Title: Perturbative Amplitudes and Ultraviolet Behavior of Supergravity Theories

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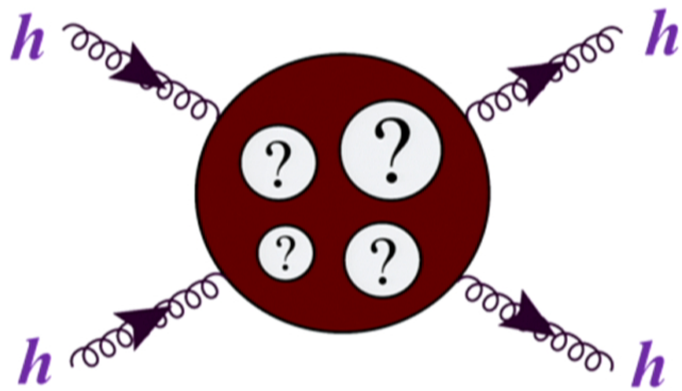
Abstract: Recently powerful techniques have emerged for performing multi-loop computations of scattering amplitudes in quantum gravity and supergravity. These techniques include generalized unitarity and the double-copy property, related to color-kinematics duality in gauge theory. Using these techniques, the ultraviolet divergence structure of N=8 supergravity, and more recently pure N=4 supergravity, have been assessed, not only in four space-time dimensions but also in higher dimensions. The results can be compared to expectations based on potential counterterms that can be constructed using (conjectured) superspace formalisms or nonlinear symmetry constraints. Interestingly, the critical ultraviolet dimension in which N=8 supergravity first diverges is equal to that for N=4 super-Yang-Mills theory through four loops. If this statement were to hold to all loop orders, then N=8 supergravity would represent a perturbatively finite, point-like theory of quantum gravity in four dimensions. In this talk, I will review all of this recent progress.

Graviton Scattering: a Gedanken Experiment

"Mathematics is the part of physics where experiments are cheap"

– V.I. Arnold



Introduction

- Quantum gravity nonrenormalizable by power counting: Newton's constant, $G_N = 1/M_{\rm Pl}^2$ is dimensionful

Counterterm Basics

- Divergences associated with local counterterms
- On-shell counterterms are generally covariant,
built out of products of Riemann tensor $R_{\mu\nu\sigma\rho}$ (& derivatives $\mathcal{D}_\mu$)

- Terms containing Ricci tensor $R_{\mu\nu}$ and scalar R
removable by nonlinear field redefinition in Einstein action

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$R^{\mu}_{\nu\sigma\rho} \sim \partial_\rho \Gamma^{\mu}_{\nu\sigma} \sim g^{\mu\kappa}\partial_\rho\partial_\nu g_{\kappa\sigma}$ has mass dimension 2

$G_N = 1/M_{Pl}^2$ has mass dimension -2

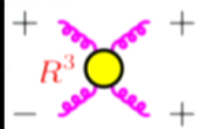
Each additional $R_{\mu\nu\sigma\rho}$ or $\mathcal{D}^2$ ←→ 1 more loop (in D=4)

One-loop → $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$
However, $R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$
is Gauss-Bonnet term, total derivative in four dimensions.
So pure gravity is UV finite at one loop (but not with matter)

't Hooft, Veltman (1974)

Pure supergravity ($\mathcal{N} \geq 1$):

Divergences deferred to at least three loops



$R^3 \equiv R^{\lambda\rho}{}_{\mu\nu} R^{\mu\nu}{}_{\sigma\tau} R^{\sigma\tau}{}_{\lambda\rho}$ cannot be supersymmetrized

produces helicity amplitude (-+++) incompatible with SUSY Ward identitites

Grisaru (1977); Deser, Kay, Stelle (1977); Tomboulis (1977)

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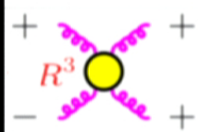

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However, at three loops, there is an N=8 supersymmetric counterterm, abbreviated R^4, plus (many) other terms containing other fields in N=8 multiplet.

Deser, Kay, Stelle (1977); Howe, Lindström (1981); Kallosh (1981); Howe, Stelle, Townsend (1981)

R^4 produces first subleading term in low-energy limit of 4-graviton scattering in type II string theory:

$$\alpha'^3 R^4 \;\Rightarrow\; \alpha'^3 stu \underbrace{M_4^{\text{tree}}(1,2,3,4)}_{\text{4-graviton amplitude in (super)gravity}}$$

Gross, Witten (1986)

$E_{7(7)}$ Constraints on Counterterms

- N=8 SUGRA has continuous symmetries: noncompact form of E_7.

