Title: Quantum measurements constrained by symmetry

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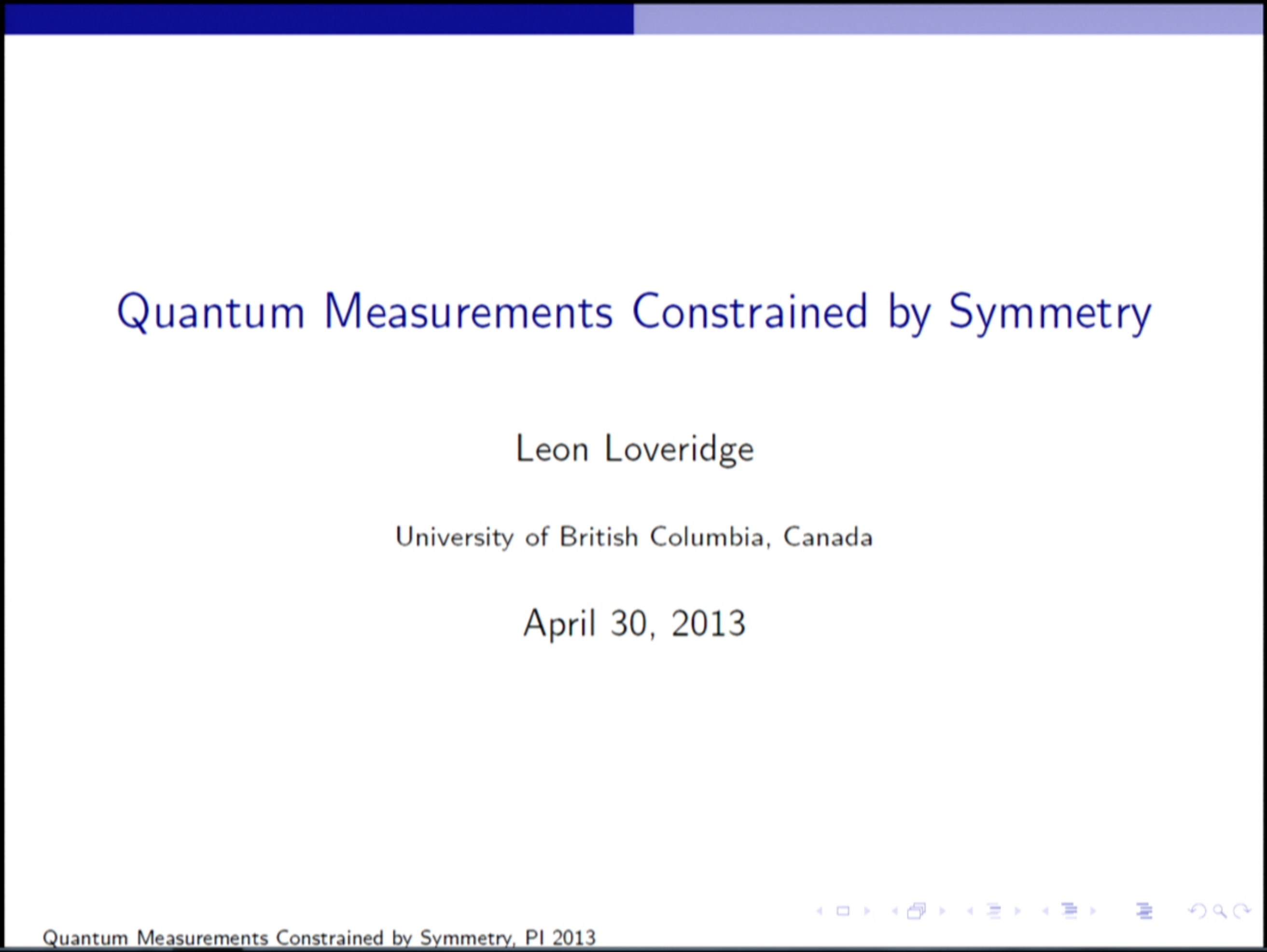
Abstract: The Wigner-Araki-Yanase (WAY) theorem delineates circumstances under which a class of quantum measurements is ruled out. Specifically, it states that any observable (given as a self adjoint operator) not commuting with an additive conserved quantity of a quantum system and measuring apparatus combined admits no repeatable measurements. I'll review the content of this theorem and present some new work which generalises and strengthens the existing results.

The observation that the WAY constraint vanishes if the observable-to-be-measured is ``relativised" (in a suitable sense) points to interesting links with quantum reference frames and superselection rules. I'll discuss some of these connections and raise some open questions.



Quantum Measurements Constrained by Symmetry

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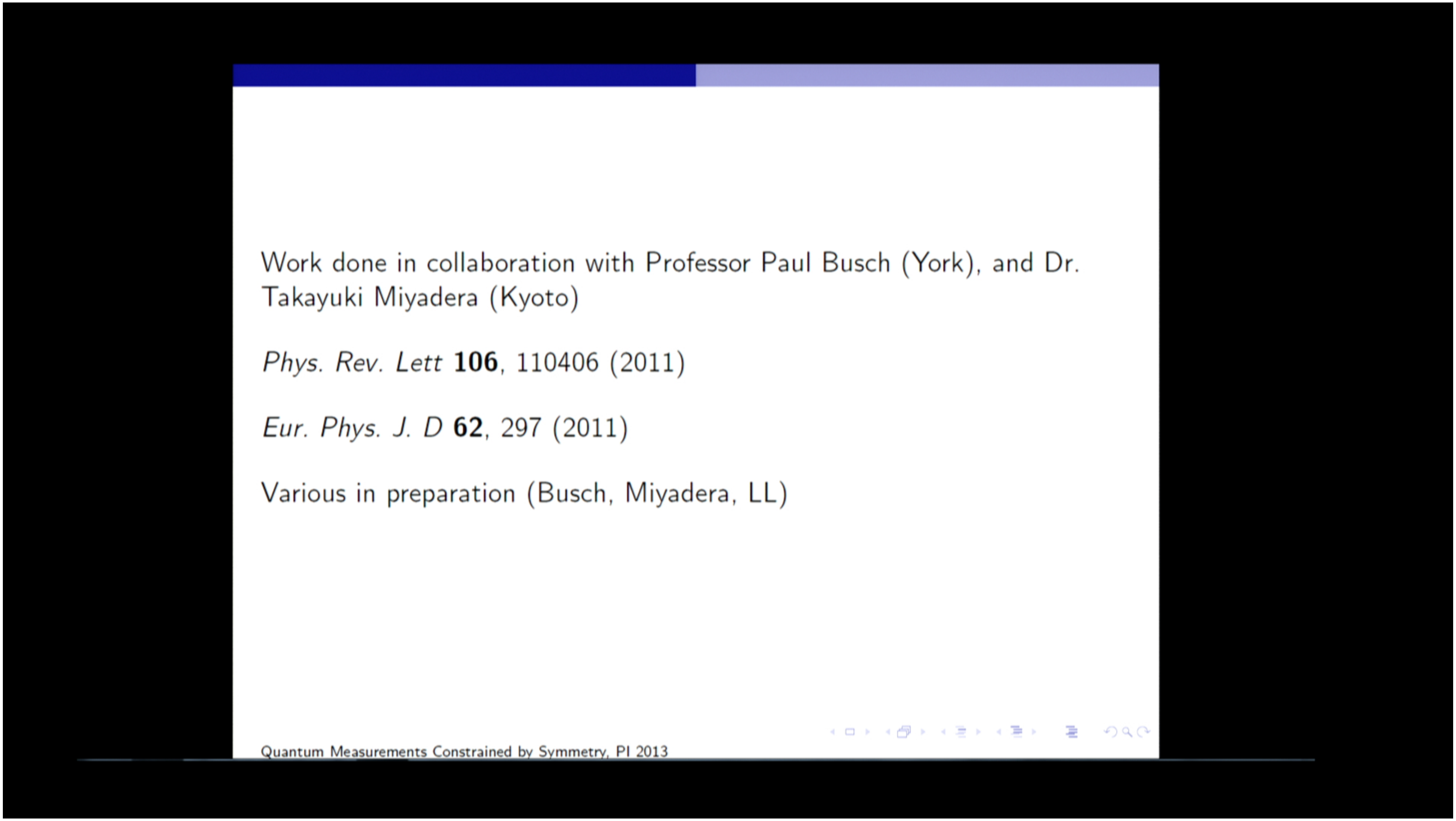
April 30, 2013

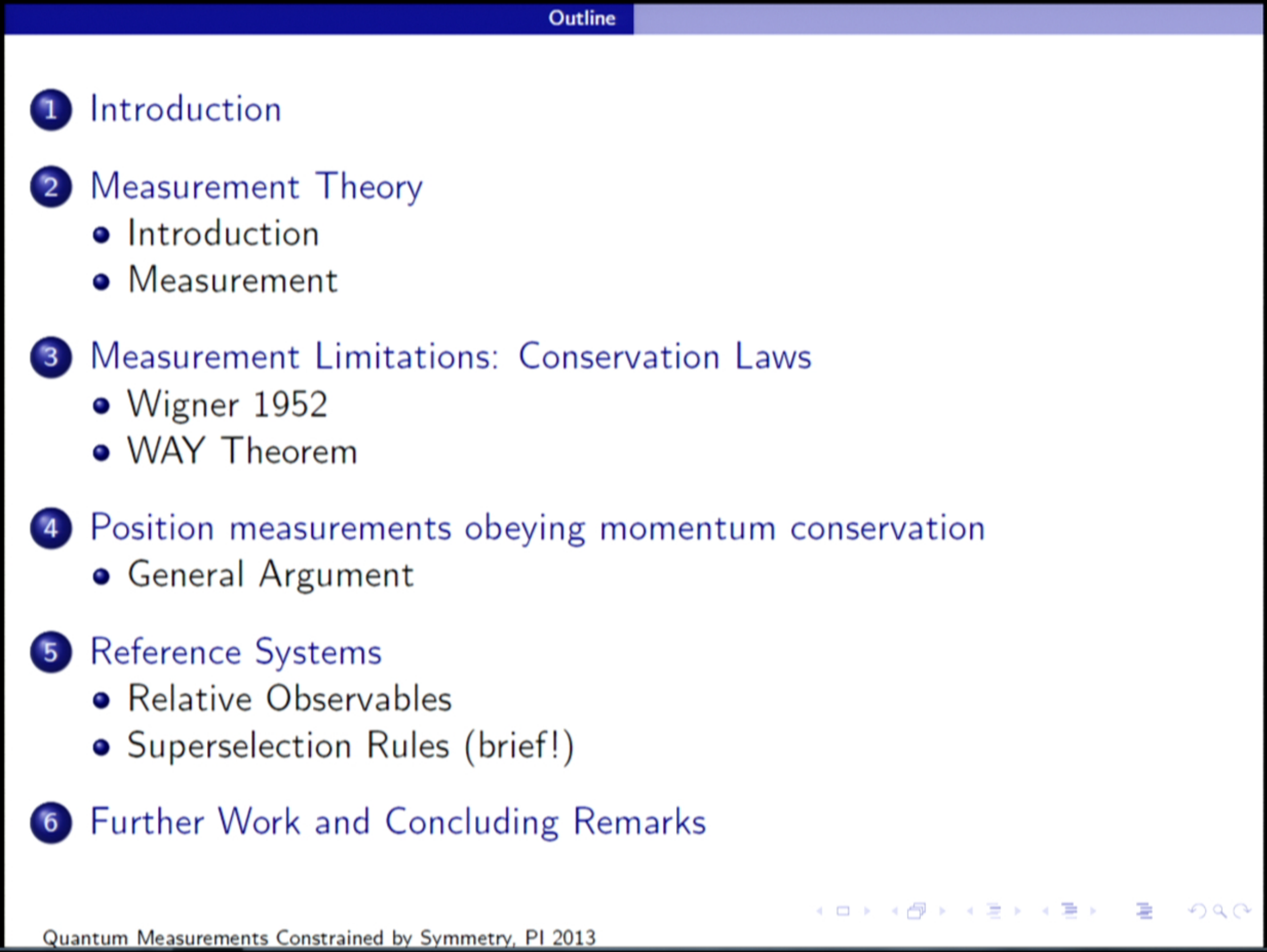


Work done in collaboration with Professor Paul Busch (York), and Dr. Takayuki Miyadera (Kyoto)

Phys. Rev. Lett **106**, 110406 (2011)

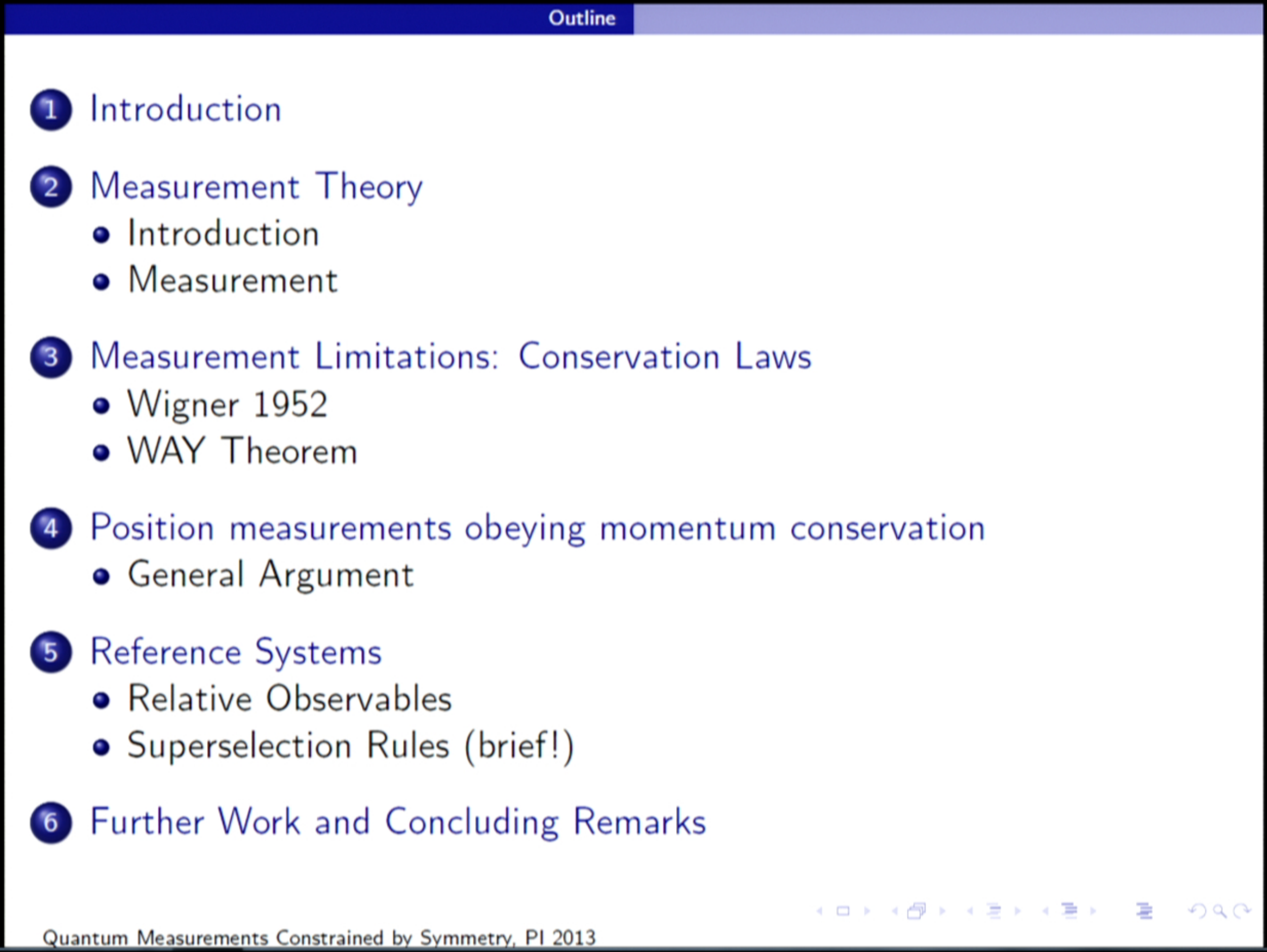
Eur. Phys. J. D **62**, 297 (2011)

