Title: Bipartite Graphs, Quivers and Gauge Theories

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Abstract: We discuss how bipartite graphs on Riemann surfaces encapture a wealth of information about the physics of large classes of supersymmetric gauge theories, especially those with quiver structure and arising from the AdS/CFT context. The correspondence between the gauge theory, the underlying algebraic geometry of it space of vacua, the combinatorics of dimers and toric varieties, as well as the number theory of dessin d'enfants becomes particular intricate under this light.

Bipartite Graphs, Quivers and Gauge Theories

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School of Physics, NanKai University;
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Perimeter Institute, May, 2013

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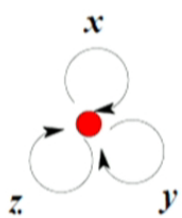
- 0511287, B. Feng, YHH, K. Kennaway, C. Vafa
- 0608050, 0701063: S. Benvenuti, B. Feng, A. Hanany, YHH J. Gray, YHH, V. Jejjala, B. Nelson
- 0909.2879: J. Hewlett, YHH
- 1204.1065, 1107.4101, 1104.5490 Hanany, YHH, Jejjala, Pasukonis, Ramgoolam, Rodriguez-Gomez
- 1201.3633 YHH, J. McKay and 1211.1931 YHH, J. McKay J. Read

Two most famous SCFTs in 4-D

- $\mathcal{N} = 4\ U(N)$ Yang-Mills
    - 3 adjoint fields X, Y, Z with superpotential
      $W = \text{Tr}(X[Y, Z]) = \text{Tr}(XYZ - XZY)$
    - Original AdS/CFT: N D3-branes transverse to flat $\mathbb{R}^6$

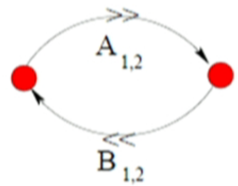


NODE = Gauge Group

ARROW = Bi−fundamental (Adj) Field

- QUIVER = Finite graph (label = rk(gauge factor)) + relations from SUSY
    - Matter Content: Nodes + arrows
    - Relations (F-Terms): $D_i W = 0 \rightsquigarrow [X, Y] = [Y, Z] = [X, Z] = 0$
- # gauge factors = $N_g = 1$; # fields = $N_f = 3$; # terms in W = $N_w = 2$

- Klebanov-Witten $\mathcal{N} = 1$ "conifold" Theory
  - $SU(N) \times SU(N)$ gauge theory with 4 bi-fundamental fields



QUIVER

	$SU(N)$	$SU(N)$
$A_{i=1,2}$	$\square$	$\overline{\square}$
$B_{j=1,2}$	$\overline{\square}$	$\square$

$W = \mathrm{Tr}(\epsilon_{il}\epsilon_{jk} A_i B_j A_l B_k)$

  - string-theory realization: N D3-branes transverse to the conifold singularity: $\{uv = wz\} \subset \mathbb{C}^4$

- $\#$ gauge factors $= N_g = 2$; $\#$ fields $= N_f = 4$; $\#$ terms in $W = N_w = 2$
- Observatio Curiosa: $N_g - N_f + N_w = 0$
- true for almost all known cases in AdS_5/CFT_4

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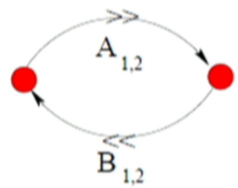

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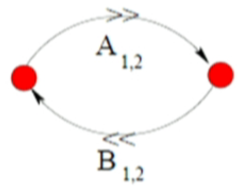

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Gauge Theory Moduli Space & CY Geometry

$$S = \int d^4x \, [\int d^4\theta \, \Phi_i^\dagger e^V \Phi_i + \left(\frac{1}{4g^2} \int d^2\theta \; \mathrm{Tr}\, \mathcal{W}_\alpha \mathcal{W}^\alpha + \int d^2\theta \, W(\Phi) + \text{h.c.}\right)]$$

W = superpotential $\quad V(\phi_i, \bar{\phi}_i) = \sum_i \left|\frac{\partial W}{\partial \phi_i}\right|^2 + \frac{g^2}{4}(\sum_i q_i |\phi_i|^2)^2$

- VACUUM $\sim \boxed{V(\phi_i, \bar{\phi}_i) = 0} \Rightarrow \begin{cases} \frac{\partial W}{\partial \phi_i} = 0 & \text{F-TERMS} \\ \sum_i q_i |\phi_i|^2 = 0 & \text{D-TERMS} \end{cases}$
- $\mathcal{M}$ **:= vacuum moduli space = space of solutions to F and D-flatness**

  = affine (singular) algebraic variety $\rightsquigarrow$ Quiver Variety
- If $\mathcal{M}$ CY3, can realize in string theory as D3-brane probing $\mathcal{M}$
  - $\mathcal{M} \longrightarrow$ Gauge Theory: Geometrical Engineering (Vafa, Cachazo, et al)
  - Gauge Theory $\longrightarrow \mathcal{M}$: Forward Algorithm
- for N-branes, get $Sym^N \mathcal{M} = \mathcal{M}^N / \Sigma_N$

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An Computational Approach

- Programme to study the computational algebraic geometry of $\mathcal{M}$: w/ Gray, Jejjala, Nelson; Hauenstein, Mehta; Hanany, Mekareeya; Chuang, Hewlett;
  1. n-fields: start with polynomial ring $\mathbb{C}[\phi_1, \ldots, \phi_n]$
  2. D = set of k GIO's: a ring map $\mathbb{C}[\phi_1, \ldots, \phi_n] \xrightarrow{D} \mathbb{C}[D_1, \ldots, D_k]$
  3. Now incorporate superpotential: F-flatness