  Cremmer, Julia (1978,1979) quantum level: Bossard, Hillmann, Nicolai, 1007.5472

- 70 scalars → coset $E_{7(7)}/\mathrm{SU}(8)$. Non-$\mathrm{SU}(8)$ part realized nonlinearly.
- $E_{7(7)}$ symmetry implies Ward identities, associated with limits of amplitudes as one or two scalars become soft Bianchi, Elvang, Freedman, 0805.0757; Arkani-Hamed, Cachazo, Kaplan, 0808.1446; Kallosh, Kugo, 0811.3414
- Soft limit of NMHV 6-point matrix element of R^4 doesn't vanish; violates $E_{7(7)}$ Elvang, Kiermaier, 1007.4813

$E_{7(7)}$ Constraints (cont.)

- Similar arguments rule out $\mathcal{D}^4 R^4$ and $\mathcal{D}^6 R^4$
- However, $\mathcal{D}^8 R^4$ is allowed (L=7 for D=4)

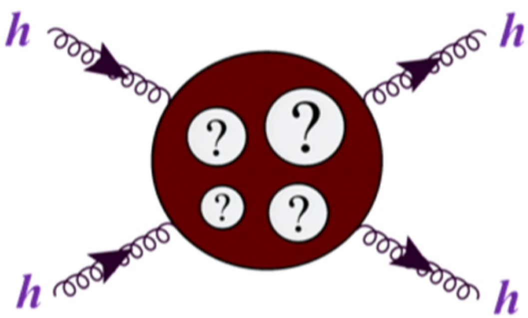
Beisert et al., 1009.1643

- Same conclusions reached by other methods

Bossard, Howe, Stelle, 1009.0743

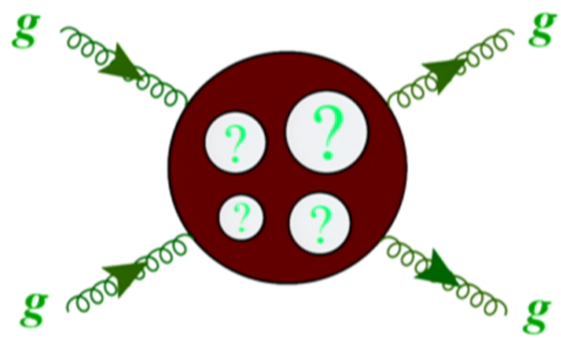
- Volume of full N=8 superspace has same dimension as $\mathcal{D}^8 R^4$ – but it vanishes!

Strategy for Assessing UV Behavior of N=8 Supergravity



N=8 SUGRA

vs.



N=4 Super-Yang-Mills

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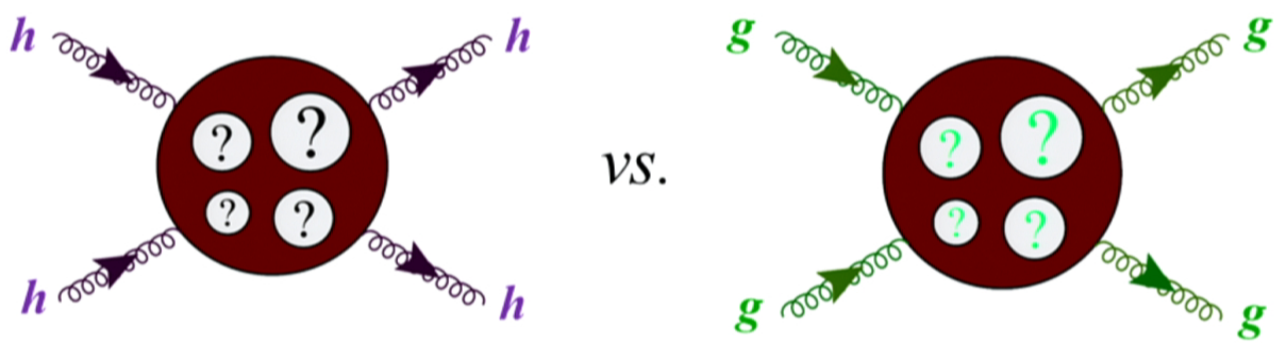

N=8 SUGRA

N=4 Super-Yang-Mills

A "mere" gauge theory. UV finite in $D=4$.

Strong evidence that it's also finite at L loops for

$$D < 4 + \frac{6}{L}$$

Full color N=4 SYM and N=8 SUGRA

- Compute N=4 SYM amplitudes for two reasons:

1. Relations between gauge theory and gravity
   (KLT → BCJ/color kinematic duality + double copy) a huge help in constructing gravity amplitudes
   Kawai, Lewellen, Tye (1986); Bern, Carrasco, Johansson, 0805.3993

2. Assess how N=8 SUGRA is doing by comparing UV behavior in D > 4 to N=4 SYM critical dimension,

$$D_c = 4 + \frac{6}{L}$$

- Need full color N=4 SYM for task 1, but it also provides interesting information for task 2.