Various in preparation (Busch, Miyadera, LL)



Outline

1. Introduction
2. Measurement Theory
    - Introduction
    - Measurement
3. Measurement Limitations: Conservation Laws
    - Wigner 1952
    - WAY Theorem
4. Position measurements obeying momentum conservation
    - General Argument
5. Reference Systems
    - Relative Observables
    - Superselection Rules (brief!)
6. Further Work and Concluding Remarks



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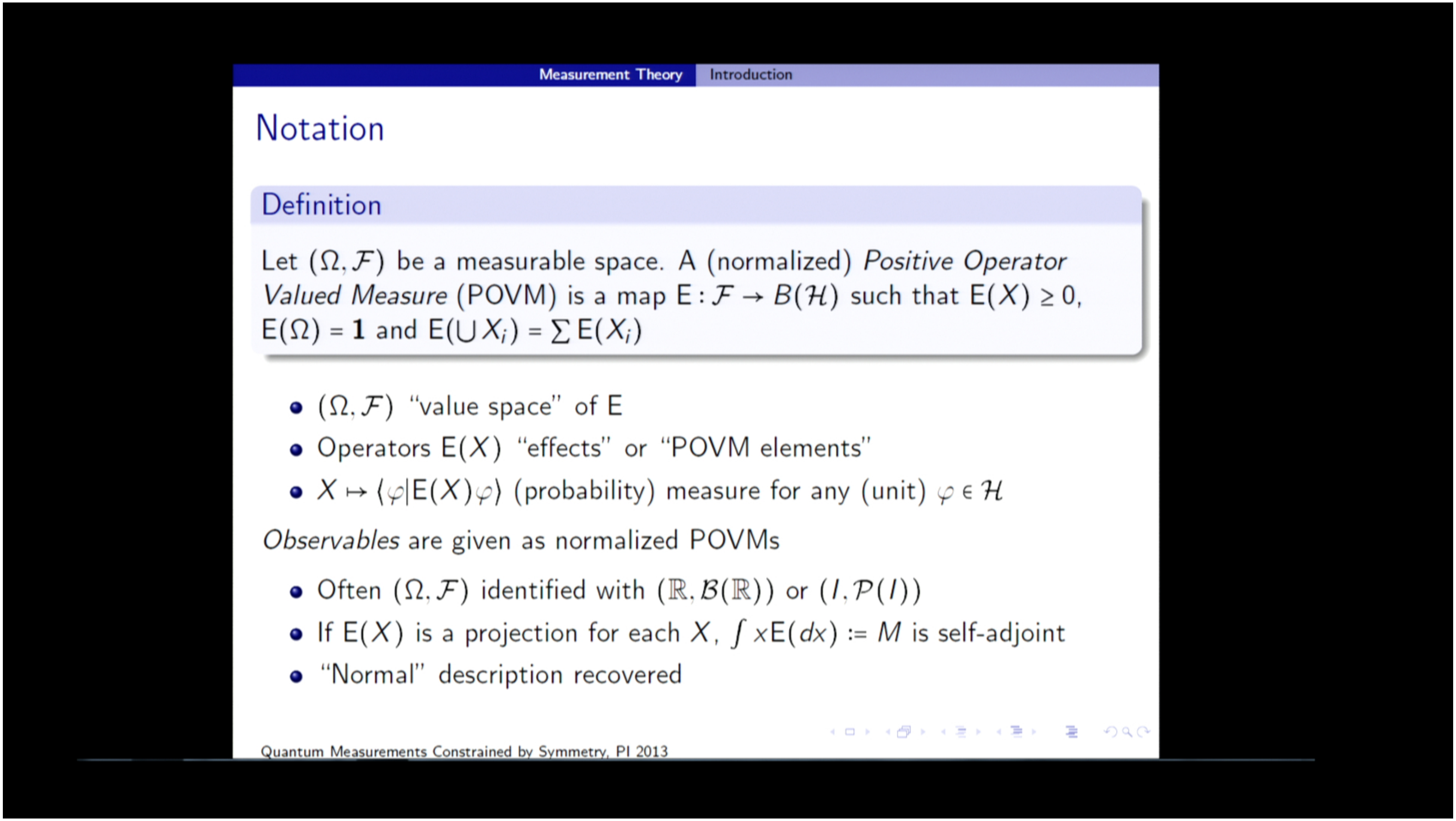


Notation

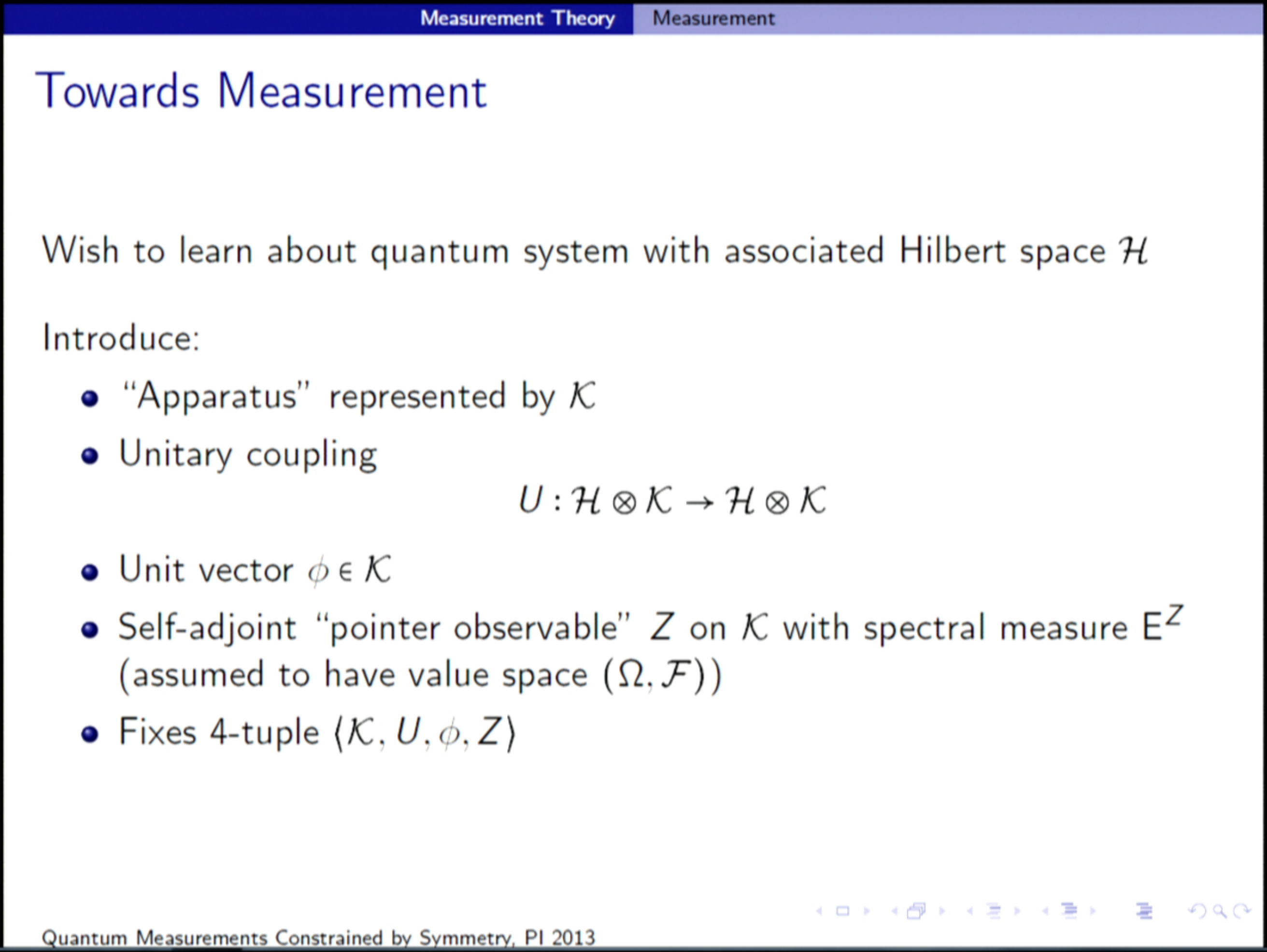
Definition

Let $(\Omega, \mathcal{F})$ be a measurable space. A (normalized) *Positive Operator Valued Measure* (POVM) is a map $\mathrm{E} : \mathcal{F} \to B(\mathcal{H})$ such that $\mathrm{E}(X) \geq 0$, $\mathrm{E}(\Omega) = \mathbf{1}$ and $\mathrm{E}(\bigcup X_i) = \sum \mathrm{E}(X_i)$

- $(\Omega, \mathcal{F})$ "value space" of E
- Operators $\mathrm{E}(X)$ "effects" or "POVM elements"
- $X \mapsto \langle \varphi | \mathrm{E}(X) \varphi \rangle$ (probability) measure for any (unit) $\varphi \in \mathcal{H}$

Observables are given as normalized POVMs

- Often $(\Omega, \mathcal{F})$ identified with $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(I, \mathcal{P}(I))$
- If $\mathrm{E}(X)$ is a projection for each X, $\int x \mathrm{E}(dx) := M$ is self-adjoint
- "Normal" description recovered



Towards Measurement

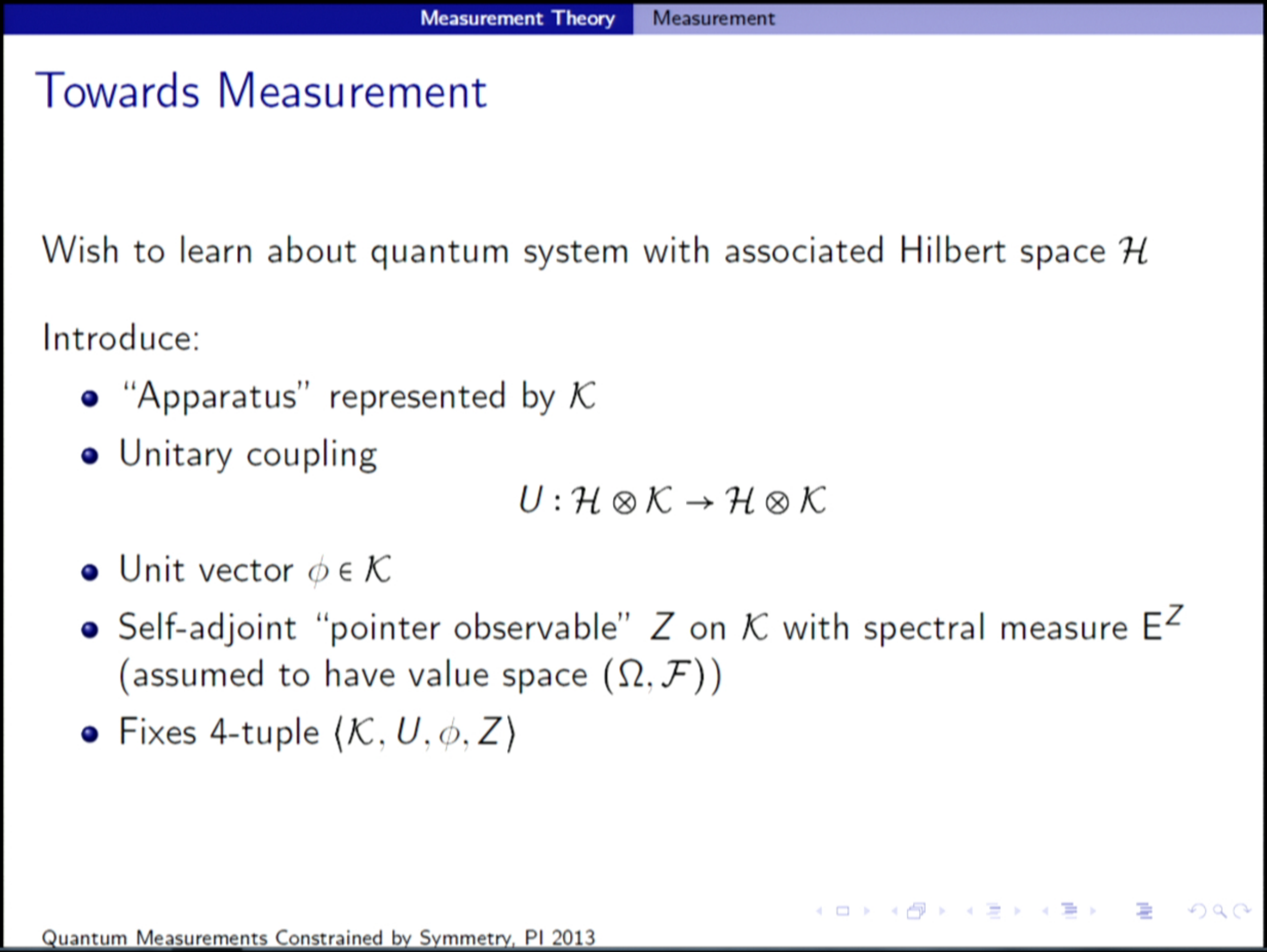
Wish to learn about quantum system with associated Hilbert space $\mathcal{H}$

Introduce:

- "Apparatus" represented by $\mathcal{K}$
- Unitary coupling

$$U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$$

- Unit vector $\phi \in \mathcal{K}$
- Self-adjoint "pointer observable" Z on $\mathcal{K}$ with spectral measure E^Z (assumed to have value space $(\Omega, \mathcal{F})$)
- Fixes 4-tuple $\langle \mathcal{K}, U, \phi, Z \rangle$



