Title: Three-point functions: SFT, integrability, and perturbative calculations.

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Abstract: I discuss several recent efforts in relating string field theory calculations of BMN BMN BMN and BMN BMN BPS correlation functions to direct perturbative calculations and integrability-assisted methods. I review the next-to-leading order agreement between strings and perturbation theory in the SO(6) sector, a conjectured extension of the integrability techniques by Escobedo, Gromov, Sever, Vieira from the SU(2) to the full SO(6) sector and agreement with SFT and PT in it at the leading order; finally, I discuss the issue of equating exactly extremal and non-extremal correlators at NLO in the integrability-assisted calculation.

Three-point functions of BMN operators at weak and strong coupling from PT, SFT and integrability

Andrew V. Zayakin

in collaboration with A.Bissi, G.Grignani, W.Schulgin
University of Santiago de Compostela
Spain

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Three-point functions

- Two-point functions/anomalous dimensions have been studied thoroughly (to all loops)
- Knowledge of three-point functions would have vested us with full knowledge of the $\mathcal{N} = 4$ SYM.
- More or less everything is known on correlators of "light" states; less is known about heavy-heavy-light operators; very little is known about heavy-heavy-heavy correlators.

Plan of the talk

- Introductions: basic objects and motivation.
- Strings vs. Perturbation theory
- Perturbation theory vs. Integrability
- Integrability vs. Strings

Motivation

- Three-point functions are non-protected objects, thus they can have all kinds of corrections
- There is no guarantee that the standard strings-to-fields equivalence methods will work
- The operator-to-string equivalence definition for purposes of three-point function construction is not unambiguous
- There are some discrepancies at one-loop level *[Bissi, Harmark, Orselli 2011]*
- Thus it is important to classify all the sectors in which there are discrepancies so that their nature could be established.

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How to obtain the three-point functions

- From field theory *[Beisert, Kristjansen, Plefka, Semenoff, Staudacher 2002]*: by direct perturbative calculation
- From Bethe Ansatz*[Gromov, Vieira 2012]*
- From string field theory: as matrix elements of the Dobashi-Yoneya vertex *[Dobashi, Yoneya 2004]*
- From string theory in the semiclassic regime: as action value on a worldsheet solution with three delta-sources *[Zarembo 2010]*

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Asymptotics vs. Theories

Three-point functions can be obtained from field theory, field theory assisted by Bethe Ansatz, string field theory, string theory semiclassics in the following regimes:

Based on	λ	R-charge J	$\lambda' = \frac{\lambda}{J^2}$	Nr. of magnons
FT	small	any	any	small
Bethe Ansatz	small	any	any	any
SFT	large	any	any	small
ST semiclassics	large	large	any	large

Thus the unique possibility to compare these objects is the λ' expansion in Frolov-Tseytlin limit, where asymptotically large-coupling and small-coupling limits can be unified.

Definitions

We consider two-magnon BMN operators

$$\mathcal{O}^J_{ij,n} = \frac{1}{\sqrt{JN^{J+2}}} \sum_{l=0}^{J} \mathrm{tr}\left(\phi_i Z^l \phi_j Z^{J-l}\right) \psi_{n,l},$$

which fall into the three irreducible representations of $SO(4)$; we choose the symmetric one for which

$$\psi^S_{n,l} = \cos\frac{(2l+1)\pi n}{J+1}$$

We consider three operators: $\mathcal{O}_1 = \mathcal{O}^{J_1,12}_{n_1}$, $\mathcal{O}_2 = \mathcal{O}^{J_2,23}_{n_2}$, $\mathcal{O} = \mathcal{O}^{J,31}_{n}$, where n_1, n_2, n_3 are the magnon momenta, J_1, J_2, J_3 are their R-charges R_3, $J = J_1 + J_2$, $J_1 = Jy$, $J_2 = J(1-y)$.

We shall be looking for the quantity

$$C_{123} = \langle \bar{\mathcal{O}}_3 \mathcal{O}_1 \mathcal{O}_2 \rangle$$

as a function of y, J, n_1, n_2, n_3, and compare it at one loop in FT with ST in Penrose limit.

Problems on our way

Let us point out some of the obstacles that may be encountered on the way to three-point functions:

1. Double-trace admixture
2. Fermionic operators admixture
3. Magnon momentum nonconserving admixture

Happily enough,

- problem (1) is resolved by withdrawing ourselves to the non-extremal sector;
- problem (2) resolved by choosing the symmetric sector operators in $SO(6)$;
- problem (3) is resolved by invoking the large-J limit.

