Title: Classifying fractionalization: symmetry classification of gapped Z2 spin liquids in two dimensions

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Abstract: Quantum number fractionalization is a remarkable property of topologically ordered states of matter, such as fractional quantum Hall liquids, and quantum spin liquids. For a given type of topological order, there are generally many ways to fractionalize the quantum numbers of a given symmetry. What does it mean to have different types of fractionalization? Are different types of fractionalization a universal property that can be used to distinguish phases of matter? In this talk, I will answer these questions, focusing on a simple class of topologically ordered phases, namely two-dimensional gapped Z2 spin liquids, and I will present a symmetry classification of these phases. I will
also discuss efforts in progress to find microscopic models realizing different symmetry classes.

Classifying fractionalization:
Symmetry classification of gapped Z_2 spin liquids in two dimensions

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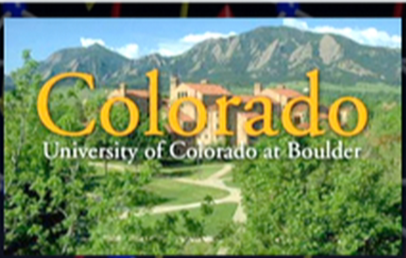
Michael Hermele



Andrew Essin & MH, PRB **87**, 104406 (2013)
Hao Song & MH, in progress

Perimeter Institute
March 26, 2013

the David & Lucile Packard FOUNDATION

Thanks to...

- Andrew Essin (postdoc @ Boulder → postdoc @ Caltech IQI)
- Hao Song (student @ Boulder)

the David & Lucile Packard FOUNDATION

Themes / Questions

1. What is a quantum phase of matter?
2. What is quantum number fractionalization?

Goal: try to better answer these questions by developing classifications.

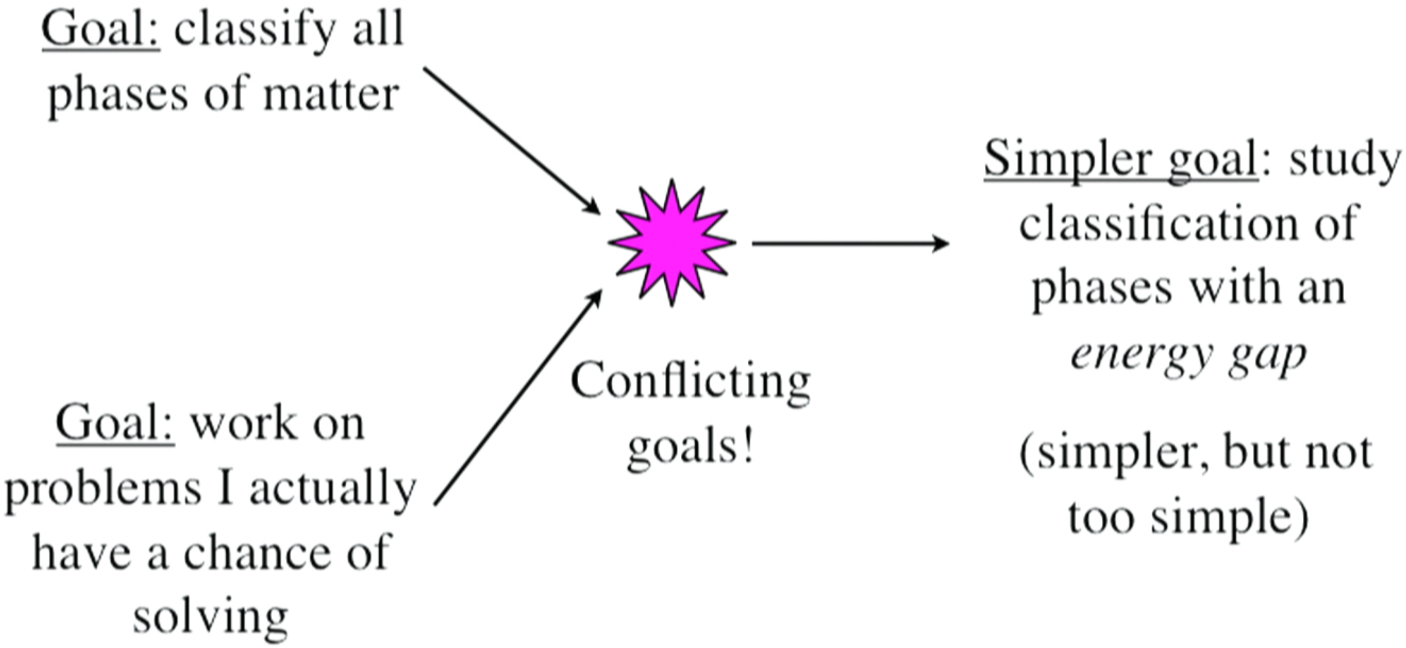
Outline

1. Motivation/background
2. Symmetry classification for non-point group symmetry
3. General symmetry classification
4. Realization of (some) symmetry classes in microscopic models

} with Andrew Essin (items 2–3)

} with Hao Song and Andrew Essin (item 4)

Quantum phases of matter



<u>Goal:</u> classify all phases of matter

<u>Goal:</u> work on problems I actually have a chance of solving

Conflicting goals!

<u>Simpler goal</u>: study classification of phases with an *energy gap*

(simpler, but not too simple)

Gapped quantum matter (no symmetry)

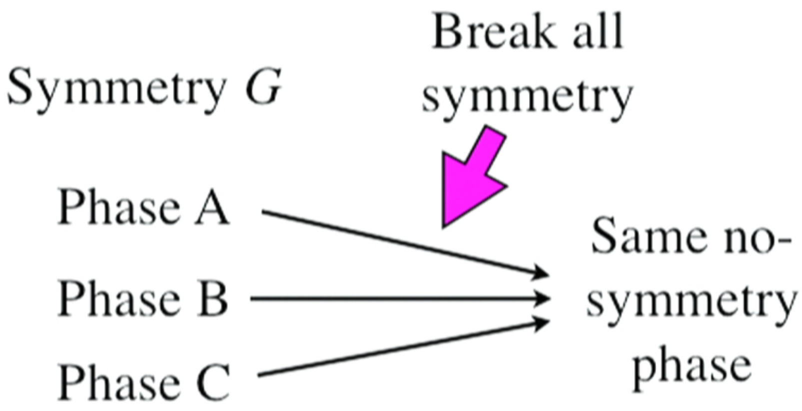
- Throughout this talk: consider local bosonic models (*i.e.* generalized spin models with finite-range interactions)

Energy gap, no symmetry present

- d=1: Only one trivial phase
- d=2:
  - Trivial phase
  - Kitaev E_8 state
  - Topologically ordered phases (anyons)
  - More?
- d=3:
  - Trivial phase
  - Stack of d=2 states
  - Topologically ordered phases (point and line “anyons”)
  - More?

Gapped quantum matter with symmetry

- Assume no spontaneous symmetry breaking.
- Notion of *symmetry enrichment*:



Symmetry G

Break all symmetry

Phase A

Phase B

Phase C

Same no-symmetry phase

- For a fixed no-symmetry phase and fixed symmetry group G, we say that A, B, C are distinct *symmetry enrichments*.

Gapped quantum matter with symmetry

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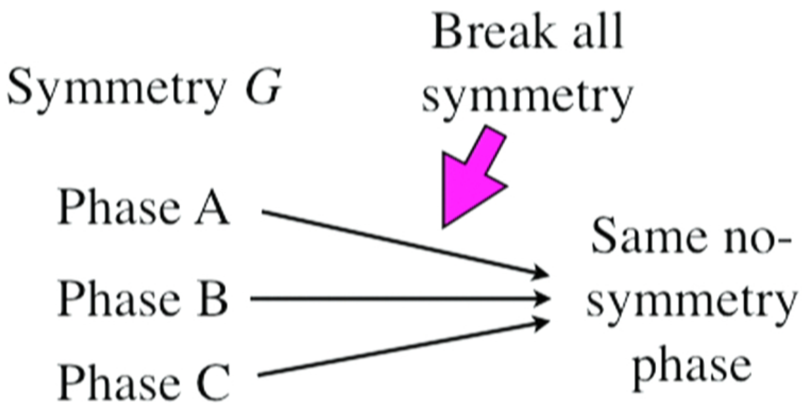

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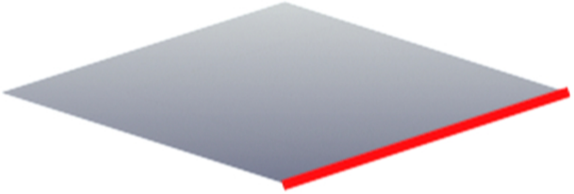
Symmetry protected topological (SPT) phases

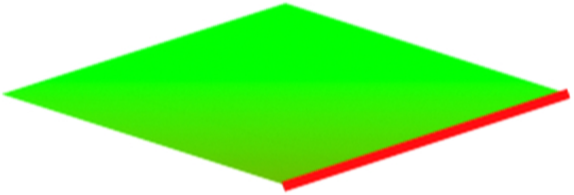
- SPT phases are symmetry enrichments of the trivial phase
- Classic examples:
  - Haldane phase of S=1 chain
  - Z_2 topological band insulators (d=2,3)
    (Kane & Mele; J. E. Moore & Balents; R. Roy; Fu, Kane & Mele)
- Recently, many more examples + developing systematic understanding



- Key physical property: non-trivial end/edge/surface states that are gapless, spontaneously break symmetry, or are otherwise non-trivial ... this is most robust for internal symmetries (*e.g.* time reversal, spin rotation)

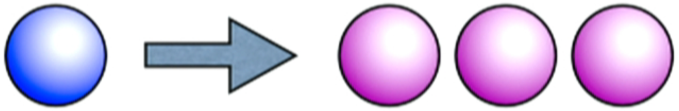
Symmetry enriched topological (SET) phases

- SET phases are symmetry enrichments of topologically ordered phases
- Many examples of SET phases in models, but systematic understanding of how to classify SET phases is less developed than for SPT phases
- But many very recent works: M. Levin & A. Stern; A. Mesaros & Y. Ran; A. M. Essin & MH; L.-Y. Hung & X.-G. Wen; L.-Y. Hung and Y. Wan; Y.-M. Lu & A. Vishwanath; X.-G. Wen; C. Wang & T. Senthil



- In *d*=2 SET phases: anyon excitations → non-trivial bulk properties. *Space group symmetry* is thus more important than for SPT phases.