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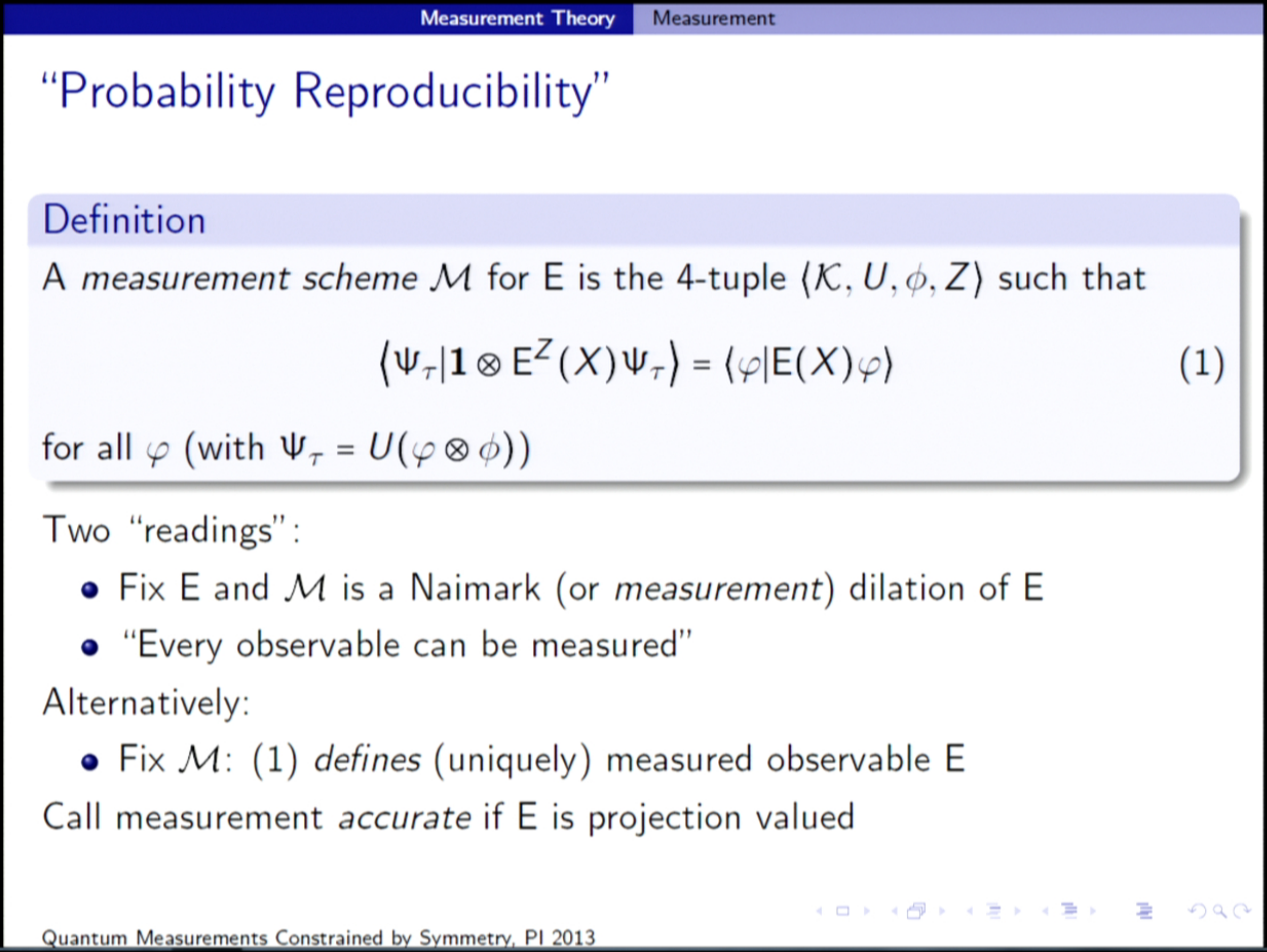


"Probability Reproducibility"

Definition

A *measurement scheme* $\mathcal{M}$ for E is the 4-tuple $\langle \mathcal{K}, U, \phi, Z \rangle$ such that

$$\left\langle \Psi_\tau | \mathbf{1} \otimes \mathrm{E}^Z(X) \Psi_\tau \right\rangle = \langle \varphi | \mathrm{E}(X) \varphi \rangle \qquad (1)$$

for all φ (with $\Psi_\tau = U(\varphi \otimes \phi)$)

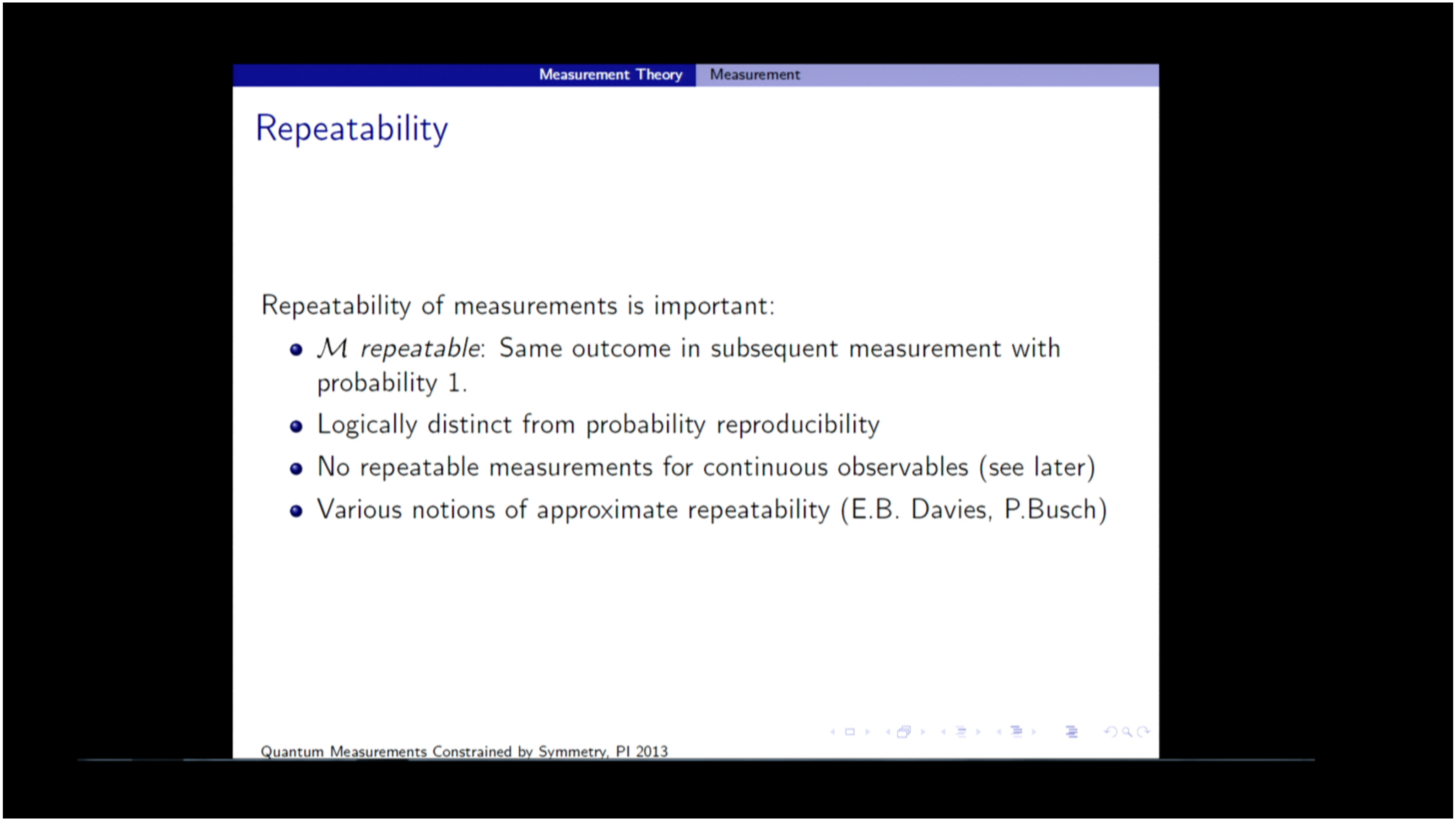
Two "readings":

- Fix E and $\mathcal{M}$ is a Naimark (or *measurement*) dilation of E
- "Every observable can be measured"

Alternatively:

- Fix $\mathcal{M}$: (1) *defines* (uniquely) measured observable E

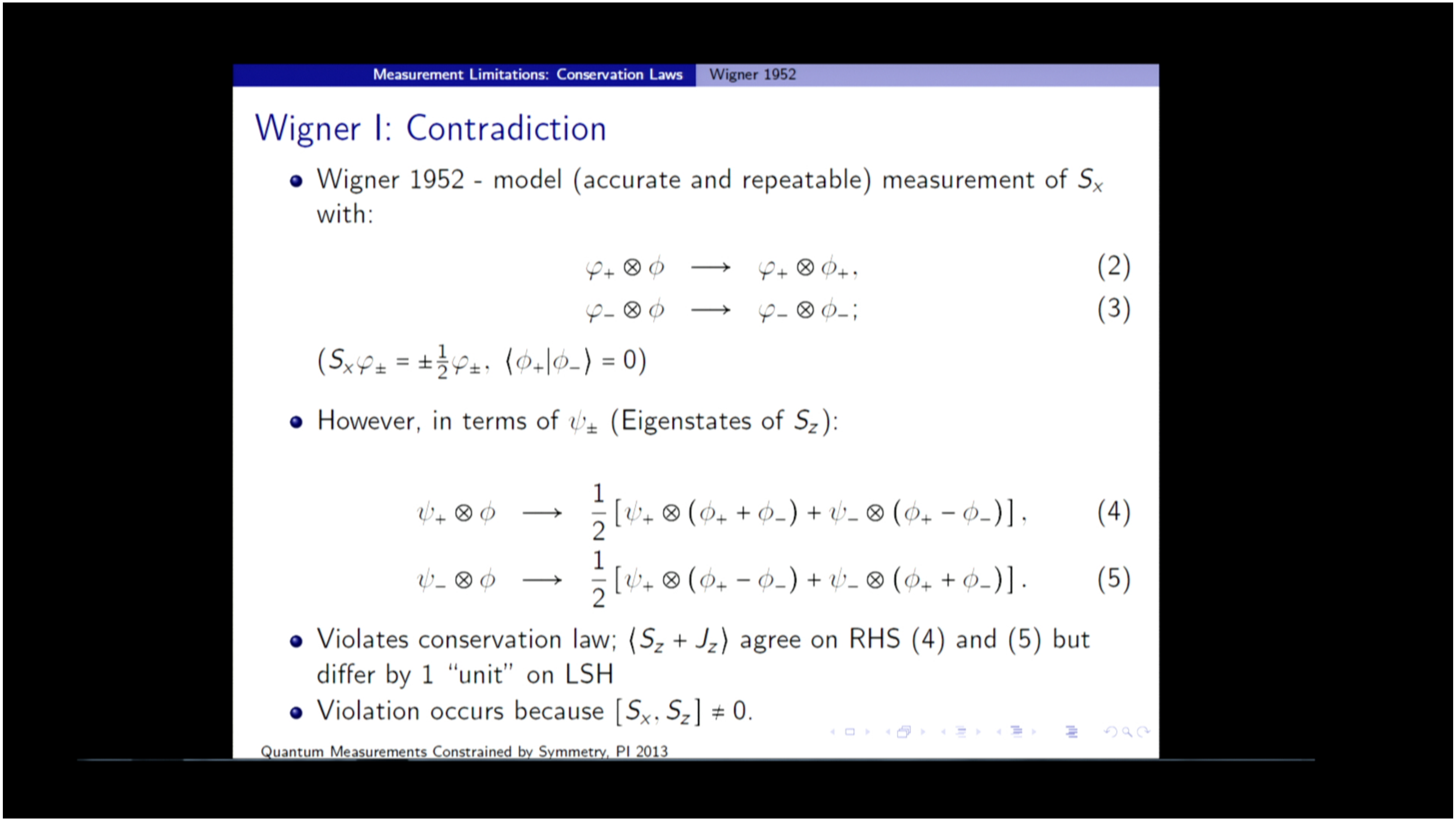
Call measurement *accurate* if E is projection valued



Repeatability

Repeatability of measurements is important:

- $\mathcal{M}$ *repeatable*: Same outcome in subsequent measurement with probability 1.
- Logically distinct from probability reproducibility
- No repeatable measurements for continuous observables (see later)
- Various notions of approximate repeatability (E.B. Davies, P.Busch)



Wigner I: Contradiction

- Wigner 1952 - model (accurate and repeatable) measurement of S_x with:

$$\varphi_+ \otimes \phi \longrightarrow \varphi_+ \otimes \phi_+, \tag{2}$$

$$\varphi_- \otimes \phi \longrightarrow \varphi_- \otimes \phi_-; \tag{3}$$

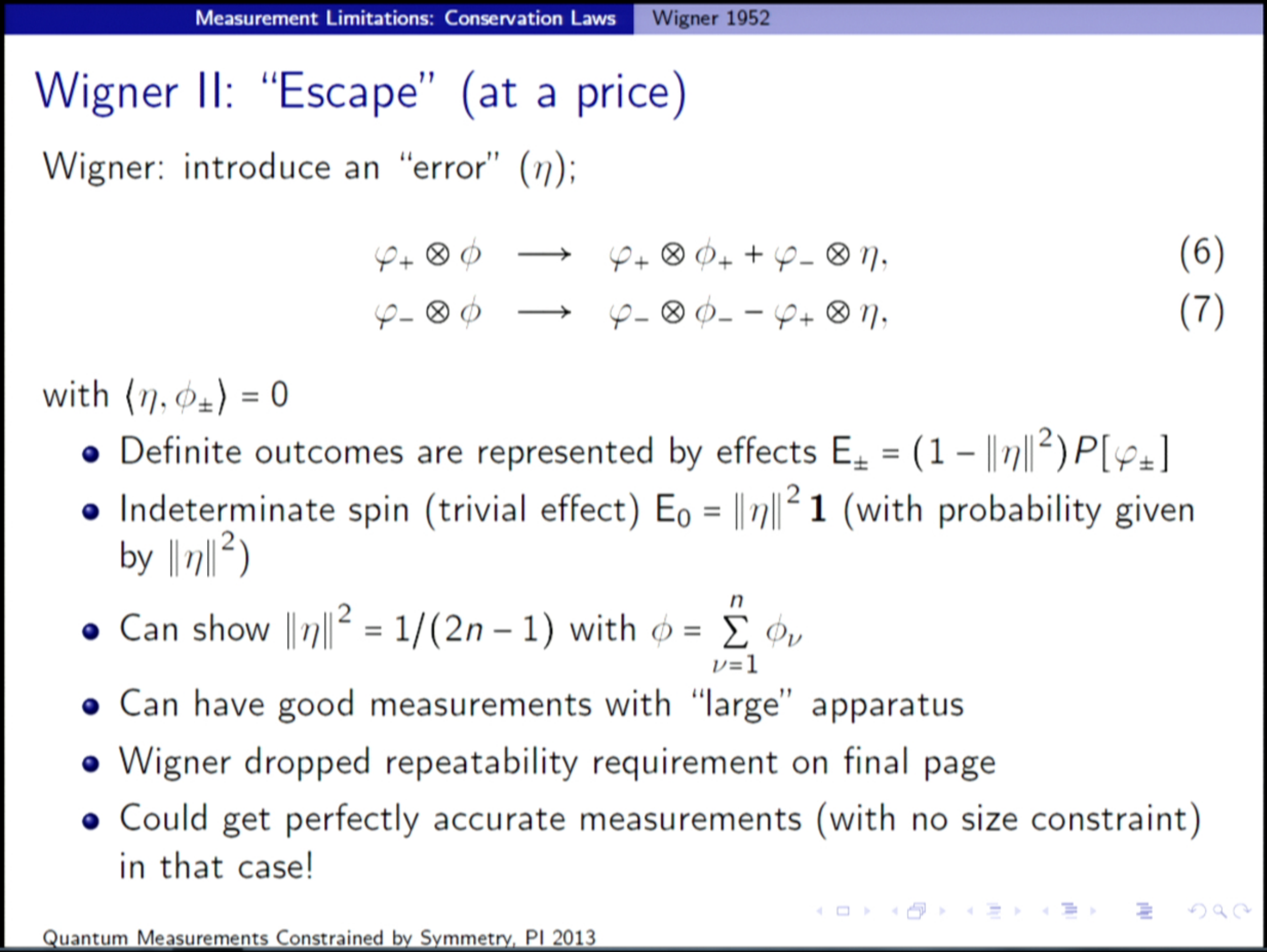
$(S_x \varphi_\pm = \pm\frac{1}{2}\varphi_\pm, \ \langle \phi_+ | \phi_- \rangle = 0)$

- However, in terms of $\psi_\pm$ (Eigenstates of S_z):

$$\psi_+ \otimes \phi \longrightarrow \frac{1}{2}\left[\psi_+ \otimes (\phi_+ + \phi_-) + \psi_- \otimes (\phi_+ - \phi_-)\right], \tag{4}$$

$$\psi_- \otimes \phi \longrightarrow \frac{1}{2}\left[\psi_+ \otimes (\phi_+ - \phi_-) + \psi_- \otimes (\phi_+ + \phi_-)\right]. \tag{5}$$

- Violates conservation law; $\langle S_z + J_z \rangle$ agree on RHS (4) and (5) but differ by 1 "unit" on LSH
- Violation occurs because $[S_x, S_z] \neq 0$.