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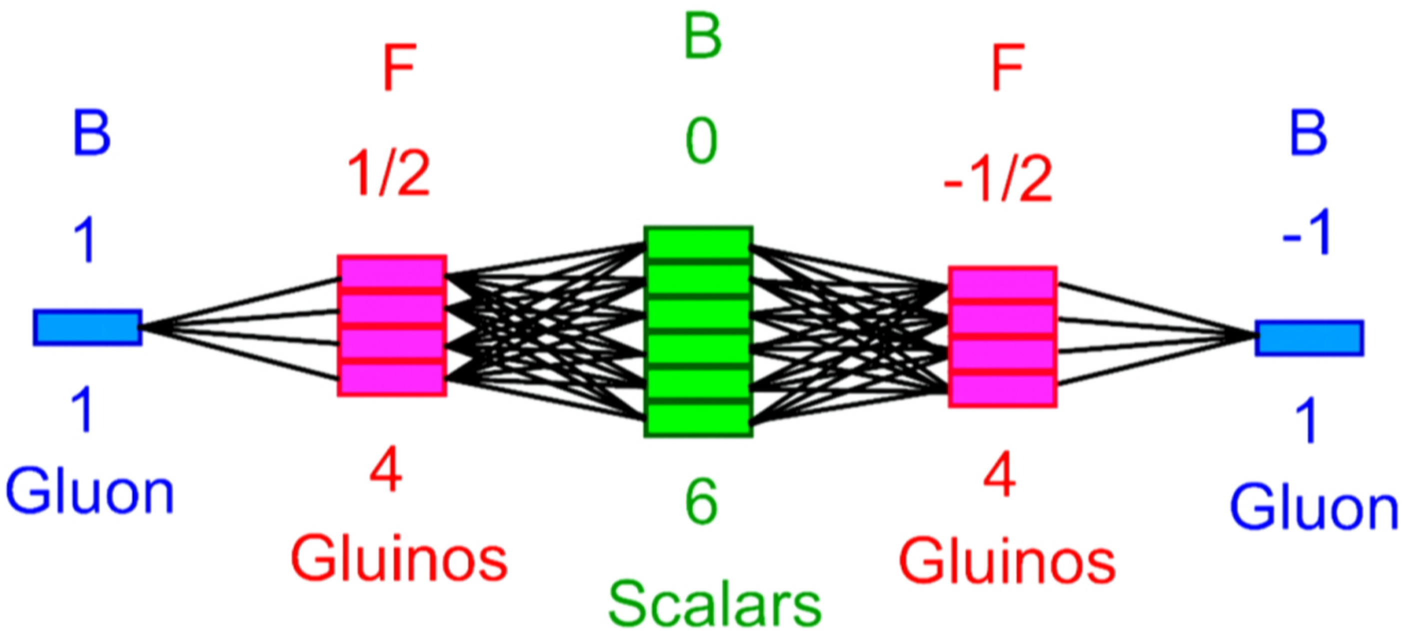
N=4 spectrum much simpler



Color-kinematics duality

- First found in 4-point non-Abelian gauge theory amplitudes Zhu (1980), Goebel, Halzen, Leveille (1981)
- Massless adjoint gauge theory, color factors

$C \sim f^{abe} f^{cde}$:

$$\mathcal{A}_4^{\rm tree} = \frac{n_s C_s}{s} + \frac{n_t C_t}{t} + \frac{n_u C_u}{u}$$

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- Group theory → 3 terms are not independent

Jacobi identity: $C_t - C_u = C_s$

- In suitable gauge, kinematic numerators obey:

$$n_t - n_u = n_s$$

- Same structure extends to arbitrary number of legs (and loops?), and provides a way to get gravity as a "double copy":

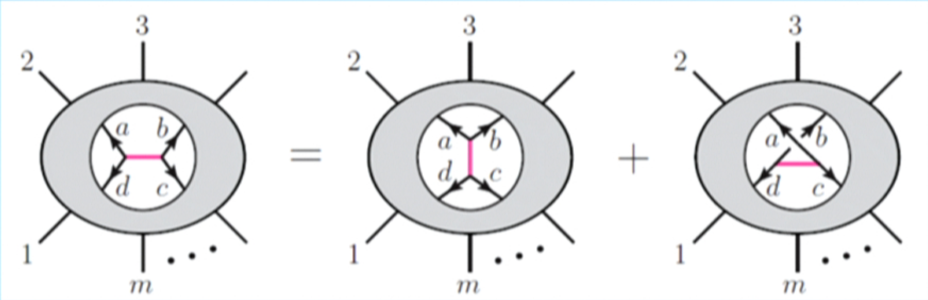
$$M_4^{\text{tree}} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

Bern, Carrasco, Johansson, 0805.3993

Color-kinematics at loop level

BCJ, 1004.0476

- Consider any 3 graphs connected by a Jacobi identity



- Color factors obey

$$C_s = C_t - C_u$$

- Duality requires

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- Very strong constraint on structure of integrands; only a handful of independent integral numerators left after imposing it.