Fractionalization

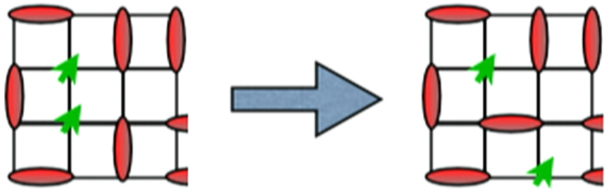


Charge e electron

Charge $e/3$ Laughlin quasiparticles

$\nu = 1/3$ Fractional quantum Hall liquid

Gapped Z_2 spin liquid



S=1 “magnon”

S=1/2 spinons

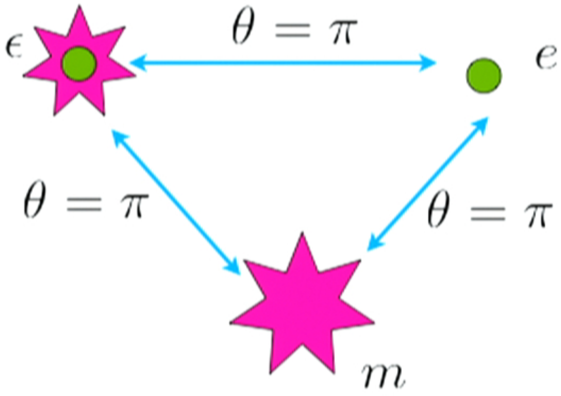
- What are distinct types of fractionalization?
- How to describe/classify?
- Can classifying fractionalization help classify SET phases?

Gapped Z_2 spin liquids in 2d

- Gapped Z_2 spin liquids = Z_2 topological order + no spontaneous symmetry breaking

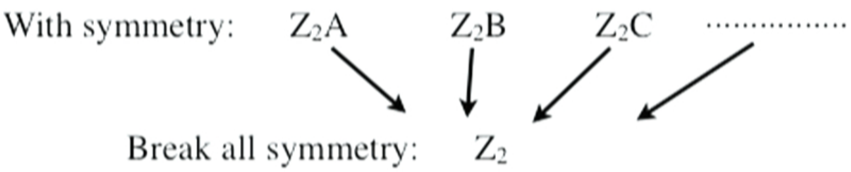
Z_2 topological order: particle types (anyons)

- Two bosons (e and m). One fermion (ε). Also one "trivial" boson (1).
    - Fusion rules: $\epsilon \times \epsilon = m \times m = e \times e = 1$
      $\epsilon \times m = e \ , \ \epsilon \times e = m \ , \ e \times m = \epsilon$
    - Mutual statistics:



Symmetry classification

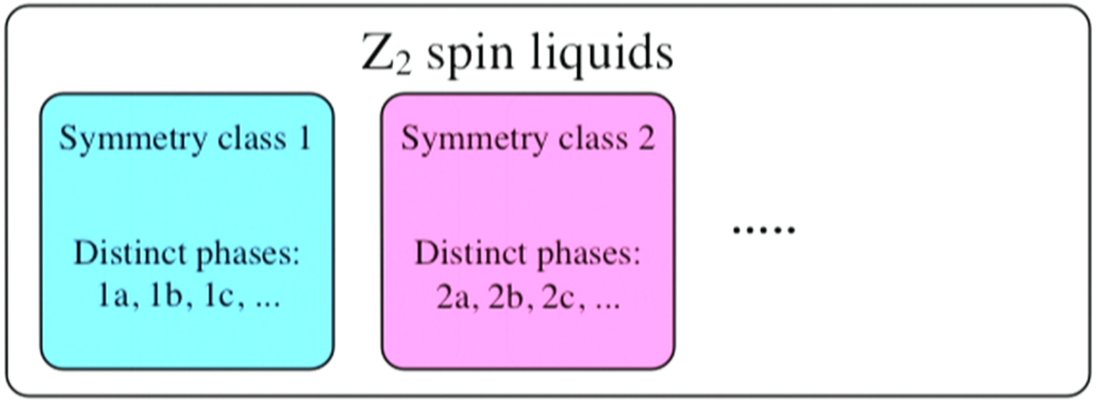
- In presence of symmetry, there are many gapped Z_2 spin liquids (X.-G. Wen, ...)



With symmetry: Z_2A Z_2B Z_2C ....

Break all symmetry: Z_2

- Can we classify such distinct Z2 spin liquids?
- Simpler: symmetry classification



Z_2 spin liquids

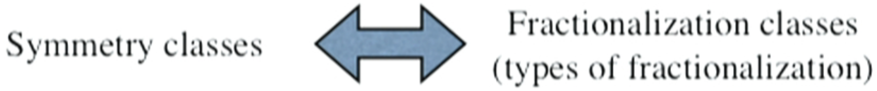
Symmetry class 1

Distinct phases:
1a, 1b, 1c, ...

Symmetry class 2

Distinct phases:
2a, 2b, 2c, ...

.....

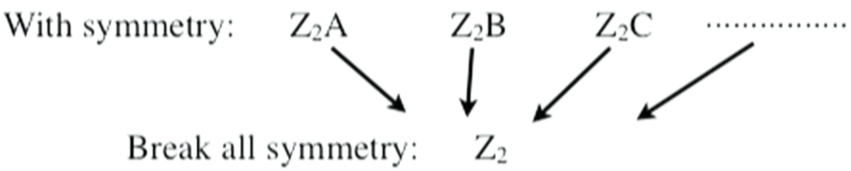


Symmetry classes

Fractionalization classes
(types of fractionalization)

Symmetry classification

- In presence of symmetry, there are many gapped Z_2 spin liquids (X.-G. Wen, ...)



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Break all symmetry: Z_2

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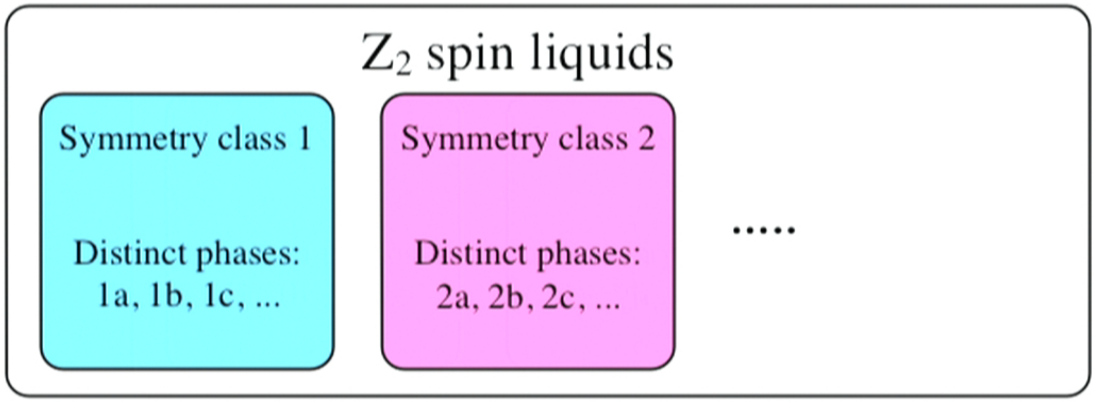
- Simpler: symmetry classification



Z_2 spin liquids

Symmetry class 1

Distinct phases:
1a, 1b, 1c, ...

Symmetry class 2

Distinct phases:
2a, 2b, 2c, ...

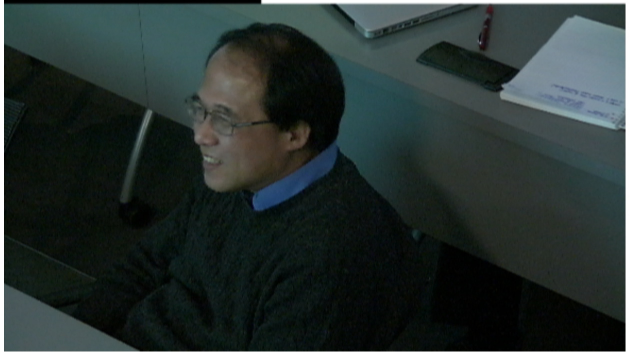
.....

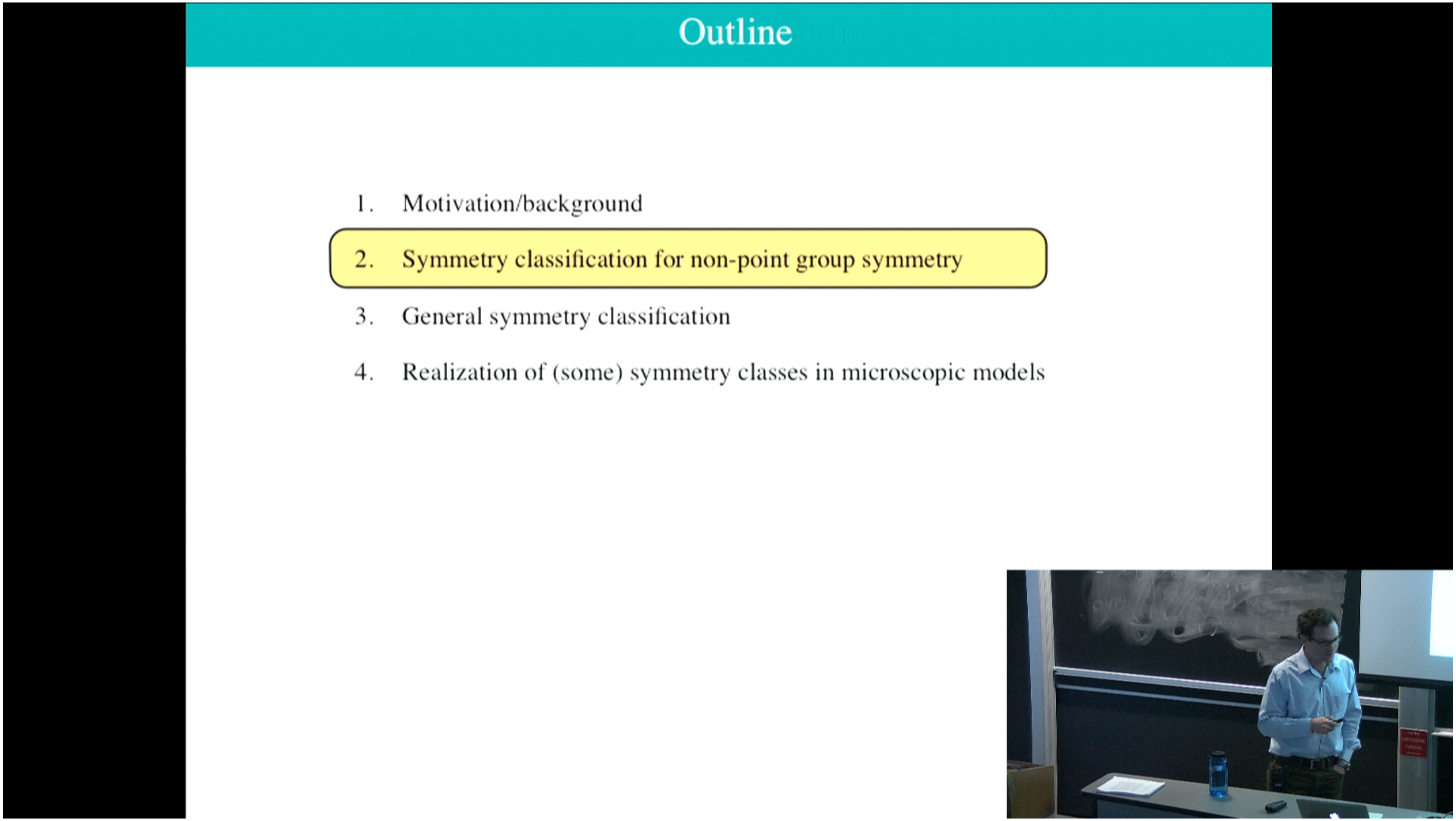
Symmetry classes ⟷ Fractionalization classes (types of fractionalization)

Not in this talk: symmetry classes “beyond fractionalization.”