Mixing: multitrace

Multi-trace operator redefinition is organized e.g. as

$$\mathcal{O}_n^{J,12\prime} = \mathcal{O}_{12,n}^{J,12} - \frac{J^2}{N} \sum_{k,r} \frac{r^{3/2}\sqrt{1-r}\sin^2(\pi n r)k}{\sqrt{J}\pi^2(k-nr)^2(k+nr)} \mathcal{T}_{12,k}^{J,r}$$

where $\mathcal{T}_{12,n}^{J,r} = \mathcal{O}_n^{rJ,12}\mathcal{O}^{(1-r)J}$, $\mathcal{O}^J$ being the normalized vacuum operator of length J. For non-extremal kinematics the multi-trace mixing becomes significant only in the next-order corrections in $1/N$.

Mixing: magnon nonconserving

The magnon mode number nonpreserving BMN operator $\mathcal{O}_n$ redefinition in the order λ' is organized as

$$\mathcal{O}_n^{J,12\prime} = \mathcal{O}_n^{J,12} - \frac{\lambda}{(J+1)\pi^2} \sum_{m=1}^{[J/2]} \delta_{m \neq n} \frac{\sin^2 \frac{\pi n}{J+1} \cos \frac{\pi n}{J+1} \sin^2 \frac{\pi m}{J+1} \cos \frac{\pi m}{J+1}}{\sin^2 \frac{\pi n}{J+1} - \sin^2 \frac{\pi m}{J+1}} \mathcal{O}_m^{J,12},$$

here $[J/2]$ denotes the integer part of $J/2$. This operator redefinition can been shown not to contribute due to suppression by higher-order powers of $\frac{1}{J}$.

Mixing: fermions

The admixture with fermionic operators is the most difficult to handle. At order λ it is not yet known for the class of symmetric traceless operators considered in this work. The mixing for the trace class operators is derived in *[Georgiou, Gili, Russo 2009]*:

$$
\begin{aligned}
\mathcal{O}_n^J &= \sqrt{\frac{N_0^{-J-2}}{(J+3)}}\,\mathcal{Z} \sum_{i=1}^{3}\sum_{p=0}^{J} \cos\frac{\pi n(2p+3)}{J+3}\, \mathrm{tr}\, Z_i Z^p \bar{Z}_i Z^{J-p} \qquad (1)\\
&+ \frac{g\sqrt{N}}{4\pi}\sin\frac{\pi n}{J+3}\sqrt{\frac{N_0^{-J-1}}{(J+3)}}\sum_{p=0}^{J-1}\sin\frac{\pi n(2p+4)}{J+3}\,\mathrm{tr}\,\psi^{1\alpha} Z^p \psi^2_\alpha Z^{J-1-p}\\
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&+ \frac{g^2 N}{16\pi^2}\sin^2\frac{\pi n}{J+3}\sqrt{\frac{N_0^{-J}}{(J+3)}}\sum_{p=0}^{J-2}\cos\frac{\pi n(2p+5)}{J+3}\,\mathrm{tr}\,D_\mu Z Z^p D^\mu Z Z^{J-p-2} + \mathcal{O}(g^3)\,.
\end{aligned}
$$

If the mixing for the symmetric traceless sector was described by a formula of the same type as there, a rough estimate yields that the mixing might contribute in our case at the order g^2/J^2. However, the tree-level contribution is of order J^2 and the one-loop goes as $g^2 J^0$. Thus the mixing correction will appear at the next $1/J^2$ order while holding λ' order fixed, so that it would not contribute.

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String field theory

By "string field theory" here we mean an effective quantum mechanics emerging for BMN excitations for strings in PP-wave limit at large R-charge. This approach was developed by Spradlin, Volovich, Schwarz, Klebanov, Roiban et al., Dobashi and Yoneya, and other people in 2001-2004. By considering a configuration of three merged strings in the pp-background, the Hamiltonian for string oscillations modes can be figured out, represented by the so-called Neumann matrices. In the leading $1/N$ order this approach is exact to all orders in λ', which is advantageous for doing non-perturbative calculations. Below we present the results of calculating three-point functions with these effective Hamiltonians.