Wigner II: "Escape" (at a price)

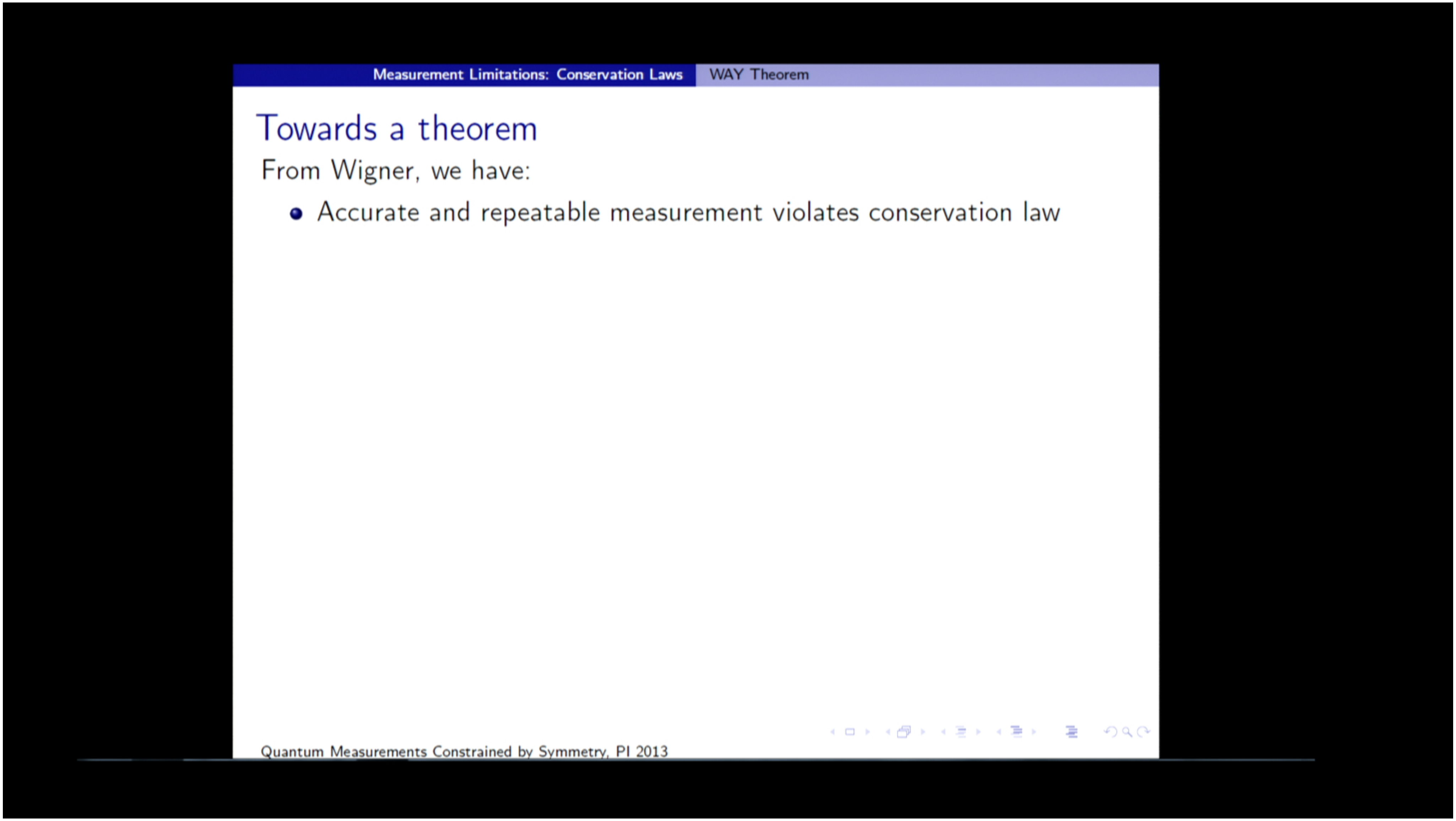
Wigner: introduce an "error" (η);

$$\varphi_+ \otimes \phi \longrightarrow \varphi_+ \otimes \phi_+ + \varphi_- \otimes \eta, \tag{6}$$

$$\varphi_- \otimes \phi \longrightarrow \varphi_- \otimes \phi_- - \varphi_+ \otimes \eta, \tag{7}$$

with $\langle \eta, \phi_\pm \rangle = 0$

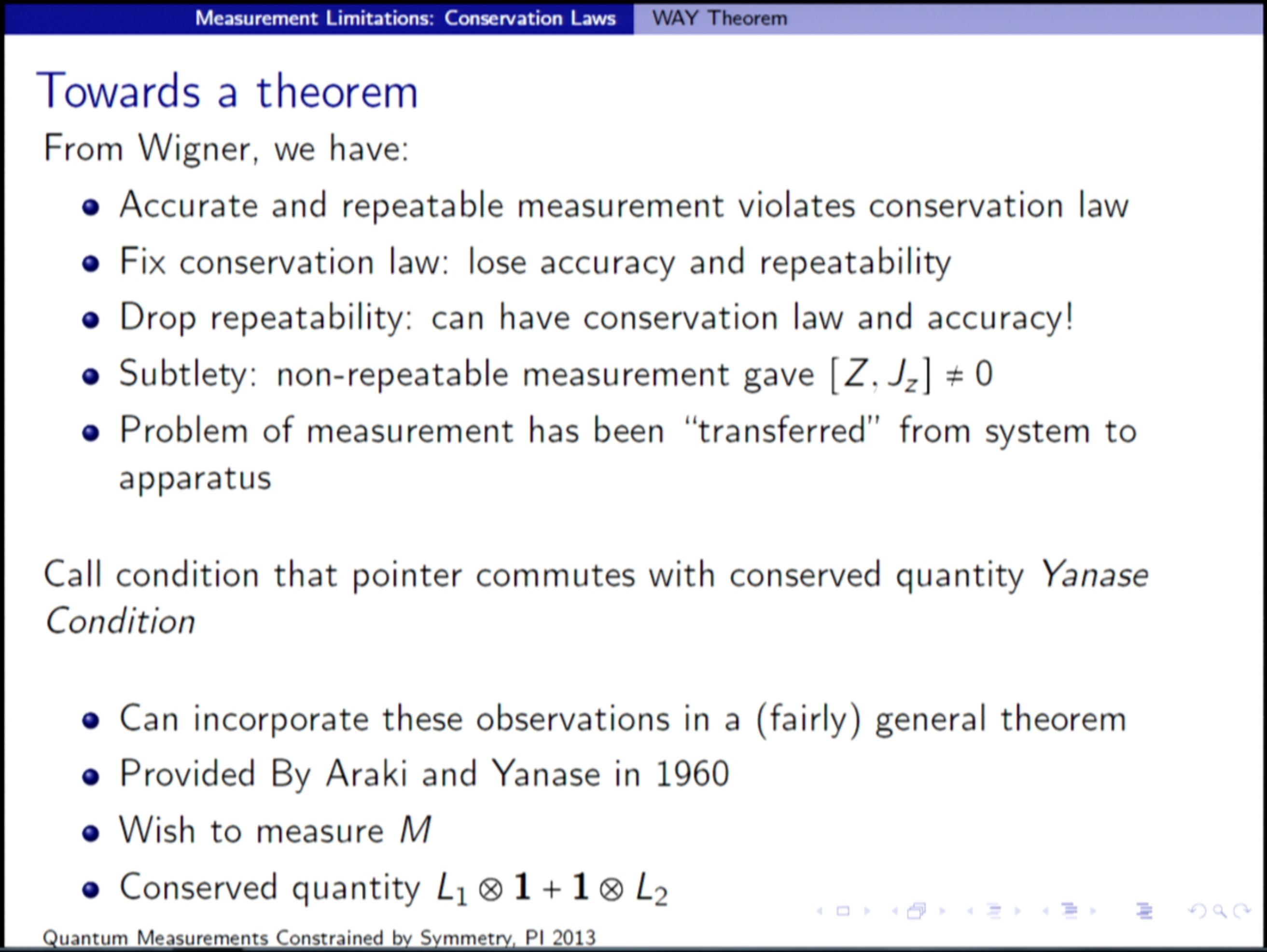
- Definite outcomes are represented by effects $\mathsf{E}_\pm = (1 - \|\eta\|^2) P[\varphi_\pm]$
- Indeterminate spin (trivial effect) $\mathsf{E}_0 = \|\eta\|^2 \mathbf{1}$ (with probability given by $\|\eta\|^2$)
- Can show $\|\eta\|^2 = 1/(2n-1)$ with $\phi = \sum_{\nu=1}^{n} \phi_\nu$
- Can have good measurements with "large" apparatus
- Wigner dropped repeatability requirement on final page
- Could get perfectly accurate measurements (with no size constraint) in that case!



Towards a theorem

From Wigner, we have:

- Accurate and repeatable measurement violates conservation law



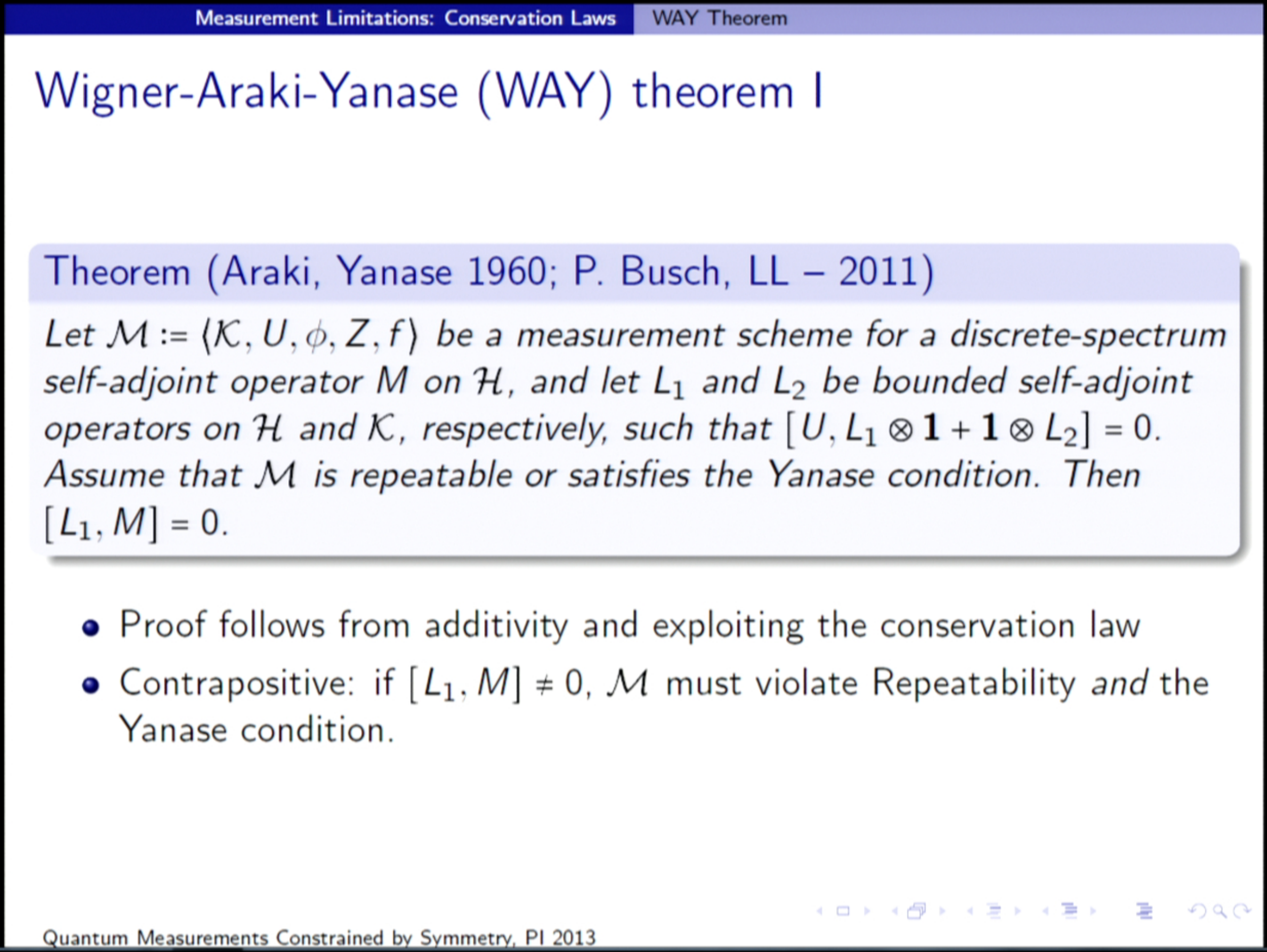
Towards a theorem

From Wigner, we have:

- Accurate and repeatable measurement violates conservation law
- Fix conservation law: lose accuracy and repeatability
- Drop repeatability: can have conservation law and accuracy!
- Subtlety: non-repeatable measurement gave $[Z, J_z] \neq 0$
- Problem of measurement has been “transferred” from system to apparatus