Prior work

- Projective symmetry group classification (Xiao-Gang Wen, 2001)
- Ying Ran & Xiao-Gang Wen, 2002, unpublished
- Alexei Kitaev, Ann. Phys. 2006, Appendix F





Outline

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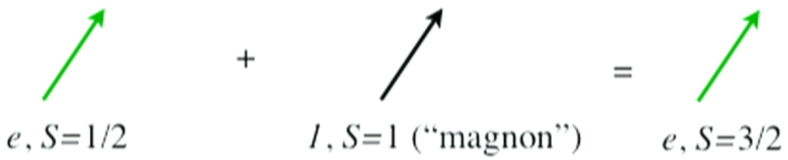
Spin rotation symmetry

- *e*-particle could have S=0, 1/2, 1, 3/2, ...



- Only integer vs. half-odd-integer spin matters → two fractionalization classes
- Therefore, we *don't* want to classify by irreducible representations. Coarser classification is needed.

e	m	ε
1/2	0	1/2
0	1/2	1/2
1/2	1/2	0
0	0	0

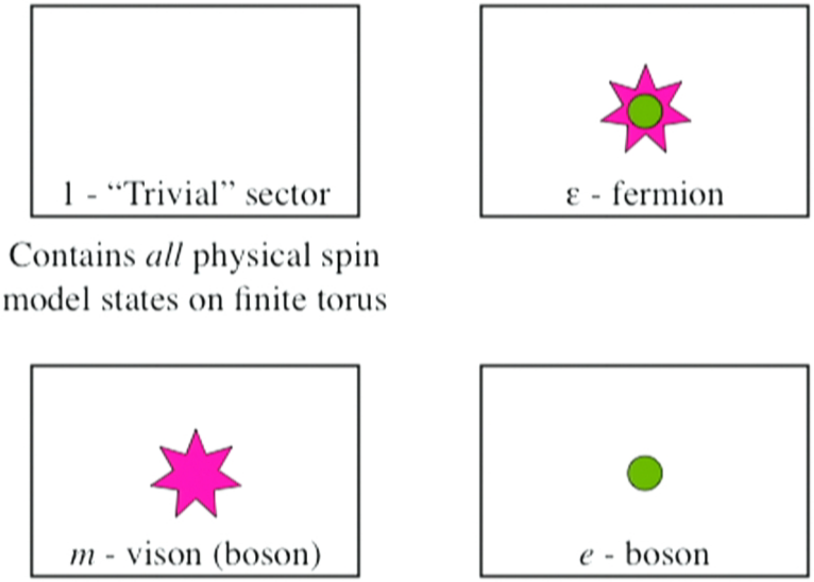
S mod 1

Same, under *relabeling*
e↔m

- Three symmetry classes if *only* SO(3) spin rotation symmetry present

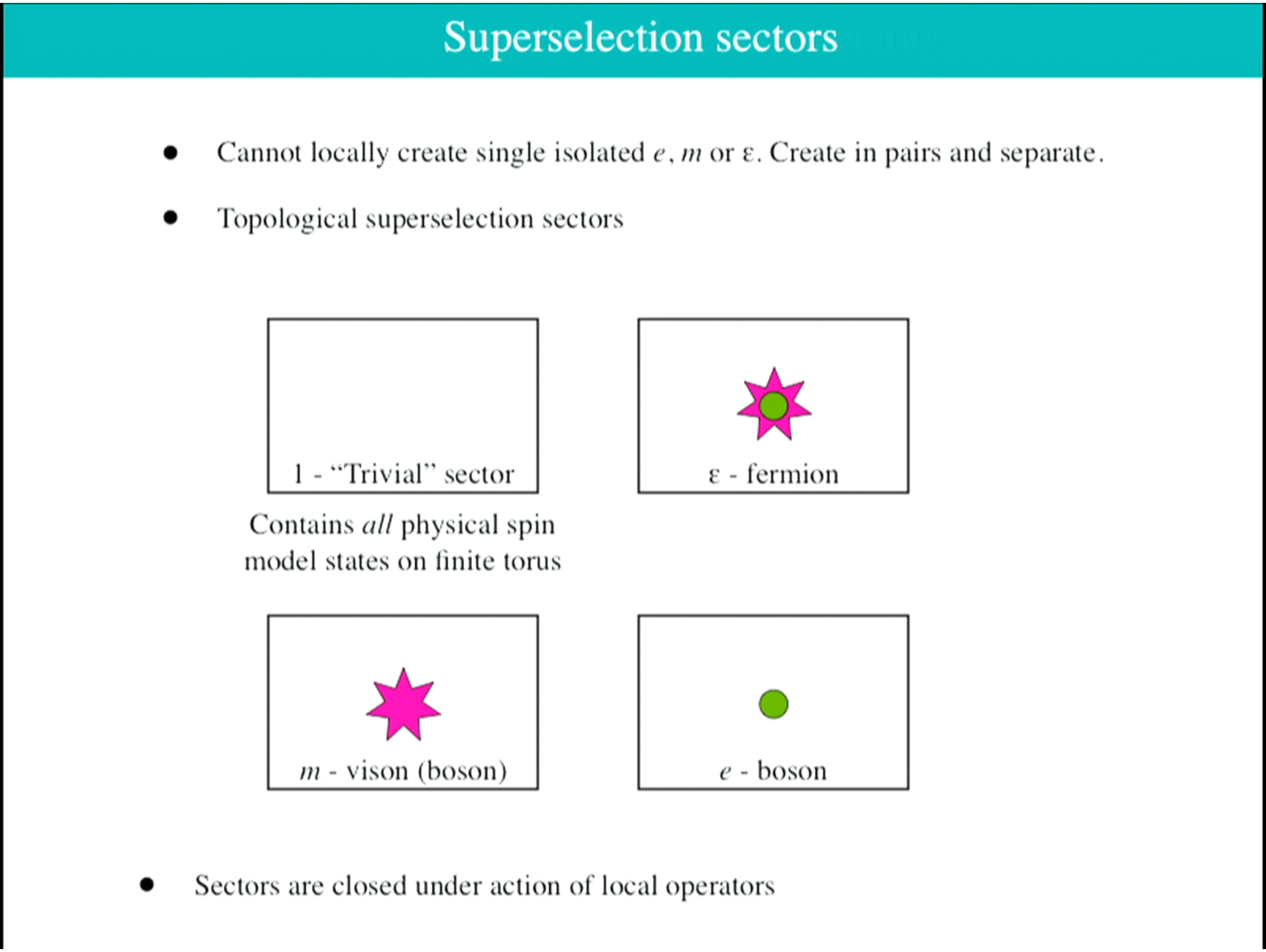
Superselection sectors

- Cannot locally create single isolated e, m or ε. Create in pairs and separate.
- Topological superselection sectors



Contains *all* physical spin model states on finite torus

- Sectors are closed under action of local operators



Superselection sectors

- Cannot locally create single isolated e, m or ε. Create in pairs and separate.
- Topological superselection sectors

1 - "Trivial" sector

ε - fermion

Contains *all* physical spin
model states on finite torus

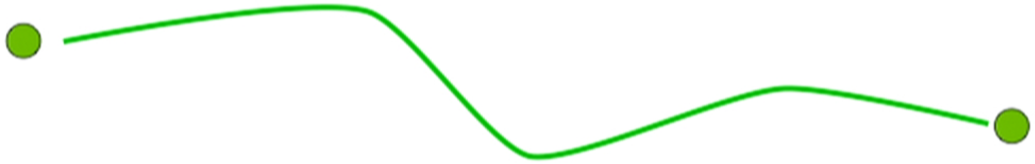
m - vison (boson)

e - boson

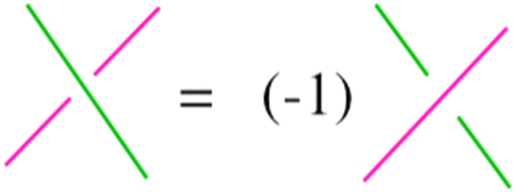
- Sectors are closed under action of local operators

String operators

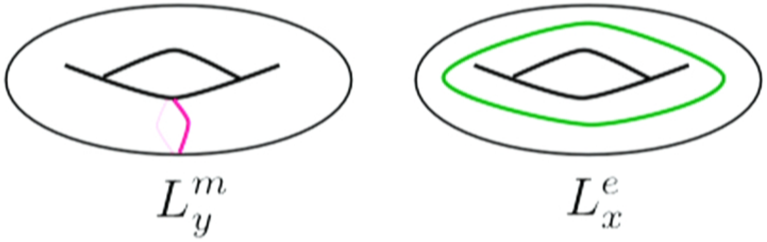
- To move an e-particle, or to create two isolated e's, act with string operator:



- e- and m-strings anti-commute at crossing points:



- Loop operators/algebra:



$$\{L_x^e, L_y^m\} = 0$$

$$\{L_y^e, L_x^m\} = 0$$

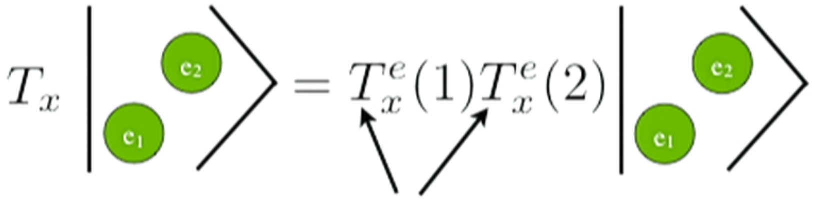
$$(L_{x,y}^e)^2 = (L_{x,y}^m)^2 = 1$$

⇒ D=4 irrep (4-fold ground state degeneracy)

