Title: Completing Canonical Quantization

Date: May 21, 2013 03:30 PM

URL: http://pirsa.org/13050006

Abstract: The process of canonical quantization:is reexamined with the goal of ensuring there is only one reality, where $\hbar>0$, in which classical and quantum theories coexist. Two results are a clarification of the effect of canonical coordinate transformations and the role of Cartesian coordinates. Other results provide validation

of alternative quantum procedures that allow acceptable solutions for models
that exhibit unacceptable solutions using traditional procedures. Several examples will illustrate advantages of the new classical/quantum connection including one case where conventional methods lead to quantum triviality while the enhanced procedures lead to an acceptable nonlinear quantum behaviour.

This seminar uses only simple physical and mathematical concepts, and should be widely accessible.

Completing Canonical Quantization

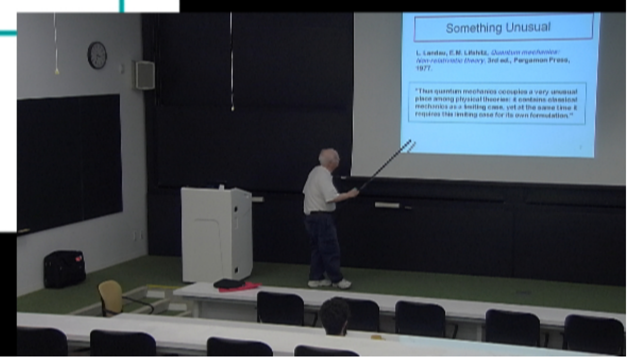
John R. Klauder
University of Florida
Gainesville, FL

1

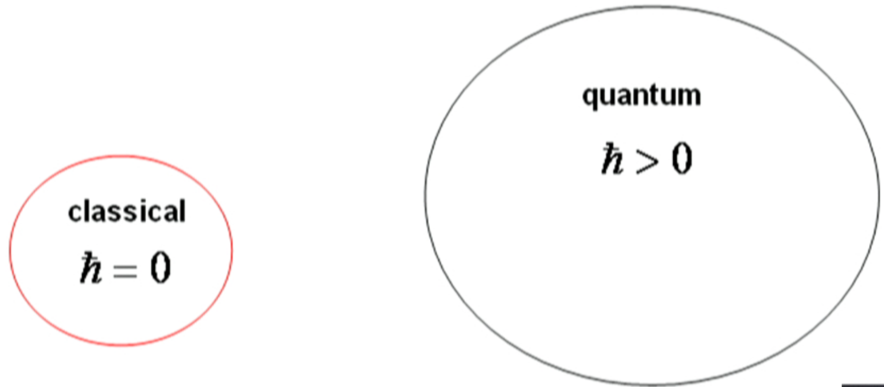
Something Unusual

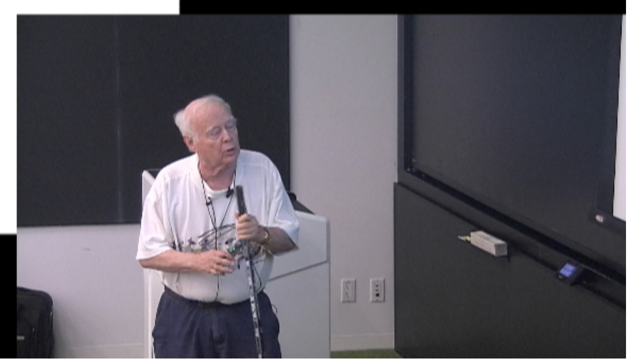
L. Landau, E.M. Lifshitz, *Quantum mechanics: Non-relativistic theory*, 3rd ed., Pergamon Press, 1977.

"Thus quantum mechanics occupies a very unusual place among physical theories: it contains classical mechanics as a limiting case, yet at the same time it requires this limiting case for its own formulation."

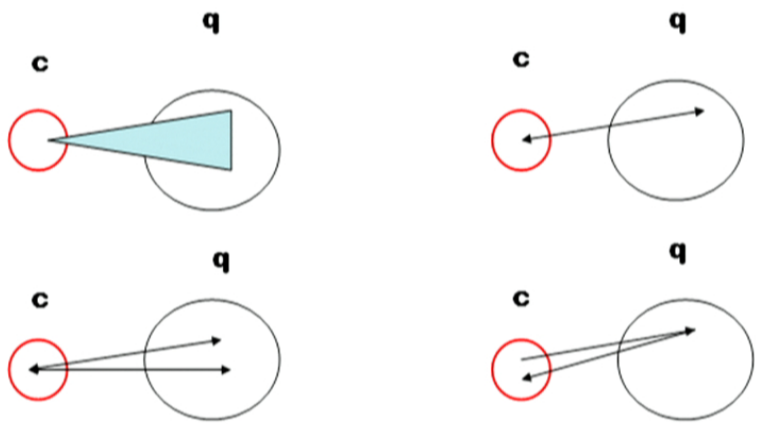


Classical & Quantum



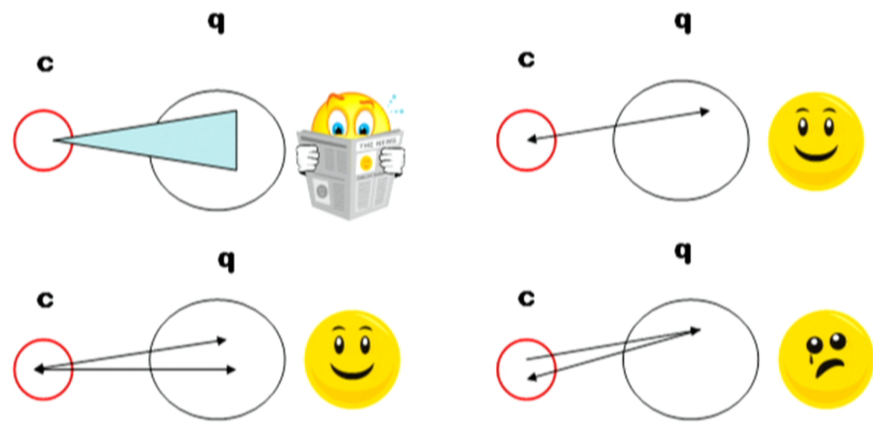


Class. & Quant. Possibilities



c = classical   q = quantum

Class. & Quant. Possibilities



c = classical   q = quantum

that the linear operator H introduced in the preceding section is the energy of the system in quantum mechanics.

In classical mechanics a dynamical system is defined mathematically when the Hamiltonian is given, i.e. when the energy is given in terms of a set of canonical coordinates and momenta, as this is sufficient to fix the equations of motion. In quantum mechanics a dynamical system is defined mathematically when the energy is given in terms of dynamical variables whose commutation relations are known, as this is then sufficient to fix the equations of motion, in both Schrödinger's and Heisenberg's form. We need to have either H expressed in terms of the Schrödinger dynamical variables or H_t expressed in terms of the corresponding Heisenberg dynamical variables, the functional relationship being, of course, the same in both cases. We call the energy expressed in this way the *Hamiltonian* of the dynamical system in quantum mechanics, to keep up the analogy with the classical theory.

A system in quantum mechanics always has a Hamiltonian, whether the system is one that has a classical analogue and is describable in terms of canonical coordinates and momenta or not. However, if the system does have a classical analogue, its connexion with classical mechanics is specially close and one can usually assume that the Hamiltonian is the same function of the canonical coordinates and momenta in the quantum theory as in the classical theory.† There would be a difficulty in this, of course, if the classical Hamiltonian involved a product of factors whose quantum analogues do not commute, as one would not know in which order to put these factors in the quantum Hamiltonian, but this does not happen for most of the elementary dynamical systems whose study is important for atomic physics. In consequence we are able also largely to use the same language for describing dynamical systems in the quantum theory as in the classical theory (e.g. to talk about particles with given masses moving through given fields of force), and when given a system in classical mechanics, can usually give a meaning to 'the same' system in quantum mechanics.

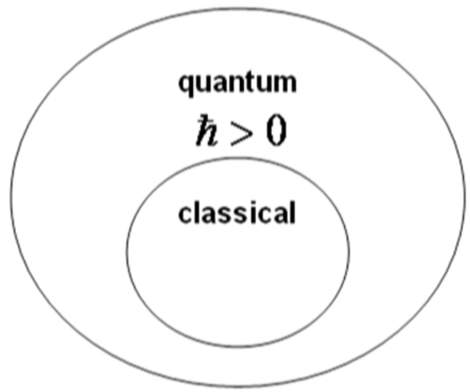
Equation (13) holds for v_t any function of the Heisenberg dynamical variables not involving the time explicitly, i.e. for v any constant

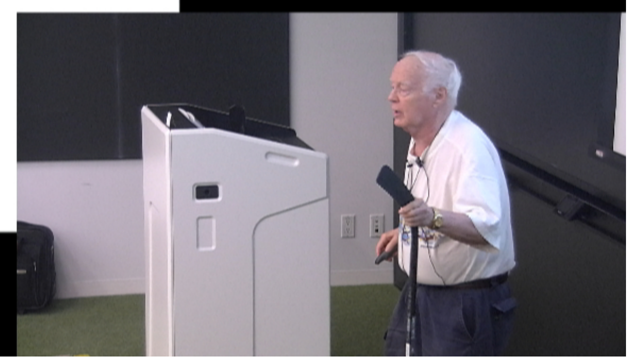
† This assumption is found in practice to be successful only when applied with the dynamical coordinates and momenta referring to a Cartesian system of axes and not to more general curvilinear coordinates.