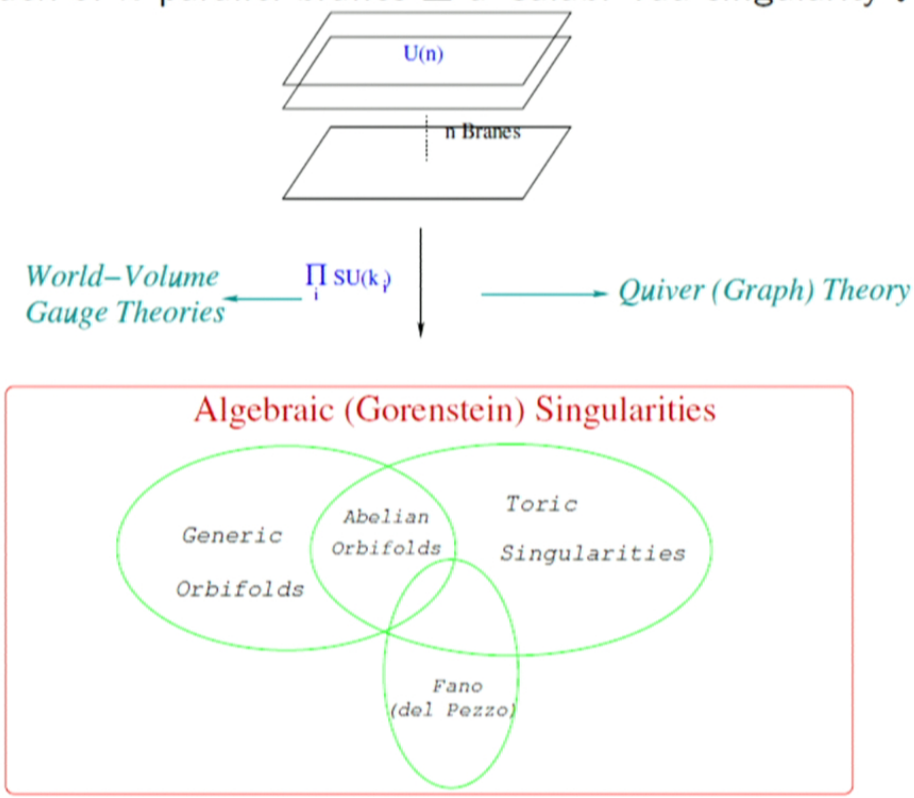
     $\langle f_{i=1,\ldots,n} = \frac{\partial W(\phi_i)}{\partial \phi_i} = 0 \rangle \simeq$ ideal of $\mathbb{C}[\phi_1, \ldots, \phi_k]$
  4. Moduli space = image of the ring map

     $$\frac{\mathbb{C}[\phi_1,\ldots,\phi_n]}{\{F = \langle f_1,\ldots,f_n \rangle\}} \xrightarrow{D=GIO} \mathbb{C}[D_1, \ldots, D_k], \quad \mathcal{M} \simeq \mathrm{Im}(D)$$
- Image is an ideal of $\mathbb{C}[D_1, \ldots, D_k]$, i.e.,

  $\mathcal{M}$ explicitly realised as an affine variety in $\mathbb{C}^k$
- Gröbner Bases: implemented in MACAULAY2, Singular...
- Numerical AG: implemented in Bertini...

Gauge Theory & CY correpondence: Paradigm

Stack of n parallel branes $\perp$ a Calabi-Yau singularity $\mathcal{M}$:



Orbifolds:

$\mathbb{C}^3/(\Gamma \subset SU(3))$

Toric:

e.g., conifold, $Y^{p,q}$, $L^{p,q}$ ...

Fano (del Pezzo):

e.g., $dP_{0,\dots,8}$

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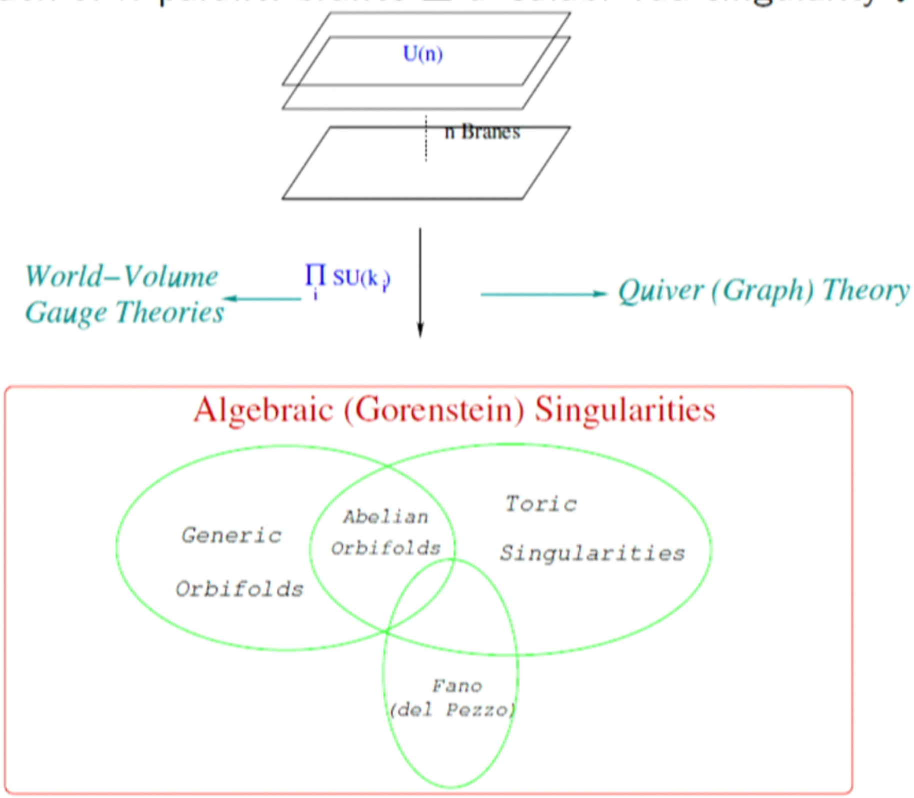

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Toric AdS_5/CFT_4, or Gauge Theories with $\mathcal{M}$ Toric CY3

- By far the largest class known and studied
- World-volume physics and Geometry of $\mathcal{M} \sim$ Combinatorical Data $\sim$ (integer cones σ in $\mathbb{Z}^r$-lattice); (CY3 $\rightsquigarrow$ planar toric diag)
  - Computing $\mathcal{M}$: (Aspinwall, Diaconescu, Douglas, Greene, Morrison, Plesser, et al.) from Witten's GLSM;
  - Feng-Hanany-YHH (Inverse Algorithm toric diag $\rightsquigarrow$ gauge theory) 0003085
- Explicit Ricci-flat metric known for infinite families $Y^{p,q}$, L^{abc} (conifold, special case); [Candelas-de la Ossa, Cvectic, Hanany, Pope, Sparks, Waldram ...]
- Dual Sasaki-Einstein Cone has $U(1)^3$ isometry
- No known *compact* CY3 metrics, all known affine CY3 metrics are toric so far.