Translation symmetry

- Translation symmetry: $T_xT_y = T_yT_x$ $T_xT_yT_x^{-1}T_y^{-1} = 1$ → Holds for physical states (1-sector)

- Acting on state with two *e*-particles:



These operators localized near corresponding *e*-particles

"Symmetry localization"

- Note: we assume *e* and *m* particles not exchanged under translation. This is "beyond fractionalization," and is incompatible with symmetry localization

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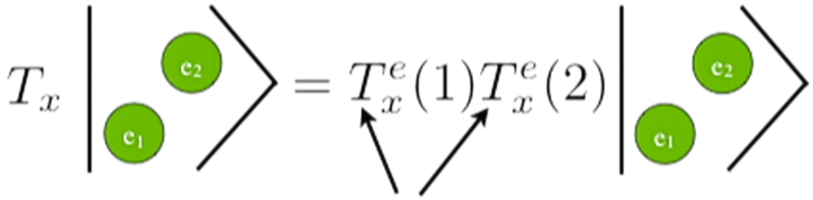

$$T_x \left| \; e_1 \; e_2 \; \right\rangle = T^e_x(1) T^e_x(2) \left| \; e_1 \; e_2 \; \right\rangle$$

"Symmetry localization"

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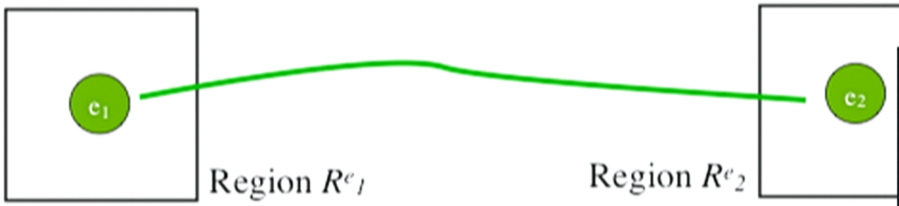

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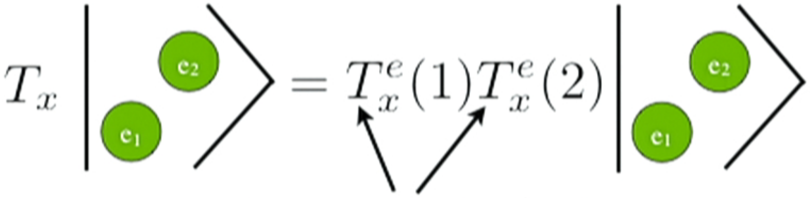

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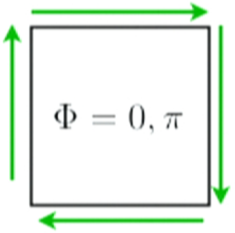
“Symmetry localization”

$$T_xT_yT_x^{-1}T_y^{-1} = 1$$

$$\Downarrow$$

$$T_x^eT_y^e(T_x^e)^{-1}(T_y^e)^{-1} = \pm 1$$



Interpretation: *e*-particle feels 0 or π flux per plaquette

Translation symmetry

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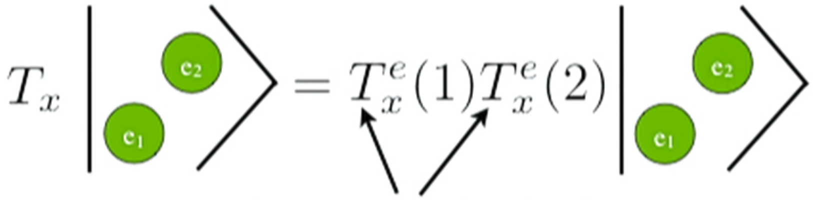

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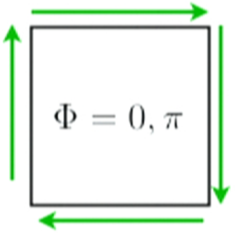

Interpretation: e-particle feels 0 or π flux per plaquette

This is a consequence of $e \times e = 1$ fusion rule

Translation symmetry

$T_x^e T_y^e T_x^{e-1} T_y^{e-1} = \sigma^e = \pm 1$ is *constant* on the *e*-sector

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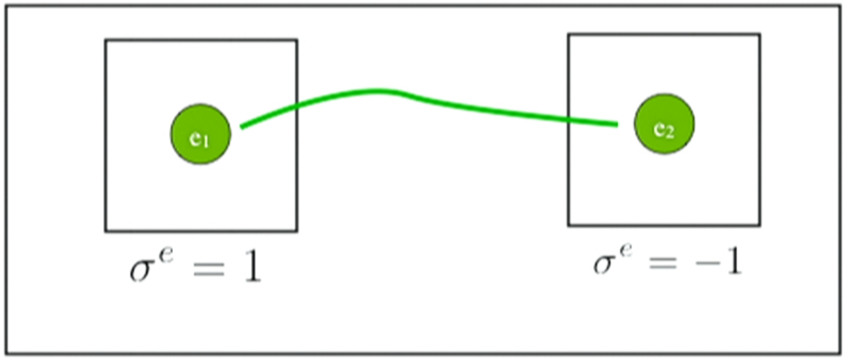
Argument: Suppose the contrary...



1-sector region on which $T_x T_y T_x^{-1} T_y^{-1} = -1$

Contradiction!

Translation symmetry

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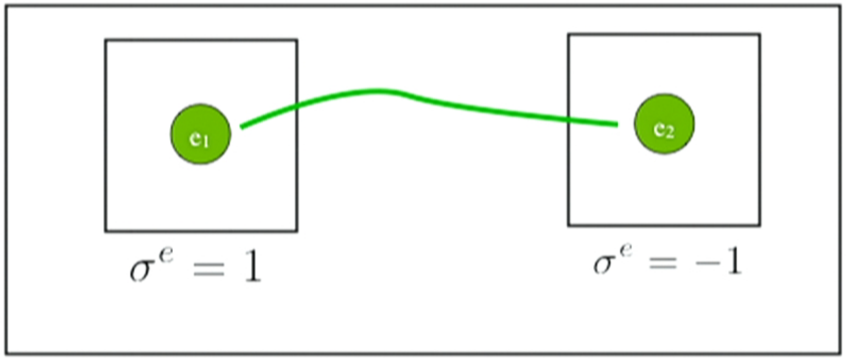

1-sector region on which $T_x T_y T_x^{-1} T_y^{-1} = -1$

Contradiction!

This implies that σ^e is a robust property of a Z_2 spin liquid phase, as long as gap remains open and translation symmetry is preserved.

Translation symmetry: classes

$T_x T_y T_x^{-1} T_y^{-1}$

e	m	ε
-1	1	-1
-1	-1	1
1	1	1

- Translation symmetry: 2 fractionalization classes & 3 symmetry classes
- These classes all realized in Kitaev toric code model (vary signs of vertex & plaquette terms)

- This is *not* a classification of irreps, but instead is the coarser classification desired.

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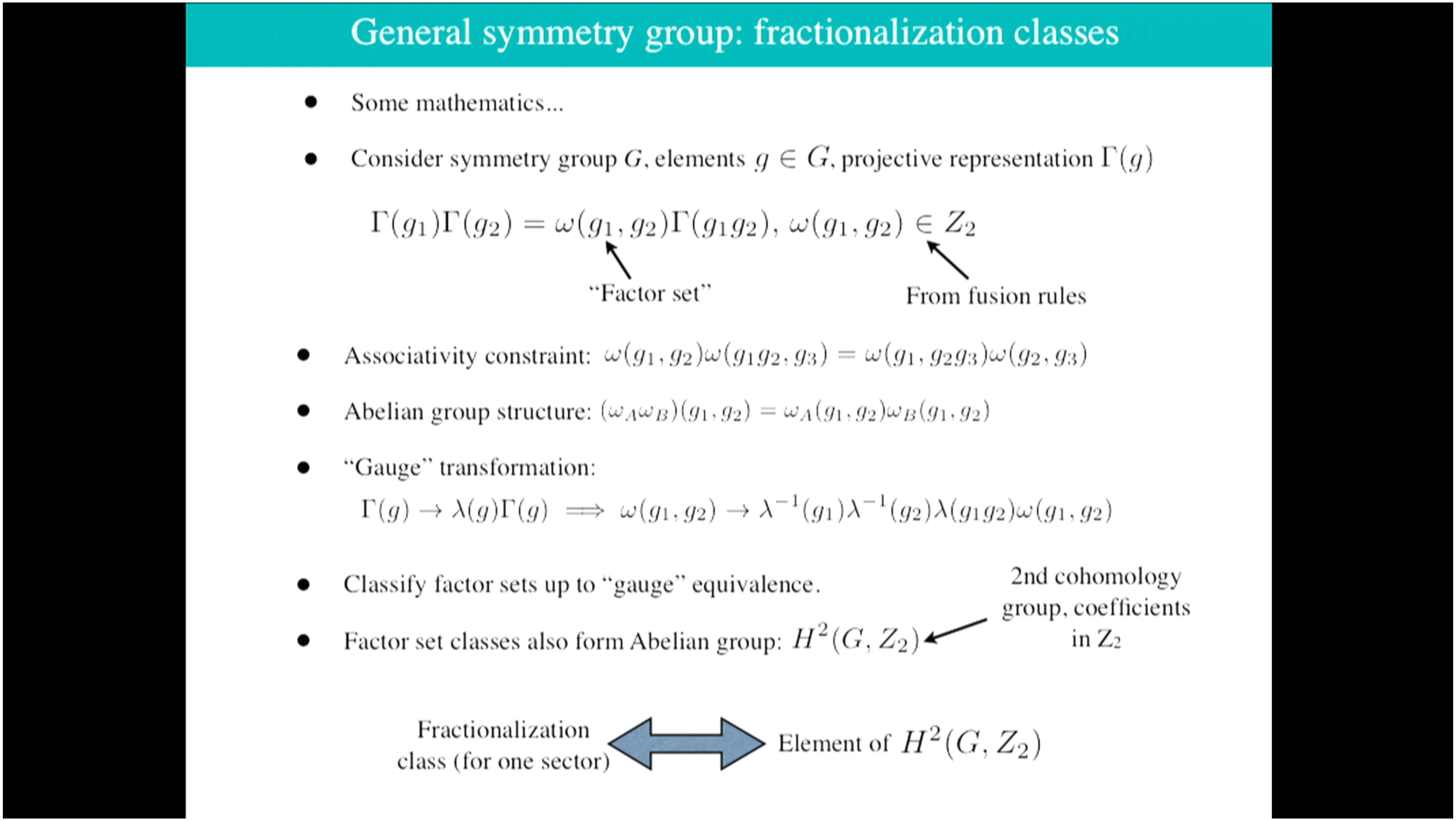