Call condition that pointer commutes with conserved quantity *Yanase Condition*

- Can incorporate these observations in a (fairly) general theorem
- Provided By Araki and Yanase in 1960
- Wish to measure M
- Conserved quantity $L_1 \otimes \mathbf{1} + \mathbf{1} \otimes L_2$

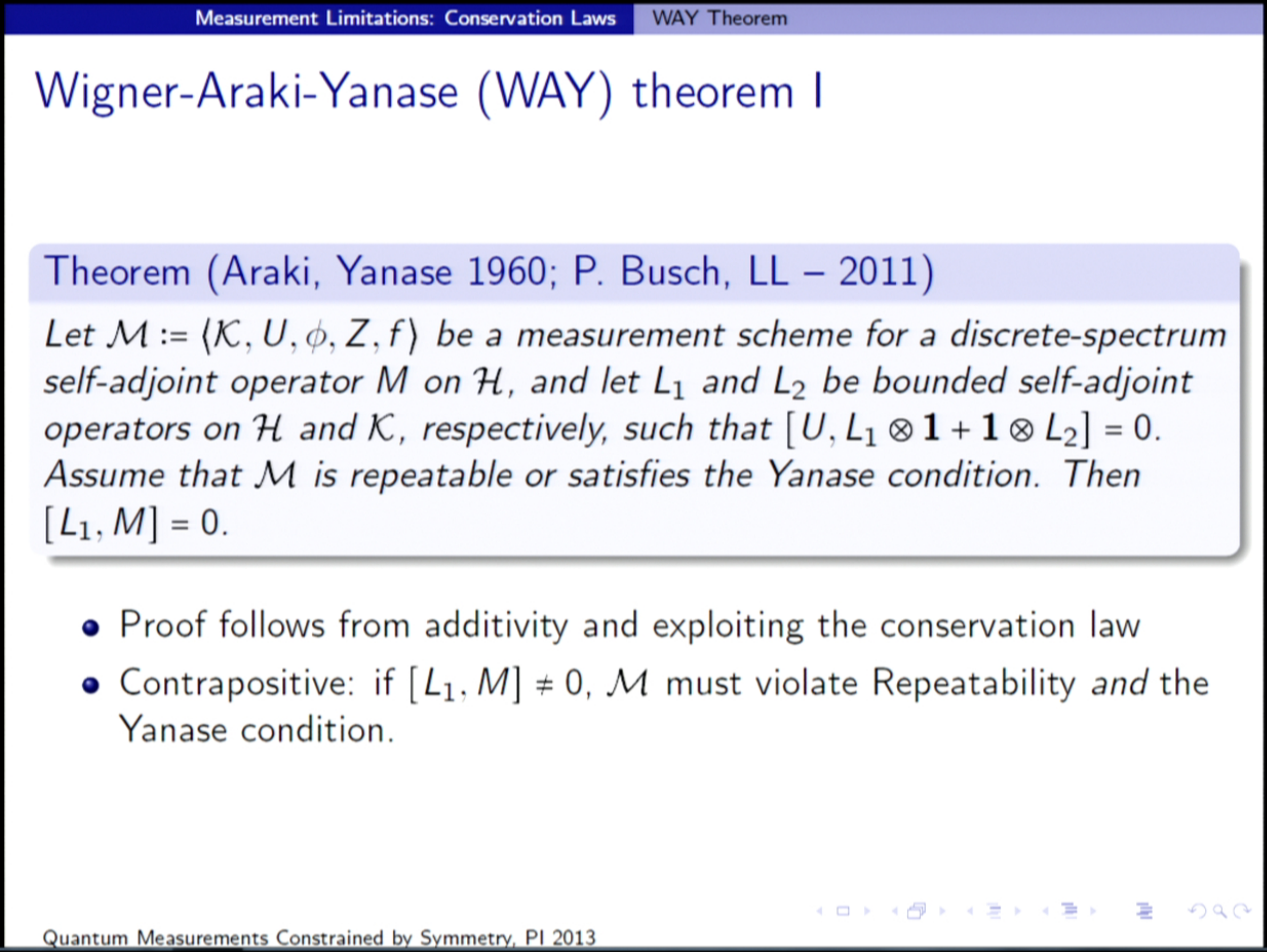


Wigner-Araki-Yanase (WAY) theorem I

Theorem (Araki, Yanase 1960; P. Busch, LL – 2011)

Let $\mathcal{M} := \langle \mathcal{K}, U, \phi, Z, f \rangle$ *be a measurement scheme for a discrete-spectrum self-adjoint operator* M *on* $\mathcal{H}$, *and let* L_1 *and* L_2 *be bounded self-adjoint operators on* $\mathcal{H}$ *and* $\mathcal{K}$, *respectively, such that* $[U, L_1 \otimes \mathbf{1} + \mathbf{1} \otimes L_2] = 0$. *Assume that* $\mathcal{M}$ *is repeatable or satisfies the Yanase condition. Then* $[L_1, M] = 0$.

- Proof follows from additivity and exploiting the conservation law
- Contrapositive: if $[L_1, M] \neq 0$, $\mathcal{M}$ must violate Repeatability *and* the Yanase condition.



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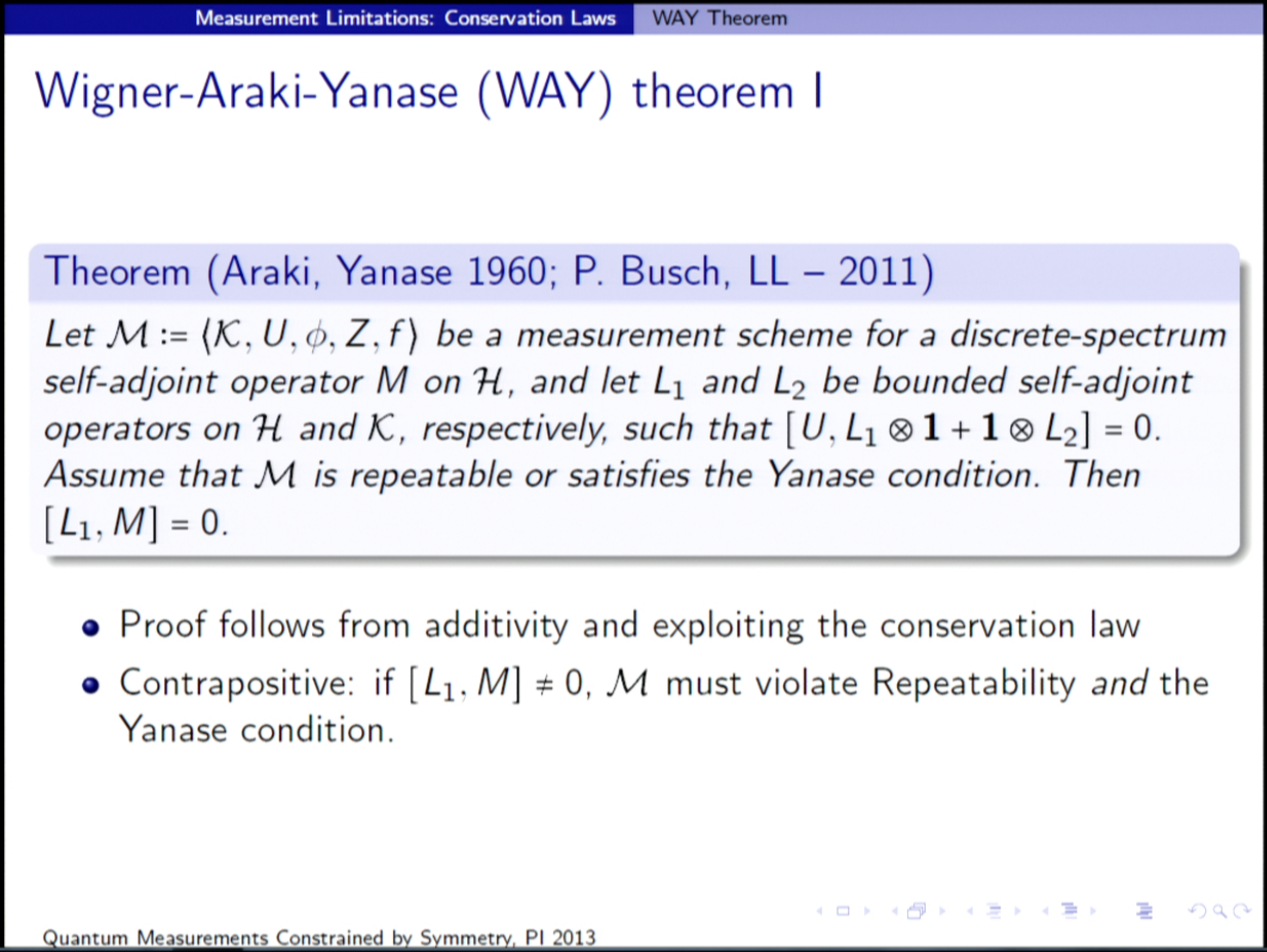

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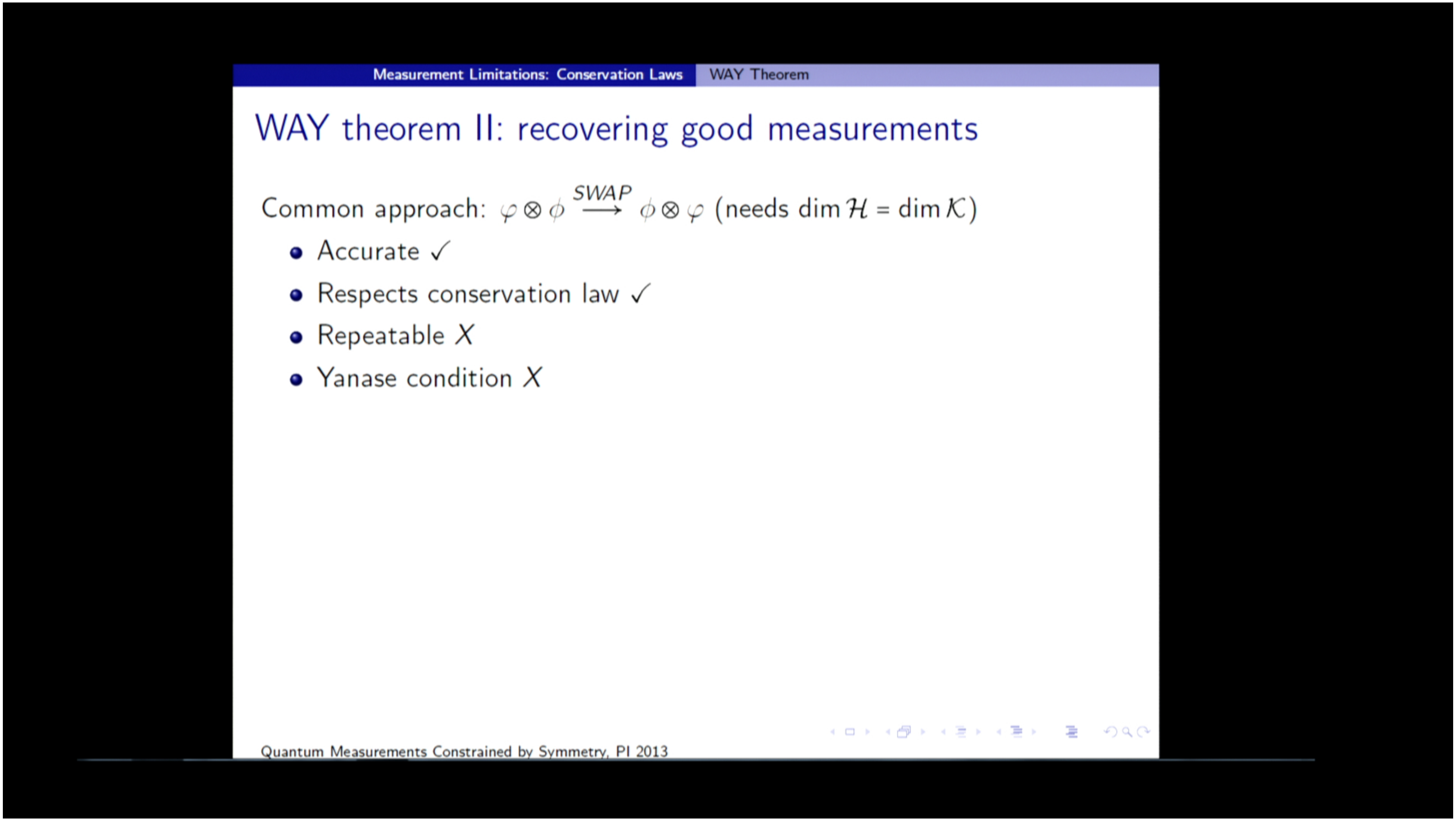
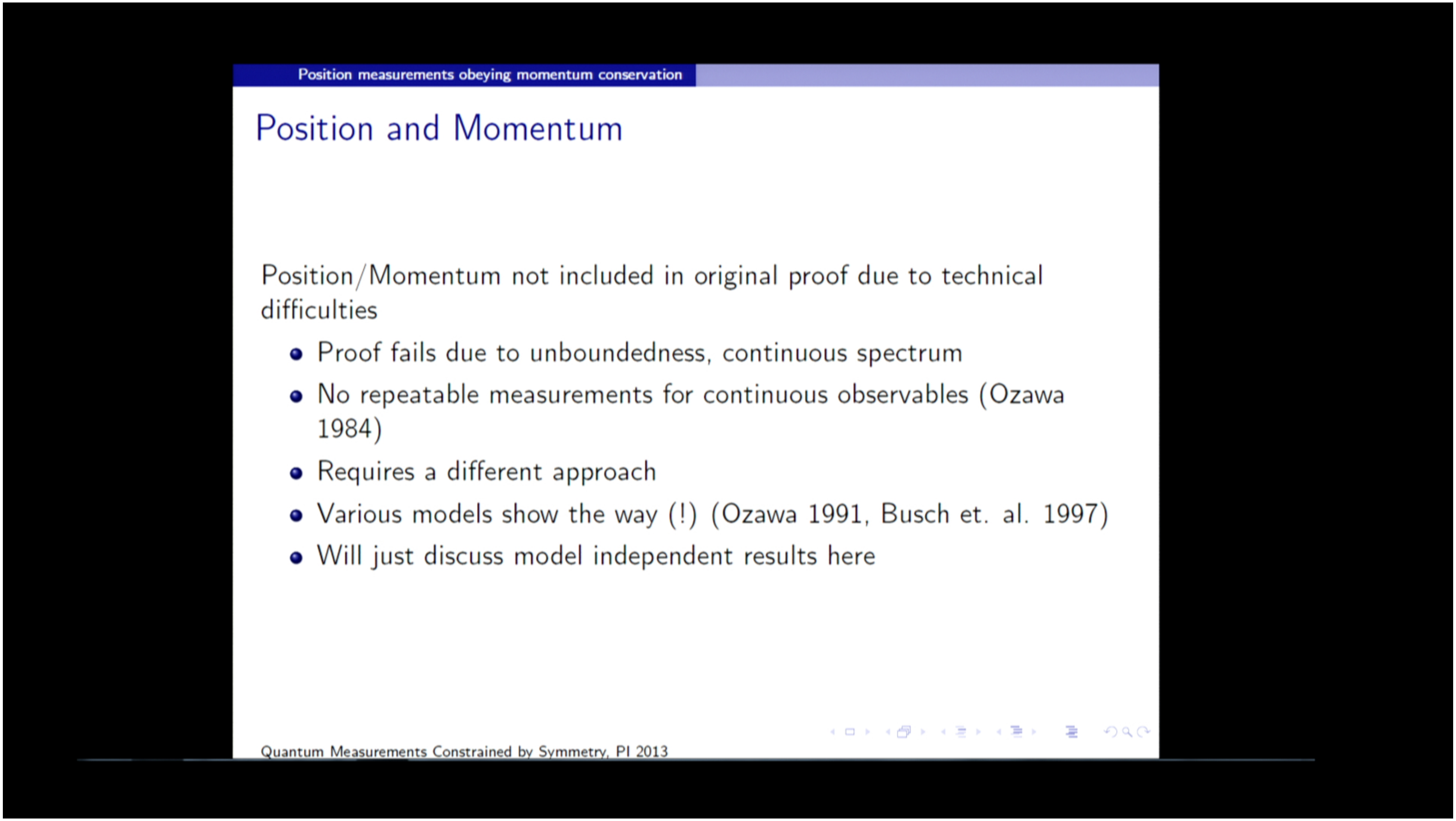


