Title: Loop quantization of a weak-coupling limit of Euclidean gravity

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Abstract: I will describe recent work in collaboration with Adam Henderson, Alok Laddha, and Madhavan Varadarajan on the loop quantization of a certain $G_{\mathrm{N}}\rightarrow 0$ limit of Euclidean gravity, introduced by Smolin. The model allows one to test various quantization choices one is faced with in loop quantum gravity, but in a simplified setting. The main results are the construction of finite-triangulation Hamiltonian and diffeomorphism constraint operators whose continuum limits can be evaluated in a precise sense, such that the quantum Dirac algebra of constraints closes nontrivially and free of anomalies. The construction relies heavily on techniques of Thiemann's QSD treatment, and lessons learned applying such techniques to the loop quantization of parameterized scalar field theory and the diffeomorphism constraint in loop quantum gravity. I will also briefly discuss the status of the quantum constraint algebra in full LQG, and how some of the lessons learned from the present model may guide us in that setting.

Loop Quantization of Abelian Euclidean Gravity

Casey Tomlin
with Adam Henderson, Alok Laddha, and Madhavan Varadarajan

IGC PSU & AEI

What I am going to say

- Motivation: Constraint algebra & LQG dynamics
- The $U(1)^3$ model
- Constraint algebra revisited
- Hints from toy models
- A few details
- Outlook

Motivation: Constraint algebra and dynamics

- Classically, constraints generate the "hypersurface deformation" algebra

$$\begin{aligned} \{D[\vec{N}], D[\vec{M}]\} &= D[\mathcal{L}_{\vec{N}}\vec{M}] \\ \{D[\vec{N}], H[N]\} &= H[\mathcal{L}_{\vec{N}}N] \\ \{H[N], H[M]\} &= D[q^{ab}(M\partial_b N - N\partial_b M)] \end{aligned} \qquad (\star)$$

encoding 4D spacetime covariance in 3+1 form.[1]

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- **Philosophy:** Representing $(\star)$ via quantum operators is *a defining property* to be satisfied by *any* (canonical) theory of quantum geometry, and defines a notion of *quantum spacetime covariance.* E.g.,

$$[\hat{H}[N], \hat{H}[M]] = \mathrm{i}\hbar\hat{D}[\hat{\vec{\omega}}] \qquad (\triangle)$$

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- Practically, demanding $(\triangle)$ can reduce quantization ambiguities

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Constraint algebra and dynamics—LQG

"State of the art"—QSD (Thiemann 1996)

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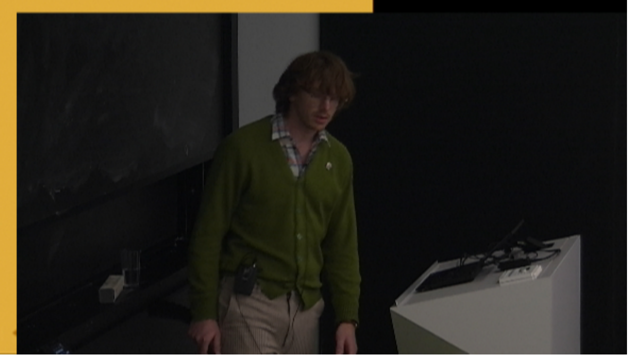

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$$[\hat{H}[N], \hat{H}[M]]\Phi = 0.$$

  - However, the most straightforward quantization of the RHS gives $\hat{D}[\hat{\vec{\omega}}]\Phi = 0$, so no inconsistency.

