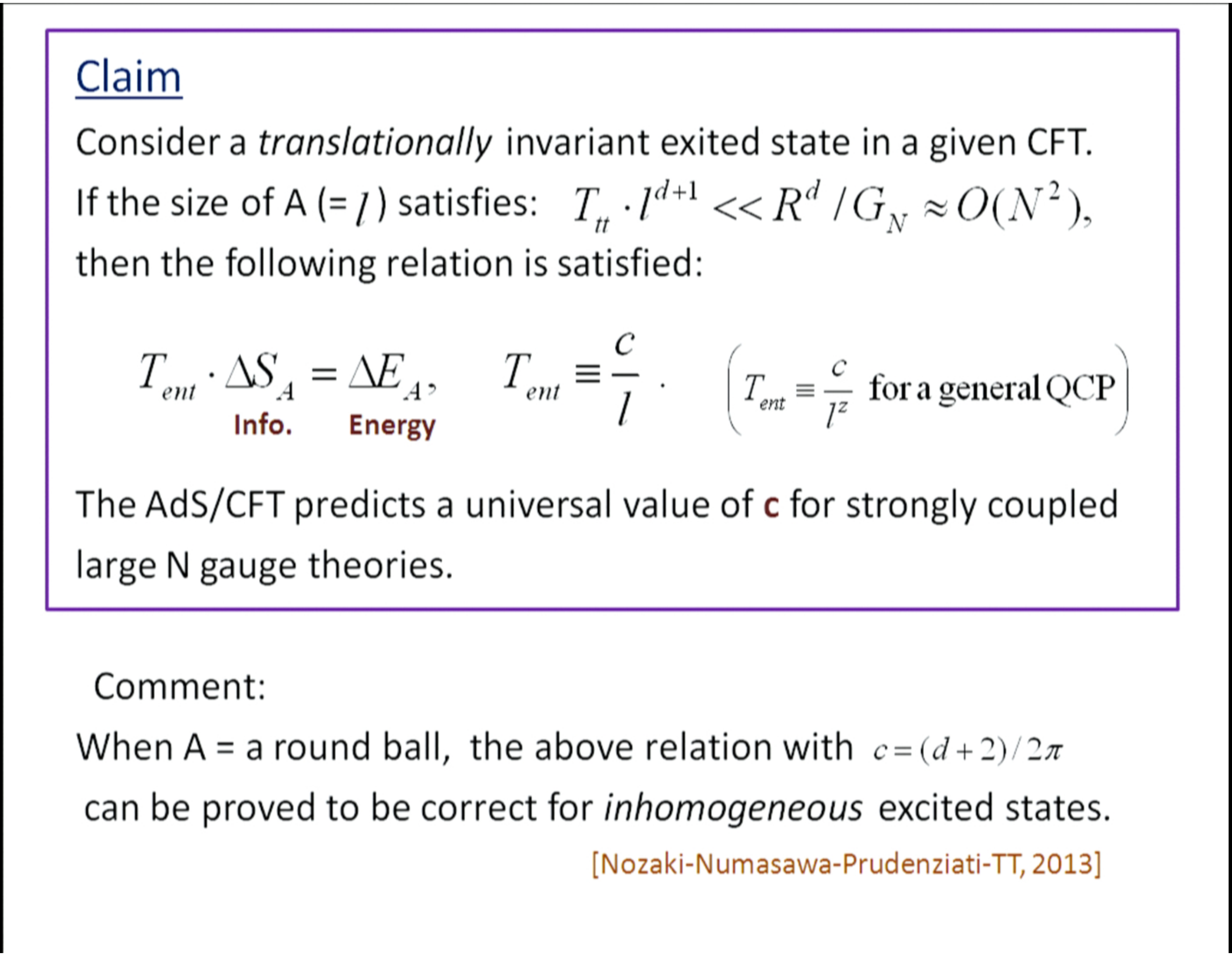
$l_{tot} =$ the total length of the system

Example 2. 3d CFT at finite temp.

$$T_{ent}(T_{BH}, l) \equiv \left.\frac{d\Delta E_A(T_{BH}, l)}{d\Delta S_A(T_{BH}, l)}\right|_{l=fixed} ,$$

$$\Rightarrow \begin{cases} T_{ent} \to c \cdot l^{-1} & (l \to 0) \\ T_{ent} \to T_{BH} & (l \to \infty) \end{cases} .$$





Claim

Consider a *translationally* invariant exited state in a given CFT.