WAY theorem II: recovering good measurements

Common approach: $\varphi \otimes \phi \xrightarrow{SWAP} \phi \otimes \varphi$ (needs $\dim \mathcal{H} = \dim \mathcal{K}$)

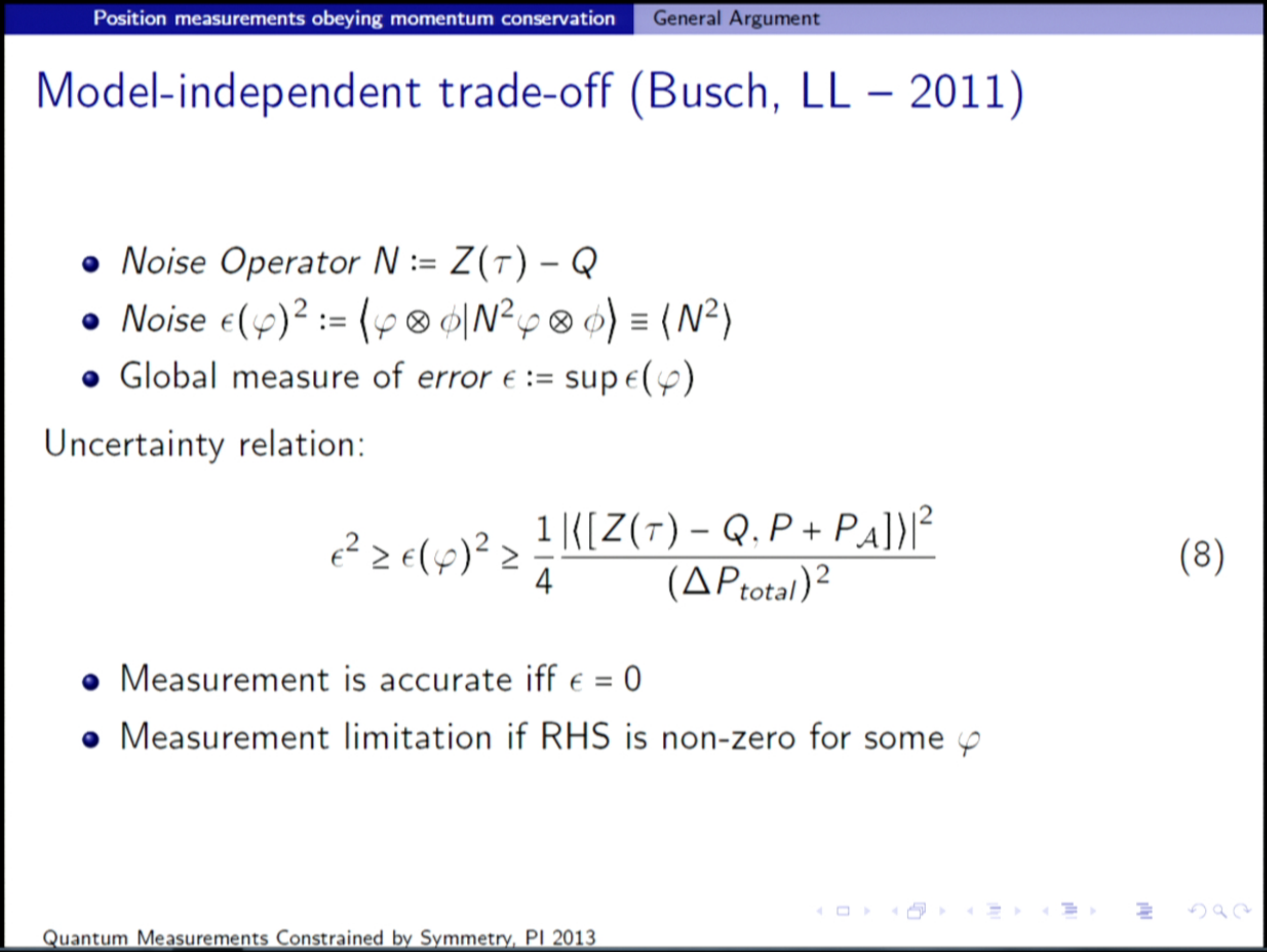
- Accurate ✓
- Respects conservation law ✓
- Repeatable X
- Yanase condition X



Position and Momentum

Position/Momentum not included in original proof due to technical difficulties

- Proof fails due to unboundedness, continuous spectrum
- No repeatable measurements for continuous observables (Ozawa 1984)
- Requires a different approach
- Various models show the way (!) (Ozawa 1991, Busch et. al. 1997)
- Will just discuss model independent results here



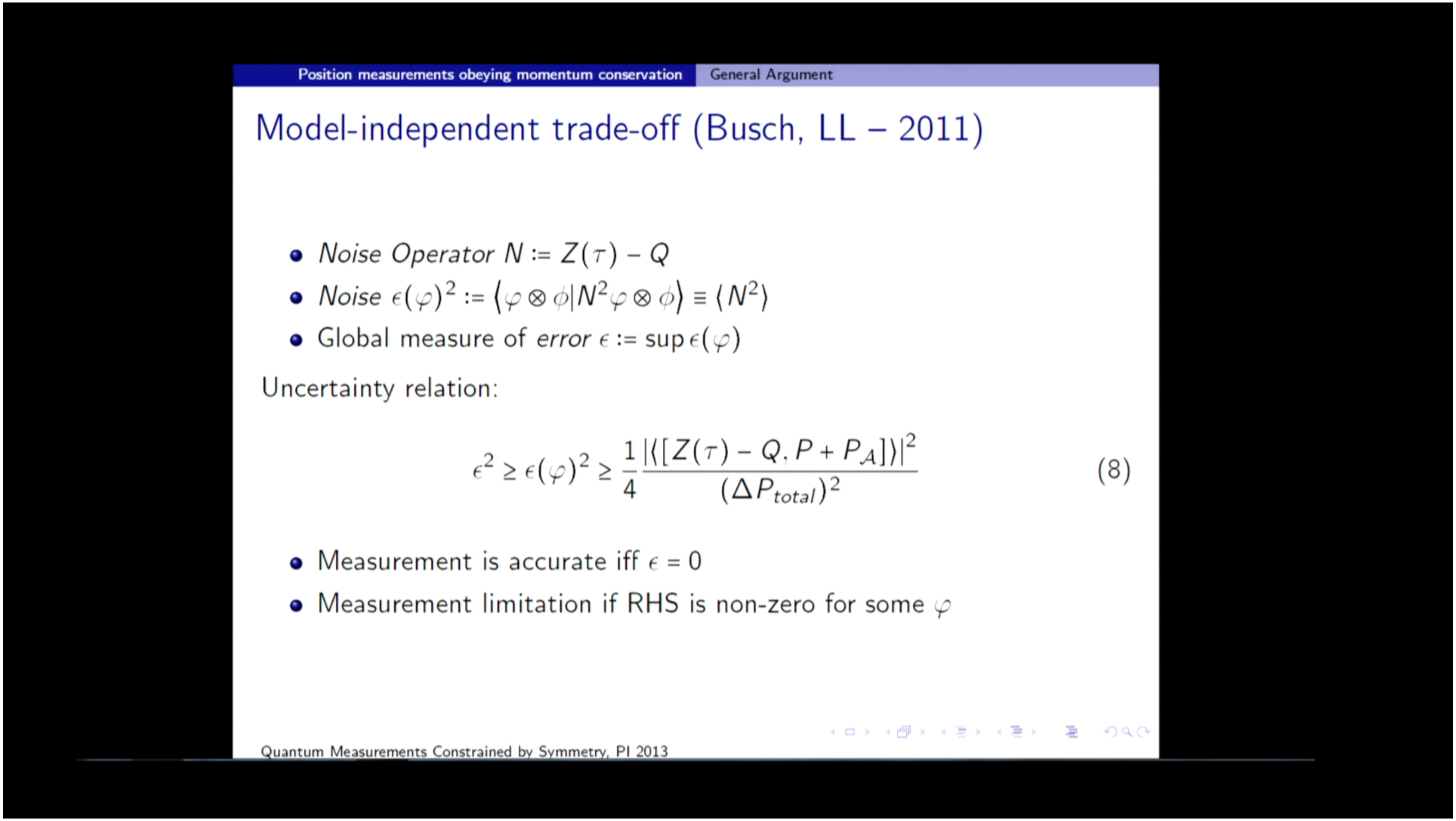
Model-independent trade-off (Busch, LL – 2011)

- *Noise Operator* $N := Z(\tau) - Q$
- *Noise* $\epsilon(\varphi)^2 := \left\langle \varphi \otimes \phi | N^2 \varphi \otimes \phi \right\rangle \equiv \langle N^2 \rangle$
- Global measure of *error* $\epsilon := \sup \epsilon(\varphi)$

Uncertainty relation:

$$\epsilon^2 \geq \epsilon(\varphi)^2 \geq \frac{1}{4} \frac{|\langle [Z(\tau) - Q, P + P_{\mathcal{A}}] \rangle|^2}{(\Delta P_{total})^2} \tag{8}$$

- Measurement is accurate iff $\epsilon = 0$
- Measurement limitation if RHS is non-zero for some φ



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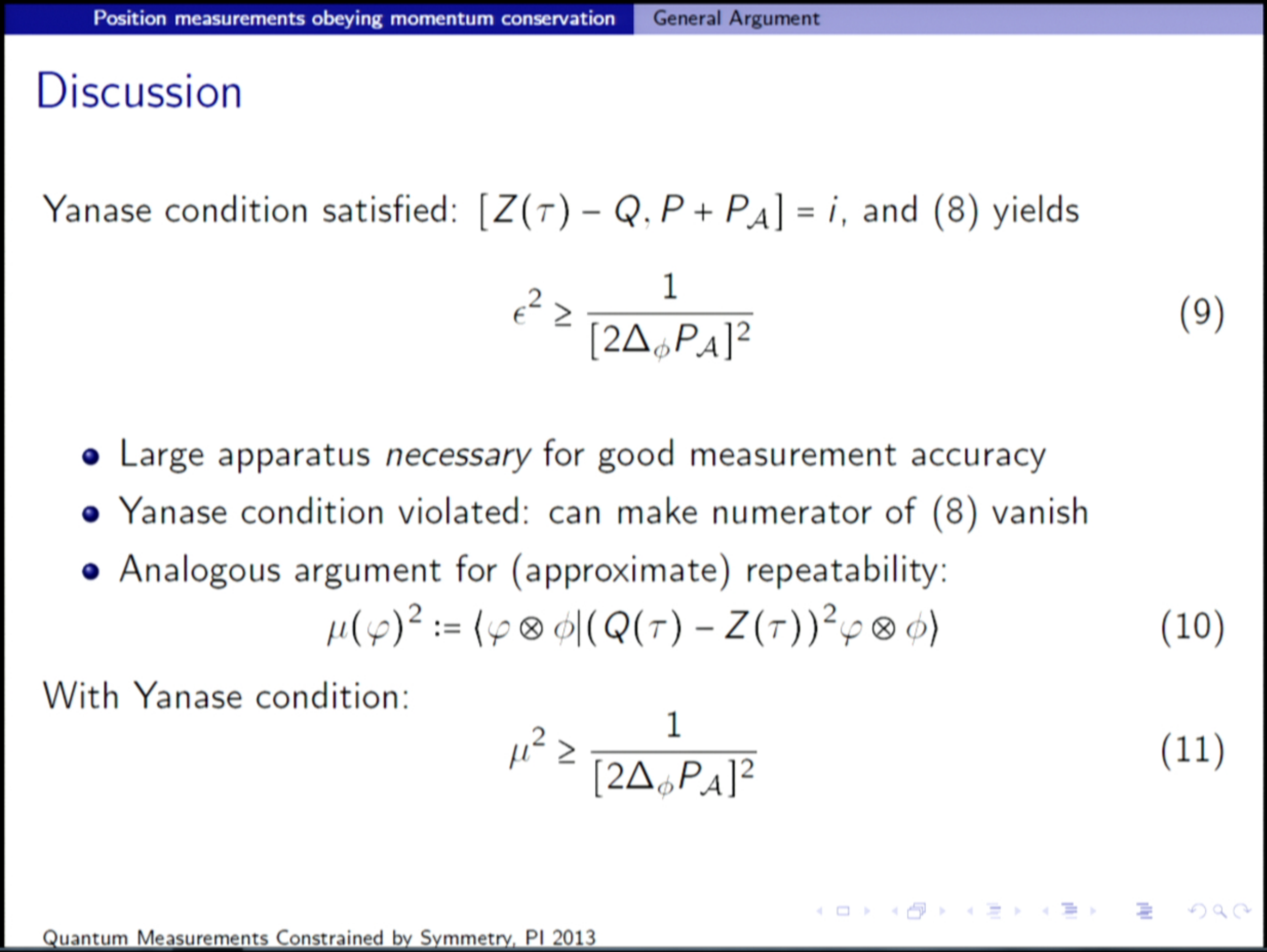


Discussion

Yanase condition satisfied: $[Z(\tau) - Q, P + P_{\mathcal{A}}] = i$, and (8) yields