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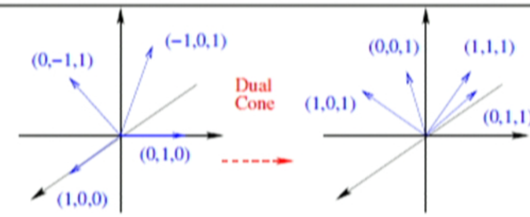
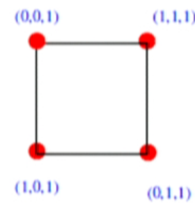
Notation for Affine Toric Variety

Def		Example (Conifold)
Comb.:	Convex Cone $\sigma \in \mathbb{Z}^d \rightsquigarrow$ Dual Cone $\sigma^\vee \rightsquigarrow X =$ $\text{Spec}_{Max}\mathbb{C}[S_\sigma = x_i^{\text{gen}(\sigma^\vee)\cap\mathbb{Z}^d}]$ Toric Diagram $= S_\sigma$	 $S_\sigma = \langle a = z, c = yz, b = xyz, d = xz \rangle$ $ab = cd$ in $\mathbb{C}^4[a, b, c, d]$
Symp:	Generalise $\mathbb{P}^n$: a $(\mathbb{C}^*)^{q-d}$ action on $\mathbb{C}^q_{[x_i]}$ $x_i \mapsto \lambda_a^{Q_{i=1\ldots q}^{a=1\ldots q-d}} x_i$ with Relations: $\sum_{i=1}^{d} Q_i^a v_i = 0$ Toric Diagram $= v_i$	 $Q = [-1, -1, 1, 1]$ $\mathbb{C}^*$ on $\mathbb{C}^4 \rightsquigarrow$ $\ker Q = G_t = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$
Comp:	Binomial Ideal $\langle \prod p_i = \prod q_j \rangle$	$ab = cd$ in $\mathbb{C}^4$

toric gauge theory $\simeq$ dimer/tiling configurations

- $G - E + N_T = 0$ is Euler relation for a torus! G nodes (vol(toric diag)), E edges (fields), N_T faces (loops in the quiver, W-terms)
- Quiver: loops / F-term relations $\sim \mathcal{M}$ = CY3 (planar toric diag D)
  - dual of $D = (p, q)$-configuration of branes
  - repetition in D = multiplicities in homogeneous coord
- Dual Periodic Quiver: (coloured graph dual) black (white) nodes $\sim W$ terms with + (-) sign; (Hanany, Kennaway, Feng, Franco, YHH, Uranga, Vegh, Wecht)
  - bi-partite: each field appears twice with opposite sign; periodic: torus
  - Dimer Model on a torus (i.e., periodic planar tiling)
  - Kasteleyn matrix = weighted adjacency matrix of a dimer model
  - $P(z, w) = Det(Kas)$ = 2-var poly $\rightsquigarrow$ Perfect matchings = coefficients of $Det(Kas)$ = multiplicities in GLSM fields = repetitions of the nodes in D

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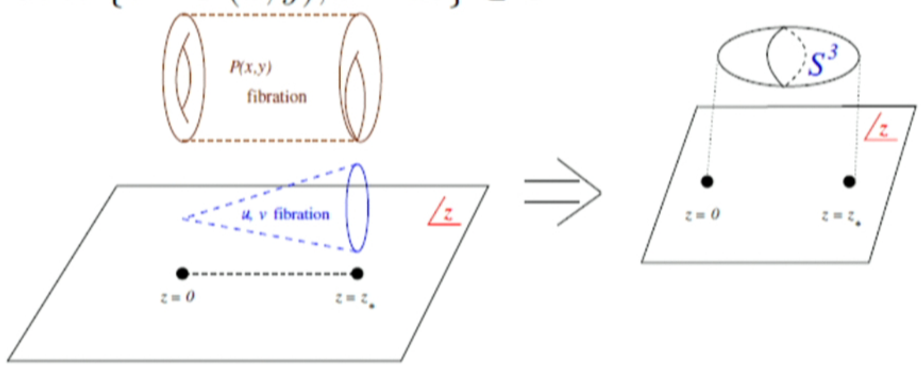
Toric CY3, Mirror Symmetry & Dimers/Tilings

- [Feng-Kennaway-YHH-Vafa] torus T^2 lives in T^3 of mirror symmetry; [Heckman-Vafa-Xie-Yamazaki] stringy origins of BFT [Franco]
- Mirror Symmetry:
  - Thrice T-duality: D3-brane on CY3 $\rightsquigarrow$ D6-branes wrapping 3-cycles in mirror CY3: $\{z = P(x,y), z = uv\} \subset \mathbb{C}^5$



  - Intersection in H_3 of mirror gives quiver: $a_{ij} = S_j \cap S_j$
- zig-zag paths in $T^2 \sim (p,q)$-cycle $\sim$ winding paths in Σ (show by Pick's)

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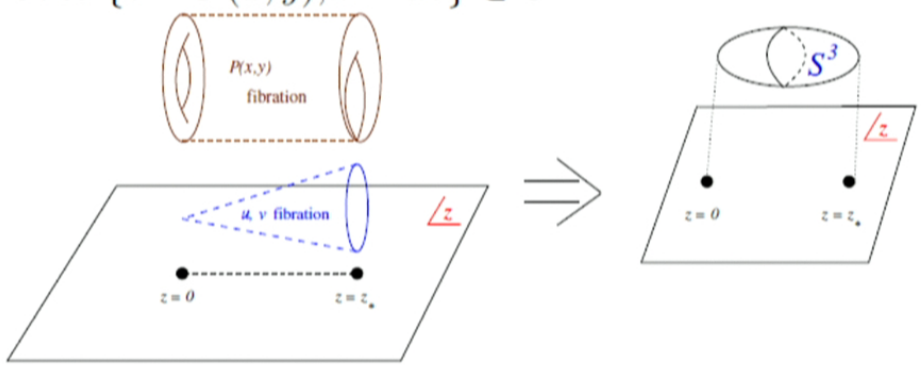