String field theory calculation

In terms of the BMN basis $\{\alpha_m\}$ our operators look like

$$\mathcal{O}_m = \alpha^\dagger_m \alpha^\dagger_{-m}|0\rangle$$

The three-point function is related to the matrix element of the string field Hamiltonian as follows

$$\langle \bar{\mathcal{O}}_3 \mathcal{O}_1 \mathcal{O}_2 \rangle = \frac{4\pi}{-\Delta_3 + \Delta_1 + \Delta_2} \sqrt{\frac{J_1 J_2}{J}} H_{123}$$

where

$$\Delta_i = J_i + 2\sqrt{1 + \lambda' n_i^2},$$

and the matrix element is defined as

$$H_{123} = \langle 123 | V \rangle.$$

Dobashi–Yoneya prefactor

We use the findings of [Grignani et al. 2006] to start with the Dobashi–Yoneya prefactor [Dobashi, Yoneya 2004] in the natural string basis $\{a^r_m\}$.

$$|V\rangle = P\, e^{\frac{1}{2}\sum_{m,n} N^{rs}_{mn}\delta^{IJ} a^{rI\dagger}_m a^{sJ\dagger}_n}|0\rangle.$$

Here I, J are $SU(4)$ flavour indices, r, s run within $1, 2, 3$ and refer to the first, second and third operator. The natural string basis is related to the BMN basis for $m > 0$ as follows

$$\alpha_m = \frac{a_m + i a_{-m}}{\sqrt{2}}, \alpha_{-m} = \frac{a_m - i a_{-m}}{\sqrt{2}}$$

The Neumann matrices are given as

$$N^{rs}_{m,n} = \frac{1}{2\pi}\frac{(-1)^{r(m+1)+s(n+1)}}{x_s\omega_{rm} + x_r\omega_{sn}}\sqrt{\frac{x_r x_s(\omega_{rm} + \mu x_r)(\omega_{sn} + \mu x_s)s_{rm}s_{qn}}{\omega_{rm}\omega_{sn}}},$$

$$N^{rs}_{-m,-n} = -\frac{1}{2\pi}\frac{(-1)^{r(m+1)+s(n+1)}}{x_s\omega_{rm} + x_r\omega_{sn}}\sqrt{\frac{x_r x_s(\omega_{rm} - \mu x_r)(\omega_{sn} - \mu x_s)s_{rm}s_{qn}}{\omega_{rm}\omega_{sn}}},$$

where m, n are always meant positive, $s_{1m} = 1$, $s_{2m} = 1$, $s_{3m} = -2\sin(\pi m y)$, $x_1 = y$, $x_2 = 1 - y$, $x_3 = -1$, the frequencies are $\omega_{r,m} = \sqrt{m^2 + \mu^2 x_r^2}$,, and the expansion parameter is $\mu = \frac{1}{\sqrt{\lambda'}}$. The Dobashi-Yoneya prefactor we are using is the prefactor supported with positive modes only:

$$P = \sum_{m>0}\sum_{r,I}\frac{\omega_r}{\mu\alpha_r} a^{Ir\dagger}_m a^{Ir}_m.$$

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String result

Due to the flavour structure of C_{123} the only combinations of terms from the exponent that could contribute are $N^{12}_{n_1 n_2} N^{23}_{n_2 n_3} N^{31}_{n_3 n_1}$. The leading order contribution is

$$C^0_{123} = \frac{1}{\pi^2} \frac{\sqrt{J}}{N} \frac{n_3^2 y^{3/2} (1-y)^{3/2} \sin^2(\pi n_3 y)}{(n_3^2 y^2 - n_1^2)(n_3^2 (1-y)^2 - n_2^2)}$$

The next-order coefficient in the expansion

$$C_{123} = C^0_{123} \left(1 + \lambda' c^1_{123}\right),$$

where $c^1_{123} \equiv \frac{C^1_{123}}{C^0_{123}}$ is

$$c^1_{123} = -\frac{1}{4} \left(\frac{n_1^2}{y^2} + \frac{n_2^2}{(1-y)^2} + n_3^2 \right).$$

Let us compare this calculation to the field theory calculation.