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Smolin's Weak-Coupling Limit[4]

Euclidean, self-dual, first order action:

$$S[e,\omega] = \frac{1}{G_{\mathrm{N}}}\int \mathrm{d}^4x\ |e| e_I^\mu e_J^\nu R_{\mu\nu}{}^{IJ}[\omega], \qquad \omega_\mu{}^{IJ} = \tfrac{1}{2}\epsilon^{IJ}{}_{KL}\omega_\mu{}^{KL}$$

Define $A = G_{\mathrm{N}}^{-1}\omega$, take $G_{\mathrm{N}} \to 0$, 3+1 split, get

$$S[A,E] = \int \mathrm{d}t \left(\int_\Sigma \mathrm{d}^3x\ E_i^a \dot{A}_a^i - G[\Lambda] - D[\vec{N}] - H[N]\right)$$

where

$G[\Lambda] = \int \mathrm{d}^3x\ \Lambda^i \partial_a E_i^a$	Three **independent** U(1) Gauss law constraints
$D[\vec{N}] = \int \mathrm{d}^3x\ E_i^a \mathcal{L}_{\vec{N}} A_a^i$	Diffeomorphism constraint
$H[N] = \frac{1}{2}\int \mathrm{d}^3x\ N\epsilon^{ijk} E_i^a E_j^b F_{ab}^k[A]$	Euclidean Hamiltonian constraint with **Abelian** curvature $F_{ab}^i := 2\partial_{[a}A_{b]}^i$

Subalgebra of D and H again generates the HD algebra

[4] CQG **9** 883 1992

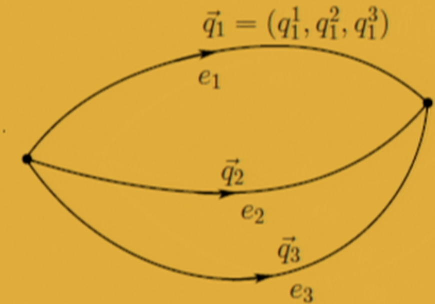
Loop quantum kinematics

Holonomies & Fluxes

$$h_e^{q^i}[A^i] = \exp\left(\mathrm{i}\kappa q^i \int_e A^i\right), \quad q^i \in \mathbb{Z}$$

$$f_i[S] = \int_S E_i^a \eta_{abc} \mathrm{d}x^b \wedge \mathrm{d}x^c$$

'Charge' networks $|c\rangle$—graphs embedded in Σ with edges labeled by 3 integers ($\mathrm{U}(1)^3$ representations)



These span a dense subset $\mathcal{D}$ whose completion wrt $\langle c|c'\rangle = \delta_{c,c'}$ is the kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}$.
$\mathrm{U}(1)^3$ gauge invariance—For each $i = 1, 2, 3$ separately, and at each vertex v, the sum of the charges on (outgoing) edges vanishes:

$$\sum_{e_I \cap \{v\}} q_I^i = 0$$

(Finite) Diffeomorphisms: For $\phi \in \mathrm{Diff}(\Sigma)$, $\hat{U}(\phi)|c\rangle := |\phi \cdot c\rangle$.

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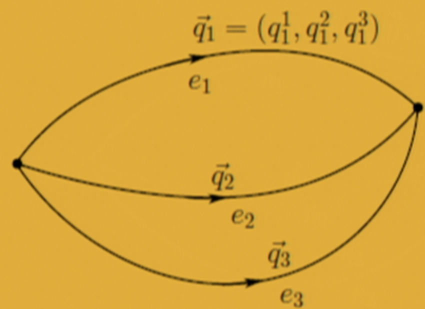

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Local operator quantization in LQG

- Given a phase space function $O[A, E]$ in terms of the local fields A, E, approximate by $O_\delta[h, f]$ in terms of holonomies and fluxes such that $\lim_{\delta \to 0} O_\delta[h, f] = O[A, E]$.

- If $\hat{O}_\delta[h, f]$ is well-defined on $\mathcal{D} \subset \mathcal{H}_{\mathrm{kin}}$, its action can be computed: $\hat{O}_\delta[h, f]|c\rangle$. The continuum limit $\lim_{\delta \to 0} \hat{O}_\delta|c\rangle$ is generally not well-defined.