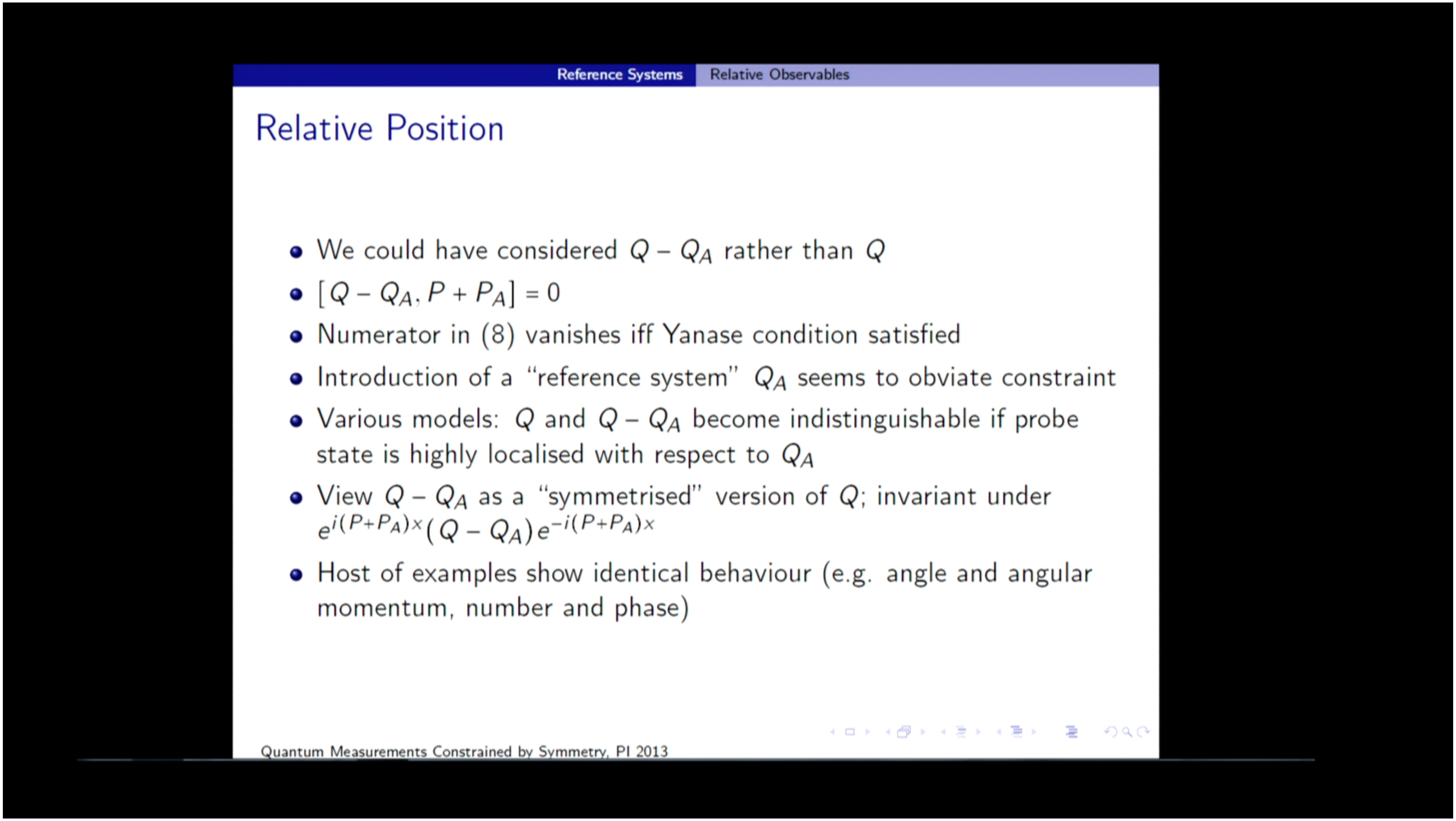
$$\epsilon^2 \geq \frac{1}{[2\Delta_\phi P_{\mathcal{A}}]^2} \tag{9}$$

- Large apparatus *necessary* for good measurement accuracy
- Yanase condition violated: can make numerator of (8) vanish
- Analogous argument for (approximate) repeatability:

$$\mu(\varphi)^2 := \langle \varphi \otimes \phi | (Q(\tau) - Z(\tau))^2 \varphi \otimes \phi \rangle \tag{10}$$

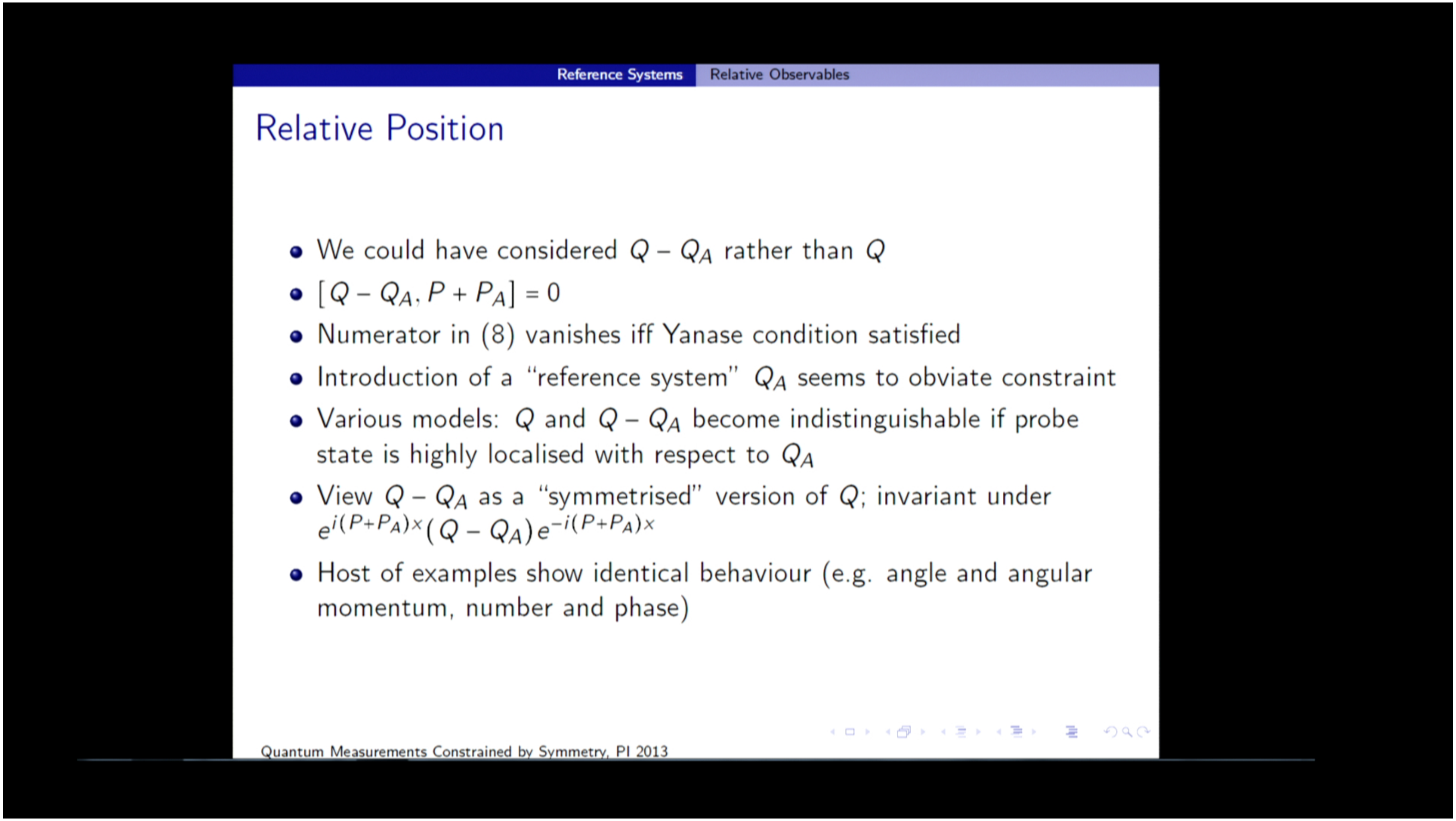
With Yanase condition:

$$\mu^2 \geq \frac{1}{[2\Delta_\phi P_{\mathcal{A}}]^2} \tag{11}$$



Relative Position

- We could have considered $Q - Q_A$ rather than Q
- $[Q - Q_A, P + P_A] = 0$
- Numerator in (8) vanishes iff Yanase condition satisfied
- Introduction of a "reference system" Q_A seems to obviate constraint
- Various models: Q and $Q - Q_A$ become indistinguishable if probe state is highly localised with respect to Q_A
- View $Q - Q_A$ as a "symmetrised" version of Q; invariant under $e^{i(P+P_A)x}(Q - Q_A)e^{-i(P+P_A)x}$
- Host of examples show identical behaviour (e.g. angle and angular momentum, number and phase)



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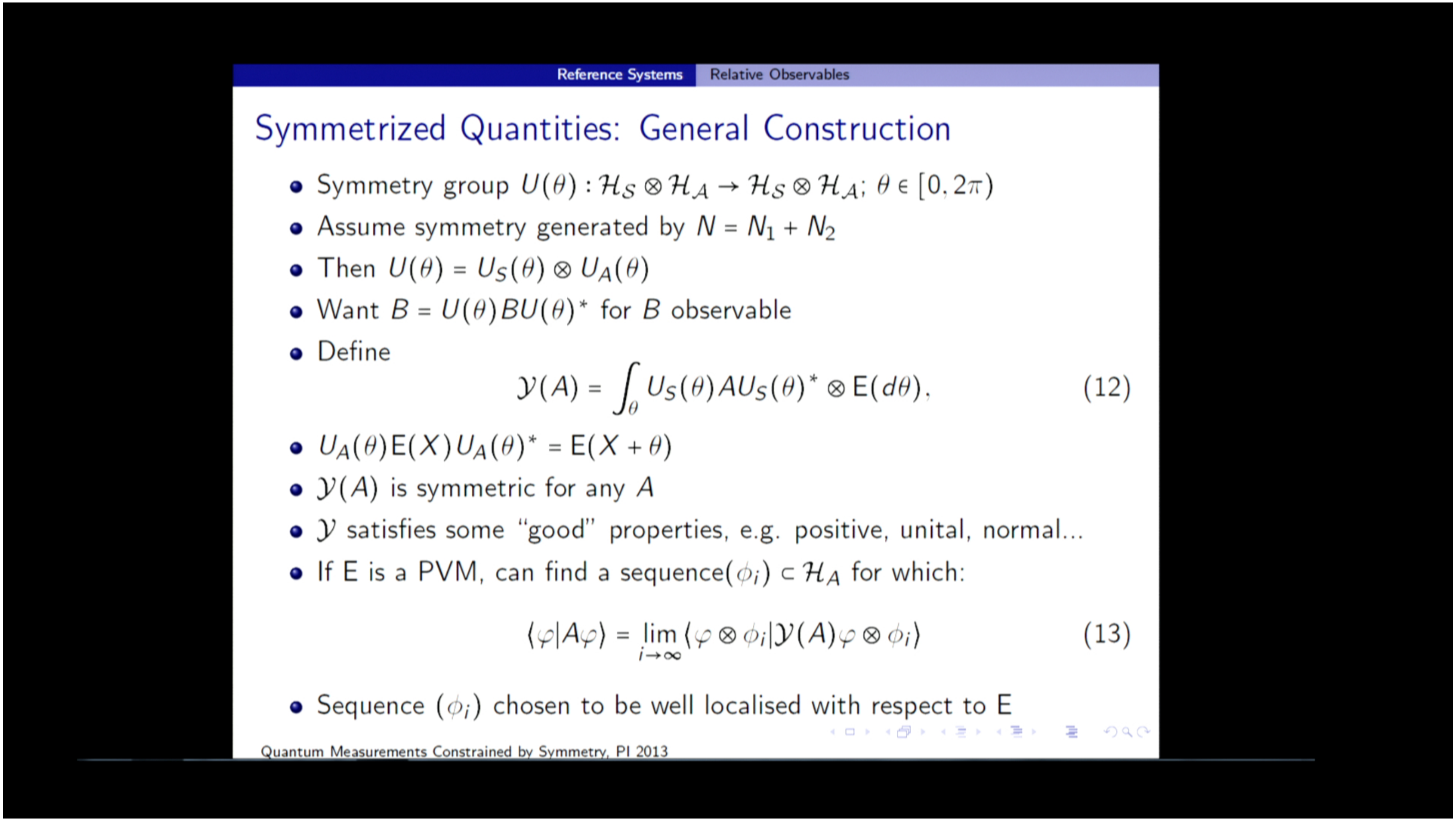

Symmetrized Quantities: General Construction

- Symmetry group $U(\theta) : \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A};\ \theta \in [0, 2\pi)$
- Assume symmetry generated by $N = N_1 + N_2$
- Then $U(\theta) = U_S(\theta) \otimes U_A(\theta)$
- Want $B = U(\theta) B U(\theta)^*$ for B observable
- Define
$$\mathcal{Y}(A) = \int_\theta U_S(\theta) A U_S(\theta)^* \otimes \mathsf{E}(d\theta), \tag{12}$$
- $U_A(\theta)\mathsf{E}(X)U_A(\theta)^* = \mathsf{E}(X + \theta)$
- $\mathcal{Y}(A)$ is symmetric for any A
- $\mathcal{Y}$ satisfies some "good" properties, e.g. positive, unital, normal...
- If E is a PVM, can find a sequence$(\phi_i) \subset \mathcal{H}_A$ for which:
$$\langle \varphi | A \varphi \rangle = \lim_{i \to \infty} \langle \varphi \otimes \phi_i | \mathcal{Y}(A) \varphi \otimes \phi_i \rangle \tag{13}$$
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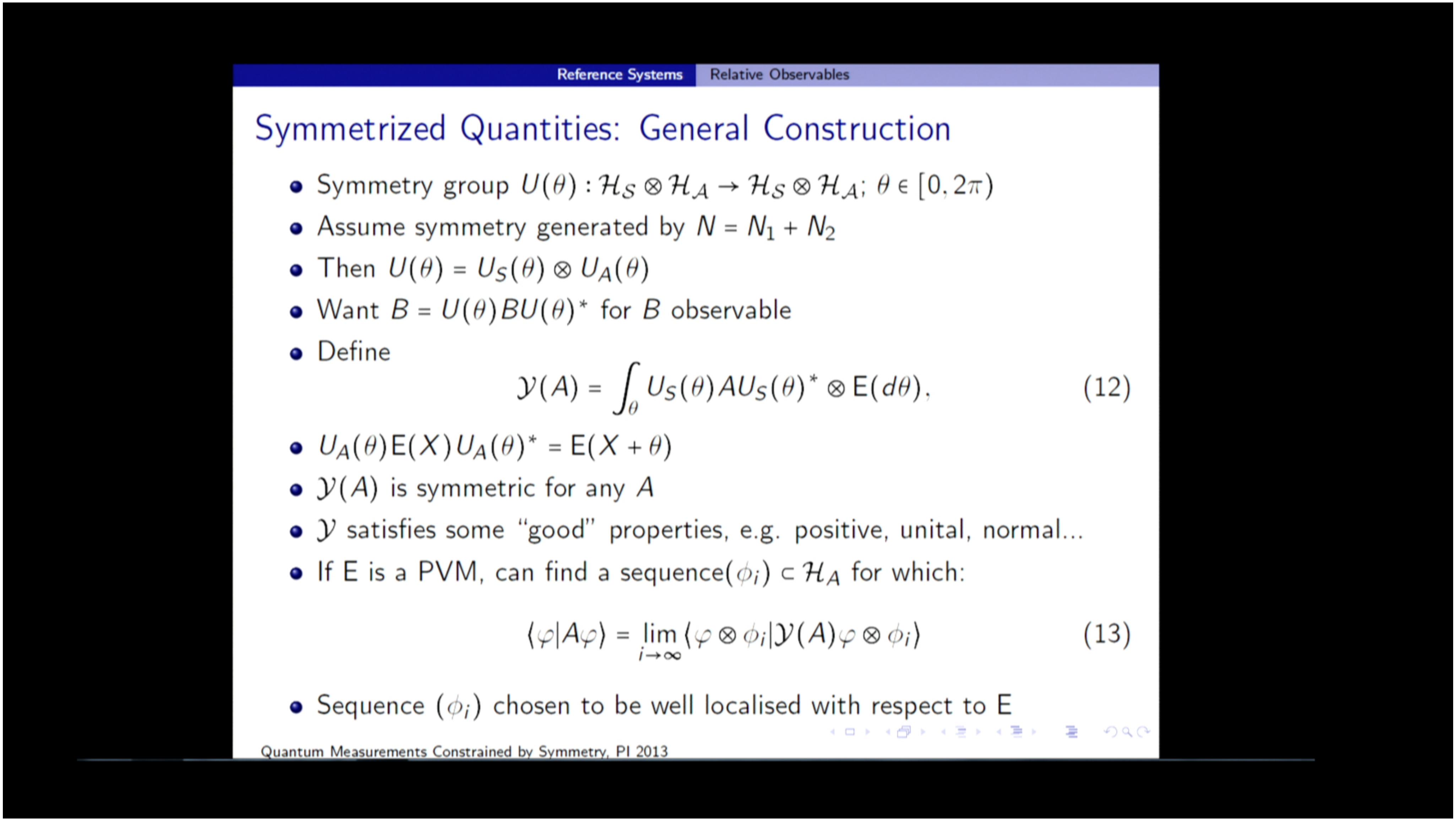