General symmetry group: fractionalization classes

- Some mathematics…
- Consider symmetry group G, elements $g \in G$, projective representation $\Gamma(g)$

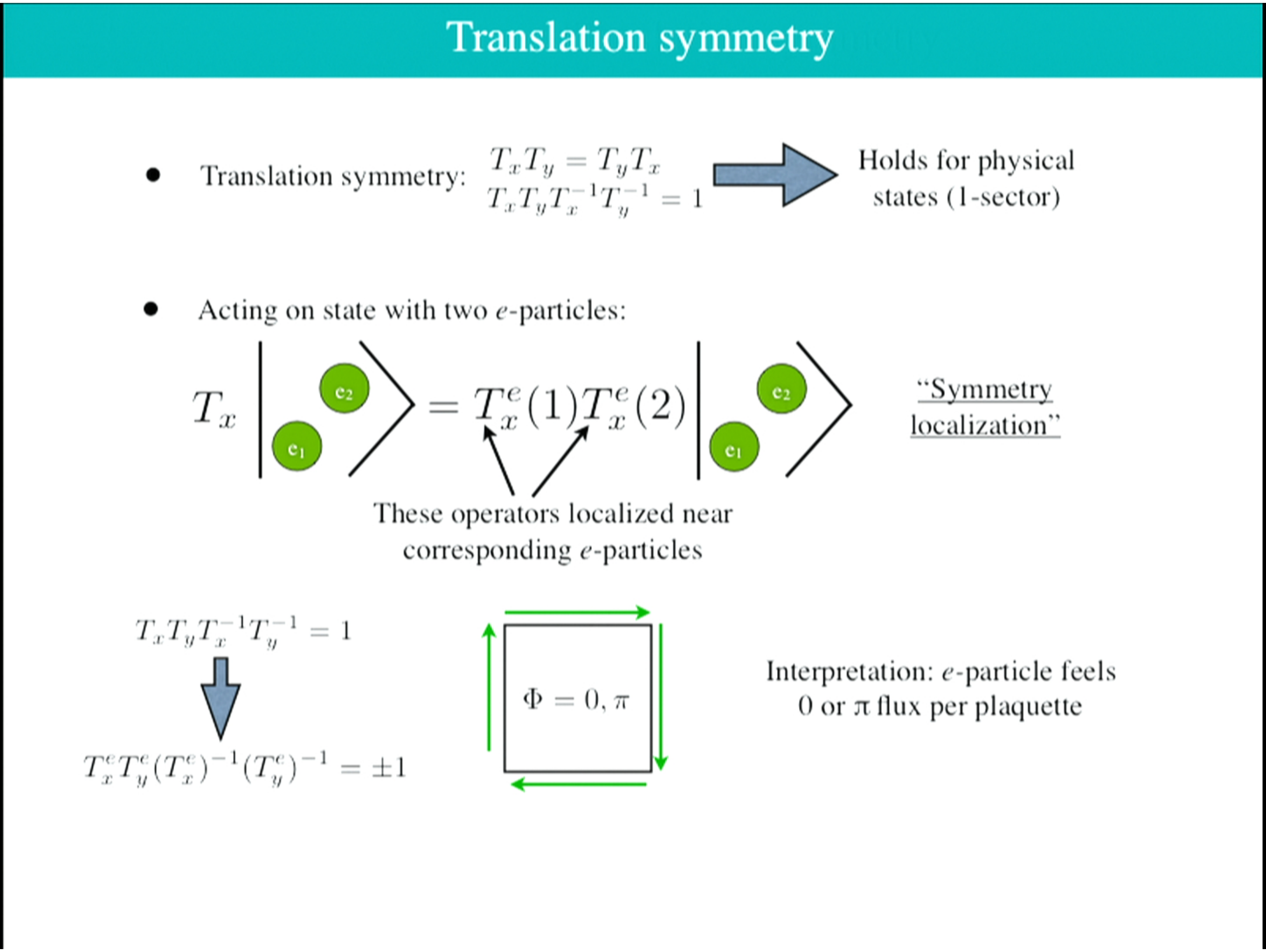
$$\Gamma(g_1)\Gamma(g_2) = \underset{\uparrow \text{ “Factor set”}}{\omega(g_1, g_2)}\Gamma(g_1 g_2),\ \omega(g_1, g_2) \underset{\uparrow \text{ From fusion rules}}{\in Z_2}$$

- Associativity constraint: $\omega(g_1, g_2)\omega(g_1 g_2, g_3) = \omega(g_1, g_2 g_3)\omega(g_2, g_3)$
- Abelian group structure: $(\omega_A \omega_B)(g_1, g_2) = \omega_A(g_1, g_2)\omega_B(g_1, g_2)$
- “Gauge” transformation:

$$\Gamma(g) \to \lambda(g)\Gamma(g) \implies \omega(g_1, g_2) \to \lambda^{-1}(g_1)\lambda^{-1}(g_2)\lambda(g_1 g_2)\omega(g_1, g_2)$$

- Classify factor sets up to “gauge” equivalence.
- Factor set classes also form Abelian group: $H^2(G, Z_2) \leftarrow$ 2nd cohomology group, coefficients in Z_2

Fractionalization class (for one sector) ⟺ Element of $H^2(G, Z_2)$



Translation symmetry

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- Acting on state with two *e*-particles:

$$T_x \Big| \; \Big\rangle = T_x^e(1) T_x^e(2) \Big| \; \Big\rangle$$

"Symmetry localization"

These operators localized near corresponding *e*-particles

$$T_x T_y T_x^{-1} T_y^{-1} = 1$$

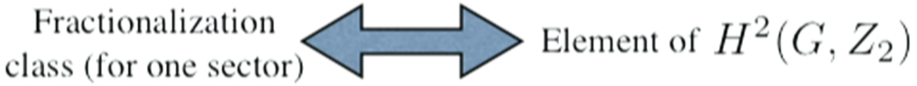
$$\Downarrow$$

$$T_x^e T_y^e (T_x^e)^{-1} (T_y^e)^{-1} = \pm 1$$

$\Phi = 0, \pi$

Interpretation: *e*-particle feels 0 or π flux per plaquette

General symmetry group: symmetry classes



Fractionalization class (for one sector)

Element of $H^2(G, Z_2)$

- Symmetry class given by specifying fractionalization class for each non-trivial superselection sector: $\omega_e, \omega_m, \omega_\epsilon \in H^2(G, Z_2)$
- But only two are independent: $\omega_\epsilon = \omega_e \omega_m$ (From $\epsilon = e \times m$ fusion rule.)
- Pair (ω_e, ω_m) can also be viewed as element of $H^2(G, Z_2 \times Z_2)$, since $H^2(G, Z_2 \times Z_2) \simeq H^2(G, Z_2) \times H^2(G, Z_2)$
- Symmetry classes are elements of $H^2(G, Z_2 \times Z_2)$, up to $e \leftrightarrow m$ relabeling

General symmetry group: symmetry classes

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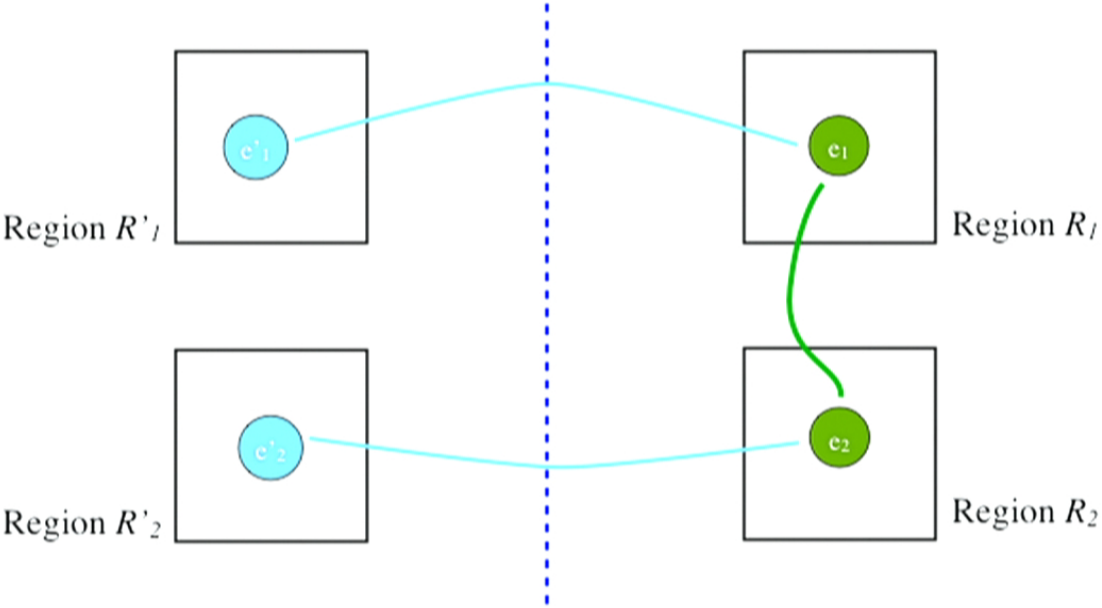
- Symmetry classes are elements of $H^2(G, Z_2 \times Z_2)$, up to $e \leftrightarrow m$ relabeling

Space group fractionalization classes

- General space group operations move some points large distances. Notion of symmetry localization needs to be modified.
- Example: P_x ($x \rightarrow -x$ reflection symmetry)
- Still have $P_x|\psi\rangle = P^e_x(1)P^e_x(2)|\psi\rangle$



- But now $P^e_x(i)$ has support on the union of regions R_i and R'_i and a linear region connecting the two. On this linear region, $P^e_x(i)$ is an e-string.