DIRAC

The Principles of Quantum Mechanics

Classical ⊂ Quantum





List of Topics

1 Classical/Quantum connection

“Enhanced Quantization”

Canonical & Affine quantization

Enhanced classical theories

2 Two toy models

3 Rotationally symmetric models

TOPIC 1

- Classical & Quantum formalism
- Canonical coherent states
- Classical $\subset$ Quantum formalism
- Canonical transformations
- Cartesian coordinates
- Affine vs. canonical variables
- Affine quantization as canonical quantization

Action Principle Formulations

Classical action : $A_C = \int [\, p(t)\dot{q}(t) - H_c(\,p(t), q(t))]\ dt$

Variation : $\delta A_C = 0$ yields : $\dot{q} = \partial H_c / \partial p, \ \ \dot{p} = -\partial H_c / \partial q$

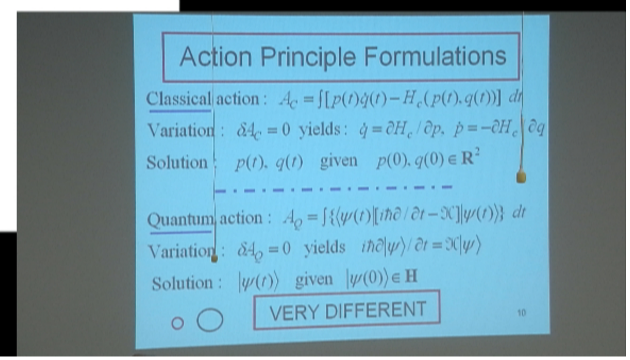
Solution : $p(t),\ q(t)$ given $p(0), q(0) \in \mathbf{R}^2$

Quantum action : $A_Q = \int \{ \langle \psi(t) | [\, i\hbar \partial / \partial t - \mathcal{H}\,] | \psi(t) \rangle \}\ dt$

Variation : $\delta A_Q = 0$ yields $i\hbar \partial |\psi\rangle / \partial t = \mathcal{H} |\psi\rangle$

Solution : $|\psi(t)\rangle$ given $|\psi(0)\rangle \in \mathbf{H}$

VERY DIFFERENT



Unification of Classical and Quantum (1)

Quantum action : $A_Q = \int \langle \psi(t) | [i\hbar\partial / \partial t - \mathcal{H}] | \psi(t) \rangle \, dt$

Restricted variation : $|\psi(t)\rangle \to |\,?\,(t)\,\rangle \in \mathbf{S} \subset \mathbf{H}$

sun

Macroscopic variations of **Microscopic** states:

Basic state: $\langle x | \eta \rangle = \eta(x)$

Translated basic state: $\langle x | \eta; q \rangle = \eta(x - q)$

Translated Fourier state: $\langle k | \eta; p \rangle = \tilde{\eta}(k - p)$

Coherent states: $\langle x | p, q \rangle = e^{ip(x-q)/\hbar} \, \eta(x - q)$

$$|p, q\rangle \equiv e^{-iqP/\hbar} e^{ipQ/\hbar} |\eta\rangle \; ; \quad |\eta\rangle = |0\rangle \; ; \quad (Q + iP)|0\rangle = 0$$

12 s.a.

Unification of Classical and Quantum (2)

Quantum action : $A_Q = \int \langle \psi(t) \| [i\hbar \partial / \partial t - \mathcal{H}] \| \psi(t) \rangle \; dt$

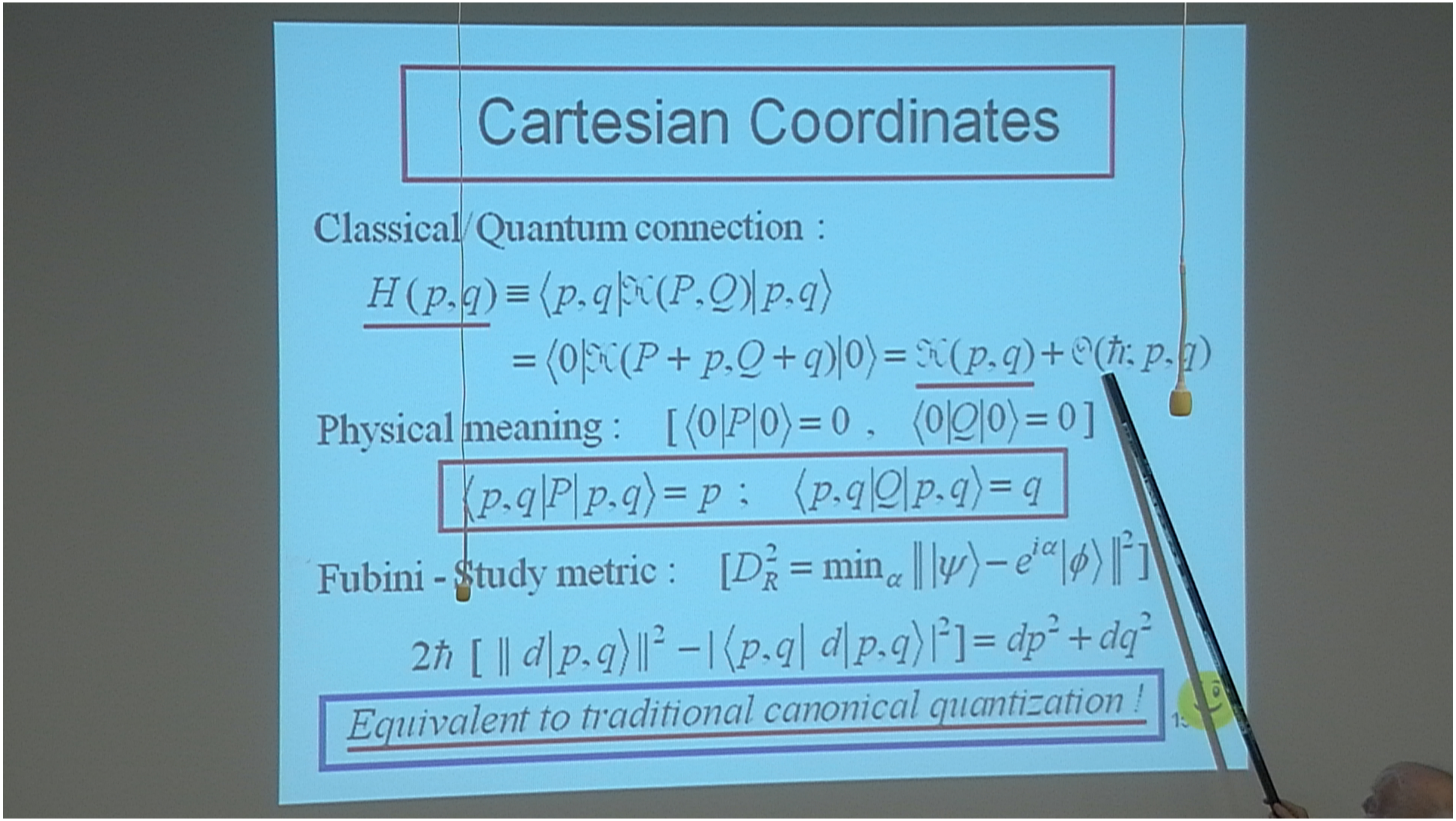
Restricted variation : $|\psi(t)\rangle \to |p(t), q(t)\rangle$ **subset**

New action : $A_R = \int \langle p(t), q(t) \| [i\hbar \partial / \partial t - \mathcal{H}] \| p(t), q(t) \rangle \; dt$

$$A_R = \int [\, p(t)\dot{q}(t) - H(\, p(t), q(t))] \; dt$$

CLASSICAL MECHANICS *IS* QUANTUM MECHANICS RESTRICTED TO A CERTAIN TWO DIMENSIONAL SURFACE IN HILBERT SPACE

13


Cartesian Coordinates
Classical/Quantum connection :
$H(p,q) \equiv \langle p,q|\mathcal{H}(P,Q)|p,q\rangle$
$= \langle 0|\mathcal{H}(P+p,Q+q)|0\rangle = \mathcal{H}(p,q) + \mathcal{O}(\hbar;p,q)$
Physical meaning : $[\langle 0|P|0\rangle = 0 \;,\; \langle 0|Q|0\rangle = 0\,]$
$\langle p,q|P|p,q\rangle = p \;;\; \langle p,q|Q|p,q\rangle = q$
Fubini - Study metric : $[D_R^2 = \min_\alpha \|\,|\psi\rangle - e^{i\alpha}|\phi\rangle\,\|^2]$
$2\hbar\;[\,\|\,d|p,q\rangle\|^2 - |\langle p,q|\;d|p,q\rangle|^2] = dp^2 + dq^2$
Equivalent to traditional canonical quantization !

Quantum/Classical Summary

Qauntum action :

$$A_Q = \int \langle \psi(t) | \, [i\hbar\partial/\partial t - \mathcal{H}(P,Q)] \, | \psi(t) \rangle \; dt$$

$$i\hbar\partial|\psi(t)\rangle/\partial t = \mathcal{H}(P,Q)|\psi(t)\rangle$$

Restricted quantum action :

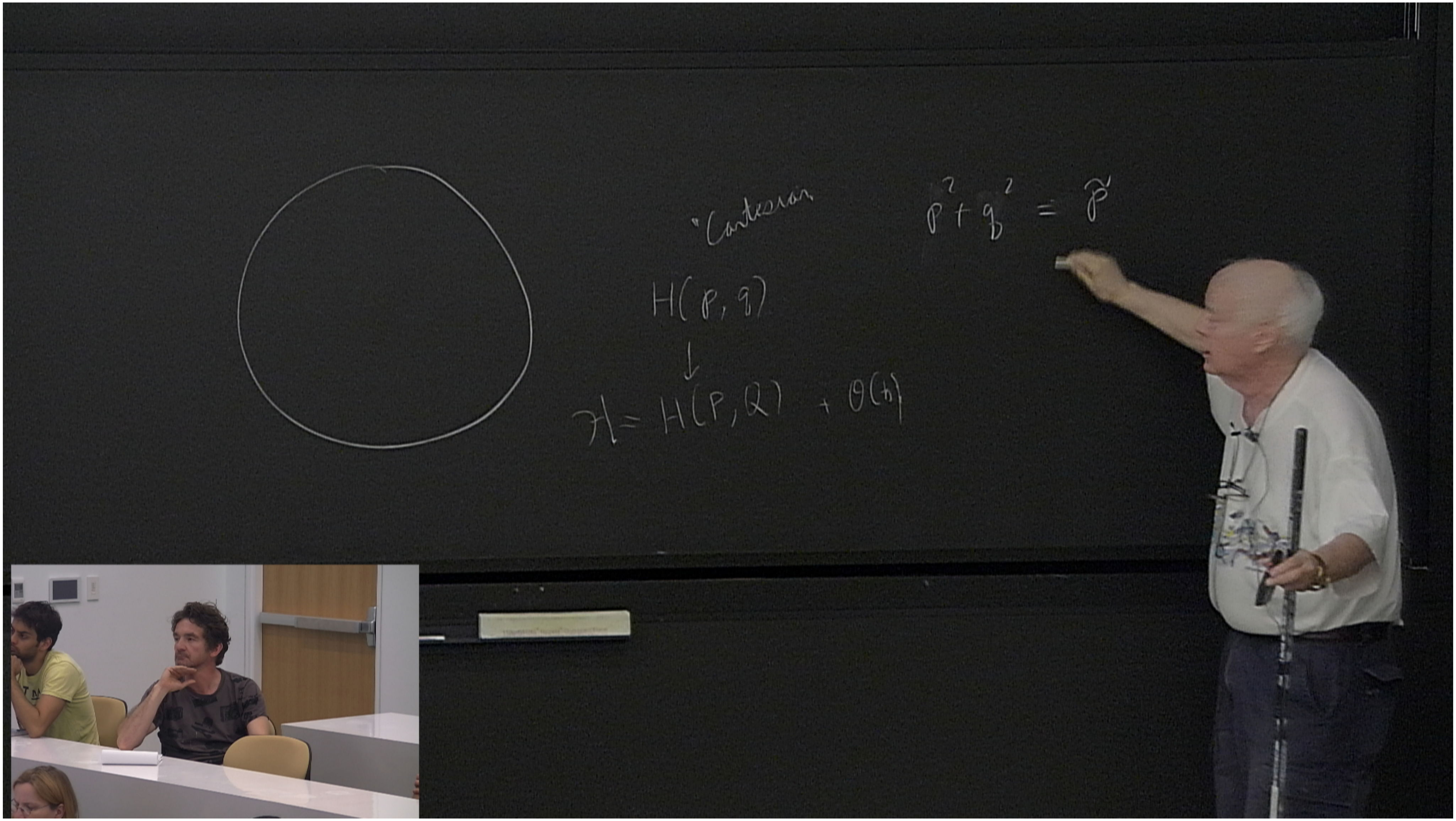
$$|\psi(t)\rangle \to |p(t),q(t)\rangle \equiv e^{-iq(t)P/\hbar} e^{ip(t)Q/\hbar} |0\rangle$$

$$A_{Q-rest.} = \int \underline{\langle p(t),q(t)|} \, [i\hbar\partial/\partial t - \underline{\mathcal{H}(P,Q)}] \, \underline{|p(t),q(t)\rangle} \; dt$$

$$= \int [p(t)\dot{q}(t) - \underline{H(p(t),q(t))}] \; dt$$

$$\dot{q} = \partial H(p,q)/\partial p \; , \quad \dot{p} = -\partial H(p,q)/\partial q$$

16


"Cartesian"
H(p, q)
ℋ = H(P, Q) + O(ħ)

Is There More?