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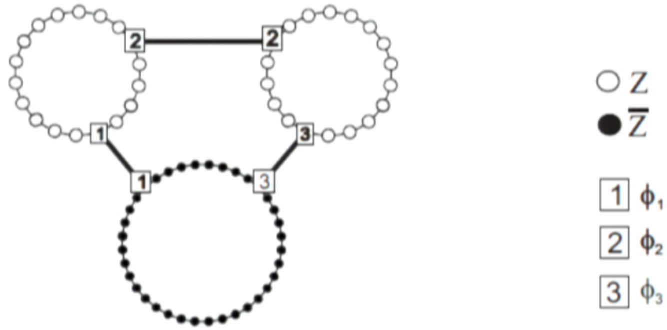
Leading Order

The tree-level diagram is shown below:



and evaluates in the leading order to

$$N\sqrt{J_1 J_2 J}\sum_{l_1,l_2}\cos\frac{\pi(2l_1+1)}{J_1+1}\cos\frac{\pi(2l_2+1)}{J_2+1}\cos\frac{\pi(2(l_1+l_2)+1)}{J+1},$$

which after the $1/J$ expansion and the due normalization of the operator to unity yields

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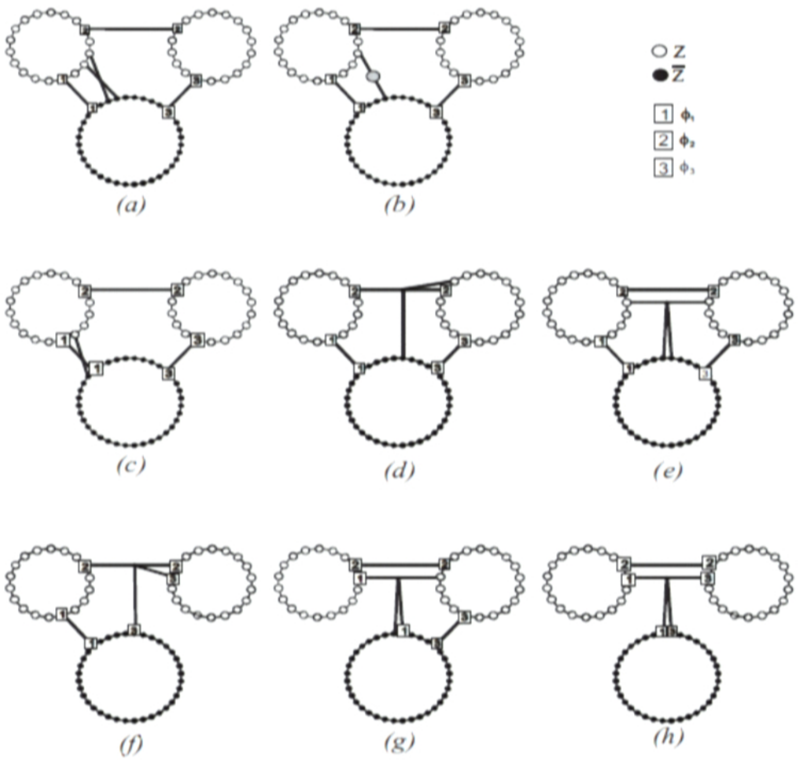
corresponding exactly to the ST result above.

One Loop

At the one loop level we estimate all possible insertions of the interaction terms of the Hamiltonian $H_2 = \frac{\lambda}{8\pi^2}(I - P)$ depicted below:



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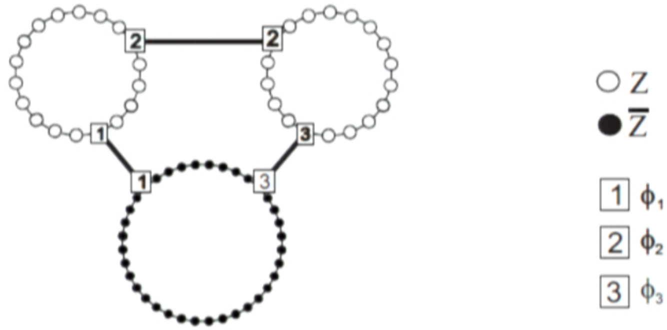

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Result

The three-point correlation function for all dynamical BMN operators matches precisely the perturbative weakly coupled planar field theory and the Penrose limit of the strongly coupled string field theory at one loop level in the Frolov–Tseytlin limit.

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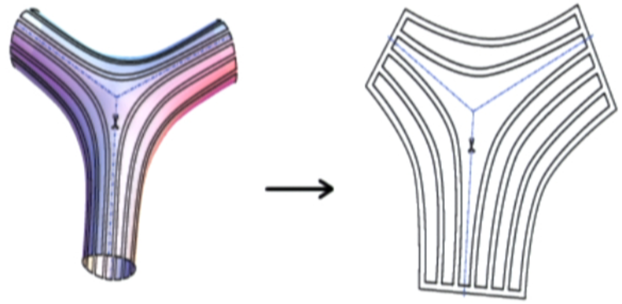
Part II. Integrability

Here the Gromov-Vieira tailoring procedure is illustrated, for details see *[Gromov, Vieira 1205.5288]*:



The procedure expresses perturbative magnon dynamics in terms of concise formulae, which is advantageous for large number of magnons reducing greatly the computer- and man-power consumption of the calculation, yet still needs a direct perturbative verification, due to the complexity of the "integrability" result derivation. Figure courtesy of Kolya Gromov.