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Amoebae & Algae

- Amoeba Projection $Log(z,w) \to (\log|z|, \log|w|)$

  $$A = Amoeba(P(z,w) \subset (\mathbb{C}^*)^2) = Log(P) \subset \mathbb{R}^2 \rightsquigarrow$$

  skeleton of A is the (p,q)-configuration
- T^2 of dimer model lives in the T^3 of mirror symmetry
  - $P(z,w) = 0$ describes fiber Σ over $s = 0$ in mirror CY3
  - $(\bigcap$ 3-cycles$)\cap\Sigma$ at a graph Γ on $T^2 \subset T^3 \rightsquigarrow$ periodic tiling
  - Alga Projection: $Arg(z,w) \to (\arg(z), \arg(w))$

    $$Alga(P(z,w) \subset (\mathbb{C}^*)^2) = Arg(P) \subset [0, 2\pi)^2 \rightsquigarrow$$

    fundamental region of dimer

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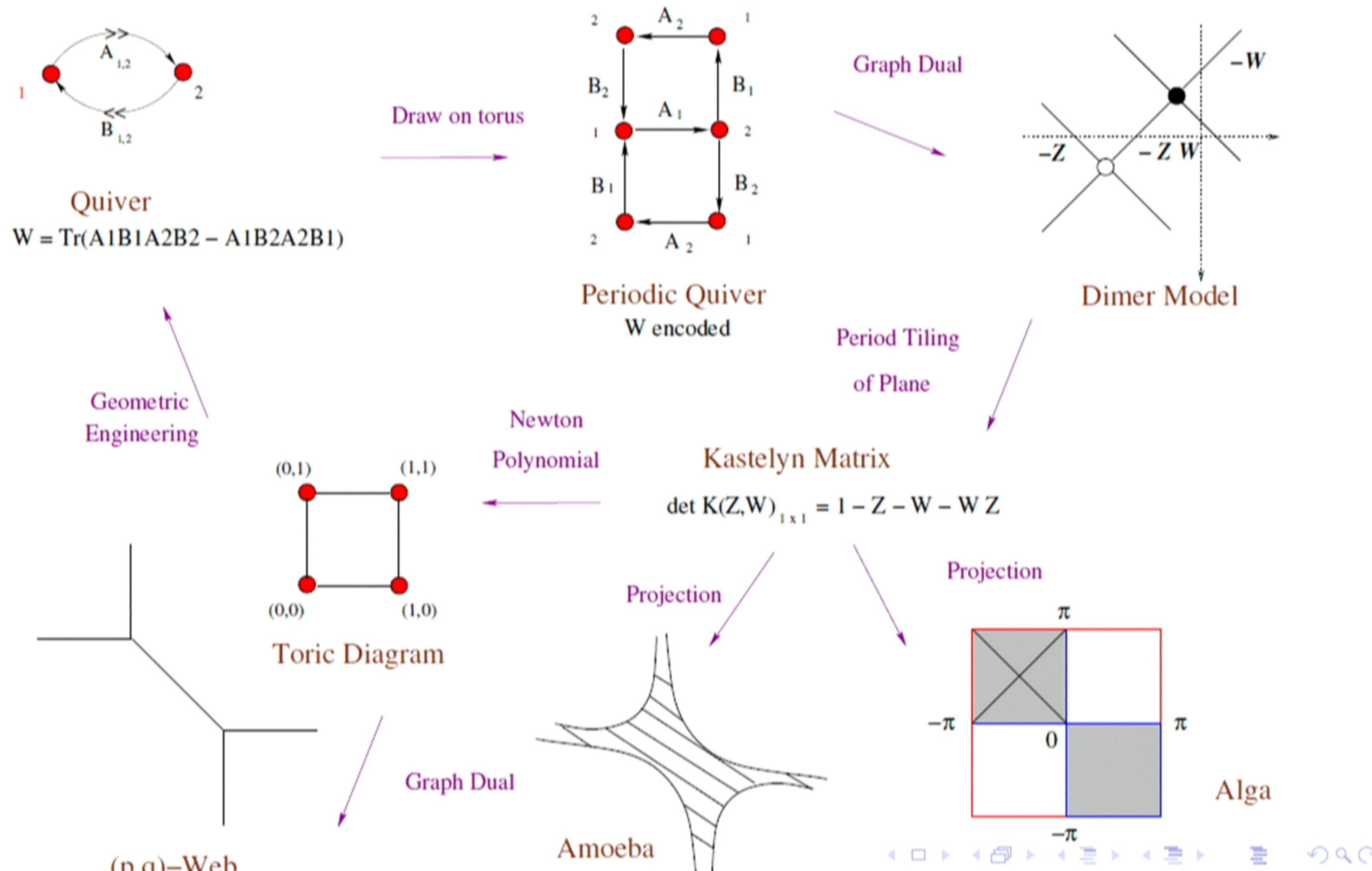

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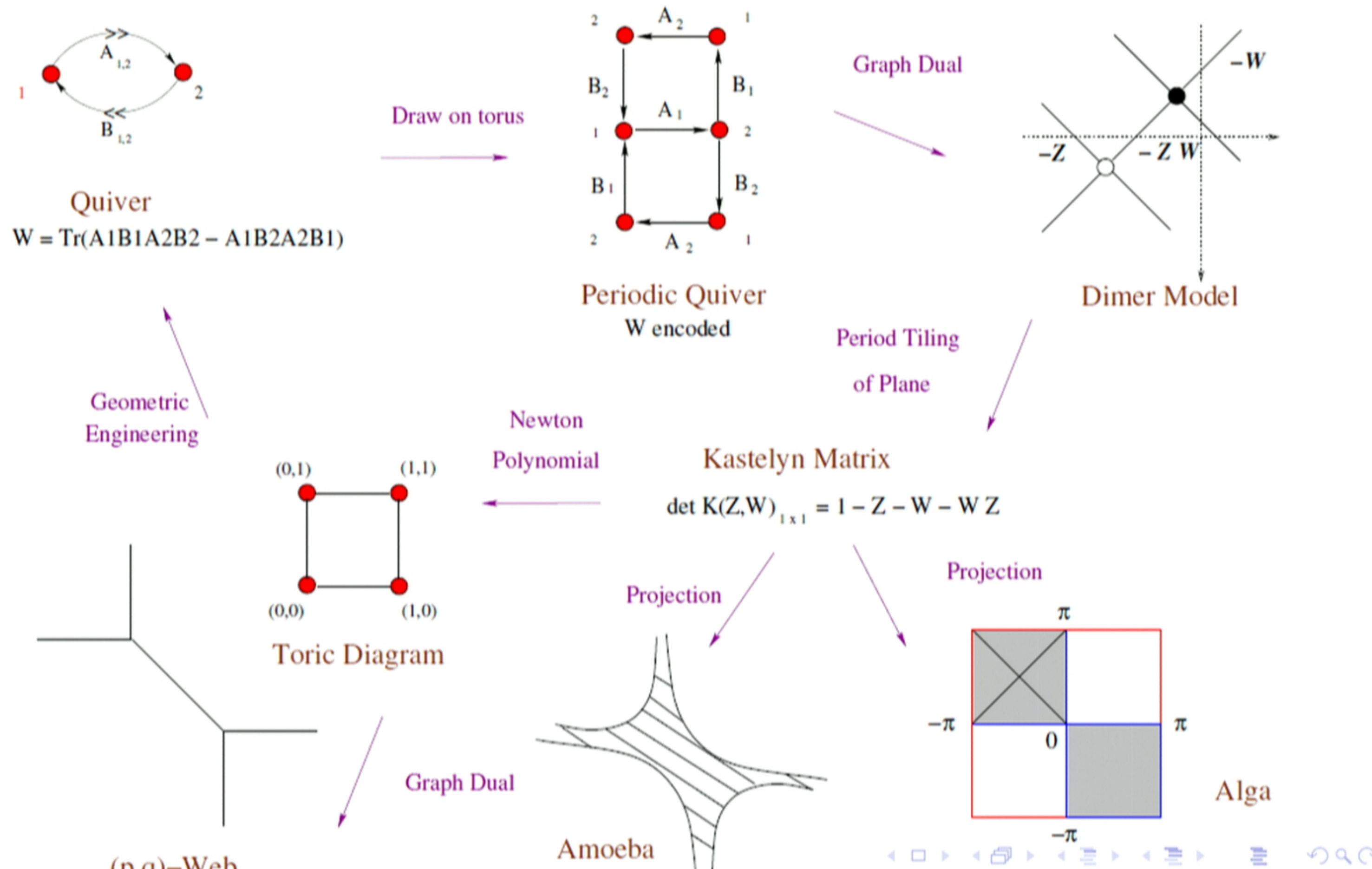

