

Title: Classifying fractionalization: symmetry classification of gapped Z_2 spin liquids in two dimensions

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Abstract: Quantum number fractionalization is a remarkable property of topologically ordered states of matter, such as fractional quantum Hall liquids, and quantum spin liquids. For a given type of topological order, there are generally many ways to fractionalize the quantum numbers of a given symmetry. What does it mean to have different types of fractionalization? Are different types of fractionalization a universal property that can be used to distinguish phases of matter? In this talk, I will answer these questions, focusing on a simple class of topologically ordered phases, namely two-dimensional gapped Z_2 spin liquids, and I will present a symmetry classification of these phases. I will also discuss efforts in progress to find microscopic models realizing different symmetry classes.

Classifying fractionalization: Symmetry classification of gapped Z_2 spin liquids in two dimensions

Michael Hermele



Andrew Essin & MH, PRB **87**, 104406 (2013)
Hao Song & MH, in progress

Perimeter Institute
March 26, 2013



Thanks to...

- Andrew Essin (postdoc @ Boulder → postdoc @ Caltech IQI)
- Hao Song (student @ Boulder)



Themes / Questions

1. What is a quantum phase of matter?
2. What is quantum number fractionalization?

Goal: try to better answer these questions by developing classifications.

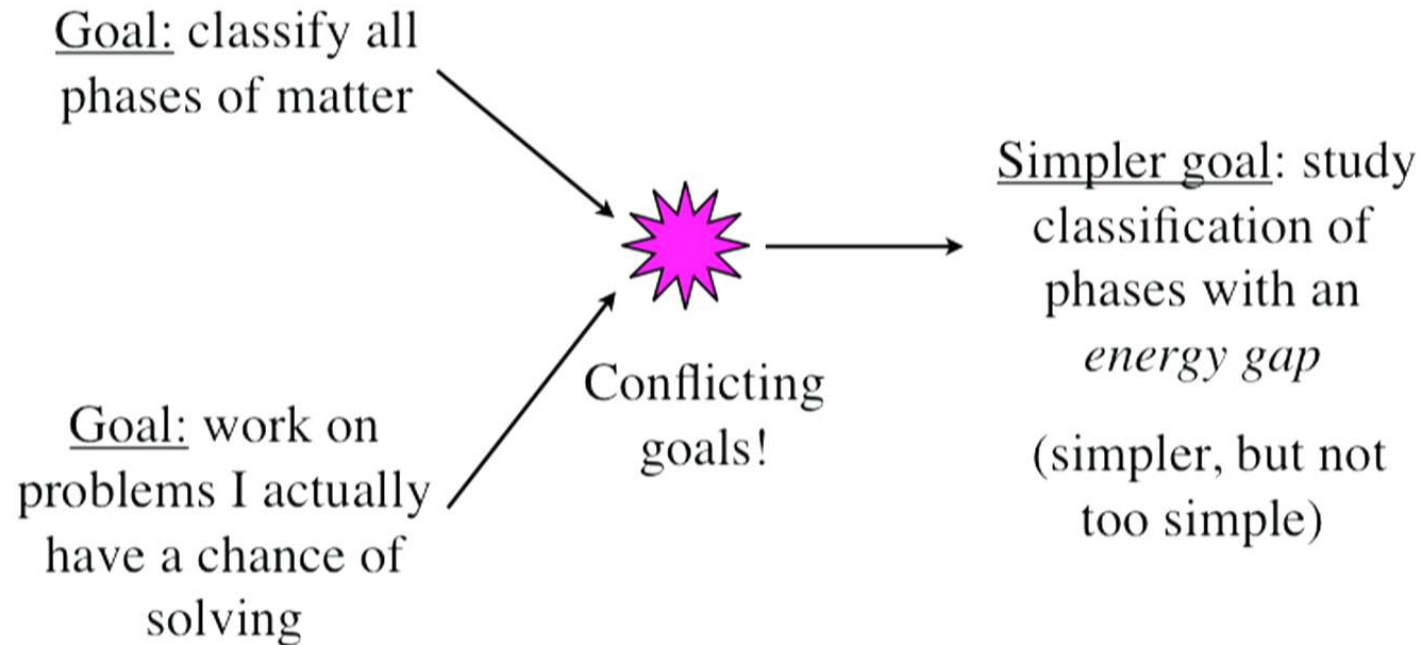


Outline

1. Motivation/background
 2. Symmetry classification for non-point group symmetry
 3. General symmetry classification
 4. Realization of (some) symmetry classes in microscopic models
- } with Andrew Essin
- } with Hao Song and Andrew Essin