If the size of A (= l) satisfies: $T_{tt} \cdot l^{d+1} << R^d / G_N \approx O(N^2),$

then the following relation is satisfied:

$$T_{ent} \cdot \underset{\textbf{Info.}}{\Delta S_A} = \underset{\textbf{Energy}}{\Delta E_A}, \qquad T_{ent} \equiv \frac{c}{l} \;. \qquad \left(T_{ent} \equiv \frac{c}{l^z} \;\; \text{for a general QCP} \right)$$

The AdS/CFT predicts a universal value of **c** for strongly coupled large N gauge theories.

Comment:

When A = a round ball, the above relation with $c = (d+2)/2\pi$ can be proved to be correct for *inhomogeneous* excited states.

[Nozaki-Numasawa-Prudenziati-TT, 2013]

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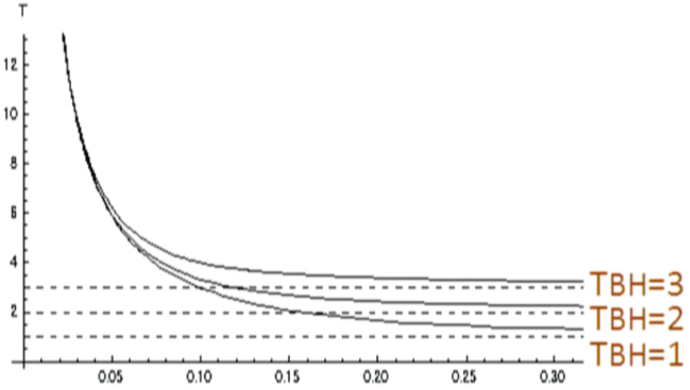

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[Nozaki-Numasawa-Prudenziati-TT, 2013]

④ Einstein equation and Entanglement Entropy

[Nozaki-Numasawa-Prudenziati-TT, 2013]

We found a universal relation which looks like the first law of thermodynamics by taking the size of subsystem A to be small.

➡ What can we say when the size of subsystem is not small ?
(though we do not expect truly universal results ….)

This is related to the following question in the AdS/CFT:

Gravity on AdS ⟷ **Entanglement in QCP(CFT)**

$g_{\mu\nu}$ AdS/CFT (Holography) $S_A(\approx \text{Area})$

Einstein eq. ⟷ What ?

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=T_{\mu\nu}$$

Some differential eq. for SA?

(4-1) Strategy

In AdS/CFT, the metric $g_{\mu\nu}$ is not a good valuable from the CFT side, because it changes under coordinate transformations.

Instead, we need to look at a diffeomorphism invariant quantity.

➡ The HEE is a useful quantity for this purpose.

Thus we would like to rewrite the Einstein eq. in terms of HEE.

To simply calculations, we consider a small perturbation around the pure AdS (=QCP) and analyze $\Delta S_A = S_A - S_A^{\text{Ground State}}$.

(4-2) AdS3/CFT2 (1 dim. QCP)

For the pure gravity on AdS3, we can show ΔS_A satisfies:

$$\left(\partial_x^2 - \partial_t^2\right)\Delta S_A(t,x,l) = 0,$$

$$\left(\partial_l^2 - \partial_t^2 - \frac{2}{l^2}\right)\Delta S_A(t,x,l) = 0.$$

~ a wave equation on the AdS spacetime.