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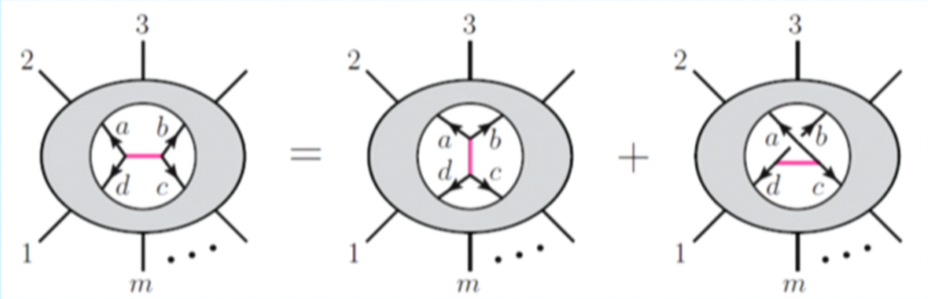

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Gravity as double copy

BCJ, 1004.0476; Bern, Dennen, Huang, Kiermaier, 1004.0476

Write all-adjoint gauge-theory amplitude in terms of cubic graphs Γ:

$$\mathcal{A}_4^{(L)} = \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i C_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

- If numerator factors n_i obey color-kinematics duality, then corresponding gravity amplitude is given by

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where $\tilde{n}_i$ are numerators for a second gauge theory [same theory for case of N=8 = $(N=4)^2$]

- Argument based on a recursion relation on the integrand.

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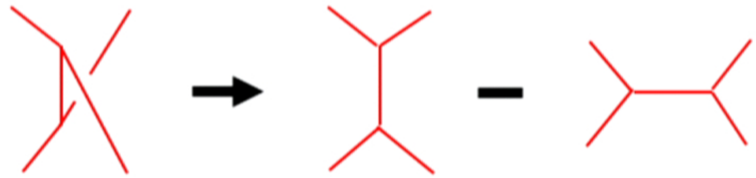
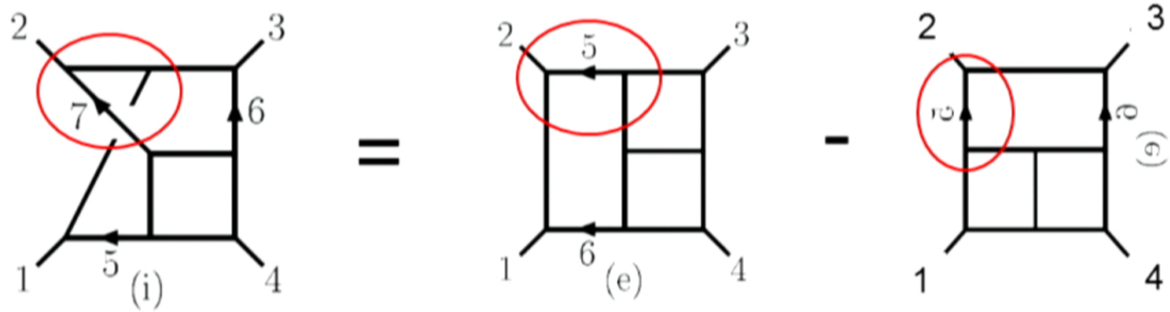
Simple 3 loop example

Using



we can relate non-planar topologies to planar ones



$$n^{(i)} = n^{(e)} - n^{(e)}|_{1 \leftrightarrow 3}$$

In fact all N=4 SYM 3 loop topologies related to (e)

(master graph)

Carrasco, Johansson, 1103.3298

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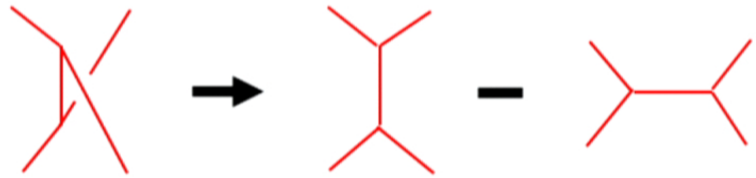

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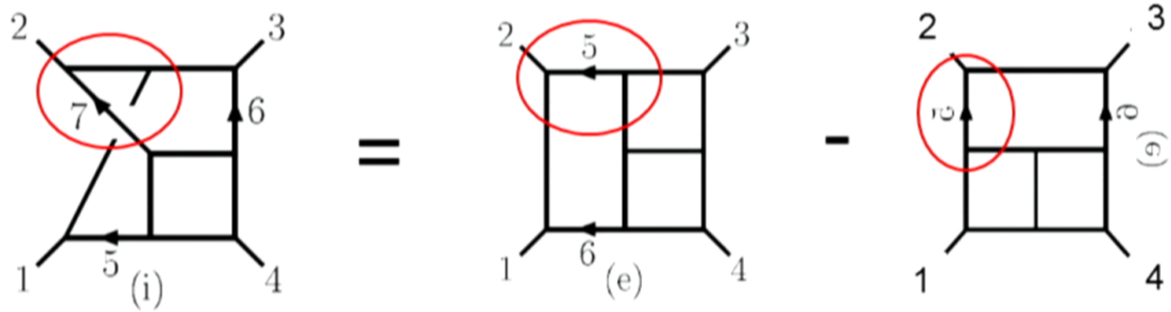