Quantum phases of matter



Gapped quantum matter (no symmetry)

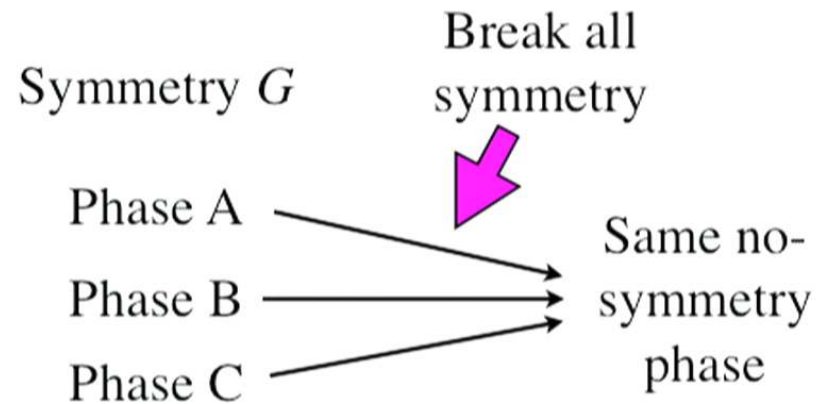
- Throughout this talk: consider local bosonic models (*i.e.* generalized spin models with finite-range interactions)

Energy gap, no symmetry present

- d=1: Only one trivial phase
- d=2:
 - Trivial phase
 - Kitaev E_8 state
 - Topologically ordered phases (anyons)
 - More?
- d=3:
 - Trivial phase
 - Stack of d=2 states
 - Topologically ordered phases (point and line “anyons”)
 - More?

Gapped quantum matter with symmetry

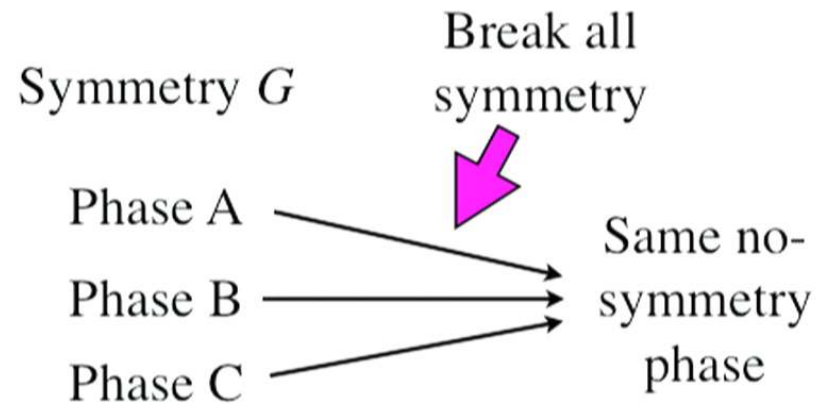
- Assume no spontaneous symmetry breaking.
- Notion of *symmetry enrichment*:



- For a fixed no-symmetry phase and fixed symmetry group G , we say that A, B, C are distinct *symmetry enrichments*.