In the presence of matter field $\phi\,(\leftrightarrow O_\phi)$ on AdS3,

they are modified as :

$$\left(\partial_x^2 - \partial_t^2\right)\Delta S_A(t,x,l) \sim \langle O_\phi \rangle \langle O_\phi \rangle,$$

$$\left(\partial_l^2 - \partial_t^2 - \frac{2}{l^2}\right)\Delta S_A(t,x,l) \sim \langle O_\phi \rangle \langle O_\phi \rangle.$$

(4-3) Pure gravity in AdS4/CFT3 (2 dim. QCP)

Let us assume a translationally invariant perturbation of AdS4.
Then we find the following equation from Einstein equation:
(A= a round ball with radius l)

$$\left(\partial_l^2 - \partial_t^2 - \frac{2}{l^2}\right)\Delta S_A(t,l) = 2\pi^2 l T_{tt}(t).$$

Remember that the energy density Ttt is related to ΔS_A in the small subsystem limit via the first law relation.

If we give the initial condition $\Delta S_A(l,t_0)$, we can determine its time evolution imposing an appropriate boundary condition.

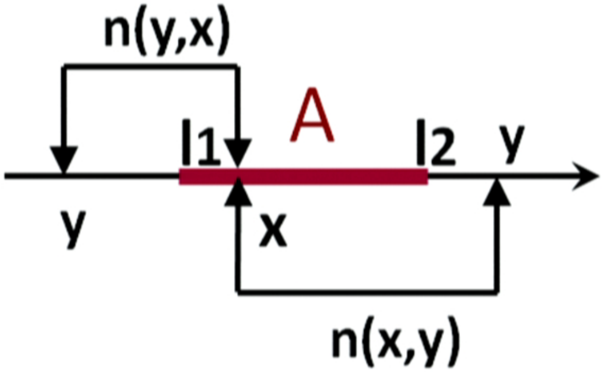
⑤ Entanglement Density [Nozaki-Numasawa-TT 2013]

(5-1) Definition and Property

We focus on the EE for a pure state in 2d CFTs for simplicity.

Let us estimate the EE for the subsystem A (=an interval) by summing all of the EE between two infinitesimal regions:

$$S_A(l_1, l_2) = \int_{l_1}^{l_2} dx \left[\int_{-\infty}^{l_1} dy \; n(y, x) + \int_{l_2}^{\infty} dy \; n(x, y) \right].$$



$$S_A = \int_{l_1}^{l_2} dx \left[\int_{-\infty}^{l_1} dy\ n(y,x) + \int_{l_2}^{\infty} dy\ n(x,y) \right].$$

$$\frac{\partial S_A}{\partial l_2} = \int_{-\infty}^{l_1} dy\ n(y,l_2) + \int_{l_2}^{\infty} dy\ n(l_2,y) - \int_{l_1}^{l_2} dxn(x,l_2).$$

Therefore we find

$$\frac{\partial^2 S_A}{\partial l_1 \partial l_2} = 2n(l_1,l_2).$$

We will call $n(l_1,l_2)$ the *entanglement density*.

Clearly this quantity should be non-negative.
As we will see, this fact comes from the SSA.

Now, let us apply the SSA relation to the intervals:

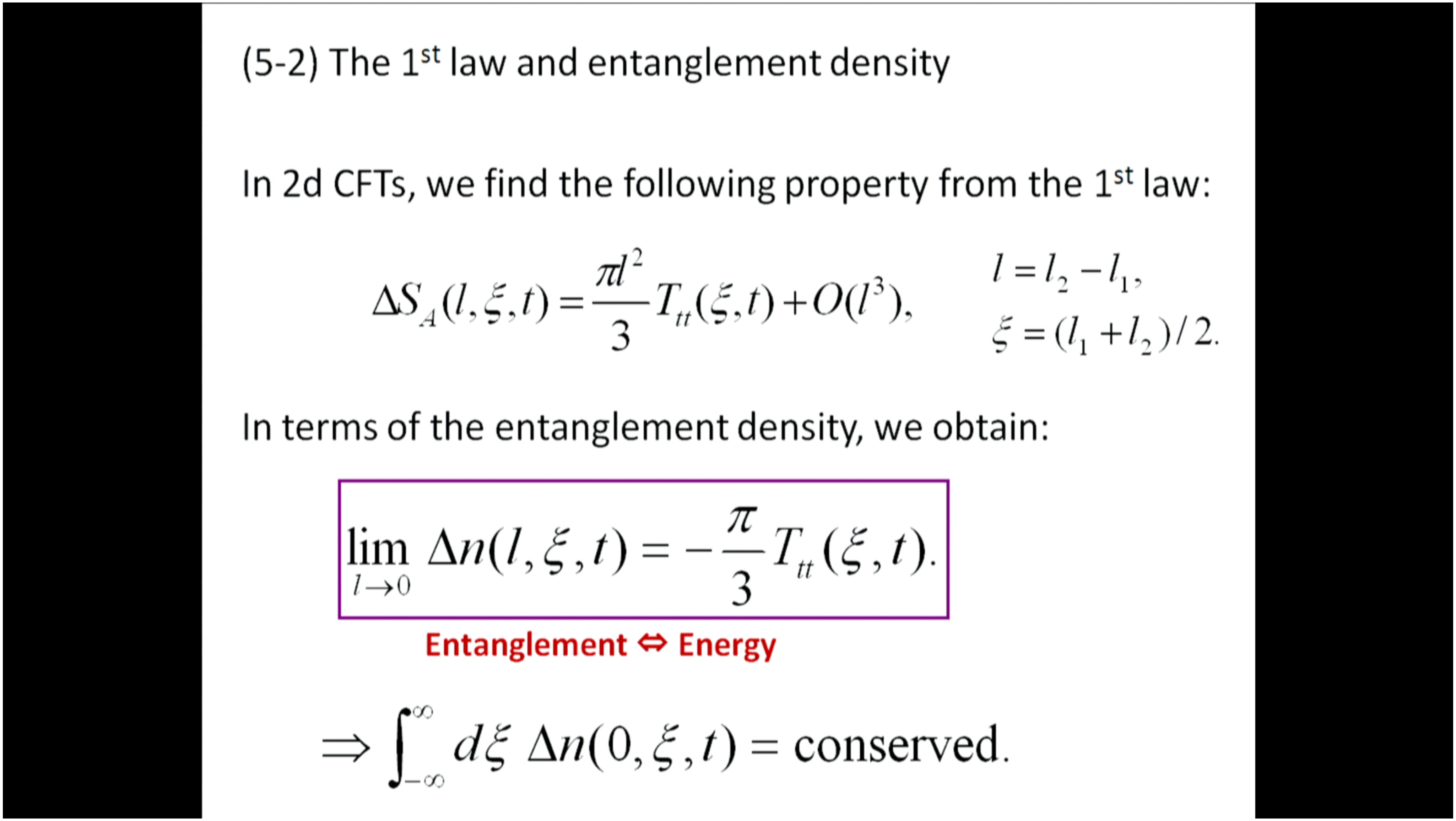
$$S_{A+B} + S_{B+C} \ge S_{A+B+C} + S_B,$$

$$\Leftrightarrow S(l_1, l_3) + S(l_2, l_4) \ge S(l_1, l_4) + S(l_2, l_3),$$

$$\Leftrightarrow \frac{\partial^2 S(x, y)}{\partial x \partial y} \ge 0,$$

$$\Leftrightarrow n(x, y) \ge 0.$$

Note: This property is true for any excited states.