- Are there ***other*** **two-dimensional sheets of normalized Hilbert space vectors** that may be used in **restricting the quantum action** and which lead to an ***enhanced classical canonical formalism***?

17

Affine Variables

Affine variables : $q = q\{q,p\} = \{q, pq\} \equiv \{q,d\}$

$i\hbar Q = Q[Q,P] = [Q,QP] = [Q,D]$; $D \equiv (PQ+QP)/2$ s.a.

Affine coherent states : $(q > 0\ ;\ Q > 0)$

$$|p,q\rangle = e^{ipQ/\hbar} e^{-i\ln(q)D/\hbar}|\eta\rangle ; \qquad [(Q-1) + iD/\tilde{\beta}]\ |\eta\rangle = 0$$

Overlap function :

$$\langle p',q'|p,q\rangle = \{\tfrac{1}{2}[\sqrt{q'/q} + \sqrt{q/q'} + i\sqrt{q'q}\,(p'-p)/\tilde{\beta}]\}^{-2\tilde{\beta}/\hbar}$$

Resolution of unity :

$$I = \int |p,q\rangle\langle p,q|\ dp\ dq/2\pi\hbar/(1-1/2\tilde{\beta}/\hbar)$$

also $(q < 0\ ,\ Q < 0) \cup (q > 0\ ,\ Q > 0)$

19

Affine Variables

Affine variables : $q = q\{q, p\} = \{q, pq\} \equiv \{q, d\}$

$$i\hbar Q = Q[Q,P] = [Q,QP] = [Q,D];\ D \equiv (PQ + QP)/2$$ s.a.

Affine coherent states : $(q > 0\ ;\ Q > 0)$

$$|p,q\rangle = e^{ipQ/\hbar} e^{-i\ln(q)D/\hbar}|\eta\rangle; \qquad [(Q-1) + iD/\tilde{\beta}]\ |\eta\rangle = 0$$

Overlap function :

$$\langle p',q'|p,q\rangle = \{\tfrac{1}{2}[\sqrt{q'/q} + \sqrt{q/q'} + i\sqrt{q'q}(p'-p)/\tilde{\beta}]\}^{-2\tilde{\beta}/\hbar}$$

Resolution of unity :

$$I = \int |p,q\rangle\langle p,q|\ dp\ dq/2\pi\hbar/(1-1/2\tilde{\beta}/\hbar)$$

also $(q < 0\ ,\ Q < 0) \cup (q > 0\ ,\ Q > 0)$

19

Affine Variables

Affine variables : $q = q\{q, p\} = \{q, pq\} \equiv \{q, d\}$

$$i\hbar Q = Q[Q, P] = [Q, QP] = [Q, D]; \quad D \equiv (PQ + QP)/2$$ s.a.

Affine coherent states : $(q > 0 \; ; \; Q > 0)$

$$|p, q\rangle = e^{ipQ/\hbar} e^{-i\ln(q)D/\hbar} |\eta\rangle; \qquad [(Q-1) + iD/\widetilde{\beta}] \; |\eta\rangle = 0$$

Overlap function :

$$\langle p', q' | p, q \rangle = \{\tfrac{1}{2}[\sqrt{q'/q} + \sqrt{q/q'} + i\sqrt{q'q}\,(p'-p)/\widetilde{\beta}]\}^{-2\widetilde{\beta}/\hbar}$$

Resolution of unity :

$$I = \int |p, q\rangle\langle p, q| \; dp \; dq / 2\pi\hbar / (1 - 1/2\widetilde{\beta}/\hbar)$$

also $(q < 0 \, , \, Q < 0) \cup (q > 0 \, , \, Q > 0)$

19

Affine Quantization (1)

Quantum action :

$$A_Q \equiv \int \langle \psi(t) | [i\hbar\partial/\partial t - \mathcal{H}(D,Q)] | \psi(t) \rangle \; dt$$

Restricted action : $|\psi(t)\rangle \rightarrow |p(t),q(t)\rangle$ **subset**

$$A_R \equiv \int \langle p(t),q(t) | [i\hbar\partial/\partial t - \mathcal{H}(D,Q)] | p(t),q(t) \rangle \; dt$$

$$= \int [-q(t)\dot{p}(t) - H(p(t),q(t))] \; dt$$

Canonical transformation : $|\tilde{p},\tilde{q}\rangle = |p,q\rangle$

$$A_R \equiv \int \langle \tilde{p}(t),\tilde{q}(t) | [i\hbar\partial/\partial t - \mathcal{H}(D,Q)] | \tilde{p}(t),\tilde{q}(t) \rangle \; dt$$

$$= \int [-\tilde{q}(t)\dot{\tilde{p}}(t) + \dot{\tilde{G}}'(\tilde{p}(t),\tilde{q}(t)) - \tilde{H}(\tilde{p}(t),\tilde{q}(t))] \; dt$$

20

Affine Quantization (2)

Classical/Quantum connection :

$$\underline{H'(pq,q)} \equiv \langle p,q|\mathcal{H}(D,Q)|p,q\rangle$$

$$= \langle\eta|\mathcal{H}(D+pqQ,qQ)|\eta\rangle = \underline{\mathcal{H}(pq,q)} + \mathcal{O}(\hbar;p,q)$$

Physical meaning : $[\langle\eta|D|\eta\rangle = 0\ ,\quad \langle\eta|Q|\eta\rangle = 1]$

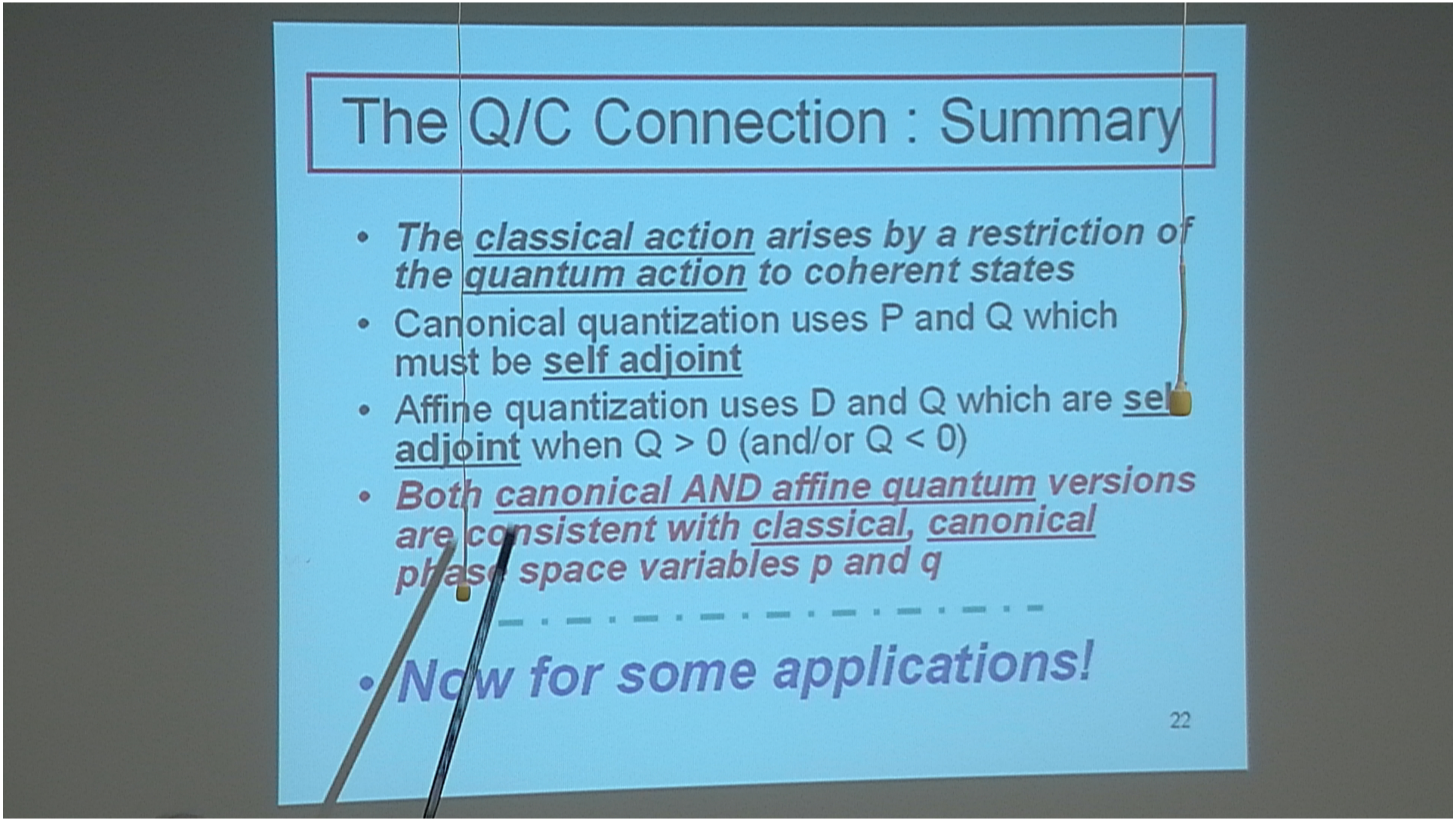
$$\boxed{\langle p,q|D|p,q\rangle = pq\ ;\quad \langle p,q|Q|p,q\rangle = q}$$