Gapped quantum matter with symmetry

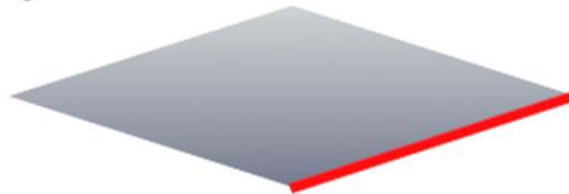
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Symmetry protected topological (SPT) phases

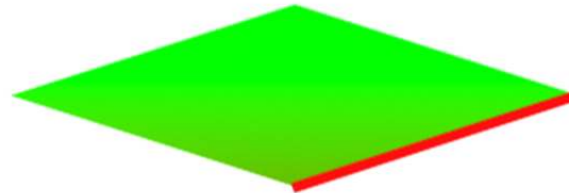
- SPT phases are symmetry enrichments of the trivial phase
- Classic examples: 🍌 Haldane phase of $S=1$ chain
🍌 Z_2 topological band insulators ($d=2,3$)
(Kane & Mele; J. E. Moore & Balents; R. Roy; Fu, Kane & Mele)
- Recently, many more examples + developing systematic understanding



- Key physical property: non-trivial end/edge/surface states that are gapless, spontaneously break symmetry, or are otherwise non-trivial ... this is most robust for internal symmetries (*e.g.* time reversal, spin rotation)

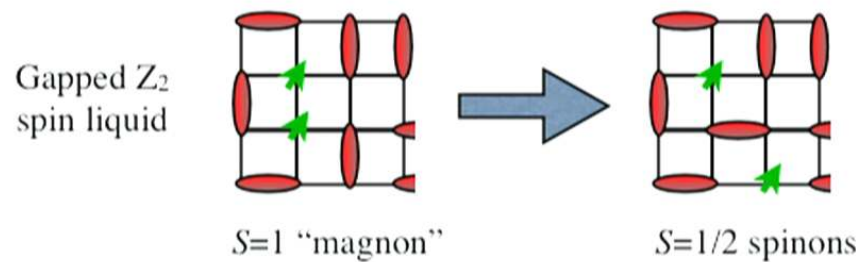
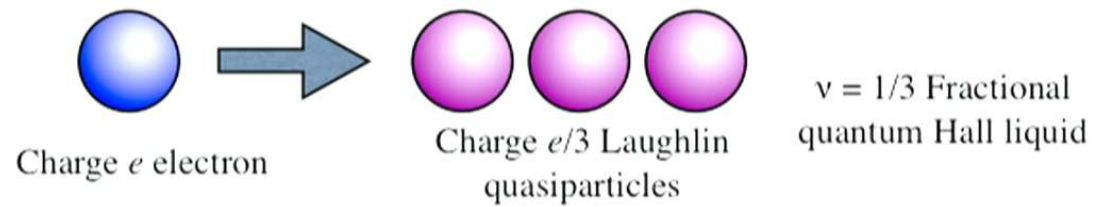
Symmetry enriched topological (SET) phases

- SET phases are symmetry enrichments of topologically ordered phases
- Many examples of SET phases in models, but systematic understanding of how to classify SET phases is less developed than for SPT phases
- But many very recent works: [M. Levin & A. Stern](#); [A. Mesaros & Y. Ran](#); [A. M. Essin & M.H. L.-Y. Hung & X.-G. Wen](#); [L.-Y. Hung and Y. Wan](#); [Y.-M. Lu & A. Vishwanath](#); [X.-G. Wen](#); [C. Wang & T. Senthil](#)



- In $d=2$ SET phases: anyon excitations \rightarrow non-trivial bulk properties. *Space group symmetry* is thus more important than for SPT phases.

Fractionalization



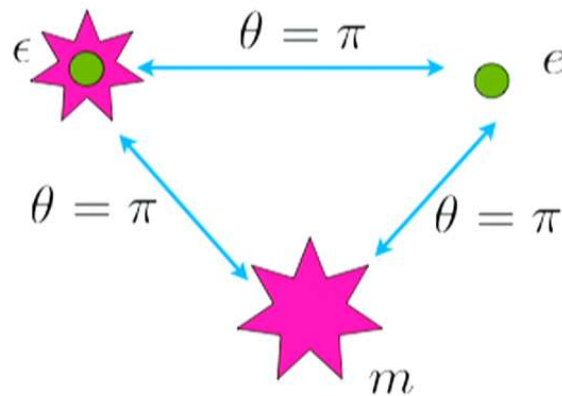
- What are distinct types of fractionalization?
- How to describe/classify?
- Can classifying fractionalization help classify SET phases?