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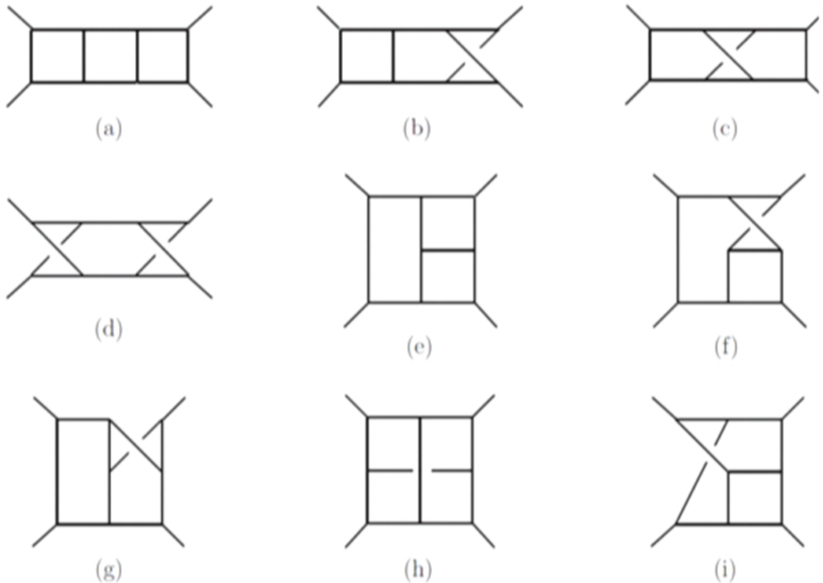
3 loop amplitude before color-kinematics duality

Nine basic integral topologies:

BCDJKR hep-th/0702112;
BCDJR, 0808.4112

- Cubic 1PI graphs only, no triangle subgraphs



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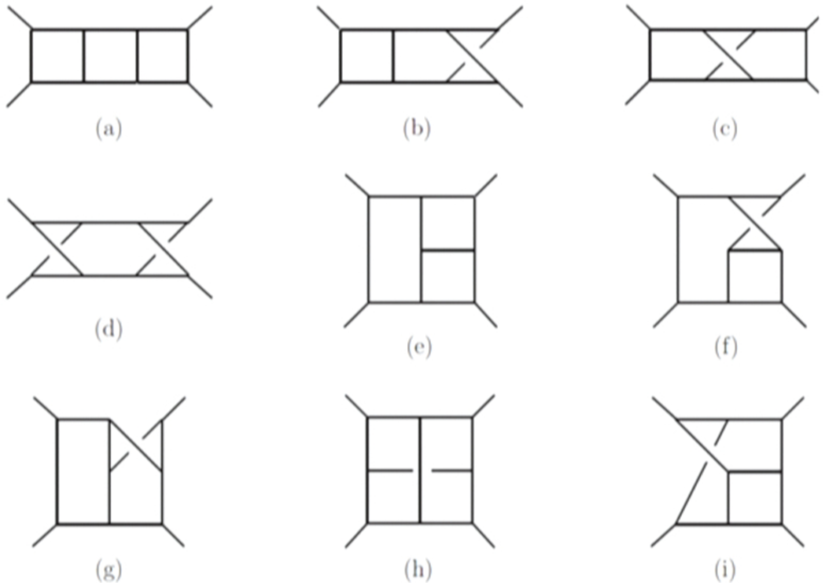
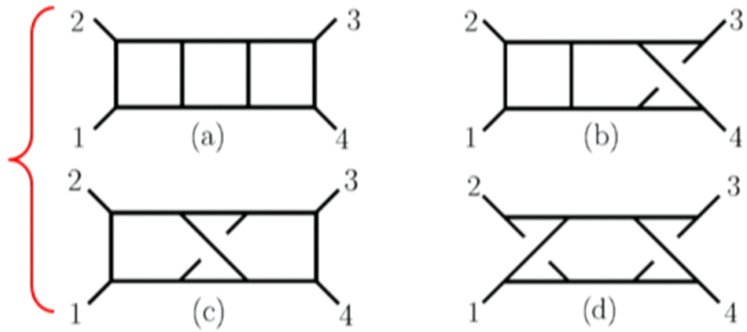