Fubini - Study metric :

$$2\hbar\ [\ \|\, d|p,q\rangle\|^2 - |\langle p,q|\,d|p,q\rangle|^2] = \tilde{\beta}^{-1}q^2dp^2 + \tilde{\beta}\ q^{-2}dq^2$$

Metric becomes Cartesian : $q \rightarrow q+\gamma\ ,\quad \tilde{\beta}/\hbar = \gamma^2 \rightarrow \infty$

21


The Q/C Connection : Summary
• The classical action arises by a restriction of the quantum action to coherent states
• Canonical quantization uses P and Q which must be self adjoint
• Affine quantization uses D and Q which are sel
adjoint when Q > 0 (and/or Q < 0)
• Both canonical AND affine quantum versions are consistent with classical, canonical
ph as space variables p and q
• Now for some applications!
22

TOPIC 2

- Solutions of the first model have singularities

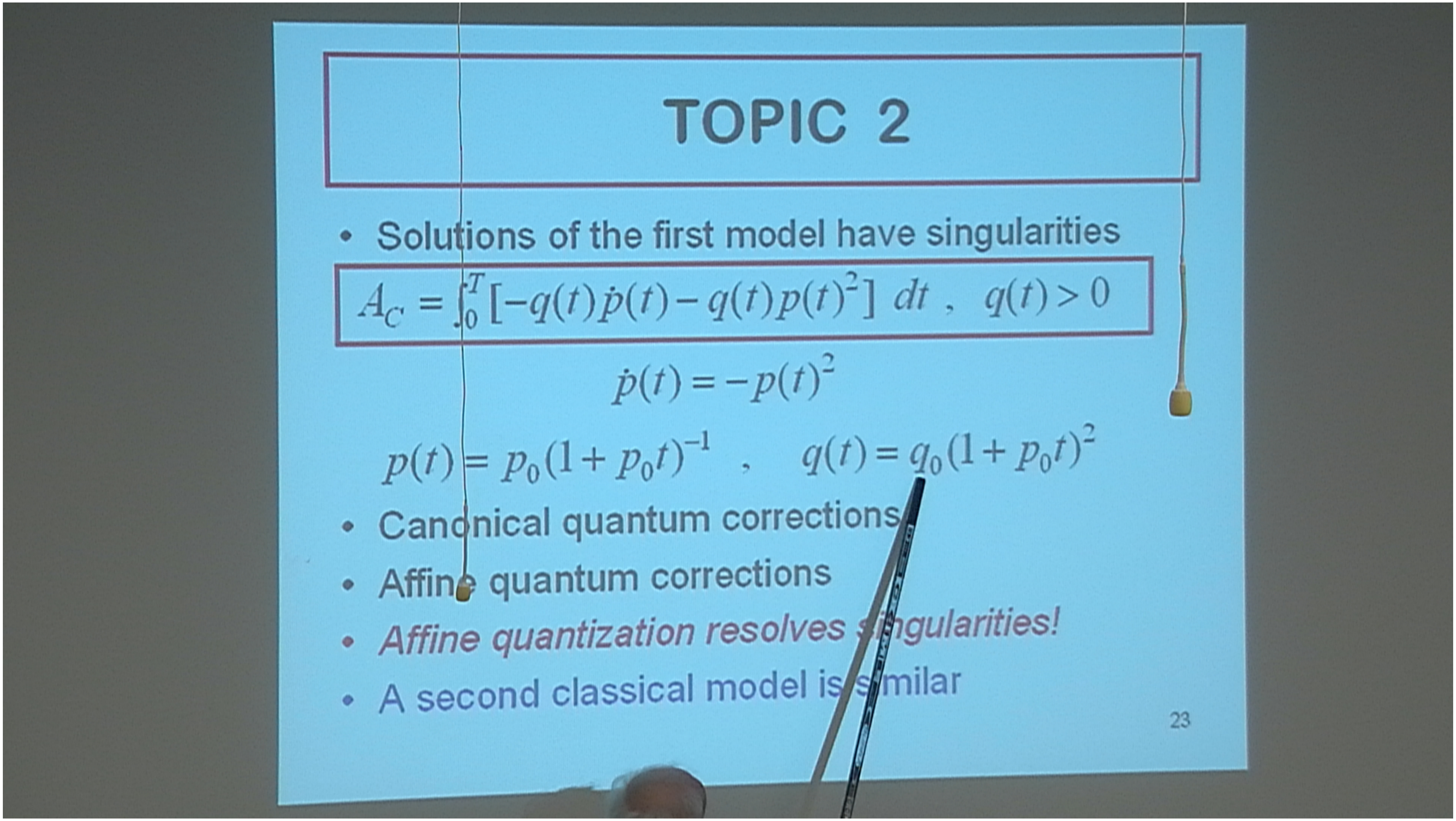
$$A_C = \int_0^T [-q(t)\dot{p}(t) - q(t)p(t)^2]\ dt\ ,\quad q(t) > 0$$

$$\dot{p}(t) = -p(t)^2$$

$$p(t) = p_0(1+p_0 t)^{-1}\quad ,\quad q(t) = q_0(1+p_0 t)^2$$

- Canonical quantum corrections
- Affine quantum corrections
- *Affine quantization resolves singularities!*
- A second classical model is similar

23


TOPIC 2
• Solutions of the first model have singularities
$A_C = \int_0^T [-q(t)\dot{p}(t) - q(t)p(t)^2]\, dt \;,\;\; q(t) > 0$
$\dot{p}(t) = -p(t)^2$
$p(t) = p_0(1+p_0 t)^{-1} \;,\;\; q(t) = q_0(1+p_0 t)^2$
• Canonical quantum corrections
• Affine quantum corrections
• Affine quantization resolves singularities!
• A second classical model is similar
23

Toy Model - 1

Classical action. : $A_C = \int [-q\dot{p} - qp^2]dt$; $q > 0$

Solution : $p(t) = p_0(1 + p_0 t)^{-1}$, $q(t) = q_0(1 + p_0 t)^2$

Canonical quant. : $\langle p,q|PQP|p,q\rangle = qp^2 + qa^2$; $a^2 = \hbar/2$

Solution : $p(t) = a\cot(a(t+\tau))$, $q(t) = (E_0/a^2)\sin(a(t+\tau))^2$

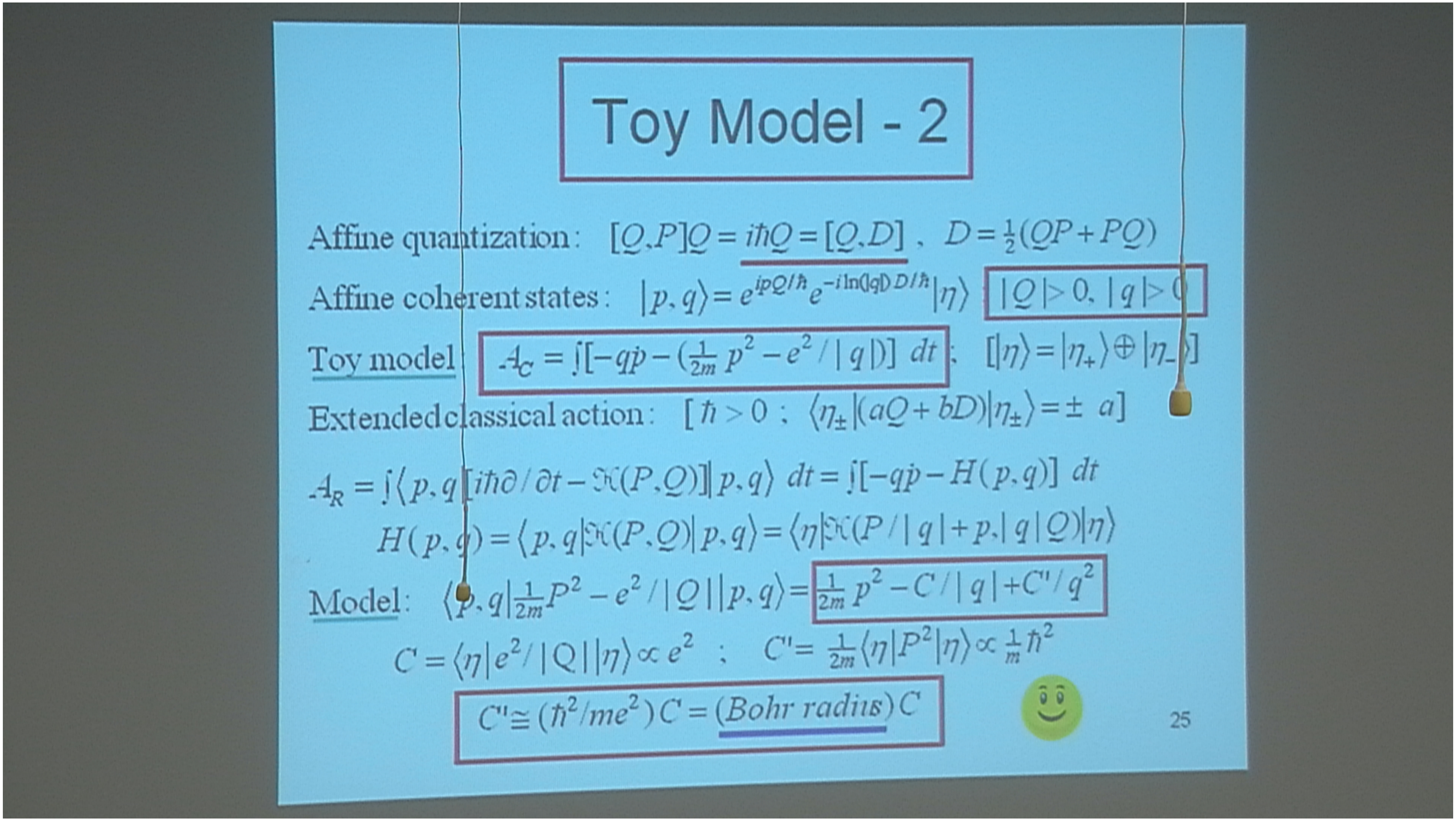
Affine quant. : $\langle p,q|DQ^{-1}D|p,q\rangle = qp^2 + \hbar^2 C/q$ ⟵