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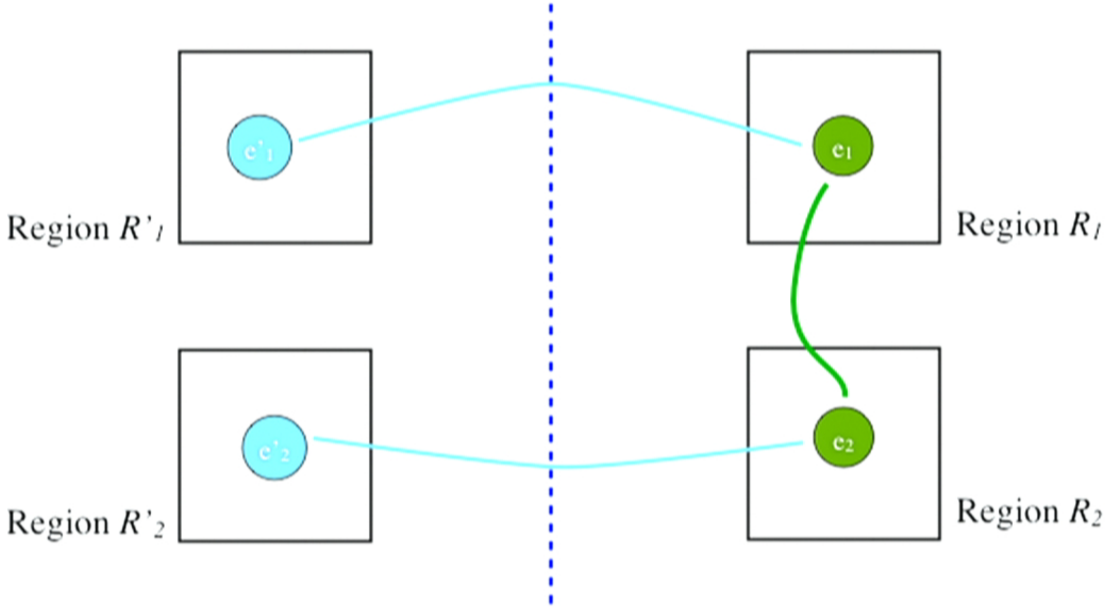

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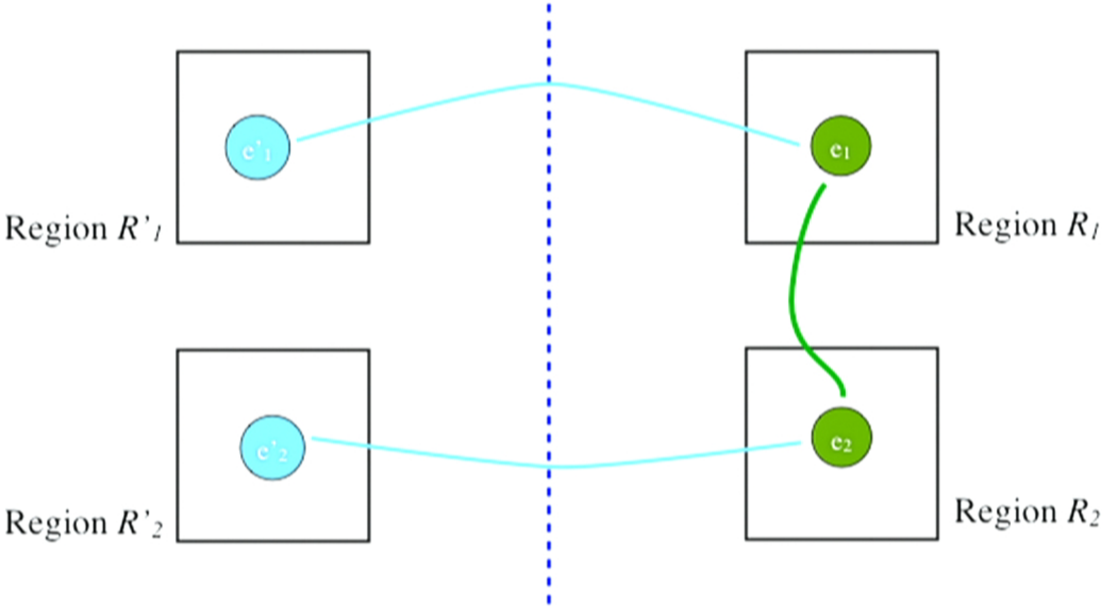

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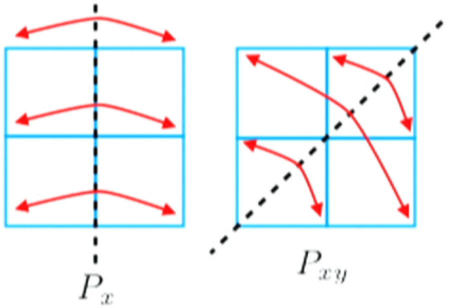
Square lattice example

- G = Square lattice space group × time reversal × spin rotation.
- Square lattice space group generators: T_x, P_x, P_{xy}
- Note that: $T_y = P_{xy} T_x P_{xy}^{-1}$
- Time reversal $\mathcal{T}$
- Spin rotation (by θ about $\hat{n}$-axis): $R(\theta\hat{n})$



- Generators + relations specify the symmetry class in each non-trivial sector:

$$\begin{aligned} P_x^2 &= \sigma_{px} \\ P_{xy}^2 &= \sigma_{pxy} \\ (P_x P_{xy})^4 &= \sigma_{pxpxy} \\ T_x T_y T_x^{-1} T_y^{-1} &= \sigma_{txty} \\ T_x P_x T_x P_x^{-1} &= \sigma_{txpx} \\ T_y P_x T_y^{-1} P_x^{-1} &= \sigma_{typx} \\ \mathcal{T}^2 &= \sigma_{\mathcal{T}} \end{aligned}$$

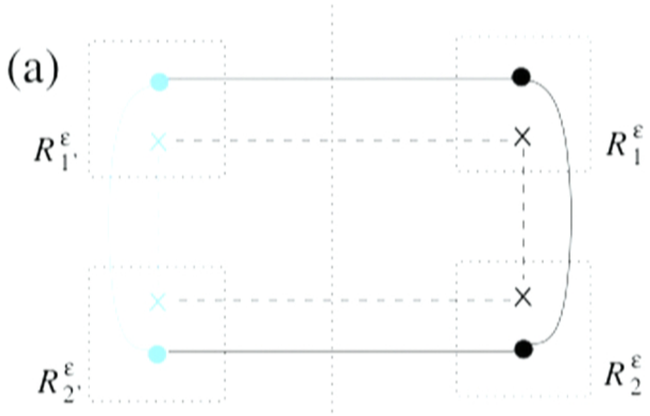
$$\begin{aligned} \mathcal{T} T_x \mathcal{T}^{-1} T_x^{-1} &= \sigma_{Ttx} \\ \mathcal{T} P_x \mathcal{T}^{-1} P_x &= \sigma_{Tpx} \\ \mathcal{T} P_{xy} \mathcal{T}^{-1} P_{xy} &= \sigma_{Tpxy} \\ R(2\pi\hat{n}) &= \sigma_R \\ R(\theta\hat{n})\mathcal{T} &= \mathcal{T} R(\theta\hat{n}) \\ R(\theta\hat{n}) P_x &= P_x R(\theta\hat{n}) \\ R(\theta\hat{n}) P_{xy} &= P_{xy} R(\theta\hat{n}) \\ R(\theta\hat{n}) T_x &= T_x R(\theta\hat{n}) \end{aligned}$$

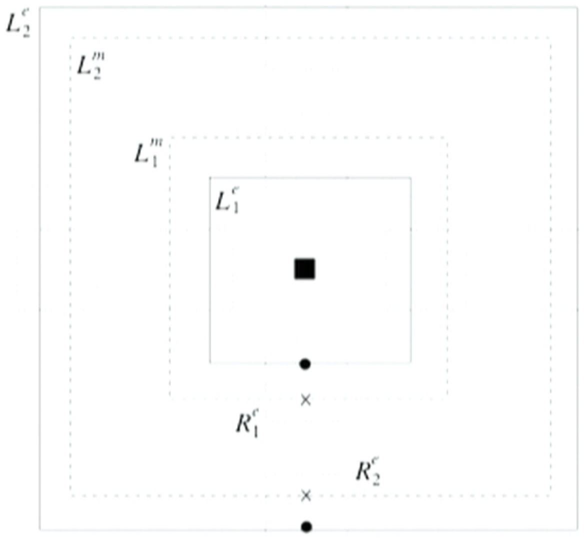
(+ Lie algebra of spin rotations)

- Here the σ's = ±1
- 11 independent Z_2 parameters → $H^2(G,Z_2) = (Z_2)^{11}$

Space group symmetry classes and braiding

- How to determine ε fractionalization class from *e* and *m* classes?
- It turns out that the H^2 product is "twisted": $\omega_\epsilon = \omega_t \omega_e \omega_m$
- Basic idea: view ε as a composite of *e* and *m*, work out symmetry-localized group relations keeping track of new statistical phase factors.





Space group symmetry classes and braiding

- Result for ω_t factor set (specify in terms of group relations):

$$(P_x P_{xy})^4 = -1 \qquad \text{Other relations trivial.}$$

$$(P^e_x P^e_{xy})^4 = \sigma^e_{pxpxy} \quad (P^m_x P^m_{xy})^4 = \sigma^m_{pxpxy} \quad (P^\epsilon_x P^\epsilon_{xy})^4 = -\sigma^e_{pxpxy}\sigma^m_{pxpxy}$$

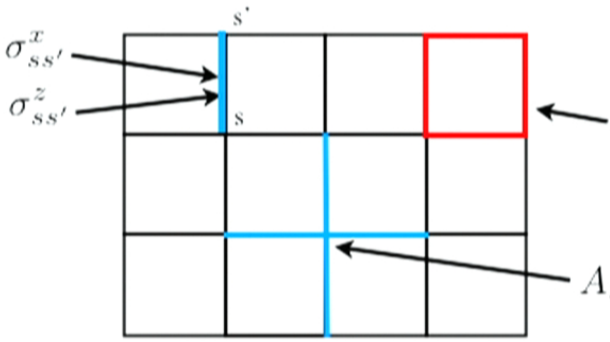
- Only the "rotation" relation is modified ... consequence of braiding statistics.