Gapped Z_2 spin liquids in 2d

- Gapped Z_2 spin liquids = Z_2 topological order + no spontaneous symmetry breaking

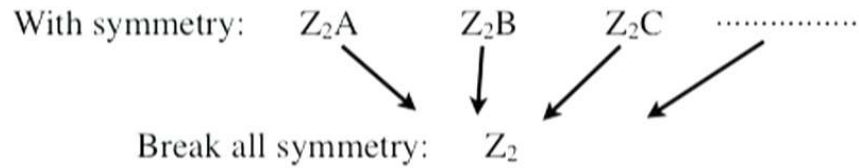
Z_2 topological order: particle types (anyons)

- Two bosons (e and m). One fermion (ϵ). Also one “trivial” boson (1).
- Fusion rules: $\epsilon \times \epsilon = m \times m = e \times e = 1$
 $\epsilon \times m = e, \epsilon \times e = m, e \times m = \epsilon$
- Mutual statistics:

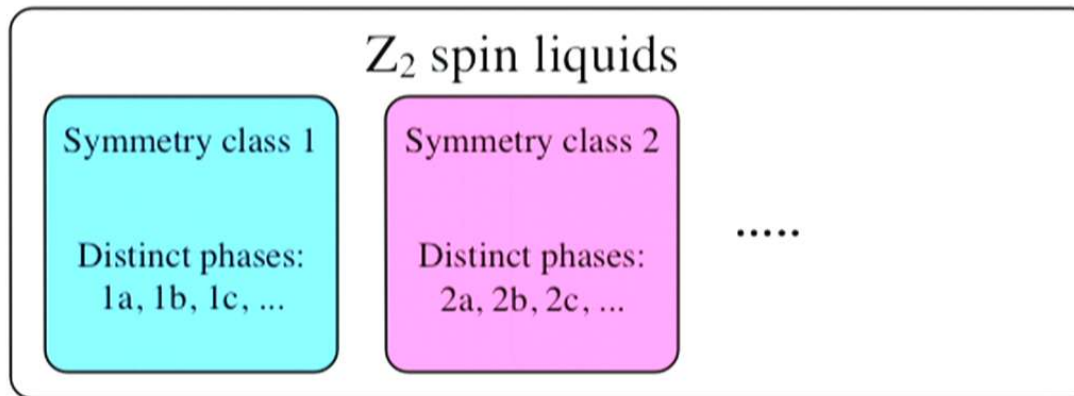


Symmetry classification

- In presence of symmetry, there are many gapped Z_2 spin liquids (X.-G. Wen, ...)

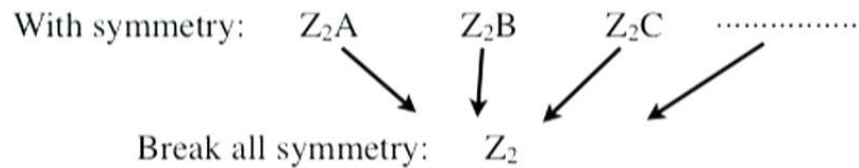


- Can we classify such distinct Z_2 spin liquids?
- Simpler: symmetry classification