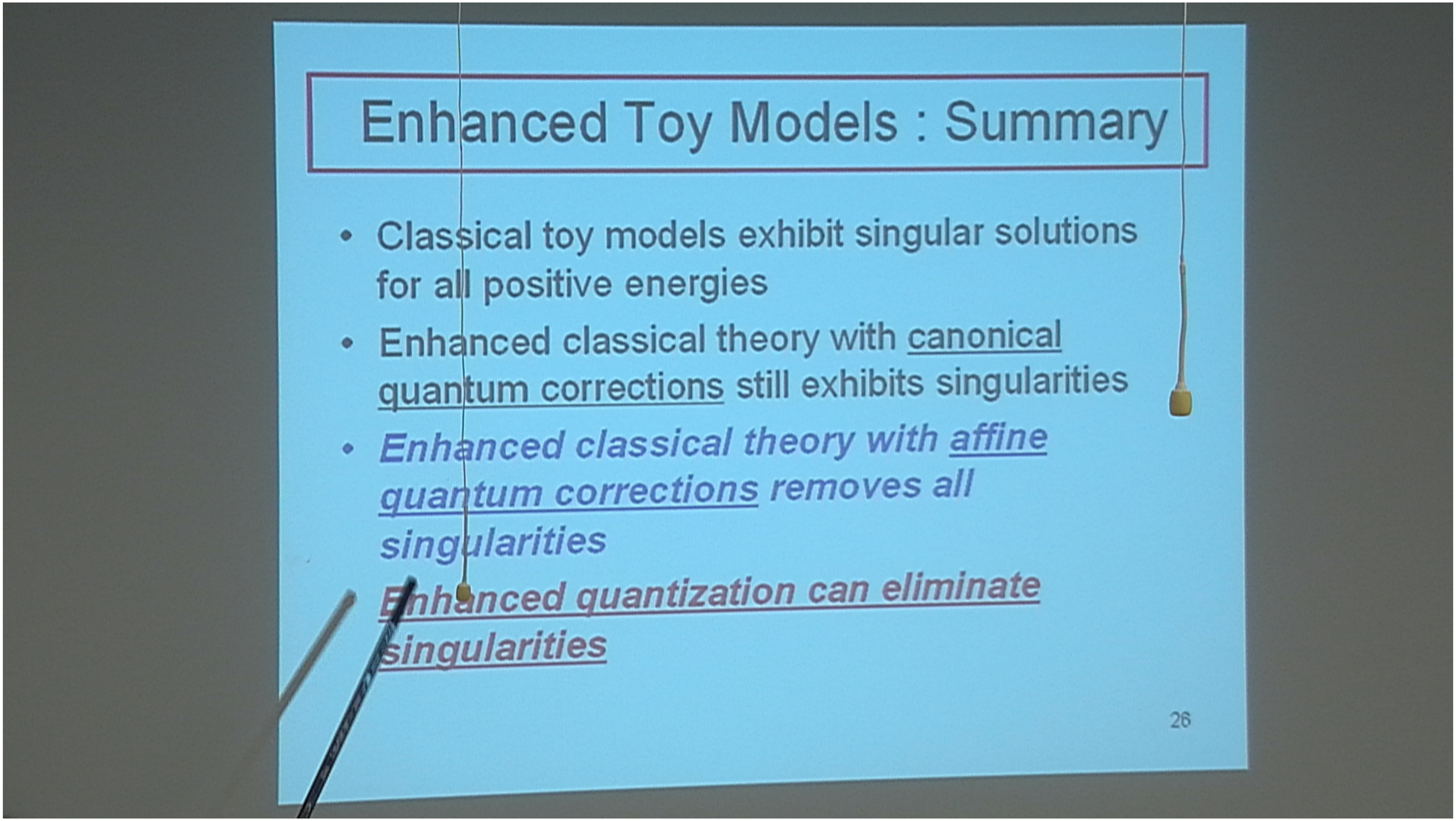
Solution : $p(t) = \dfrac{(t+\tau)}{(t+\tau)^2 + K}$, $q(t) = M[(t+\tau)^2 + K] > 0$

$$\hbar^2 C = \langle \eta|DQ^{-1}D|\eta\rangle;\ K = \hbar^2 C/4E_0^2\ ;\ M = 4E_0$$

→ $e^{iqP/\hbar} Q e^{-iqP/\hbar} = Q + q$; $e^{i\ln(q)D/\hbar} Q e^{-i\ln(q)D/\hbar} = qQ$ ⟵

24


Toy Model - 2
Affine quantization: $[Q,P]Q = i\hbar Q = [Q,D]$, $D = \frac{1}{2}(QP + PQ)$
Affine coherent states: $|p,q\rangle = e^{ipQ/\hbar} e^{-i\ln(|q|)D/\hbar}|\eta\rangle$; $|Q| > 0, |q| > 0$
Toy model: $A_C = \int[-q\dot{p} - (\frac{1}{2m}p^2 - e^2/|q|)]\,dt$; $[|\eta\rangle = |\eta_+\rangle \oplus |\eta_-\rangle]$
Extended classical action: $[\hbar > 0\ ;\ \langle\eta_\pm|(aQ + bD)|\eta_\pm\rangle = \pm\, a]$
$A_R = \int\langle p,q|[i\hbar\partial/\partial t - \mathcal{H}(P,Q)]|p,q\rangle\,dt = \int[-q\dot{p} - H(p,q)]\,dt$
$H(p,q) = \langle p,q|\mathcal{H}(P,Q)|p,q\rangle = \langle\eta|\mathcal{H}(P/|q| + p, |q|Q)|\eta\rangle$
Model: $\langle p,q|\frac{1}{2m}P^2 - e^2/|Q|\,|p,q\rangle = \frac{1}{2m}p^2 - C/|q| + C'/q^2$
$C = \langle\eta|e^2/|Q|\,|\eta\rangle \propto e^2$; $C' = \frac{1}{2m}\langle\eta|P^2|\eta\rangle \propto \frac{1}{m}\hbar^2$
$C' \cong (\hbar^2/me^2)C = (\text{Bohr radius})C$
25


Enhanced Toy Models : Summary
• Classical toy models exhibit singular solutions for all positive energies
• Enhanced classical theory with canonical quantum corrections still exhibits singularities
• Enhanced classical theory with affine quantum corrections removes all singularities
• Enhanced quantization can eliminate singularities
26

TOPIC 3

- Rotationally symmetric models $[\vec{p}^{\,2} \equiv \vec{p} \cdot \vec{p}]$

$$H(\vec{p},\vec{q}) = \tfrac{1}{2}[\vec{p}^{\,2} + m_0^2 \vec{q}^{\,2}] + \lambda_0 (\vec{q}^{\,2})^2 \quad , \quad \vec{p} = \{p_n\}_{n=1}^{N}$$

- Free quantum models for $N \leq \infty$
- Interacting quantum models for $N < \infty$

$$H(\vec{p},\vec{q}) \equiv \langle \vec{p},\vec{q} | \mathcal{H} | \vec{p},\vec{q} \rangle$$

- ***Reducible operator representation is the key***

$$H(\vec{p},\vec{q}) \equiv \langle \vec{p},\vec{q} | \mathcal{H}(\vec{P},\vec{Q},\cdots) | \vec{p},\vec{q} \rangle$$

27

TOPIC 3

- Rotationally symmetric models $[\vec{p}^{\,2} \equiv \vec{p} \cdot \vec{p}]$

$$H(\vec{p},\vec{q}) = \tfrac{1}{2}[\vec{p}^{\,2} + m_0^2\vec{q}^{\,2}] + \lambda_0(\vec{q}^{\,2})^2 \quad , \quad \vec{p} = \{p_n\}_{n=1}^{N}$$

- Free quantum models for $N \leq \infty$
- Interacting quantum models for $N < \infty$

$$H(\vec{p},\vec{q}) \equiv \langle \vec{p},\vec{q} | \mathcal{H} | \vec{p},\vec{q} \rangle$$

- ***Reducible operator representation is the key***

$$H(\vec{p},\vec{q}) \equiv \langle \vec{p},\vec{q} | \mathcal{H}(\vec{P},\vec{Q}, \cdots) | \vec{p},\vec{q} \rangle$$

27

Rotationally Sym. Models (1)

Phase space coordinates : $\vec{p} = (p_1, \ldots, p_N)$, $\vec{q} = (q_1, \ldots, q_N)$

$$H(\vec{p}, \vec{q}) = \tfrac{1}{2}[\vec{p}^{\,2} + m_0^2 \vec{q}^{\,2}] + \lambda_0 (\vec{q}^{\,2})^2 \quad ; \quad \boxed{N \le \infty}$$

Invariant under $\vec{p} \to O\vec{p}$, $\vec{q} \to O\vec{q}$; $O \in \mathrm{O}(N,R)$

Basic invariants : $X \equiv \vec{p}^{\,2}$, $Y \equiv \vec{p} \cdot \vec{q}$, $Z \equiv \vec{q}^{\,2}$

Constants of motion : E , $\vec{L}^2 = (\vec{p} \times \vec{q})^2 = XZ - Y^2$

Quantization : $\vec{p} \to \vec{P}$, $\vec{q} \to \vec{Q}$; $[Q_j, P_k] = i\hbar\delta_{jk}$

Hamiltonian : $\mathcal{H} = \tfrac{1}{2} : \vec{P}^2 + m_0^2 \vec{Q}^2 : + \lambda_0 : (\vec{Q}^2)^2 :$, $N < \infty$

When $N \to \infty$, *it is necessary that* $\lambda_0 = \lambda / N$

28

Rotationally Sym. Models (2)

Schroedinger equation with a real *unique ground state* $\varphi_N(\bar{x})$ with *full rotational symmetry*: $\psi_N(r) = \varphi_N(\bar{x})$.

Fourier transformation of the ground state distribution

$$C_N(\bar{p}) \equiv \int e^{i\bar{p}\cdot\bar{x}/\hbar}\varphi_N(\bar{x})^2 d\bar{x}$$

$$= \int e^{ipr\cos(\theta)/\hbar}\psi_N(r)^2 r^{N-1}\sin(\theta)^{N-2}\,dr\,d\theta\,d\Omega_{N-2}$$