- Let $\mathcal{D}^*$ be the algebraic dual to $\mathcal{D}$ so that every $\Psi \in \mathcal{D}^*$ is a linear map from $\mathcal{D}$ to $\mathbb{C}$. For every pair (Ψ, c), compute the one-parameter family of complex numbers $(\Psi|\hat{O}_\delta|c\rangle$. The continuum limit action is defined to be

$$\lim_{\delta \to 0} (\Psi|\hat{O}_\delta|c\rangle$$

The set of Ψ chosen determines the limit operator (c.f. URS topology)

Constraint algebra revisited

The density 1 $H[N] = \frac{1}{2}\int \mathrm{d}^3x\ N q^{-1/2} \epsilon^{ijk} E_i^a E_j^b F_{ab}^k$ must be regularized. An obvious choice is (note overall δ independence)

$$H_\delta[N] = \frac{1}{2} \sum_{\triangle \in T(\delta)} \delta^3 \cdot N(v_\triangle) \epsilon^{ijk} \cdot \frac{h^k_{\triangle_{ab}} - 1}{\mathrm{i}\delta^2} \cdot \frac{E_i(S^a_\triangle)}{\delta^2} \cdot \frac{E_j(S^b_\triangle)}{\delta^2} \cdot \delta^3 \frac{1}{\sqrt{\det q}|_\delta}$$

Its action on $|c\rangle$ is (schematically)

New vertices are trivial due to edge tangent structure and volume operator. Thus

$$\hat{H}_{\delta'}[M] \hat{H}_\delta[N] |c\rangle = M(v) N(v) \cdot$$

$+ \cdots$

Constraint algebra revisited II

- The continuum commutator is defined via diffeomorphism-invariant states; e.g.,

$$(\Psi^{c'}_{\mathrm{diff}}| := \sum_{\varphi\in \mathrm{Diff}(\Sigma)} \langle \varphi \cdot c'| \qquad \Rightarrow \qquad (\Psi^{c'}_{\mathrm{diff}}|\hat{U}(\phi) = (\Psi^{c'}_{\mathrm{diff}}|$$

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- Every term is of the form

$$M(v)N(v)(\Psi^{c'}_{\mathrm{diff}}|(\hat{U}(\phi) - 1)\Big|$$


$$\Big\rangle = 0$$

Constraint algebra revisited III

- Extend the set of Ψ_{diff} to include diffeo-non-invariant states and recheck
- Lewandowski-Marolf habitat:

$$(\Phi_f^{c'}| = \sum_{\varphi \in \mathrm{Diff}(\Sigma)} f(\varphi(v_1), \ldots, \varphi(v_n)) \langle \varphi \cdot c'|, \qquad f : \Sigma^n \to \mathbb{C}$$

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$$M(v)N(v)(\Phi_f^{c'}|(\hat{U}(\phi_\delta) - 1)\,|HHc\rangle = M(v)N(v)\,((f \circ \phi_\delta)(v) - f(v))$$

  But f is smooth, and $\lim_{\delta \to 0} \phi_\delta = \mathrm{id}$, so

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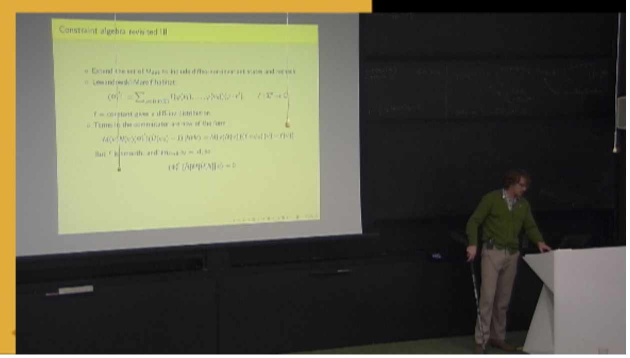

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- If there were an overall factor of δ, one could conceivably get ∂f, but not $(M\partial N - N\partial M) \Rightarrow \hat{H}$ must move vertices.