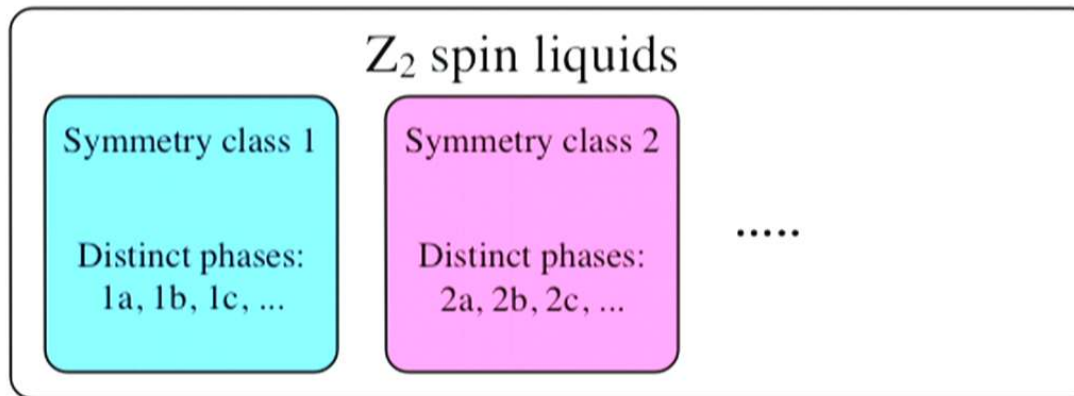


Symmetry classification

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- Can we classify such distinct Z_2 spin liquids?
- Simpler: symmetry classification



Symmetry classes  Fractionalization classes
(types of fractionalization)

Not in this talk: symmetry
classes “beyond
fractionalization.”

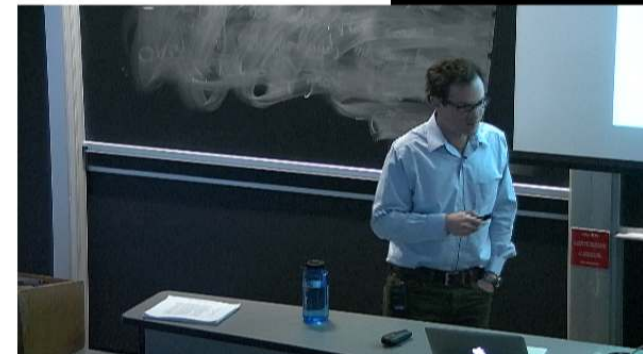
Prior work

- Projective symmetry group classification (Xiao-Gang Wen, 2001)
- Ying Ran & Xiao-Gang Wen, 2002, unpublished
- Alexei Kitaev, Ann. Phys. 2006, Appendix F



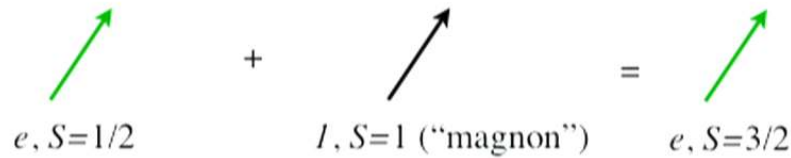
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Spin rotation symmetry

- e -particle could have $S=0, 1/2, 1, 3/2, \dots$



$$e, S=1/2 + l, S=1 \text{ ("magnon")} = e, S=3/2$$

- Only integer vs. half-odd-integer spin matters \rightarrow two fractionalization classes
- Therefore, we *don't* want to classify by irreducible representations. Coarser classification is needed.

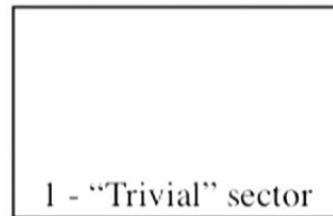
	e	m	ε
$S \bmod 1$ \rightarrow	1/2	0	1/2
	0	1/2	1/2
	1/2	1/2	0
	0	0	0

Same, under *relabeling*
 $e \leftrightarrow m$

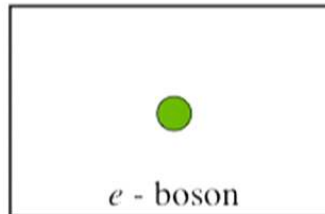
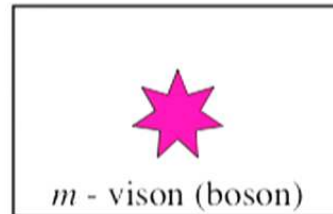
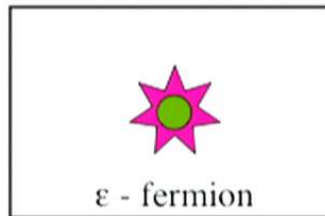
- Three symmetry classes if *only* $SO(3)$ spin rotation symmetry present

Superselection sectors

- Cannot locally create single isolated e , m or ϵ . Create in pairs and separate.
- Topological superselection sectors