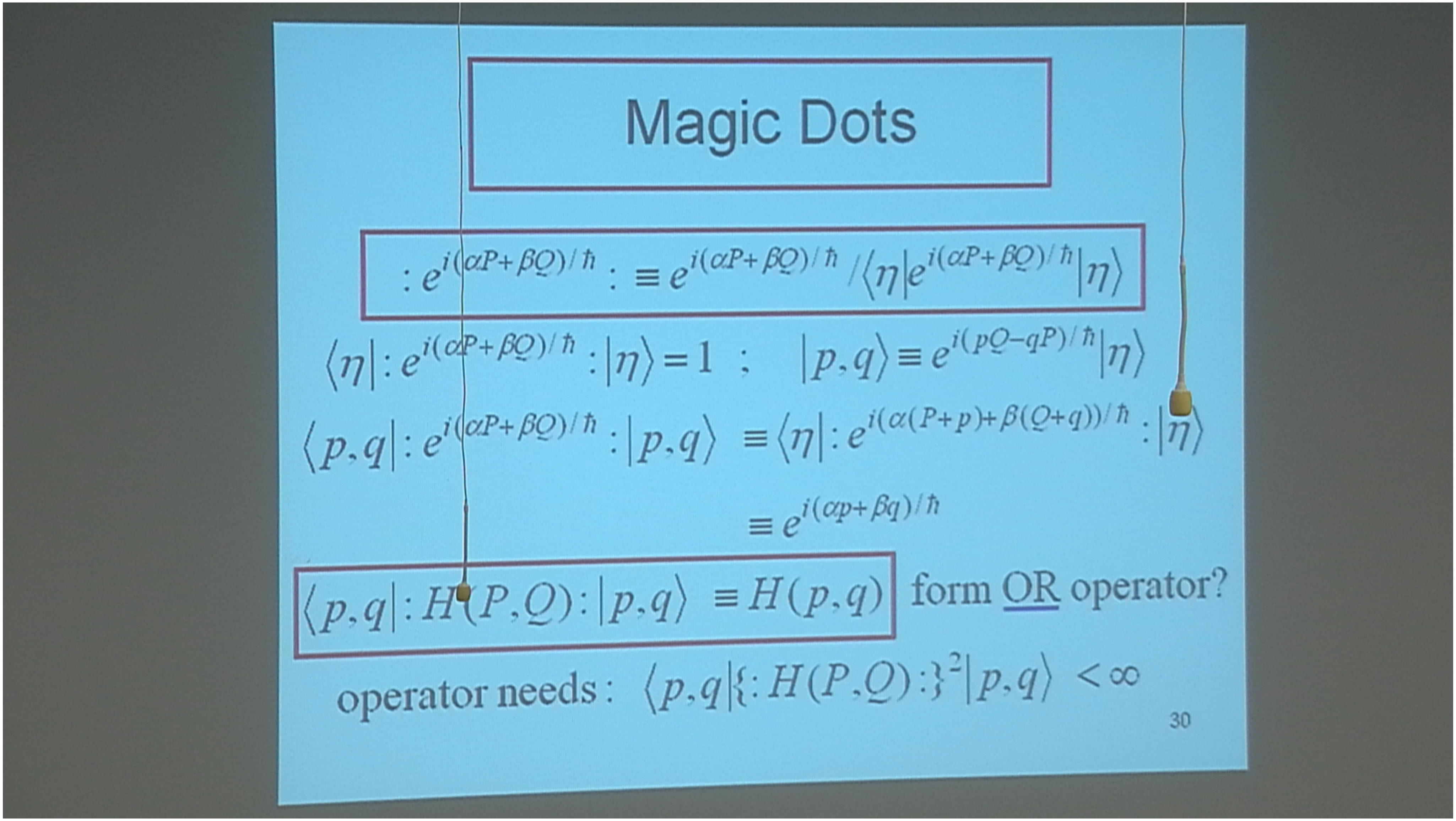
$$\approx K_N \int e^{-p^2r^2/2\hbar^2(N-2)}\psi_N(r)^2 r^{N-1}dr\,d\Omega_{N-2}$$

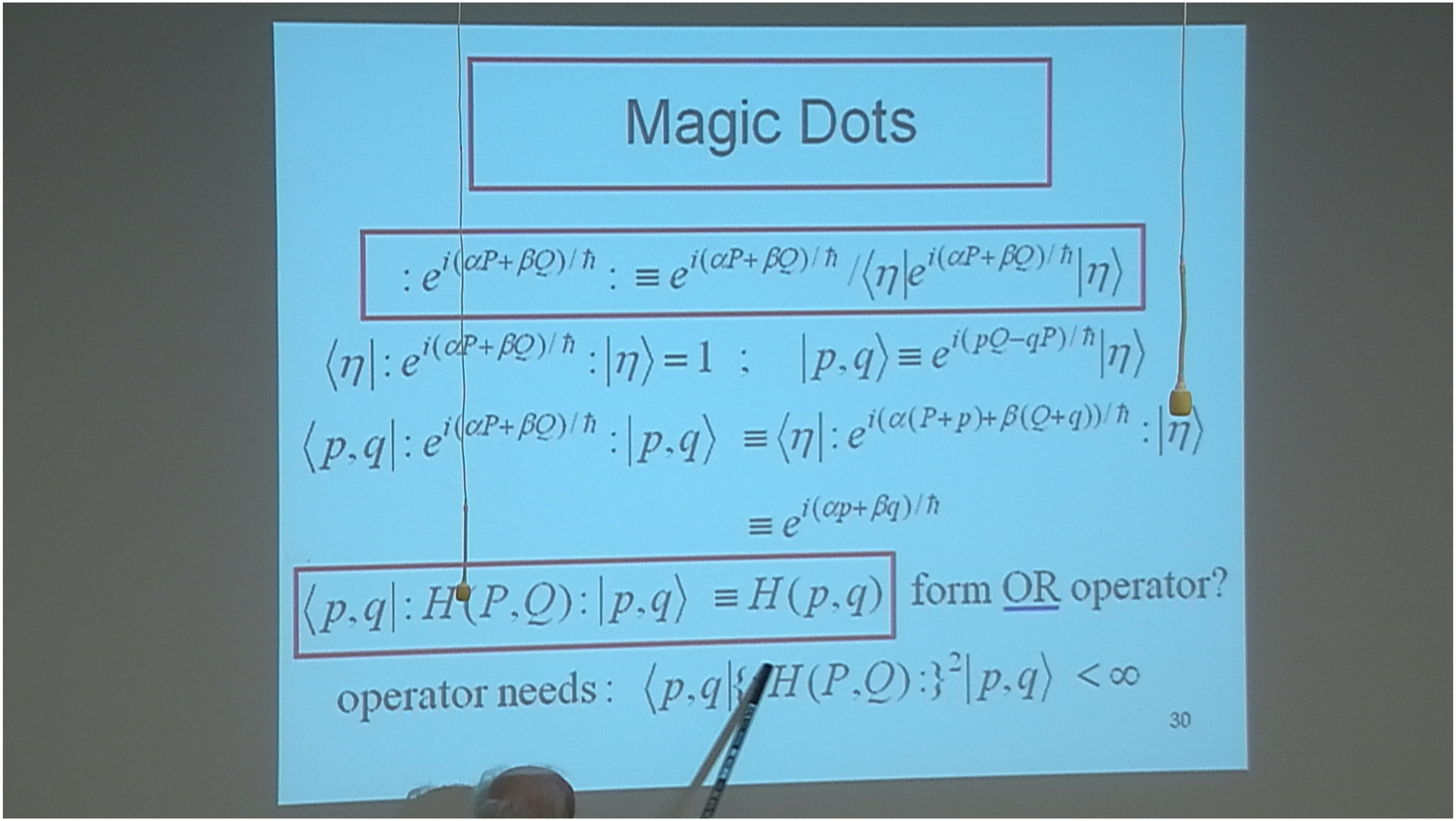
$$\rightarrow \int_0^\infty e^{-bp^2/2\hbar} f(b)\;db \;\;; \quad [\int_0^\infty f(b)\;db = 1]$$

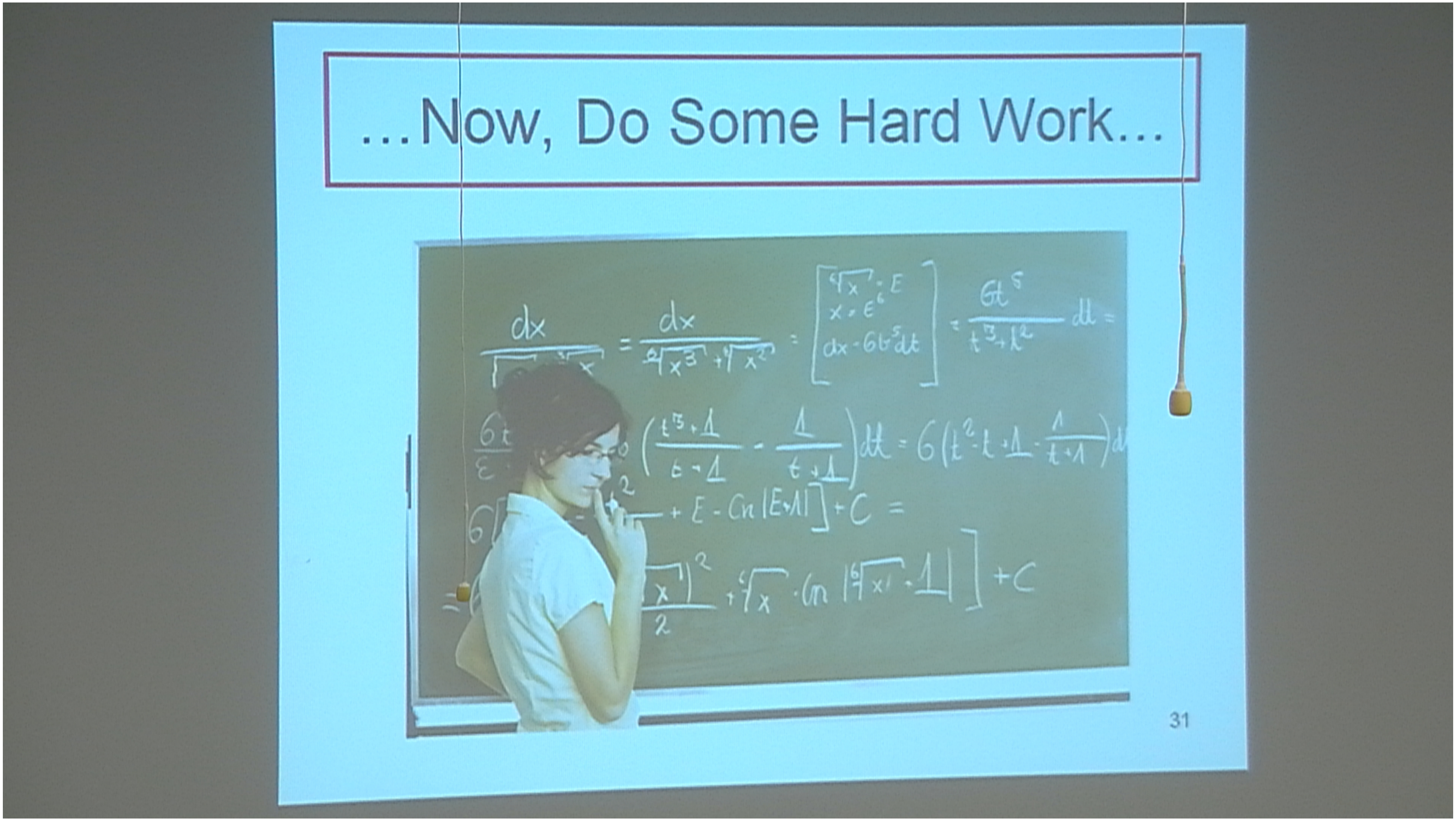
Uniqueness : $f(b) = \delta(b - 1/2\mathrm{m})$