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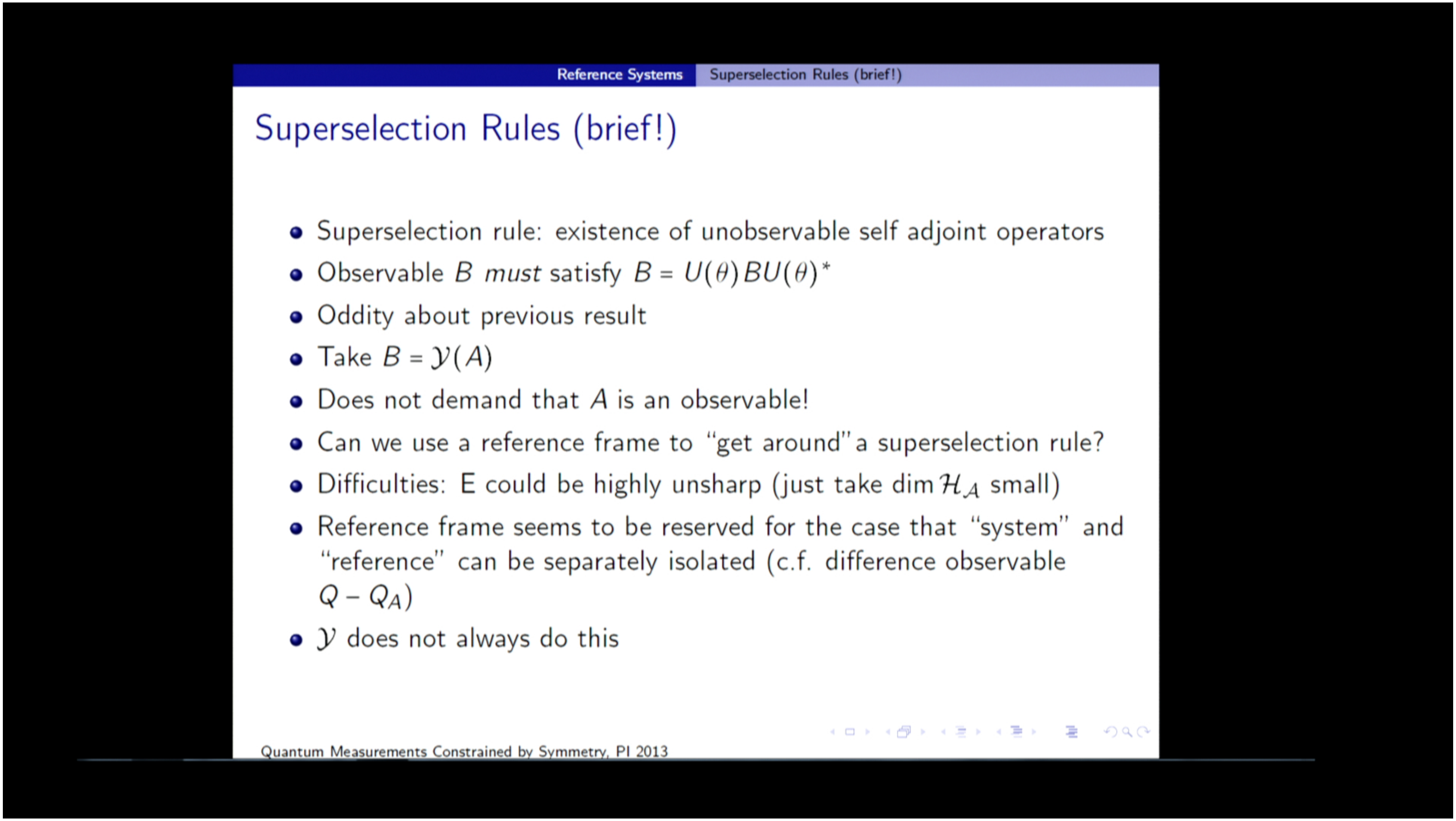
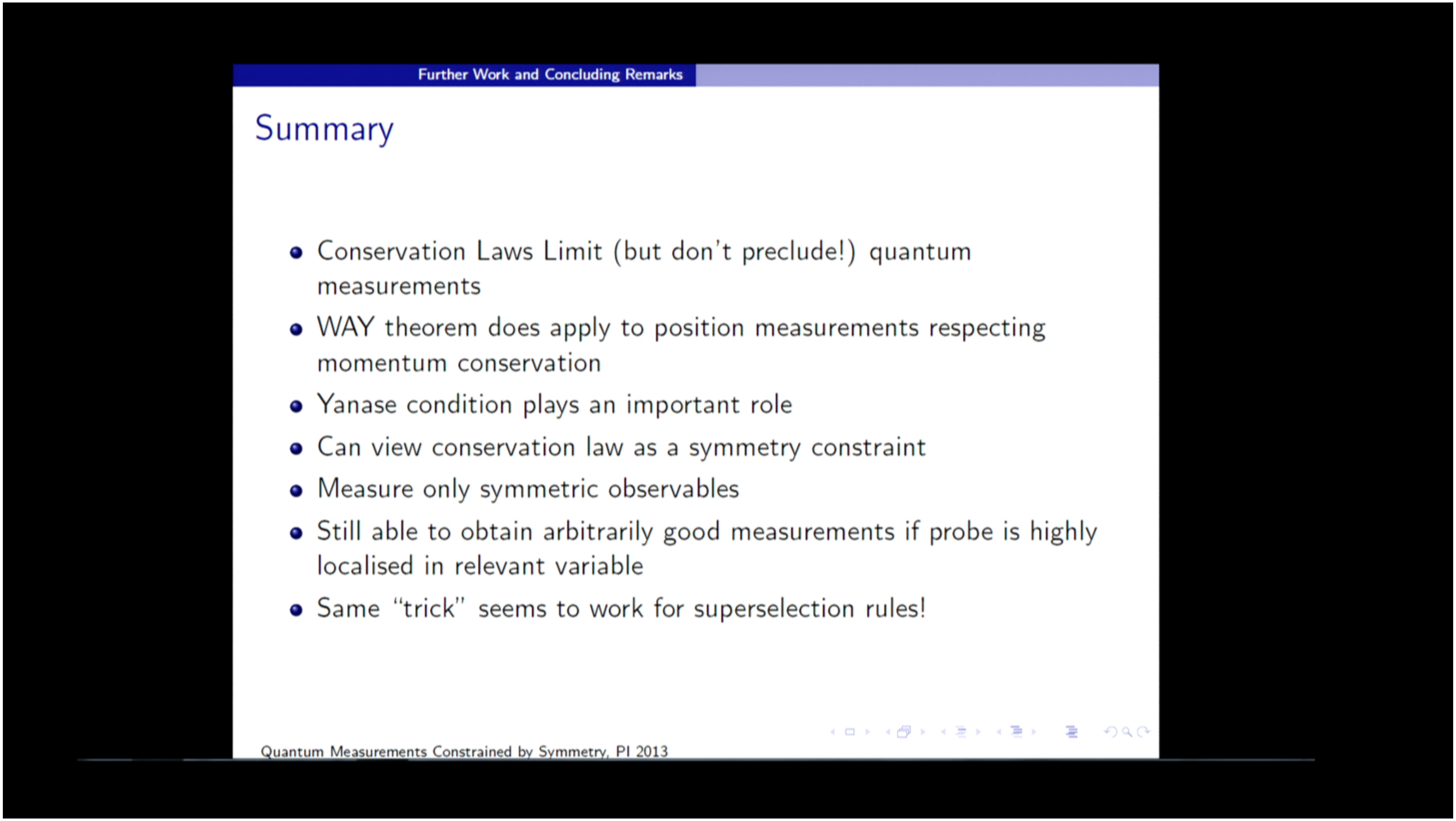


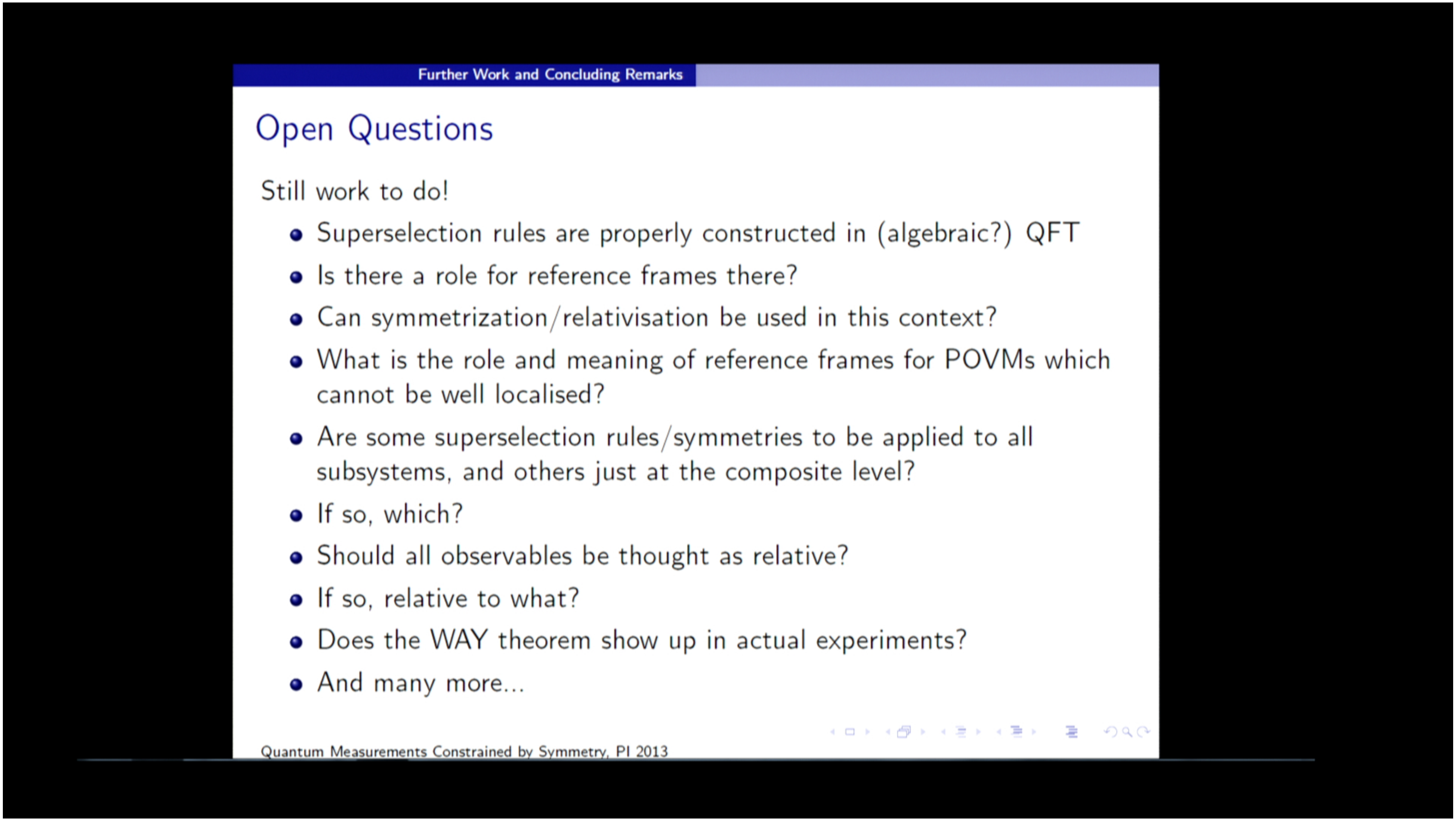
Superselection Rules (brief!)

- Superselection rule: existence of unobservable self adjoint operators
- Observable B *must* satisfy $B = U(\theta)BU(\theta)^*$
- Oddity about previous result
- Take $B = \mathcal{Y}(A)$
- Does not demand that A is an observable!
- Can we use a reference frame to "get around" a superselection rule?
- Difficulties: E could be highly unsharp (just take dim $\mathcal{H}_{\mathcal{A}}$ small)
- Reference frame seems to be reserved for the case that "system" and "reference" can be separately isolated (c.f. difference observable $Q - Q_A$)
- $\mathcal{Y}$ does not always do this



Summary

- Conservation Laws Limit (but don't preclude!) quantum measurements
- WAY theorem does apply to position measurements respecting momentum conservation
- Yanase condition plays an important role
- Can view conservation law as a symmetry constraint
- Measure only symmetric observables
- Still able to obtain arbitrarily good measurements if probe is highly localised in relevant variable
- Same "trick" seems to work for superselection rules!



Open Questions

Still work to do!

- Superselection rules are properly constructed in (algebraic?) QFT
- Is there a role for reference frames there?
- Can symmetrization/relativisation be used in this context?
- What is the role and meaning of reference frames for POVMs which cannot be well localised?
- Are some superselection rules/symmetries to be applied to all subsystems, and others just at the composite level?
- If so, which?
- Should all observables be thought as relative?
- If so, relative to what?
- Does the WAY theorem show up in actual experiments?
- And many more...