(5-2) The 1st law and entanglement density

In 2d CFTs, we find the following property from the 1st law:

$$\Delta S_A(l,\xi,t) = \frac{\pi l^2}{3} T_{tt}(\xi,t) + O(l^3), \qquad l = l_2 - l_1, \quad \xi = (l_1 + l_2)/2.$$

In terms of the entanglement density, we obtain:

$$\boxed{\lim_{l\to 0} \Delta n(l,\xi,t) = -\frac{\pi}{3} T_{tt}(\xi,t).}$$

Entanglement ⇔ Energy

$$\Rightarrow \int_{-\infty}^{\infty} d\xi \; \Delta n(0,\xi,t) = \text{conserved}.$$

(5-3) Full conservation law of entanglement density

Claim: $\int dl_1 dl_2\, \Delta n(l_1,l_2,t) = \int dl\, d\xi\, \Delta n(l_1,l_2,t) = 0.$

[Proof] We compactify the space on a circle with periodicity L.

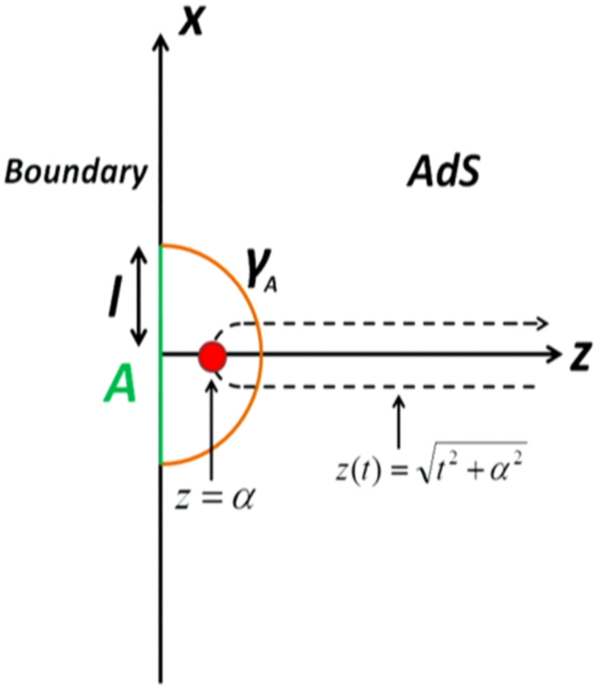
Then we find $S_A = S_B \;\Rightarrow\; \Delta n(l,\xi,t) = \Delta n(L-l, L-\xi, t).$

$$\int_0^L d\xi \int_0^{L/2} dl\, \Delta n(l,\xi,t) = \frac{1}{2}\int_0^L dl_1 \int_{l_2}^{l_1+L/2} dl_2\, \frac{\partial^2 \Delta S_A(l_1,l_2)}{\partial l_1 \partial l_2}$$

$$= \frac{1}{2}\int_0^L dl_1 \left[\underbrace{\frac{\partial}{\partial x}\Delta S_A(x,l_1+L/2)}_{\substack{=0 \\ \text{because SA=SB.}}} - \underbrace{\frac{\partial}{\partial x}\Delta S_A(x,l_1)}_{\substack{=0 \\ \text{because }1^{\text{st}}\text{ law.}}} \right]_{x=l_1} = 0.$$

(5-4) Example: Holographic Local Quenches

We argue that a simple model of holographic local quench is given by a free falling particle (mass **m**) in AdSd+2.



$$ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}$$

Trajectory: $z(t)=\sqrt{t^2+\alpha^2}$.

α~ the size of localized excitations

An analytical construction of the backreacted metric can be found in [Horowitz-Itzhaki 99].