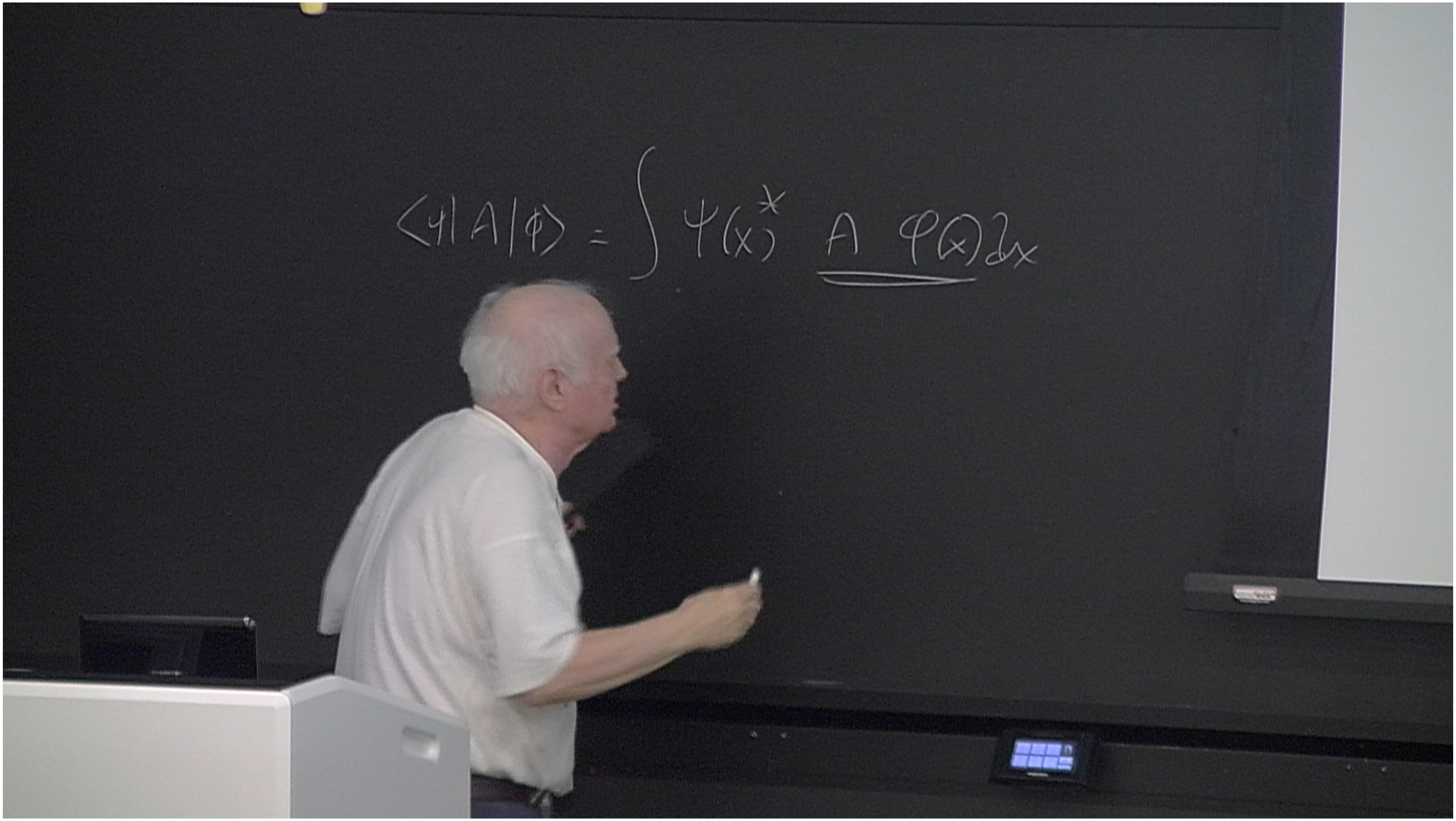
Result : $C_\infty(\bar{p}) = e^{-p^2/4\mathrm{m}\hbar}$; **A free theory!**

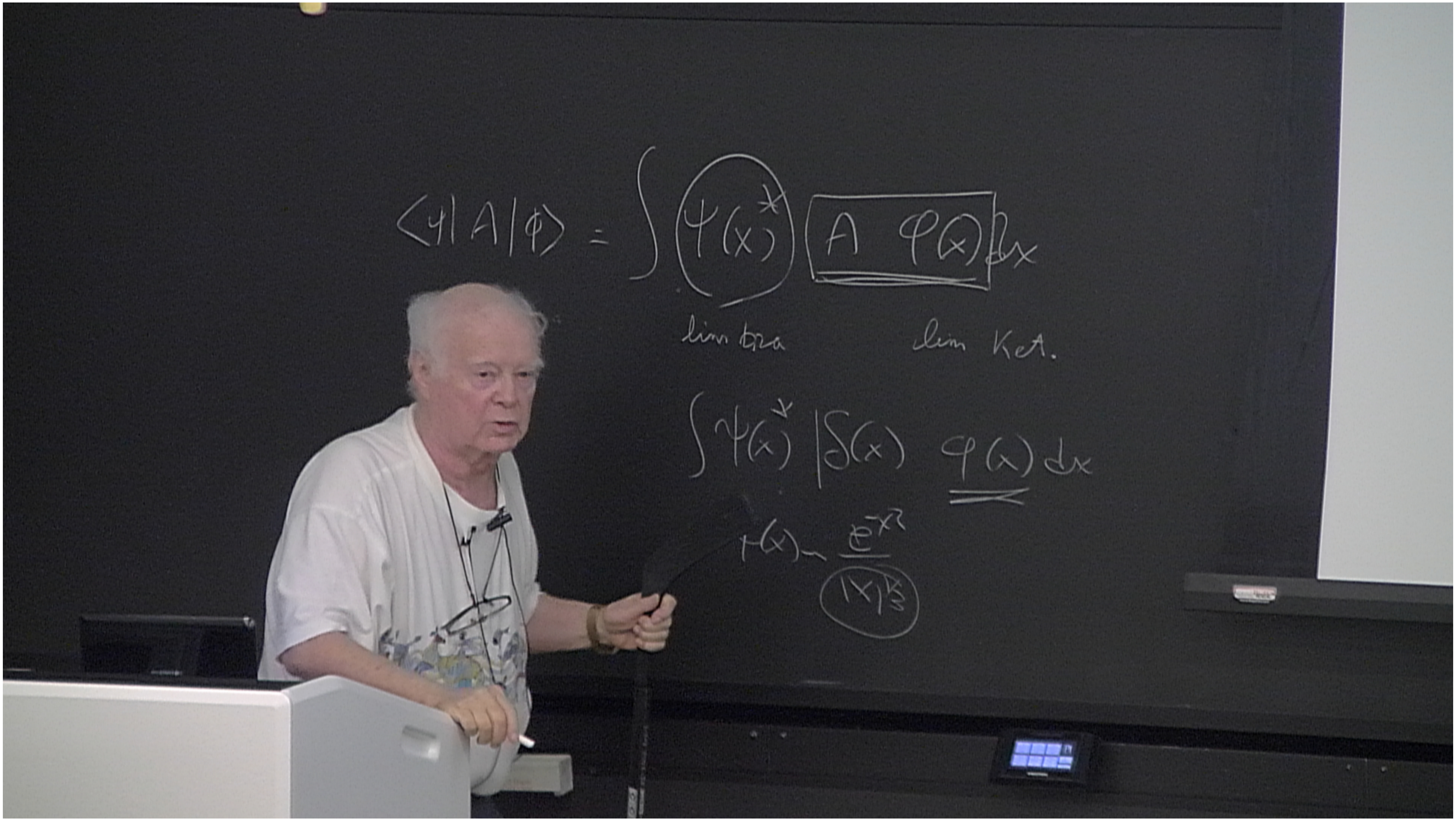
29


Magic Dots
$:e^{i(\alpha P+\beta Q)/\hbar}: \equiv e^{i(\alpha P+\beta Q)/\hbar} / \langle \eta | e^{i(\alpha P+\beta Q)/\hbar} | \eta \rangle$
$\langle \eta | :e^{i(\alpha P+\beta Q)/\hbar}: | \eta \rangle = 1 \;\; ; \quad | p,q \rangle \equiv e^{i(pQ-qP)/\hbar} | \eta \rangle$
$\langle p,q | :e^{i(\alpha P+\beta Q)/\hbar}: | p,q \rangle \equiv \langle \eta | :e^{i(\alpha (P+p)+\beta (Q+q))/\hbar}: | \eta \rangle$
$\equiv e^{i(\alpha p+\beta q)/\hbar}$
$\langle p,q | :H(P,Q): | p,q \rangle \equiv H(p,q)$ form OR operator?
operator needs: $\langle p,q | \{:H(P,Q):\}^2 | p,q \rangle < \infty$
30


Magic Dots
$:e^{i(\alpha P+\beta Q)/\hbar}: \equiv e^{i(\alpha P+\beta Q)/\hbar} / \langle\eta| e^{i(\alpha P+\beta Q)/\hbar} |\eta\rangle$
$\langle\eta| :e^{i(\alpha P+\beta Q)/\hbar}: |\eta\rangle = 1$; $|p,q\rangle \equiv e^{i(pQ-qP)/\hbar}|\eta\rangle$
$\langle p,q| :e^{i(\alpha P+\beta Q)/\hbar}: |p,q\rangle \equiv \langle\eta| :e^{i(\alpha(P+p)+\beta(Q+q))/\hbar}: |\eta\rangle$
$\equiv e^{i(\alpha p+\beta q)/\hbar}$
$\langle p,q| :H(P,Q): |p,q\rangle \equiv H(p,q)$ form OR operator?
operator needs : $\langle p,q| \{:H(P,Q):\}^2 |p,q\rangle < \infty$
30


…Now, Do Some Hard Work…
31


$\langle \psi | A | \phi \rangle = \int \psi(x)^* A \, \varphi(x) \, dx$


⟨ψ|A|φ⟩ = ∫ψ(x)* A φ(x) dx
lim bra
lim Ket.
∫ψ(x)* |δ(x) φ(x) dx

Rotationally Sym. Models (3)

$$(m_0\bar{Q} + i\bar{P})\,|0\rangle = 0 \;;\; |\bar{p},\bar{q}\rangle = \exp[i(\bar{p}\cdot\bar{Q} - \bar{q}\cdot\bar{P})/\hbar]\,|0\rangle$$

$$\underline{\langle \bar{p},\bar{q}|\mathcal{H}|\bar{p},\bar{q}\rangle} = \langle \bar{p},\bar{q}|\{\tfrac{1}{2} : \bar{P}^2 + m_0^2\bar{Q}^2 : + w : (\bar{P}^2 + m_0^2\bar{Q}^2)^2 :\}|\bar{p},\bar{q}\rangle$$

$$= \tfrac{1}{2}(\bar{p}^2 + m_0^2\bar{q}^2) + w(\bar{p}^2 + m_0^2\bar{q}^2)^2 \quad ; \quad \underline{N \le \infty}$$