Old N=4 numerators at 3 loops

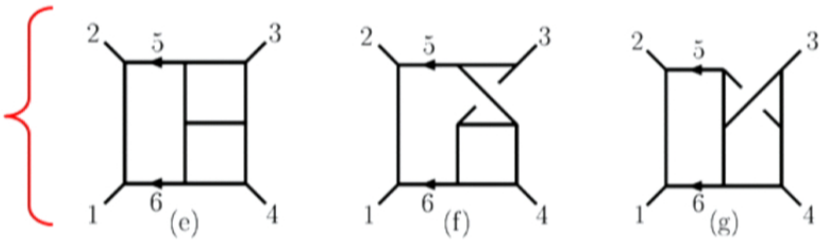
Overall
$st\,A_4^{\text{tree}}$

$$s_{iM} = (k_i + \ell_M)^2$$
$$\tau_{iM} = 2k_i \cdot \ell_M$$

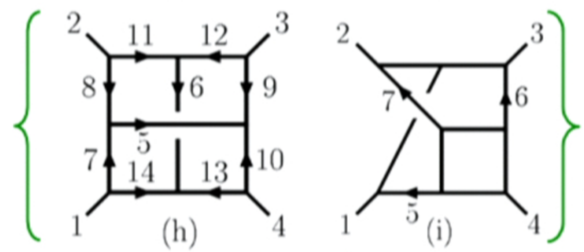
s_{12}^2



$s_{12}\, s_{46}$



$$s_{12}(\tau_{26} + \tau_{36}) + s_{14}(\tau_{15} + \tau_{25}) + s_{12}s_{14}$$



$$- s_{12}\, s_{45} - s_{14}\, s_{46} - \tfrac{1}{3}(s_{12} - s_{14})\, \ell_7^2$$

manifestly quadratic in loop momentum ℓ_M

Overall $st\,A_4^{\text{tree}}$

3 loop amplitude

BCJ, 1004.0476

$$\tau_{iM} = 2k_i \cdot \ell_M$$

N=4 SYM

s^2

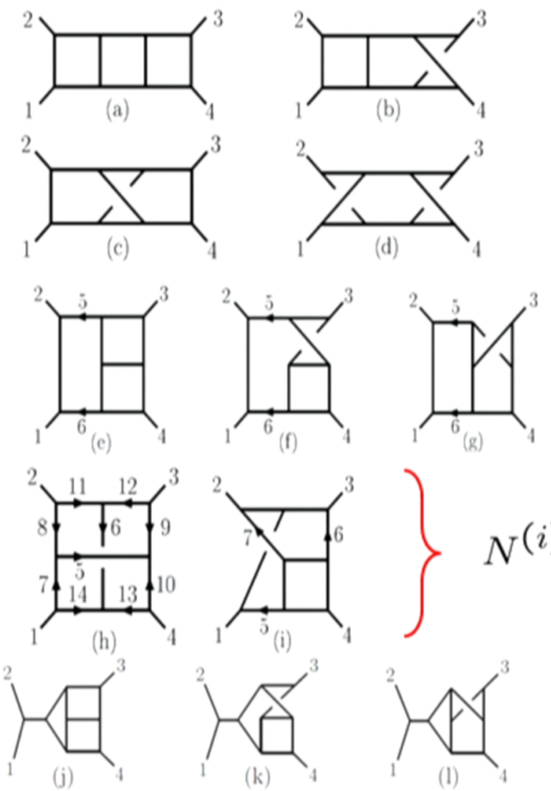
$$\frac{1}{3}\ [s(t-\tau_{36}-\tau_{46}) - t(\tau_{26}+\tau_{46}) + u(\tau_{26}+\tau_{36}) - s^2]$$