Seiberg Duality for $\mathcal{N} = 1$

- 2 theories: Direct Electric theory: N_c with N_f flavours; Dual Magnetic theory: $N_f - N_c$ (take $\frac{3}{2}N_c \leq N_f \leq 3N_c$) with N_f flavours

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$
Q	$\square$	$\square$	1
Q'	$\overline{\square}$	1	$\overline{\square}$

$W = 0$

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M	1	$\square$	$\overline{\square}$

$W = Mqq'$

- two theories with different gauge symmetries, same global symmetries and different matter content/superpotential
- have the same IR dynamics, i.e., **same vacuum moduli space**

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A Quiver Duality from Seiberg Duality

We have quiver labeled by $(N_c)_i$ and arrows a_{ij}:

1. Pick dualisation node i_0 with N_c, an define:

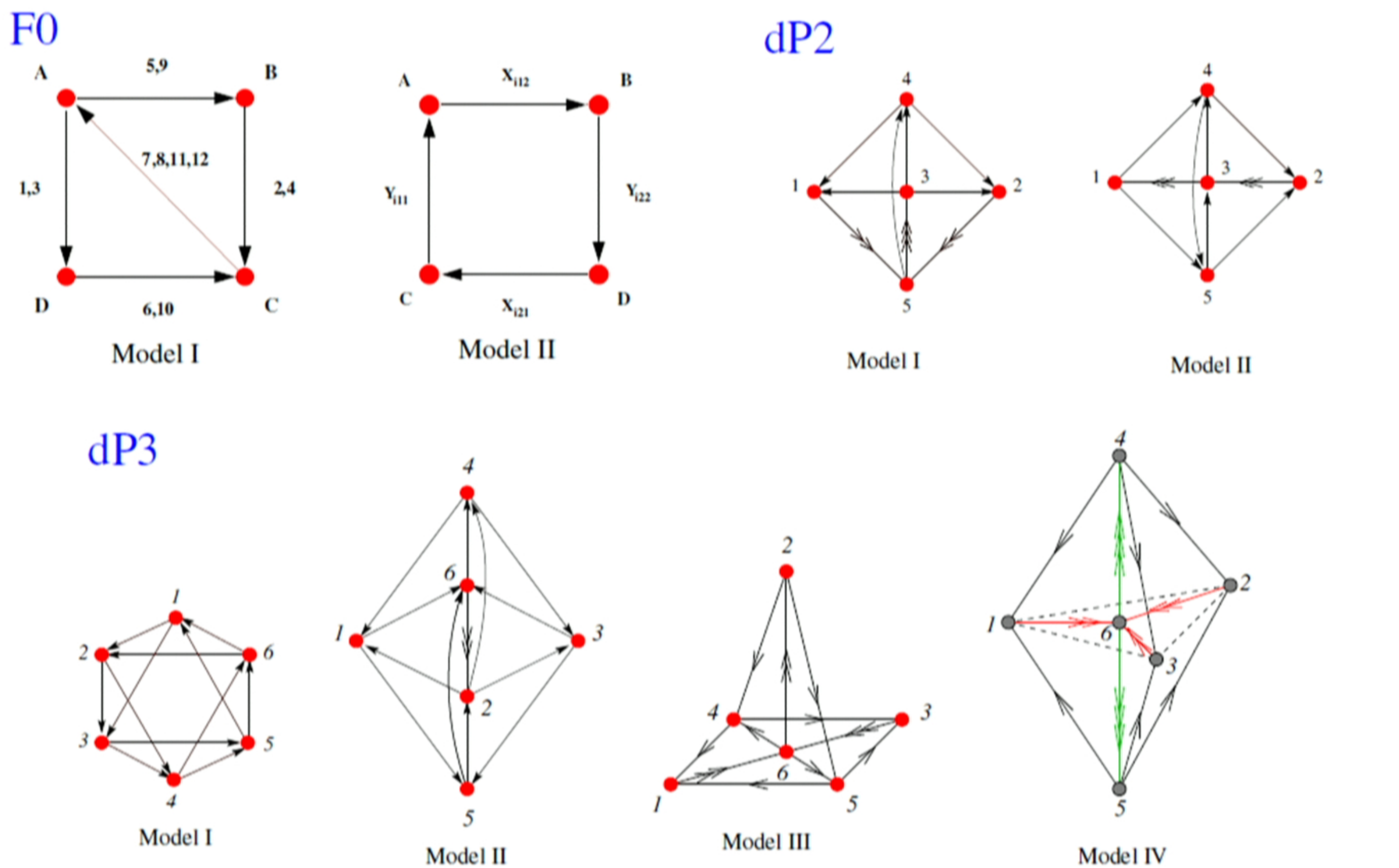
$$\begin{array}{lll} I_{in} & := & \text{nodes having arrows going into } i_0 \\ I_{out} & := & \text{nodes having arrow coming from } i_0 \\ I_{no} & := & \text{nodes unconnected with } i_0 \end{array}$$

2. $N_c \to N_f - N_c$ (where $N_f = \sum\limits_{i \in I_{in}} a_{i,i_0} = \sum\limits_{i \in I_{out}} a_{i_0,i}$)
3. Reverse arrows going in or out of i_0, leave I_{no}, and change affected nodes:

$$a_{ij}^{dual} = \begin{cases} a_{ji} & \text{if either} \quad i,j = i_0 \\ a_{ij} - a_{i_0 i} a_{j i_0} & \text{if both} \quad A \in I_{out}, B \in I_{in} \end{cases}$$

If negative, take it to mean $-a^{dual}$ arrows from j to i.