TEST

$$[m(\bar{Q} + \zeta\bar{S}) + i\bar{P}]\,|0;\zeta\rangle = [m(\bar{S} + \zeta\bar{Q}) + i\bar{R}]\,|0;\zeta\rangle = 0 \;;\; 0 < \zeta < 1$$

$$|\bar{p},\bar{q};\zeta\rangle \equiv \exp[i(\bar{p}\cdot\bar{Q} - \bar{q}\cdot\bar{P})/\hbar]\,|0;\zeta\rangle$$

$$\underline{\langle \bar{p},\bar{q};\zeta|\mathcal{H}|\bar{p},\bar{q};\zeta\rangle} = \langle \bar{p},\bar{q};\zeta|\{\tfrac{1}{2} : \bar{P}^2 + m^2(\bar{Q} + \zeta\bar{S})^2 :$$

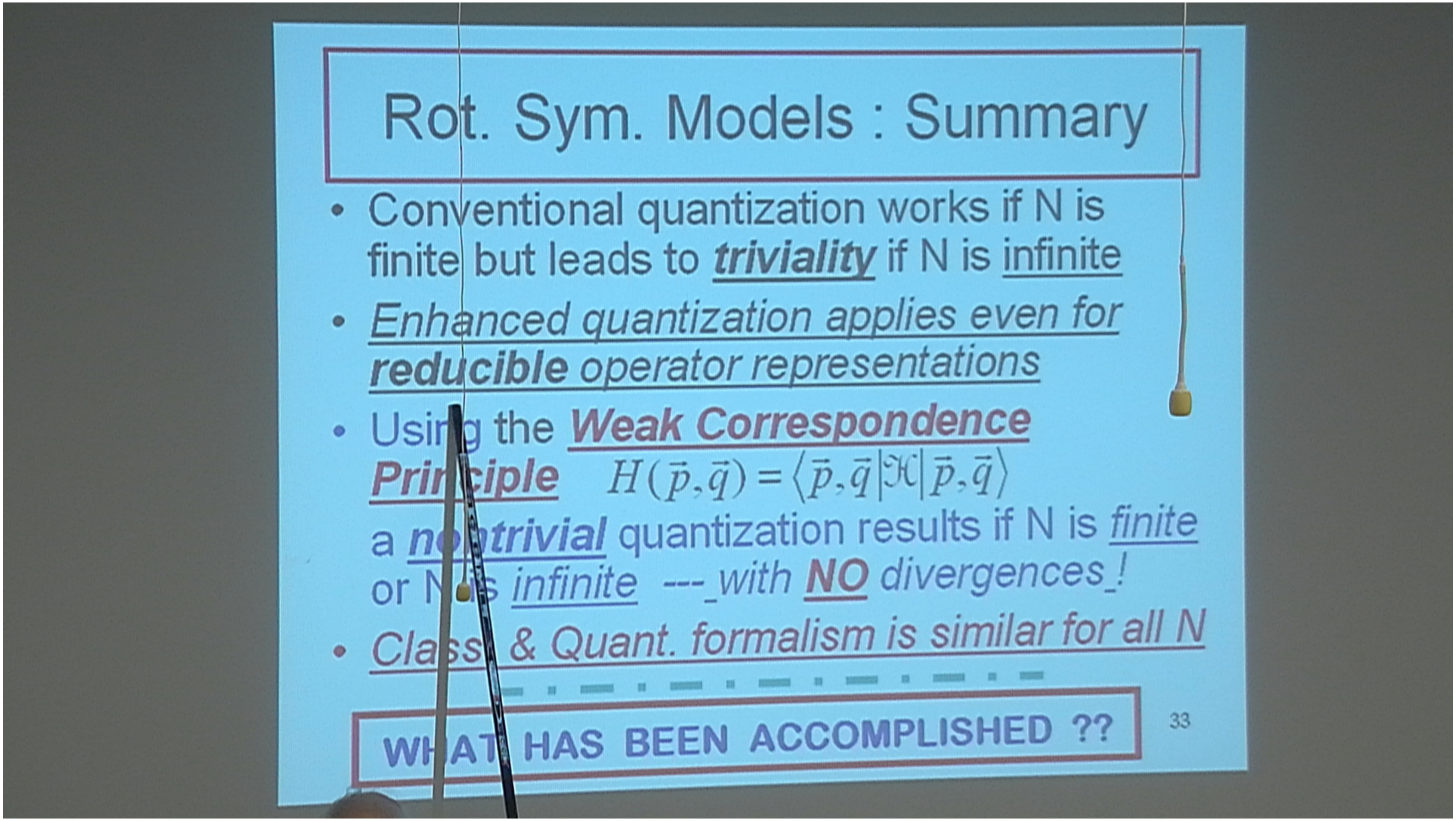
$$+ \tfrac{1}{2} : \bar{R}^2 + m^2(\bar{S} + \zeta\bar{Q})^2 : + v : [\bar{R}^2 + m^2(\bar{S} + \zeta\bar{Q})^2]^2 :\}|\bar{p},\bar{q};\zeta\rangle$$

$$= \tfrac{1}{2}(\bar{p}^2 + m^2\bar{q}^2) + \tfrac{1}{2}\zeta^2 m^2\bar{q}^2 + v\zeta^4 m^4(\bar{q}^2)^2$$

$$\boxed{\equiv \tfrac{1}{2}(\bar{p}^2 + m_0^2\bar{q}^2) + \lambda_0(\bar{q}^2)^2} \quad ; \quad \underline{N \le \infty}$$

REAL

32


Rot. Sym. Models : Summary
• Conventional quantization works if N is finite but leads to triviality if N is infinite
• Enhanced quantization applies even for reducible operator representations
• Usi g the Weak Correspondence Prin ciple $H(\vec{p},\vec{q}) = \langle \vec{p},\vec{q} | \mathcal{H} | \vec{p},\vec{q} \rangle$
a n trivial quantization results if N is finite
or N s infinite --- with NO divergences !
• Clas & Quant. formalism is similar for all N
WH AT HAS BEEN ACCOMPLISHED ??
33

Canonical *vs.* Enhanced

- Canonical quantization requires Cartesian coordinates, but ***WHY*** is not clear
- Canonical quantization works well for certain problems, but ***NOT*** for all problems

- Enhanced quantization clarifies coordinate transformations and Cartesian coordinates
- Enhanced quantization can yield canonical results -- OR provide proper results when canonical quantization fails

34

Other Enh. Quant. Projects

- Simple models of affine quantization eliminating classical singularities (on going)
- Covariant scalar models φ_n^4 (*done*)
- Affine quantum gravity (started)
- Incorporating constrained systems within enhanced quantization (started)
- Additional sheets of vectors in Hilbert space relating quan. and class. models (started)
- Extension to fermion fields (hints)

35

Rotationally Sym. Models (1)

Phase space coordinates : $\vec{p} = (p_1, \ldots, p_N)$, $\vec{q} = (q_1, \ldots, q_N)$

$$H(\vec{p}, \vec{q}) = \tfrac{1}{2}[\vec{p}^{\,2} + m_0^2 \vec{q}^{\,2}] + \lambda_0 (\vec{q}^{\,2})^2 \quad ; \quad N \leq \infty$$

Invariant under $\vec{p} \to O\vec{p}$, $\vec{q} \to O\vec{q}$; $O \in \mathbf{O}(N, R)$

Basic invariants : $X \equiv \vec{p}^{\,2}$, $Y \equiv \vec{p} \cdot \vec{q}$, $Z \equiv \vec{q}^{\,2}$

Constants of motion : E , $\vec{L}^2 = (\vec{p} \times \vec{q})^2 = XZ - Y^2$

Quantization : $\vec{p} \to \vec{P}$, $\vec{q} \to \vec{Q}$; $[Q_j, P_k] = i\hbar \delta_{jk}$

Hamiltonian : $\mathcal{H} = \tfrac{1}{2} : \vec{P}^2 + m_0^2 \vec{Q}^2 : + \lambda_0 : (\vec{Q}^2)^2 :$, $N < \infty$

When $N \to \infty$, *it is necessary that* $\lambda_0 = \lambda / N$

28

Rotationally Sym. Models (3)

TEST

$$(m_0\bar{Q} + i\bar{P})\ |0\rangle = 0\ ;\ \ |\bar{p},\bar{q}\rangle = \exp[i(\bar{p}\cdot\bar{Q} - \bar{q}\cdot\bar{P})/\hbar]\ |0\rangle$$

$$\underline{\langle\bar{p},\bar{q}|\mathcal{H}|\bar{p},\bar{q}\rangle} = \langle\bar{p},\bar{q}|\{\tfrac{1}{2}:\bar{P}^2 + m_0^2\bar{Q}^2: + w:(\bar{P}^2 + m_0^2\bar{Q}^2)^2:\}|\bar{p},\bar{q}\rangle$$

$$= \tfrac{1}{2}(\bar{p}^2 + m_0^2\bar{q}^2) + w(\bar{p}^2 + m_0^2\bar{q}^2)^2\quad ;\quad \underline{N \le \infty}$$

REAL

$$[m(\bar{Q} + \zeta\bar{S}) + i\bar{P}]\ |0;\zeta\rangle = [m(\bar{S} + \zeta\bar{Q}) + i\bar{R}]\ |0;\zeta\rangle = 0\quad ;\quad 0 < \zeta < 1$$

$$|\bar{p},\bar{q};\zeta\rangle \equiv \exp[i(\bar{p}\cdot\bar{Q} - \bar{q}\cdot\bar{P})/\hbar]\ |0;\zeta\rangle$$

$$\underline{\langle\bar{p},\bar{q};\zeta|\mathcal{H}|\bar{p},\bar{q};\zeta\rangle} = \langle\bar{p},\bar{q};\zeta|\{\tfrac{1}{2}:\bar{P}^2 + m^2(\bar{Q} + \zeta\bar{S})^2:$$

$$+ \tfrac{1}{2}:\bar{R}^2 + m^2(\bar{S} + \zeta\bar{Q})^2: + v:[\bar{R}^2 + m^2(\bar{S} + \zeta\bar{Q})^2]^2:\}|\bar{p},\bar{q};\zeta\rangle$$

$$= \tfrac{1}{2}(\bar{p}^2 + m^2\bar{q}^2) + \tfrac{1}{2}\zeta^2 m^2\bar{q}^2 + v\zeta^4 m^4(\bar{q}^2)^2$$

$$\boxed{\equiv \tfrac{1}{2}(\bar{p}^2 + m_0^2\bar{q}^2) + \lambda_0(\bar{q}^2)^2}\quad ;\quad \underline{N \le \infty}$$

32