(5-3) Full conservation law of entanglement density

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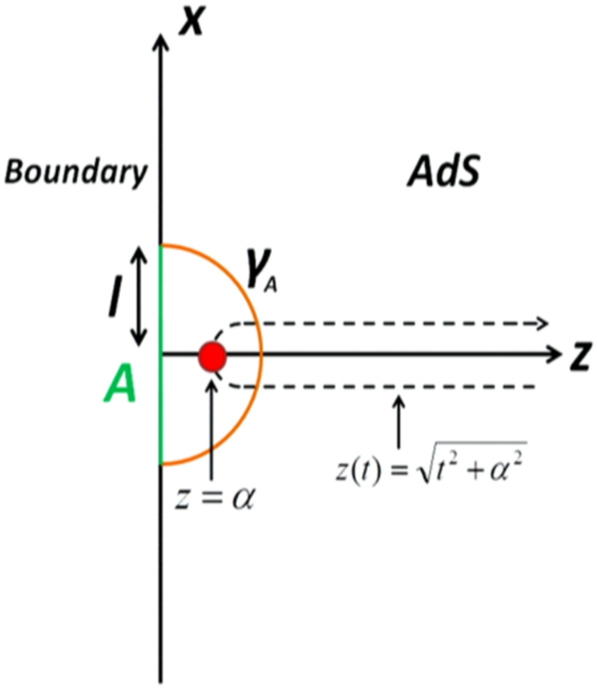

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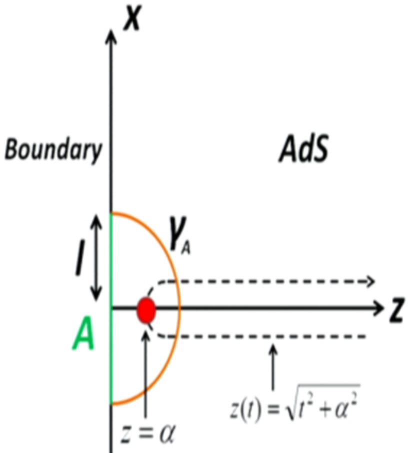
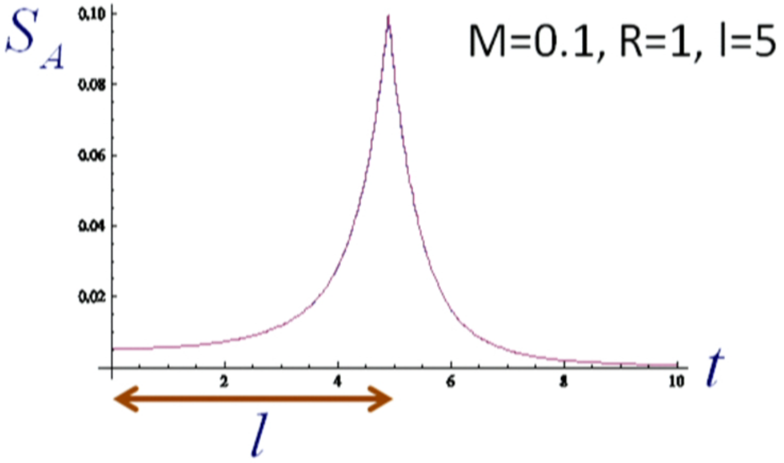
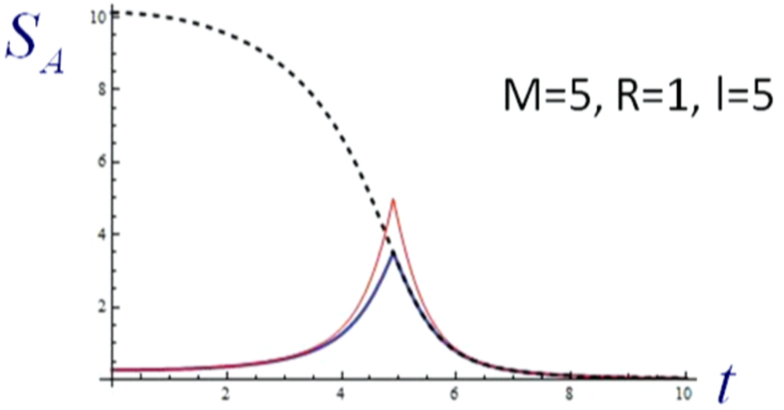
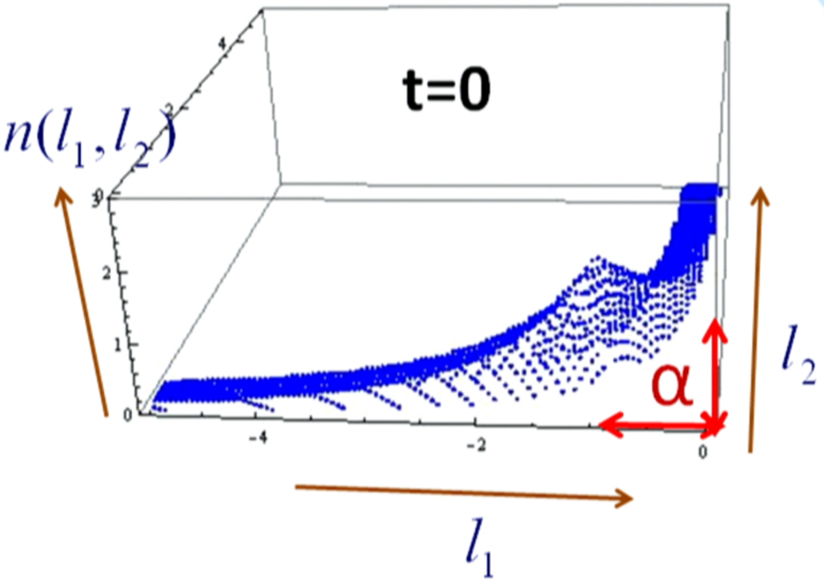
Time Evolution of HEE in AdS3/CFT2 (Case 1)