Contains *all* physical spin model states on finite torus

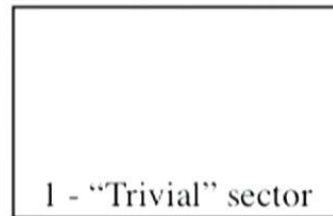


- Sectors are closed under action of local operators

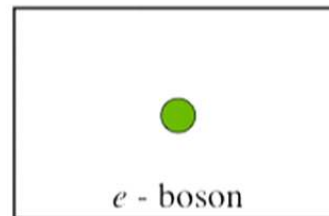
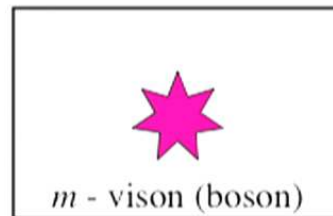
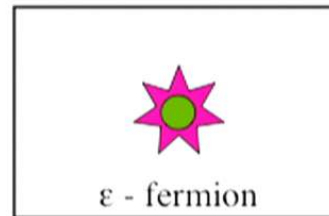


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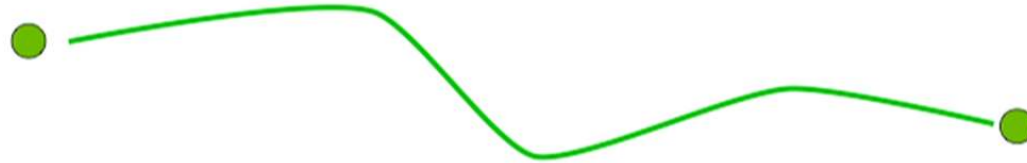
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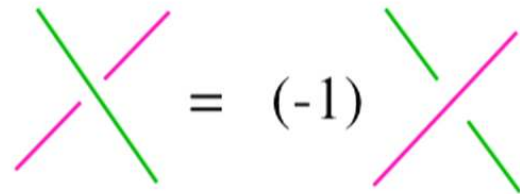
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String operators

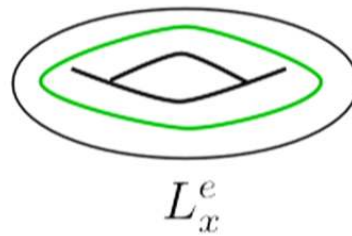
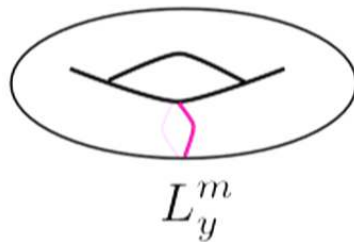
- To move an e -particle, or to create two isolated e 's, act with string operator:



- e - and m -strings anti-commute at crossing points:



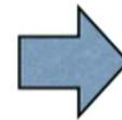
- Loop operators/algebra:



$$\{L_x^e, L_y^m\} = 0$$

$$\{L_y^e, L_x^m\} = 0$$

$$(L_{x,y}^e)^2 = (L_{x,y}^m)^2 = 1$$



$D=4$ irrep (4-fold ground state degeneracy)

Translation symmetry

- Translation symmetry: $T_x T_y = T_y T_x$
 $T_x T_y T_x^{-1} T_y^{-1} = 1 \rightarrow$ Holds for physical states (1-sector)

- Acting on state with two e -particles:

$$T_x \left| \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \right\rangle = T_x^e(1) T_x^e(2) \left| \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \right\rangle$$

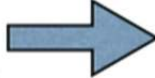
These operators localized near corresponding e -particles

“Symmetry localization”



- Note: we assume e and m particles not exchanged under translation. This is “beyond fractionalization,” and is incompatible with symmetry localization

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$$T_x \left| \begin{array}{c} \bullet c_2 \\ \bullet c_1 \end{array} \right\rangle = T_x^e(1) T_x^e(2) \left| \begin{array}{c} \bullet c_2 \\ \bullet c_1 \end{array} \right\rangle$$