Many Examples



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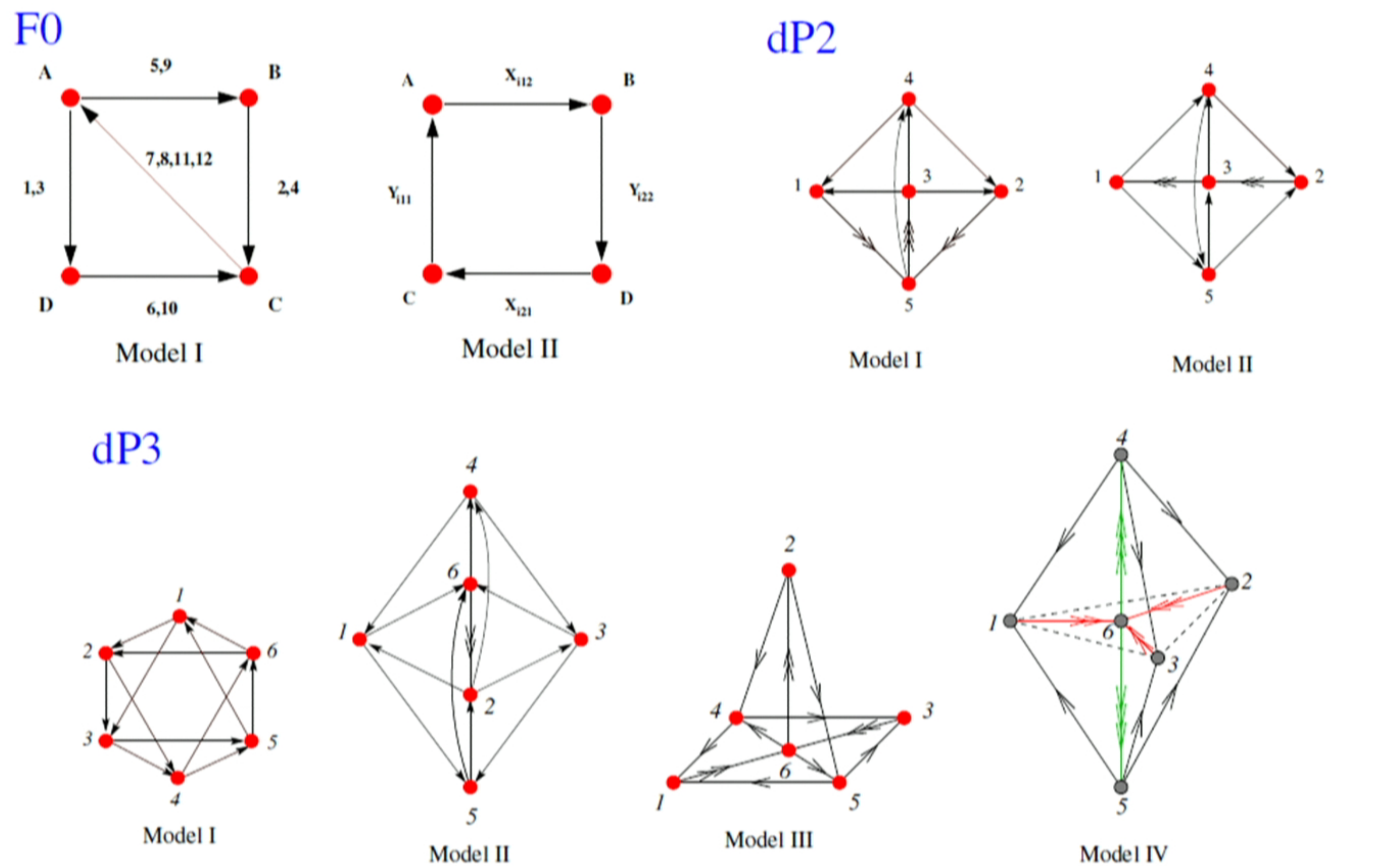

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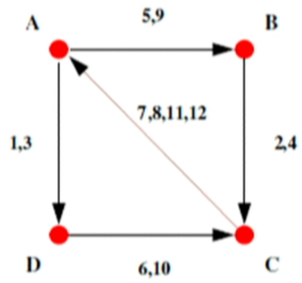
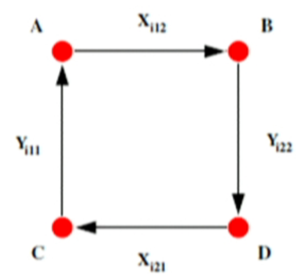
F0

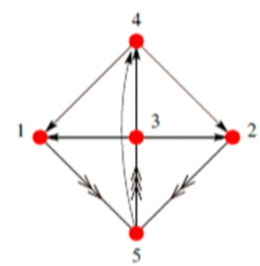


Model I



Model II

dP2



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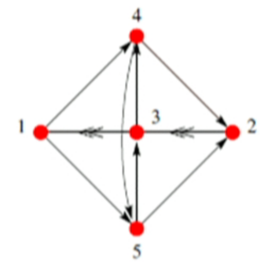
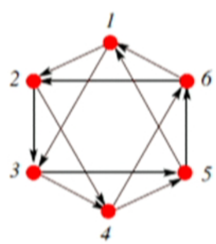


Model II

dP3



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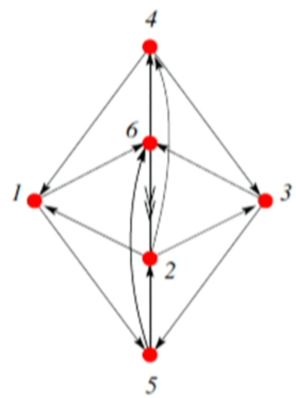