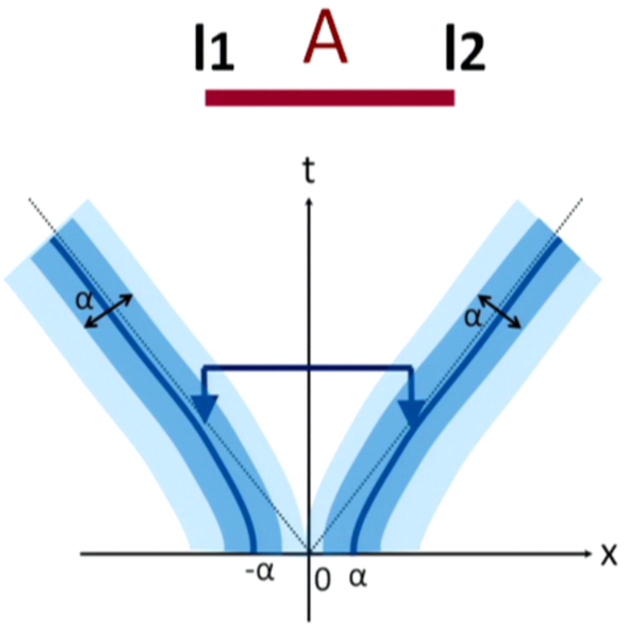


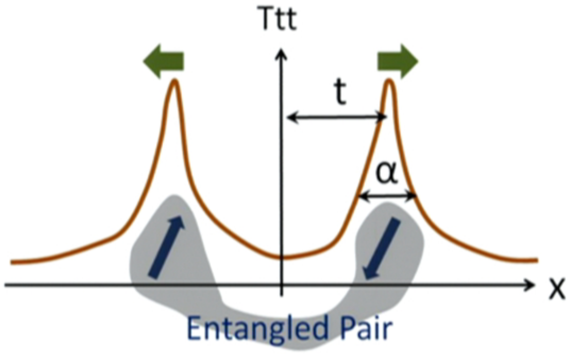




Entanglement Density







Entanglement Density