Integrability assists summation

Integrability assists greatly to simplify the perturbative calculations for a large number of magnons. The procedure by Gromov-Vieira amounts to roughly 2^K complicatedness in as opposed to C_J^K complicatedness for direct perturbative approach. This is reached by working with Bethe states givenm in terms of sets rapidities $\{u_i\}, \{v_i\}, \{w_i\}$ directly and splitting the three-point function into the following structure

$$\sum_{\text{all Bethe roots' partitions}} \text{CUT} \times \text{FLIP} \times \text{NORM} \times \text{SEW}$$

The integrable conjecture has been tested on the $SU(2)$ up to one loop by *[Gromov Vieira 2012]*, I present here a test in the $SO(6)$ sector *[Grignani, Zayakin 1208.0100]*, to compare it with the results I have just presented above*[Grignani, Zayakin 1205.5279, 1204.3096]*.

The $SU(2)$ and $SO(6)$ common structure

The $SU(2)$ three-point function looks like

$$N_c C_{123} = \sum_{\substack{\alpha\cup\bar\alpha = u\\ \beta\cup\bar\beta = v\\ \gamma\cup\bar\gamma = w}} \sqrt{L_1 L_2 L_3}\ \mathtt{Cut}(\alpha,\bar\alpha)\mathtt{Cut}(\beta,\bar\beta)\mathtt{Cut}(\gamma,\bar\gamma)\times \mathtt{Flip}(\bar\alpha)\mathtt{Flip}(\bar\beta)\mathtt{Flip}(\bar\gamma)\times$$

$$\times\ \frac{1}{\sqrt{\mathtt{Norm}(u)\mathtt{Norm}(v)\mathtt{Norm}(w)}} \times \langle\alpha\bar\beta\rangle\langle\beta\bar\gamma\rangle\langle\gamma\bar\alpha\rangle\ .$$

We work in the "coordinate" normalization, where the $\mathtt{Cut}(\alpha,\bar\alpha)$ factor is organized as

$$\mathtt{Cut}(\alpha,\bar\alpha) = \left(\frac{a^{\bar\alpha}}{d^{\bar\alpha}}\right)^{L_1} \frac{f^{\alpha\bar\alpha} f_<^{\bar\alpha\bar\alpha} f_<^{\alpha\alpha}}{f_<^{uu}}\,,$$

the factors $\mathtt{Cut}(\beta,\bar\beta)$ and $\mathtt{Cut}(\gamma,\bar\gamma)$ being analogous to the expression above. The a, d, f, g factors are defined as $a(u_j) = u_j + iV_{a_j}/2$, $d(u_j) = u_j - iV_{a_j}/2$, $e(u) = \frac{a(u)}{d(u)}$, $f(u_i,u_j) = 1 + \frac{i}{(u_i-u_j)}$, $g(u_i,u_j) = \frac{i}{2(u_i-u_j)}$.

The flip factor may now be written as

$$\mathtt{Flip}(\bar\alpha) = (e^{\bar\alpha})^{L}_{\bar\alpha}\ \frac{g^{\bar\alpha - i/2}}{g^{\bar\alpha + i/2}}\frac{f_>^{\bar\alpha\bar\alpha}}{f_<^{\bar\alpha\bar\alpha}}\,,$$

analogous expressions work for $\mathtt{Flip}(\bar\beta)$ and $\mathtt{Flip}(\bar\gamma)$. The norm is

$$\mathrm{Norm}(u) = d^u a^u f_>^{uu} f_<^{uu} \frac{1}{g^{u+i/2} g^{u-i/2}} \det(\partial_j\phi_k)\,,$$

here $\partial_j = \frac{\partial}{\partial u_j}$ and the phases are $e^{i\phi_j} = e(u_j)^{L_u}\prod_{k\neq j} S^{-1}(u_j,u_k)$.