Linear in ℓ_M

$N^{(h)}(\tau_{ij})$

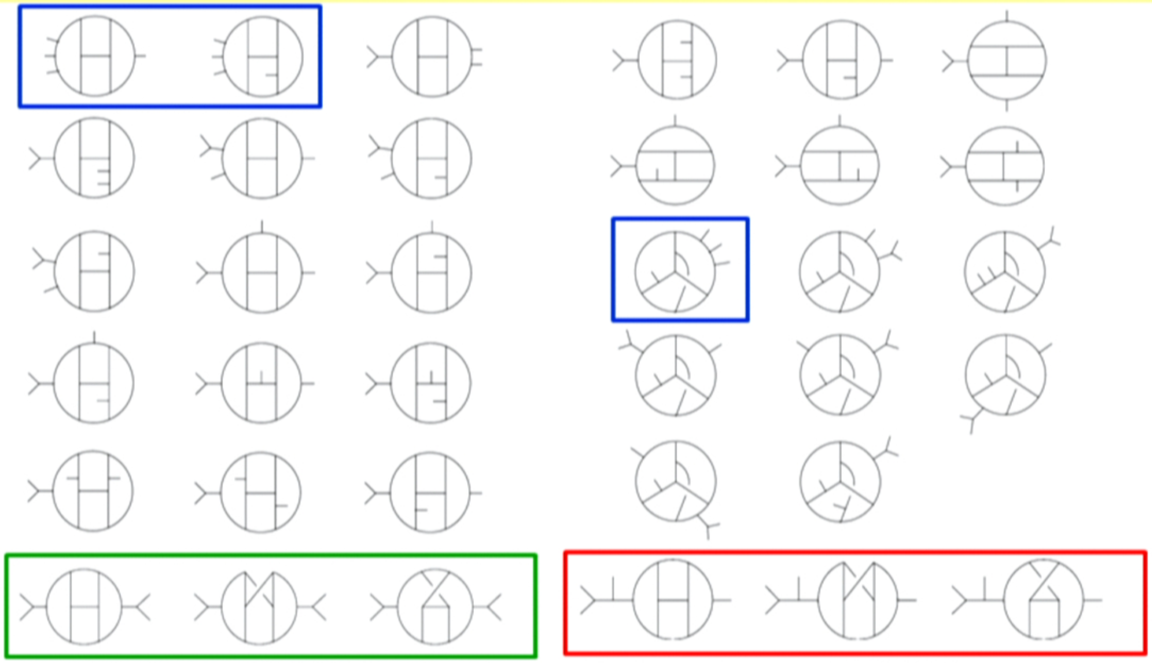
$N^{(i)}(\tau_{ij})$

$$\frac{1}{3}s(t-u)$$



4 loop amplitude after color-kinematics duality

50 nonvanishing 1PI cubic 4-point graphs BCDJR, 1201.5366
+ 3 more 1PI graphs (0 in previous representation)
+ 32 1PR graphs (6 of which are 2PR) → 85 in all



Checks and gravity amplitude

- Unitarity cuts of new N=4 SYM integrand agree with those of an old form computed without BCJ [1008.3327].
- To get N=8 SUGRA, we use double copy formula:

$$\mathcal{A}_4^{(L)} = \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i C_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

$$\mathcal{M}_4^{(L)} = \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i^2}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

- Cuts of new N=8 supergravity amplitude also agree with a previous (KLT driven) construction [0905.2326]

UV divergences at 3 loops in $D_c = 4 + 6/3 = 6$

• N=4 SYM: 1PI graphs (x) = (a), (b), … , (i) all have 10 propagators, and numerators $N^{(x)}(l_i)$ that are at most linear in loop momenta l_i.

$$\Rightarrow \quad I^{(x)} \sim \int \frac{(d^6 l_i)^3 \; l_i^\mu}{[(l_i)^2]^{10}}$$

finite in $D = 6$

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- **N=4 SYM:** 1PI graphs (x) = (a), (b), … , (i) all have 10 propagators, and numerators $N^{(x)}(l_i)$ that are at most linear in loop momenta l_i.

$$\Rightarrow \quad I^{(x)} \sim \int \frac{(d^6 l_i)^3 \; l_i^\mu}{[(l_i)^2]^{10}}$$

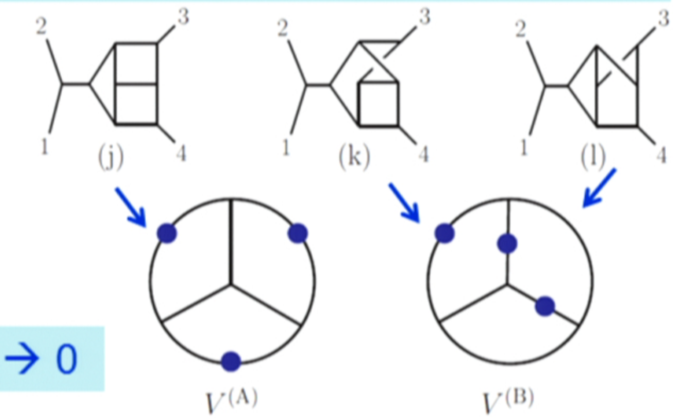
finite in $D = 6$

Only divergences come from 1PR 9 propagator graphs (y) = (j), (k), (l)

$$N^{(y)} \; = \; \frac{1}{3} s(t-u)$$

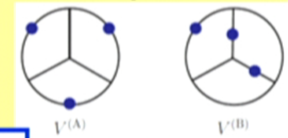
$$\Rightarrow \quad I^{(y)} \sim \int \frac{(d^{6-2\epsilon} l_i)^3}{[(l_i)^2]^9}$$

Log divergence → just set external $k_i \to 0$



3 loop N=8 SUGRA UV structure

- 1PI graphs (x) = (a) – (d) have loop-momentum independent (scalar) numerators, also after squaring → finite in $D = 6$.
- 1PI graphs (x) = (e) – (i) were linear in l_i in SYM, become quadratic in SUGRA, so they do contribute to the UV pole
- As do 1PR scalar graphs (y) = (i), (j), (k).
- Total:



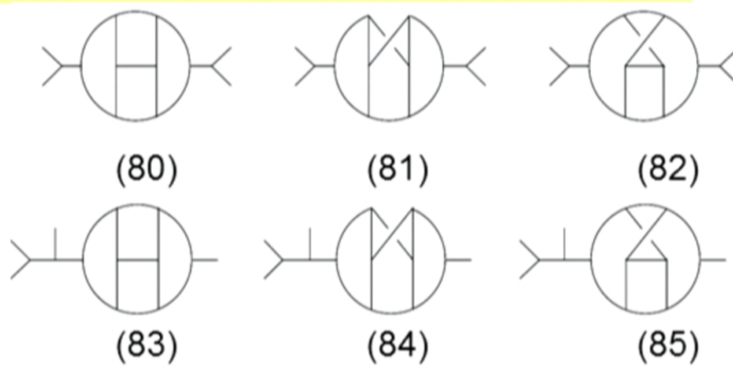
$$\mathcal{M}_4^{(3)}\Big|_{\rm pole} = -\left(\frac{\kappa}{2}\right)^8 (stu)^2 M_4^{\rm tree}\left[10\left(V^{(A)} + 3\,V^{(B)}\right)\right]$$

Curiously, **same linear combination** of $V^{(A)}$ and $V^{(B)}$ as in subleading-color part of N=4 SYM divergence! Understandable for (y) graphs, but why for 1PI ones?

UV divergences at 4 loops
in $D = 4 + 6/4 = 11/2 = 5.5$

• N=4 SYM: Master numerators N_{18} and N_{28} quadratic in l_i.
Duality relations preserve quadratic in l_i for all numerators
→ SYM divergences again only from most reducible graphs:
scalar 2PR 11-propagators

$$I \sim \int \frac{(d^{11/2-2\epsilon} l_i)^4}{[(l_i)^2]^{11}}$$



What about $L = 5$?

• Motivation: Various arguments point to 7 loops as the possible first divergence for N=8 SUGRA in D=4, associated with a D^8R^4 counterterm:

Howe, Lindstrom, NPB181, 487 (1981); Bossard, Howe, Stelle, 0908.3883; Kallosh, 0903.4630; Green, Russo, Vanhove, 1002.3805; Bjornsson, Green, 1004.2692; Bossard, Howe, Stelle, 1009.0743; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger, 1009.1643

• Same D^8R^4 counterterm shows up at $L = 4$ in $D = 5.5$

• Does 5 loops → $D^{10}R^4$ (same UV as N=4 SYM)?
  or → D^8R^4 (worse UV as N=4 SYM)?

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- Does 5 loops → $D^{10}R^4$ (same UV as N=4 SYM)?
  or → D^8R^4 (worse UV as N=4 SYM)?
- 5 loops would be a very strong indicator for 7 loops
- Now 100s of nonvanishing cubic 4-point graphs!

N=4 SYM amplitude known at $L = 5$

Bern, Carrasco, Johansson, Roiban, 1207.6666

- However, it is not in color-kinematics dual form, so one cannot yet get the N=8 SUGRA amplitude.
- Still one can inspect the full color UV divergences for N=4 SYM in D = 4 + 6/5. Very similar form to L = 3,4.

$$\mathcal{A}_4^{(5)}\Big|_{\mathrm{div}} = \frac{144}{5} g^{12} st A_4^{\mathrm{tree}} N_c^3 \Big(N_c^2 V^{(\mathrm{a})} + 12(V^{(\mathrm{a})} + 2V^{(\mathrm{b})} + V^{(\mathrm{c})})\Big) \times (t\tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4} + s\tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1})$$

(a) (b) (c)

N=4 SUGRA

- Pure N = 4 SUGRA found to be finite at 3 loops in D=4

Bern, Davies, Dennen, Huang, 1202.3423

- Rules out an "expected" R^4 counterterm, correspond to the first in a sequence of "expected" 1/N-BPS operators for N = 4,5,6,8 SUGRA at L = 3,4,5,7.

Bossard, Howe, Stelle, 1105.6087

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N=4 SUGRA

- Pure N = 4 SUGRA found to be finite at 3 loops in D=4

Bern, Davies, Dennen, Huang, 1202.3423

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- This approach predicts finiteness of N=4 SUGRA with matter multiplets in D=5 at 2 loops, and (subject to some assumptions) in D=4 at 2 and 3 loops.
- However, the cases with matter do diverge!

  Bern, Davies, Dennen, Huang, 1305.4876

Outlook

- Through 4 loops, the 4-graviton scattering amplitude of N=8 supergravity has UV behavior no worse than the corresponding 4-gluon amplitude of N=4 SYM.
- More generally, no divergence has yet been found in any pure supergravity in D=4.
- Divergences in N=4 SUGRA with matter confound potential explanations of why the pure supergravity is finite.
- Precise pole for N=8 supergravity bears a remarkable relation with subleading-color single trace pole in N=4 SYM in the same critical dimension, at 2, 3 and 4 loops.

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• Is this an accident, or could it foreshadow equal critical dimensions $D_c = 26/5$ also at 5 loops? Together with the N=4 SUGRA, might suggest that 7 loops is not where N=8 supergravity first diverges… If not there, where? L = 8? L = ∞ ?