Outline

1. Motivation/background
2. Symmetry classification for non-point group symmetry
3. General symmetry classification
4. Realization of (some) symmetry classes in microscopic models

Concrete model: toric code

A. Kitaev



$$B_p = \prod_{ss' \in p} \sigma^z_{ss'}$$

$$A_s = \prod_{s' \sim s} \sigma^x_{ss'}$$

$$H = -u \sum_s A_s - K \sum_p B_p$$

Choose $u, K = \pm 1$

- Ground state has: $A_s = u, B_p = K$
- e-particles live at vertices s where $A_s = -u$
- m-particles live at plaquettes p where $B_p = -K$
- Focus on square lattice space group symmetry. Can show *four* symmetry classes realized depending on u, K:

$$\begin{aligned} (P^e_x)^2 &= 1 & \qquad (P^m_x)^2 &= 1 \\ (P^e_{xy})^2 &= 1 & (P^m_{xy})^2 &= 1 \\ (P^e_x P^e_{xy})^4 &= 1 & (P^m_x P^m_{xy})^4 &= u \\ T^e_x T^e_y T^{e-1}_x T^{e-1}_y &= K & T^m_x T^m_y T^{m-1}_x T^{m-1}_y &= u \\ T^e_x P^e_x T^e_x P^{e-1}_x &= 1 & T^m_x P^m_x T^m_x P^{m-1}_x &= 1 \\ T^e_y P^e_x T^{e-1}_y P^{e-1}_x &= 1 & T^m_y P^m_x T^{m-1}_y P^{m-1}_x &= u. \end{aligned}$$

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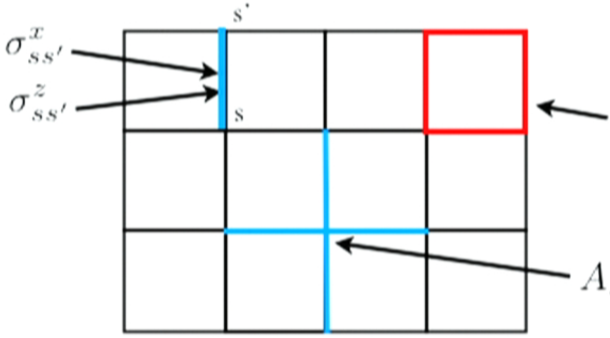

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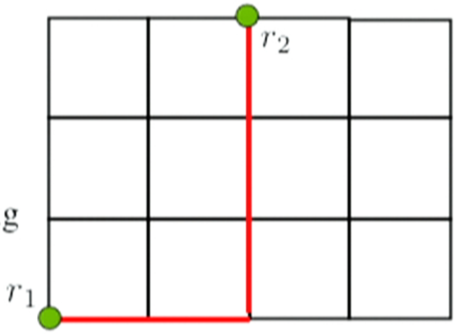
Explicit construction of symmetry localization

Initial state:
two e-particles,
connected by string



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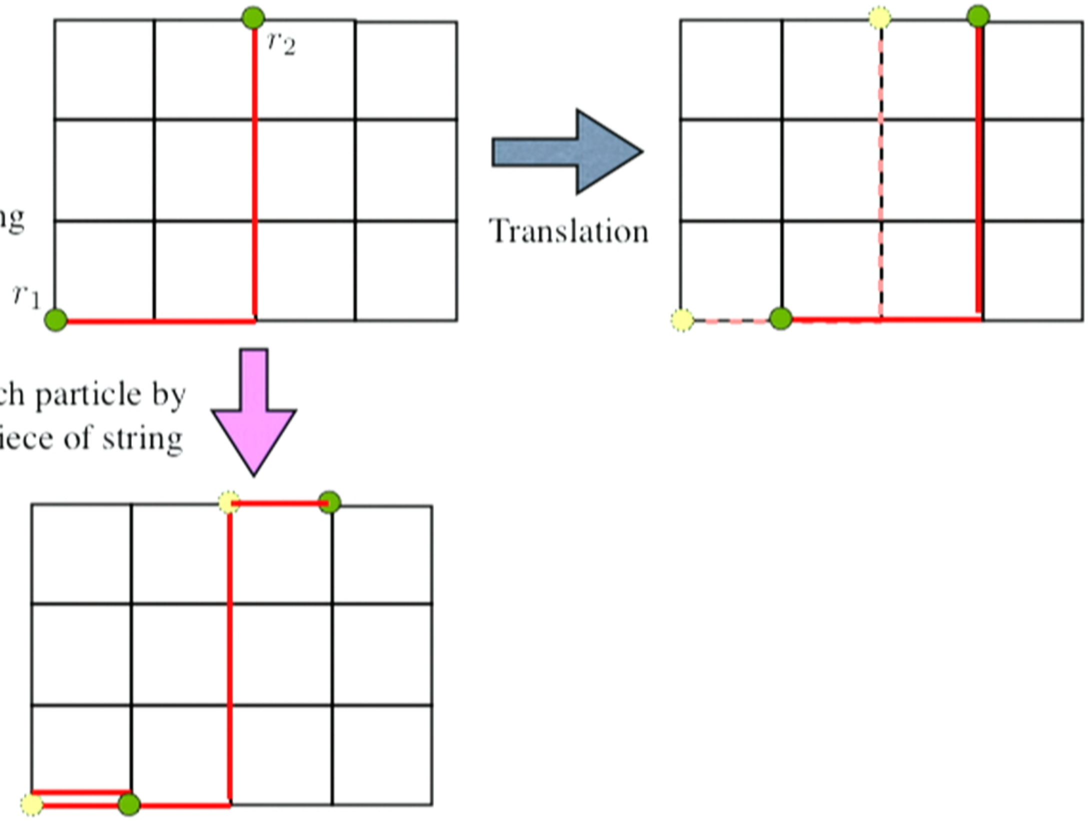

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Translation

Move each particle by
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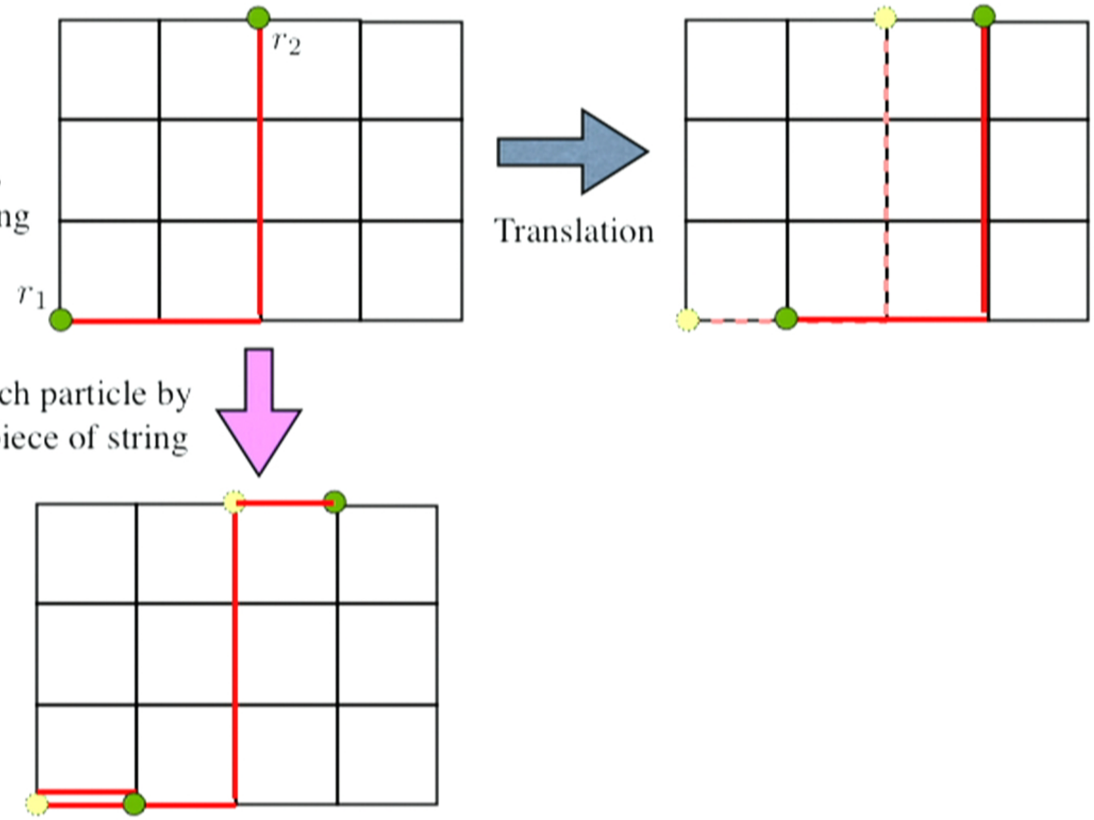

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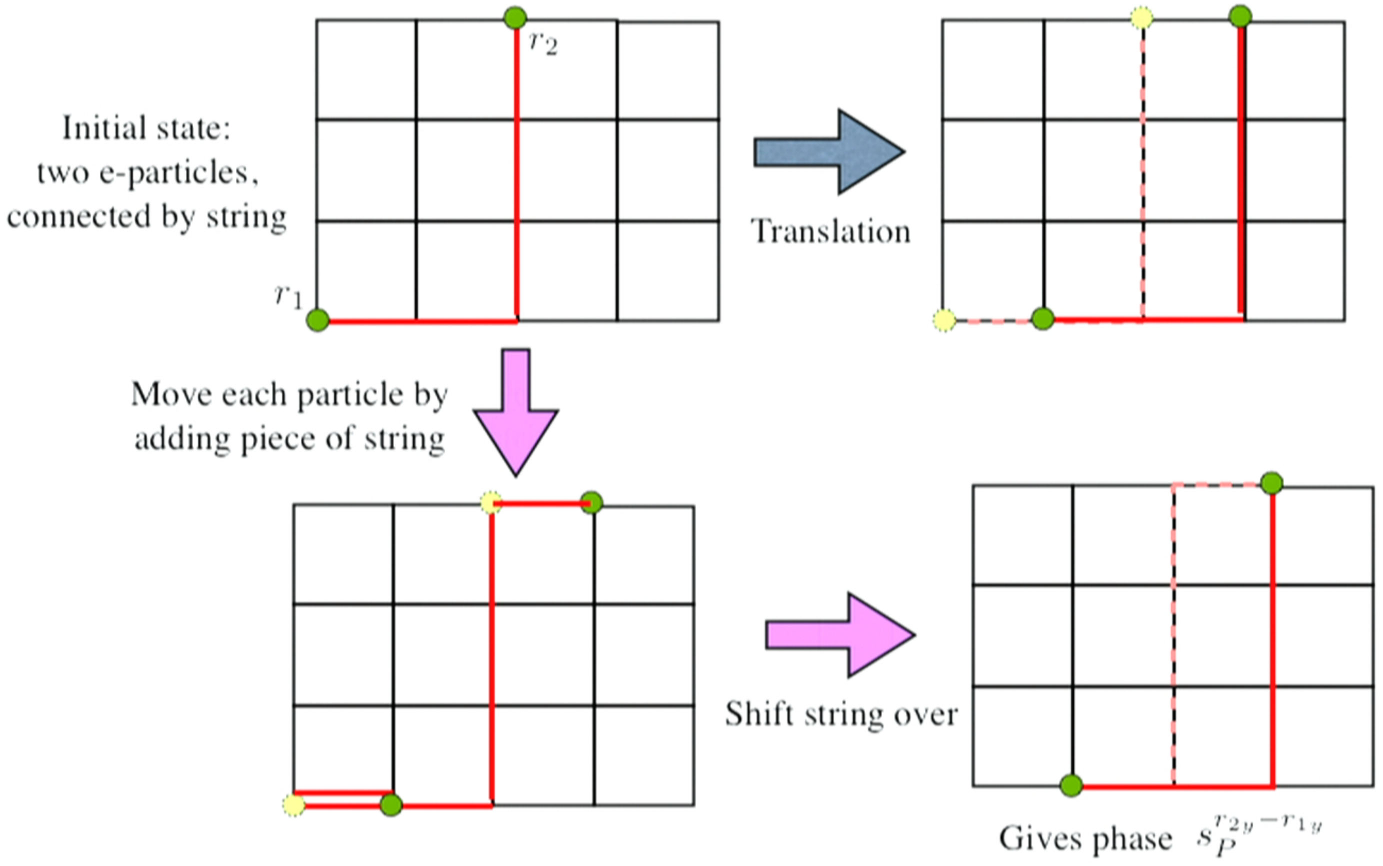