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$$\mathtt{Cut}(\alpha,\bar\alpha) = \left(\frac{a^{\bar\alpha}}{d^{\bar\alpha}}\right)^{L_1} \frac{f^{\alpha\bar\alpha} f_<^{\bar\alpha\bar\alpha} f_<^{\alpha\alpha}}{f_<^{uu}} ,$$

the factors $\mathtt{Cut}(\beta,\bar\beta)$ and $\mathtt{Cut}(\gamma,\bar\gamma)$ being analogous to the expression above. The a, d, f, g factors are defined as $a(u_j) = u_j + iV_{a_j}/2$, $d(u_j) = u_j - iV_{a_j}/2$, $e(u) = \frac{a(u)}{d(u)}$, $f(u_i,u_j) = 1 + \frac{i}{(u_i - u_j)}$, $g(u_i,u_j) = \frac{i}{2(u_i - u_j)}$.

The flip factor may now be written as

$$\mathtt{Flip}(\bar\alpha) = (e^{\bar\alpha})^{L}_{\bar\alpha} \frac{g^{\bar\alpha - i/2}}{g^{\bar\alpha + i/2}} \frac{f_>^{\bar\alpha\bar\alpha}}{f_<^{\bar\alpha\bar\alpha}} ,$$

analogous expressions work for $\mathtt{Flip}(\bar\beta)$ and $\mathtt{Flip}(\bar\gamma)$. The norm is

$$\mathrm{Norm}(u) = d^u a^u f_>^{uu} f_<^{uu} \frac{1}{g^{u+i/2} g^{u-i/2}} \det(\partial_j \phi_k) ,$$

here $\partial_j = \frac{\partial}{\partial u_j}$ and the phases are $e^{i\phi_j} = e(u_j)^{L_u} \prod_{k\neq j} S^{-1}(u_j,u_k)$.

Choice of Operators

We have mentioned that comparison of integrability results to anything else may be performed only for the non-extremal correlators. Above we chose the $SO(6)$ sector and each operator having two magnons to make our case non-extremal. Either for perturbation theory or for string field theory this was not a problem. For integrability at leading order it required a yet conjectural extension of the Escobedo-Gromov-Sever-Vieira technique to a different group-theoretical sector. To test integrability vs. strings at next-to-leading order in λ' we use the one-loop Gromov-Vieira expression which is known in the $SU(2)$ sector.

For our correlator to remain non-extremal, we choose the following three operators:

Operator	Magnons	Length
$\mathcal{O}_1$	$n_2, -n_2$	$J+2$
$\mathcal{O}_2$	$n_1, -n_1, n_4, -n_4$	$Jr+4$
$\mathcal{O}_3$	$n_3, -n_3$	$J(1-r)+2$

SFT result

The matrix element thus will be organized in terms of Neumann matrices as

$$\langle 1|\langle 2|\langle 3|H_3\rangle \sim N^{21}_{n_1 n_2} N^{21}_{-n_1 - n_2} N^{23}_{n_4 n_3} N^{23}_{-n_4 - n_3} + \text{combinatorics}$$

From where for

$$C_{123} = C^0_{123} + \lambda' C^1_{123}$$

(again assuming $n_1 = n_4$) we get

$$C^0_{123} = -\frac{16r\left(n_1^2 + 3n_2^2 r^2\right)\sin^2(\pi n_2 r)}{\pi^2\sqrt{J(1-r)r}\left(n_1^2 - n_2^2 r^2\right)^2},$$

as before for integrability, and

$$C^1_{123} = \frac{64\sin^2(\pi n_2 r)\left(5n_1^4(r-1)^2 + n_1^2 r^2\left(4n_2^2(r-1)^2 + 3n_3^2\right) + 3n_2^2 r^4\left(n_2^2(r-1)^2 + 3n_3^2\right)\right)}{J^{5/2}(r-1)^{5/2} r^{3/2}\left(n_1^2 - n_2^2 r^2\right)^2}.$$

This is different from the integrability-assisted result and therefore needs some explanation.

Discussion

For the first time in the $SO(6)$ sector we have explicitly demonstrated that for the three-point functions

- SFT at strong coupling identical with perturbation theory at small coupling in the Frolov-Tseytlin limit at one loop.
- Integrability-assisted resummation a la Escobedo-Gromov-Sever-Vieira can be successfully generalized to the $SO(6)$ case and is shown to be identical with SFT and perturbation theory.
- Yet the SFT-integrability correspondence has a discrepancy in the one-loop $SU(2)$ sector.

Given these correspondences and discrepancies, discussion can be raised:

- To which extent may the PT-SFT equalities be understood as coincidences?
- How essential is the role of Frolov-Tseytlin limit? To which order will the equalities hold beyond it?
- What is the reason of the discrepancy between the one-loop SFT and integrability results?