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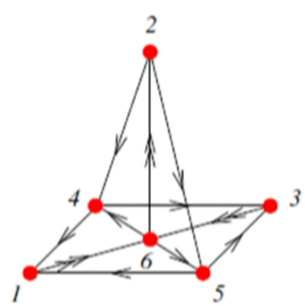
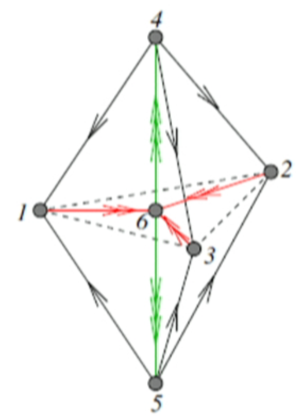


Model III



Model IV

Perspectives on Seiberg Duality

- Mirror Picture (Type IIA)
  - D6-branes wrapping SL-$k+3$ cycles S_i in the mirror Y
  - Quiver = intersection matrix $A_{ij} = S_i \circ S_j$
  - **Picard-Lefschetz** transformations about the S_{i_0} vanishing cycle:

    $S_i \to S_i - (S_i \circ S_{i_0}) S_{i_0}$
- Derived Category (Type IIB)
  - think of brane as support for coherent sheaf with Mukai vector:

    $\mathrm{ch}(F_i) := (\mathrm{rk}, c_1, c_2)$
  - Quiver: $A_{ij} = \chi(F_i, F_j) := \sum\limits_m (-1)^m \dim_{\mathbb{C}} \mathrm{Ext}^m(F_i, F_j)$
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  - cluster mutation rules on cluster (matrix) variables
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Grothendieck's Dessin d'Enfant

- Belyĭ Map: rational map $\beta : \Sigma \longrightarrow \mathbb{P}^1$ ramified only at $(0, 1, \infty)$
  - **Theorem [Belyĭ]:** β exists $\Leftrightarrow$ Σ can be defined over $\overline{\mathbb{Q}}$
  - (β, Σ) Belyĭ Pair
  - Dessin d'Enfants $= \beta^{-1}([0,1] \in \mathbb{P}^1)$ is a *bi-partite graph* on Σ: label $\beta^{-1}(0)$ black and $\beta^{-1}(1)$ white, then $\beta^{-1}(\infty)$ lives one per face
  - Ramification data: $\left\{ \begin{array}{c} r_0(1), r_0(2), \ldots, r_0(B) \\ r_1(1), r_1(2), \ldots, r_1(W) \\ r_\infty(1), r_\infty(2), \ldots, r_\infty(I) \end{array} \right\}$
- equivalently, Permutation Triple: σ_B, σ_W and $\sigma_B \sigma_W \sigma_\infty = \mathbb{I}$ (encodes how the sheets are permuted at the ramification points; cf. Ramgoolam et al.)

$$\sigma_B = (\ldots)_{r_0(1)}(\ldots)_{r_0(2)} \cdots (\ldots) r_0(B), \quad \sigma_W = (\ldots)_{r_1(1)}(\ldots)_{r_1(2)} \cdots (\ldots) r_1(W)$$

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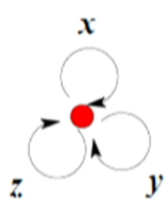
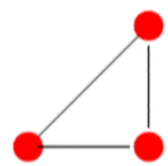
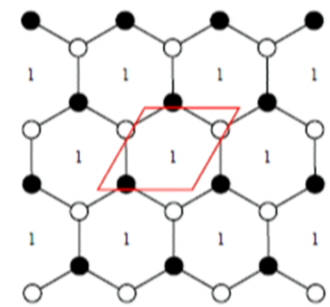
Gauge Theories and Dessins

- Matter content and interaction of SUSY gauge theory with toric moduli space is specified by Belyĭ pair
- Our most familiar example of $\mathcal{N} = 4$ super-Yang-Mills:

Theory	Toric Diag	Belyĭ Pair	Dessin on T^2 (dimer)
 $W = \mathrm{Tr}(X[Y,Z])$	 $\mathcal{M} \simeq \mathbb{C}^3$	$y^2 = x^3 + 1$ $\beta(x,y) = \frac{y+1}{2}$	

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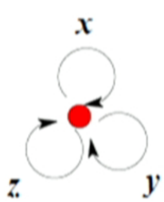
<td>
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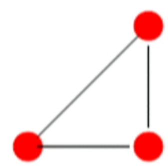
<td>
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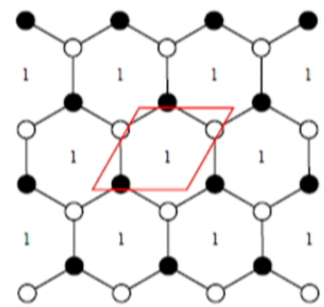
<td></td>
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- Klebanov-Witten's Conifold Theory

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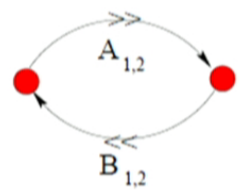
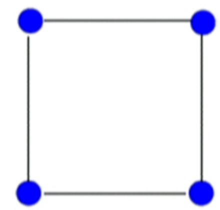
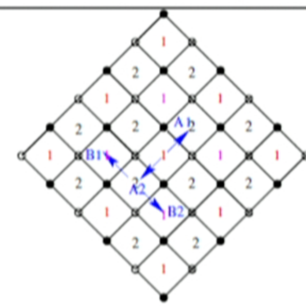
 $W = \text{Tr}(\epsilon_{il}\epsilon_{jk} A_i B_j A_l B_k)$	 $\mathcal{M} \simeq \{uv - wz = 0\} \subset \mathbb{C}^4$
Belyĭ Pair	**Dessin on T^2 (dimer)**
$y^2 = x(x-1)(x-\frac{1}{2})$ $\beta(x,y) = \frac{x^2}{2x-1}$	

Plethora of Non-Trivial Examples

e.g., Cone over $F_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (zeroth Hirzebruch surface);

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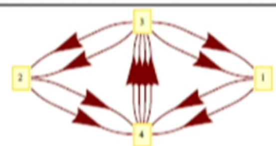
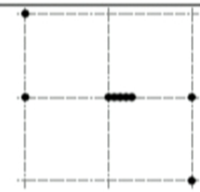
 $$W = X^1_{14}X^4_{43}X^2_{31} + X^2_{14}X^2_{43}X^1_{31} + X^1_{24}X^1_{43}X^1_{32} + X^2_{24}X^3_{43}X^2_{32} - X^1_{43}X^1_{31}X^1_{14} - X^2_{43}X^2_{32}X^1_{24} - X^3_{43}X^2_{31}X^2_{14} - X^4_{43}X^1_{32}X^2_{24}$$	 $\mathcal{M} \simeq$Hirzebruch 0
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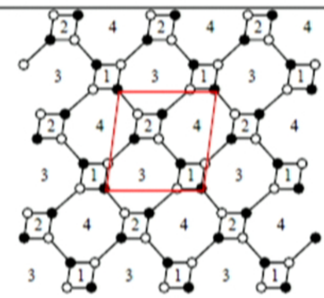
$$y^2 = x - x^3$$ $$\beta(x,y) = \frac{i\left(x^2 - \sqrt[3]{-1}\right)^3}{3\sqrt{3}x^2(x^2-1)}$$	