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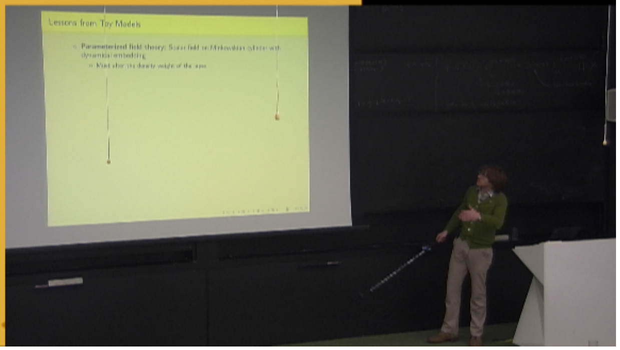

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  - Strategy: Quantize a regularized operator $\hat{D}_\delta[\vec{N}]$ on $\mathcal{H}_{\mathrm{kin}}$ such that

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  This guarantees that $(\star)$ holds in the continuum limit $\delta \to 0$ (on the LM habitat).

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    This guarantees that $(\star)$ holds in the continuum limit $\delta \to 0$ (on the LM habitat).
  - Curvature operator on $\mathcal{H}_{\rm kin}$ must be state-dependent, and requires non-perturbative corrections:

$$\hat{F}^i_\delta = \frac{\mathrm{tr}(h_\Box \tau^i)}{\delta^2} + \frac{2\mathrm{i}}{3\ell_{\rm P}^2} \frac{\mathrm{tr}(h_\Box - \mathbf{1})}{\delta^2} E_i(S_\delta)$$

    $2^{\rm nd}$ term is higher-order in δ, so $\lim_{\delta\to 0} F^i_\delta = F^i$ classically.

U(1) Ingredients

Choose the density weight to get overall factors of δ so that derivatives could be generated.

Look for a geometric interpretation of H, and ask that the finite-triangulation operator mimic that action by exploiting the availability of non-perturbative corrections.

Determine how to quantize $D[\vec{\omega}]$

Find a controllable set of distributions à la LM that allow the continuum limit to be taken non-trivially

Density weight

For $\alpha \in \mathbb{R}$, $q^{-\alpha} \sim E^{-3\alpha} \sim \delta^{6\alpha}$

$$H_\delta[N] \sim \delta^3 \cdot \frac{F^k_{ab}|_\delta}{\mathrm{i}\delta^2} \cdot \frac{E_i(S^a_\triangle)}{\delta^2} \cdot \frac{E_j(S^b_\triangle)}{\delta^2} \cdot \frac{\delta^{6\alpha}}{q^\alpha_\delta} \quad \Rightarrow \quad \alpha = \tfrac{1}{3}$$

gives $H_\delta = \delta^{-1} \times H^{\mathrm{reg}}_\delta$.

Note:

- Now the lapse has density weight $-\frac{1}{3}$
- Classically, density weight α leads to the RHS

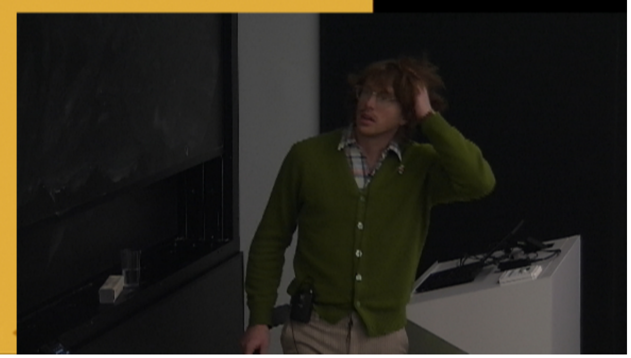
$$D[\vec{\omega}] = \int \mathrm{d}^3x \; q^{1-2\alpha} q^{ab} \left(N\partial_a M - M\partial_a N\right) F^i_{bc} E^c_i \sim \delta^{12\alpha-5}$$

$$\alpha = \tfrac{1}{3} \quad \Rightarrow \quad D_\delta[\vec{\omega}] \sim \frac{1}{\delta}$$

Can possibly generate diffeos... Note also that

$$\alpha = \tfrac{1}{2} \quad \Rightarrow \quad D_\delta[\vec{\omega}] \sim \delta$$

so trivial continuum limit is no surprise.