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Shift string over

Gives phase $s_P^{r_{2y}-r_{1y}}$

Define: $T^e_x(r) = (-1)^{r_y}\sigma^z_{r,r+\boldsymbol{x}}$ $\Rightarrow$ $T_x = T^e_x(r_1)T^e_x(r_2)$

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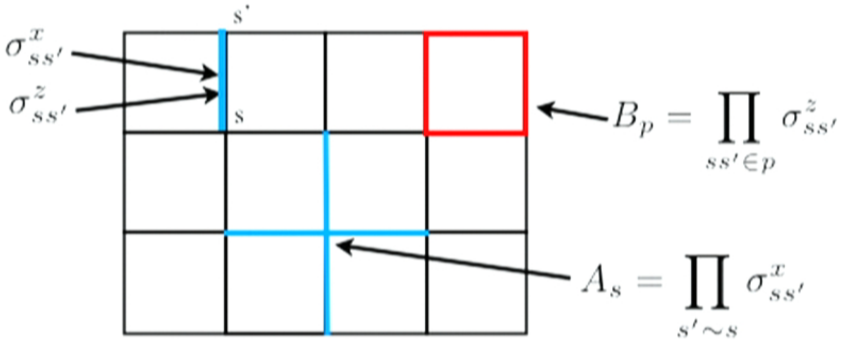

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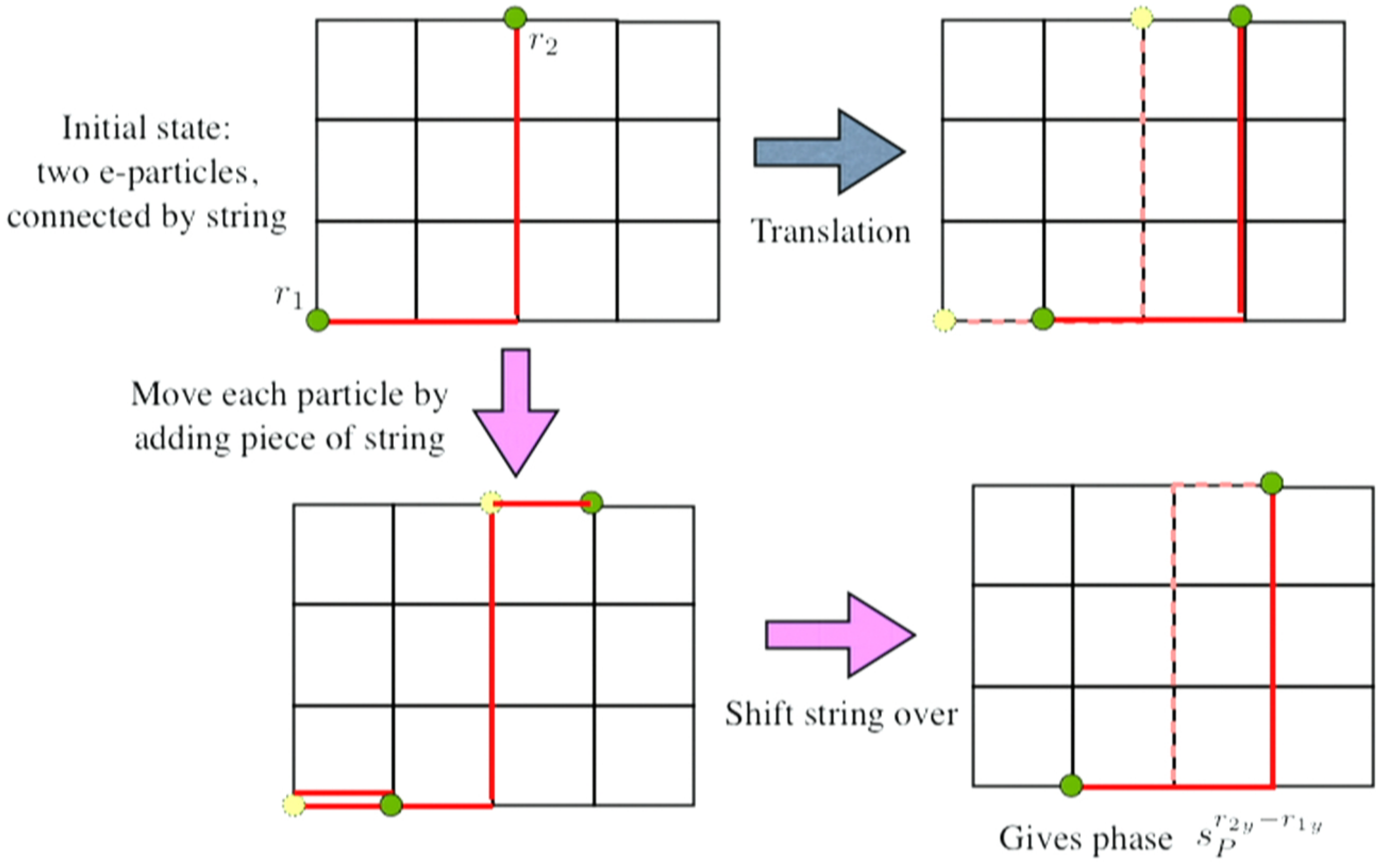

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Explicit realization of (more) symmetry classes

with Hao Song

- Can *all* symmetry classes be realized? Probably not, in strictly 2d models. For some on-site symmetries, we know some classes can only be realized as the boundary theory of a d=3 SPT phase! (Vishwanath & Senthil; C. Wang & Senthil).
- Then, which/how many classes can be realized in 2d models?
- Approach: we study a class of generalized toric code models. Within this class, we prove that most of the 2080 symmetry classes are impossible, and find explicit realizations for the 82 others.
- Class of models: toric code defined on (almost) arbitrary 2d lattice. (Links may cross, but vertices may not stack.)
- No "spin-orbit coupling." Label links by ℓ, then:

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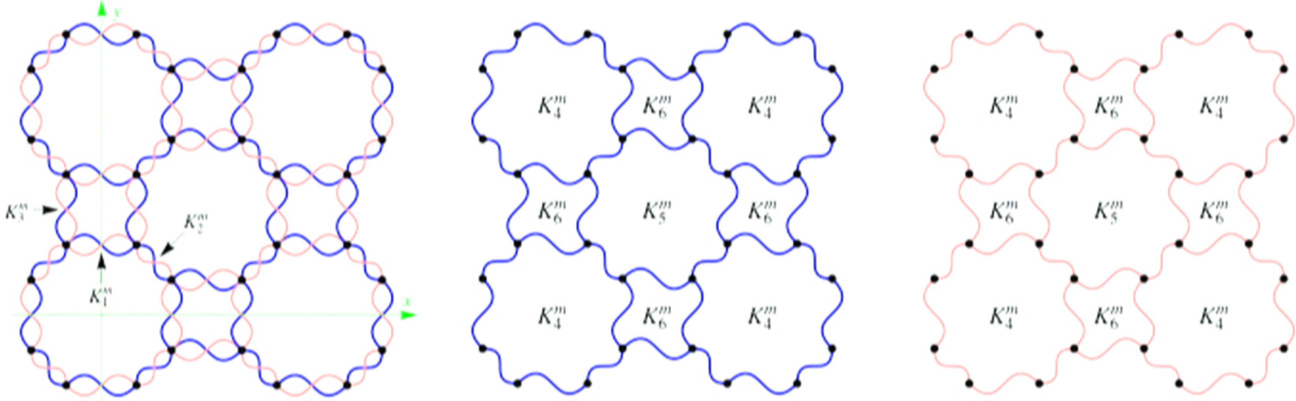
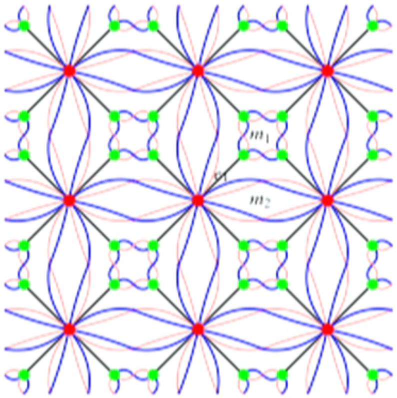
All *e*-fractionalization classes can be realized

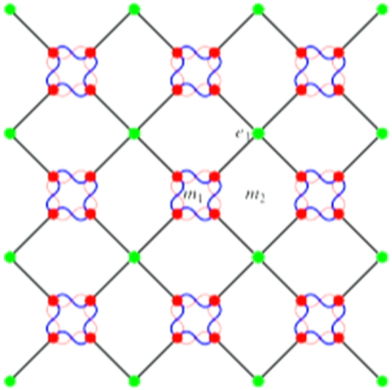


- Model on this lattice has six independent Z_2 parameters controlling the sign of fluxes in the ground state.
- All 2^6 *e*-fractionalization classes realized by varying these parameters
- *m*-fractionalization class is trivial

Other classes

- General arguments show that only 82 symmetry classes are possible (so only 18 where both e and m classes are non-trivial)





Open issues

- General understanding of which symmetry classes possible in strictly 2d? Can all classes be realized at boundary of 3d SPT phases?
- Generalization to other topological orders (this is trivial for any Abelian topological order with non-point group symmetry)
- Full symmetry classification, including "beyond fractionalization" $e \leftrightarrow m$ interchange
- Three dimensions ... connection to edge states of 2d SPT phases?
- How can symmetry class be determined given ground state wavefunction, excited states? Application to numerics on kagome & J_1-J_2 Heisenberg models?
- Experimental signatures?