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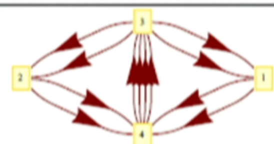
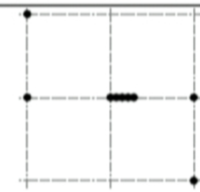
 $W = X^1_{14}X^4_{43}X^2_{31} + X^2_{14}X^2_{43}X^1_{31} + X^1_{24}X^1_{43}X^1_{32} + X^2_{24}X^3_{43}X^2_{32} - X^1_{43}X^1_{31}X^1_{14} - X^2_{43}X^2_{32}X^1_{24} - X^3_{43}X^2_{31}X^2_{14} - X^4_{43}X^1_{32}X^2_{24}$	 $\mathcal{M} \simeq$Hirzebruch 0
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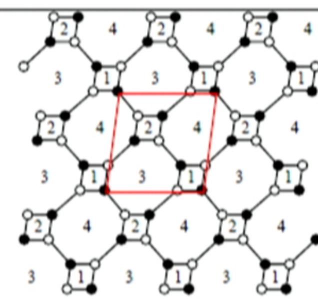
$y^2 = x - x^3$ $\beta(x,y) = \frac{i\left(x^2 - \sqrt[3]{-1}\right)^3}{3\sqrt{3}x^2(x^2-1)}$	

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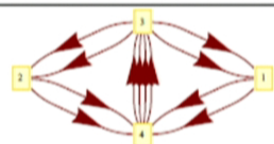

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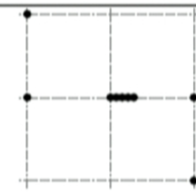

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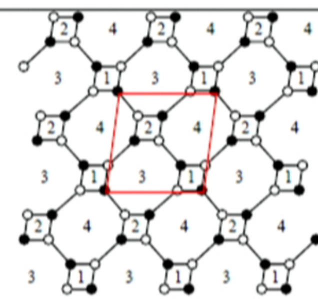
Dessin on T^2 (dimer)



Rigidity

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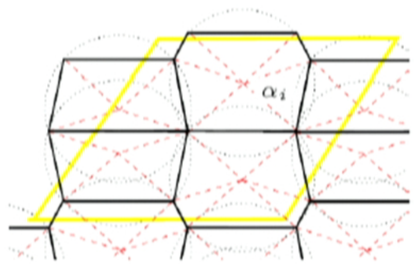

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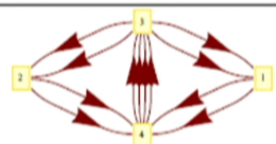
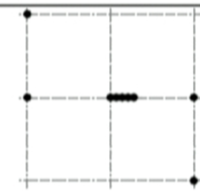
 $W = X^1_{14}X^4_{43}X^2_{31} + X^2_{14}X^2_{43}X^1_{31} + X^1_{24}X^1_{43}X^1_{32} + X^2_{24}X^3_{43}X^2_{32} - X^1_{43}X^1_{31}X^1_{14} - X^2_{43}X^2_{32}X^1_{24} - X^3_{43}X^2_{31}X^2_{14} - X^4_{43}X^1_{32}X^2_{24}$	 $\mathcal{M} \simeq$Hirzebruch 0
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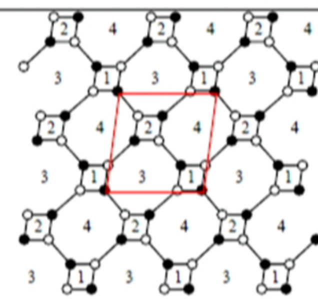
$y^2 = x - x^3$ $\beta(x,y) = \frac{i\left(x^2 - \sqrt[3]{-1}\right)^3}{3\sqrt{3}x^2(x^2-1)}$	

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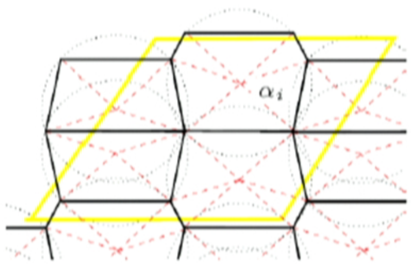

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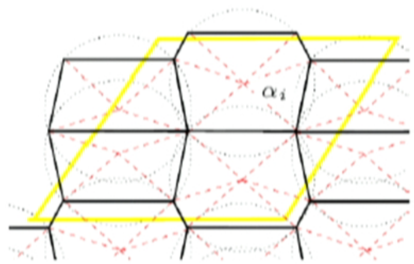

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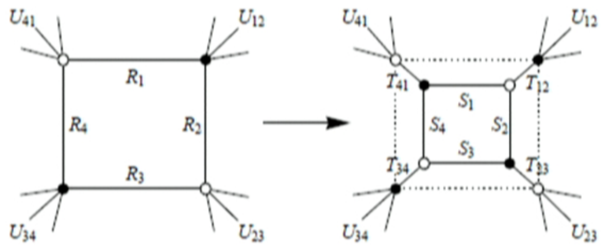
Square move, Seiberg Duality & Isogeny

- To fix R-charges: a-maximization or volume Z-minimization of SE (Intriligator-Wecht, Martelli-Sparks-Yau);
  $a(R) \sim 3\,\mathrm{Tr}\,R^3 - \mathrm{Tr}\,R \sim \sum_i (R_i - 1)^3$ need to maximize $a(R)$
  - Hanany, YHH, Jejjala, Pasukonis, Ramgoolam, Rodriguez-Gomez
  - R-charges and normalized volume of dual geometry are *algebraic numbers*
  - Many open puzzles : e.g. in some cases τ(isoradial) = τ(dessin), why? match τ(dimer) & T^2 in mirror T^3-fibration?
- Seiberg Duality = so-called "Urban Renewal"



- $j(\tau)$ of isoradial dimer invariant: elliptic curve for dessins isogenous
- transcendence deg over $\mathbb{Q}$ inv