Geometric interpretation

- Let $N_i^a := Nq^{-1/3}E_i^a$. Classically this "electric shift" is a (density weight zero) vector field. The action of H on A can be written

$$\{A_a^i, H[N]\} = Nq^{-1/3}\epsilon^{ijk}E_j^b F_{ab}^k = -\epsilon^{ijk}\mathcal{L}_{\vec{N}_j}A_a^k + \partial_a\left(\epsilon^{ijk}N_j^b A_b^k\right)$$

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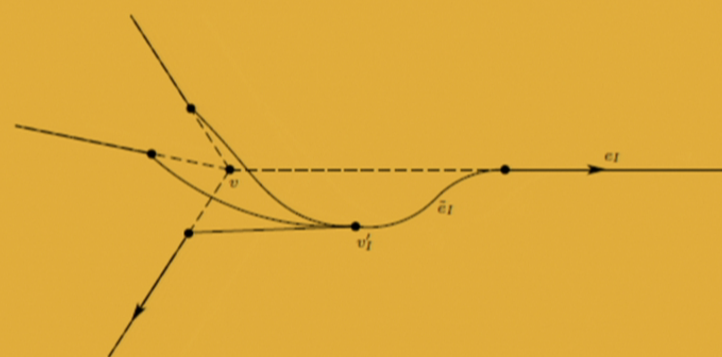
  - First term generates "diffeomorphisms" in the direction of a triad-dependent vector field.
  - Second term is a gauge transformation.
- This translates into an action on charge networks:

$$\hat{H}_{\delta,v}[N]|c\rangle \sim N(v)\lambda_q^{-1/3}\sum_I\sum_i q_I^i \frac{1}{\delta}(|c_{I,i,\delta}\rangle - |c\rangle)$$



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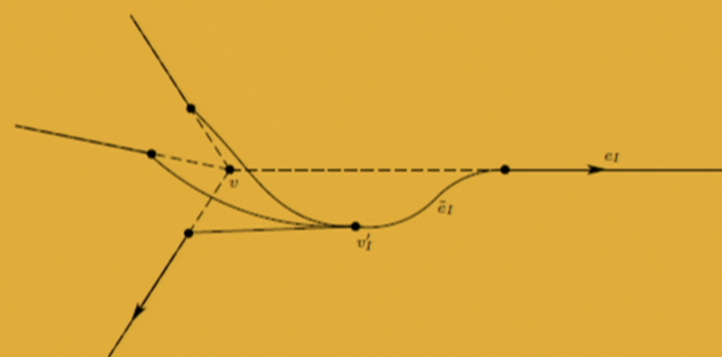
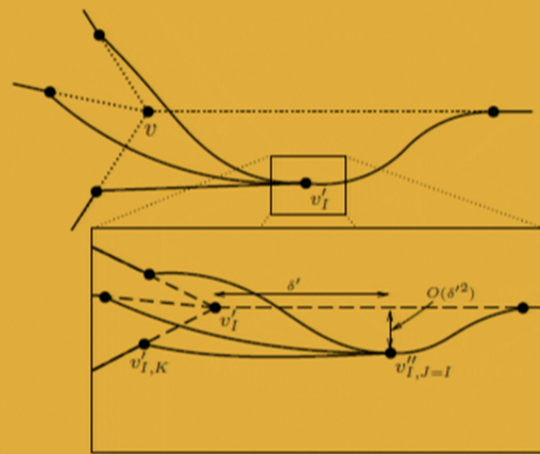


- Non-trivial vertex is moved, leaving the original vertex trivial $\Rightarrow$ 2nd $\hat{H}$ acts at displaced vertex!

Commutator and continuum limit



To take the continuum limit, we require a habitat. Construct a set B_{VSA} of charge network bras based on a given charge network c with non-trivial vertex set $V(c)$

$$B^c_{\mathrm{VSA}} \ni c' \sim \prod_\alpha (\varphi_\alpha \cdot \hat{H}_\delta) c \qquad \rightarrow \qquad (\Psi^f_{B^c_{\mathrm{VSA}}}| := \sum_{\tilde{c} \in B^{c'}_{\mathrm{VSA}}} f(V(\tilde{c})) \langle \tilde{c}|$$

Then

$$(\Psi^f_{B^{c'}_{\mathrm{VSA}}} | [\hat{H}[N], \hat{H}[M]] | c\rangle \sim_v \sum_I \sum_i (q^i_I)^2 \lambda_q^{-1/3}(v) \lambda_q^{-1/3}(v_I) \, (M\partial_I N - N\partial_I M) \, \partial_I f$$

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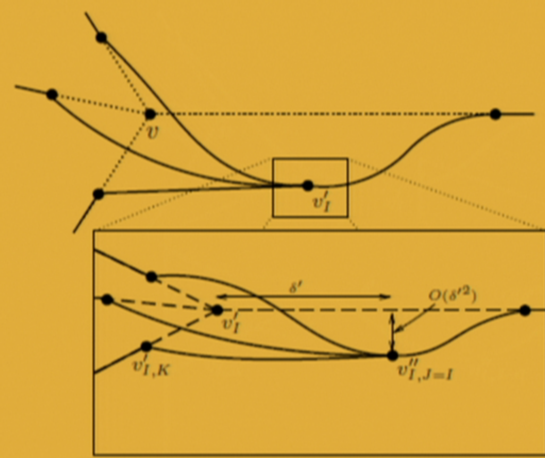

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RHS

Possible problem: Straightforward quantization of

$$D[\vec{\omega}] = \int \mathrm{d}^3x \; q^{-2/3} E_i^a E_i^b \, (N\partial_a M - M\partial_a N) \, F_{bc}^j E_j^c$$

would give an action proportional to $\lambda_q^{-2/3}(v)$, whereas LHS contains $\lambda_q^{-1/3}(v)\lambda_q^{-1/3}(v')$.

Solution: Consider the diffeomorphism generator smeared with electric shift $N_i^a = q^{-\alpha} N E_i^a$:

$$D[\vec{N}_i] = \int \mathrm{d}^3x \; N_i^a F_{ab}^j E_j^b$$

Remarkable identity:

$$\textstyle\sum_i \{D[\vec{N}_i], D[\vec{M}_i]\} = (2\alpha - 1)\, D[\vec{\omega}] \qquad (\blacktriangledown)$$

Instead quantize $D[\vec{\omega}]$ as

$$\hat{D}[\vec{\omega}] := \frac{1}{2\alpha - 1} \frac{1}{\mathrm{i}\hbar} \textstyle\sum_i [D[\vec{N}_i], D[\vec{M}_i]]$$

Naturally gives the same structure as $[\hat{H}, \hat{H}]$ and matches in continuum limit

Interesting feature: $(\blacktriangledown)$ trivializes for the usual density 1 choice $\alpha = \frac{1}{2}$

Summary (hopefully I did not bore you into slumber)

- This $G_\mathrm{N} \to 0$ theory is a nice toy model for testing LQG constructions
- Applying the lessons of previous work
  - QSD—general framework and Thiemann tricks
  - LM—habitats
  - PFT—kinematically singular operators necessary
  - diff constraint—geometric interpretation is key

  leads to an off-shell representation of the HD algebra

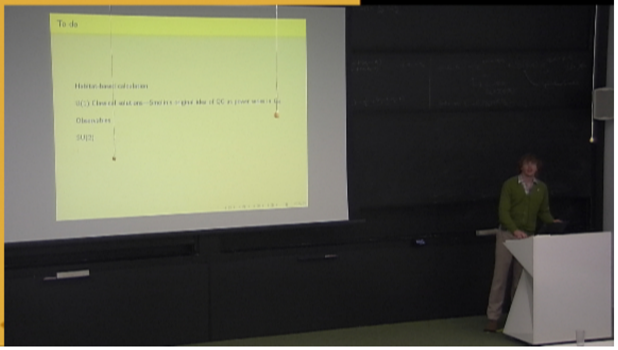
To do

Habitat-based calculation

U(1) Classical solutions—Smolin's original idea of QG as power series in G_N

Observables

SU(2)



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