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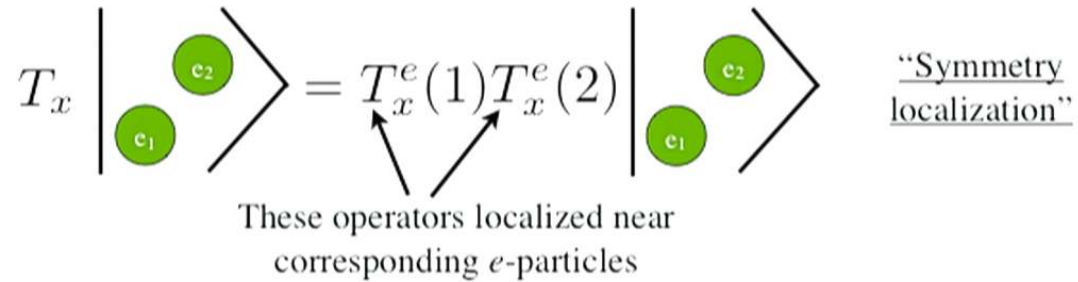
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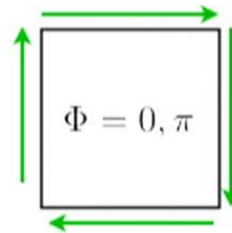
- Acting on state with two e -particles:



$$T_x T_y T_x^{-1} T_y^{-1} = 1$$

\downarrow

$$T_x^e T_y^e (T_x^e)^{-1} (T_y^e)^{-1} = \pm 1$$

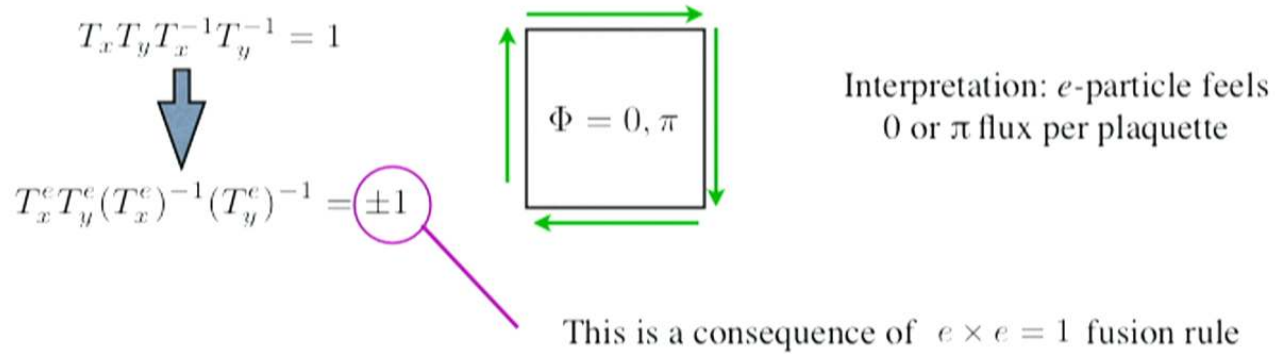
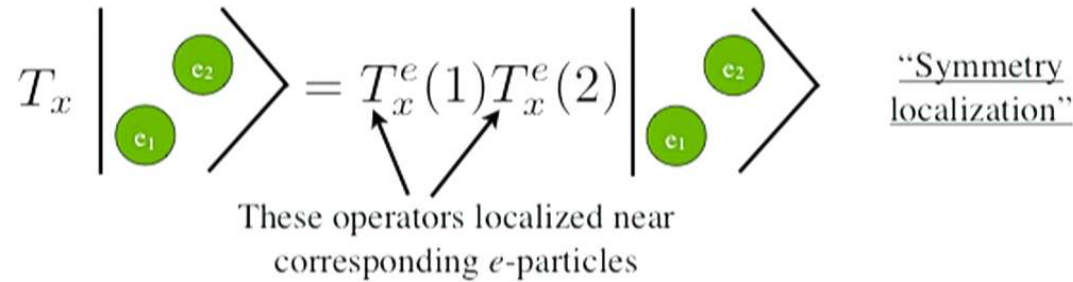


Interpretation: e -particle feels 0 or π flux per plaquette

Translation symmetry

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Translation symmetry

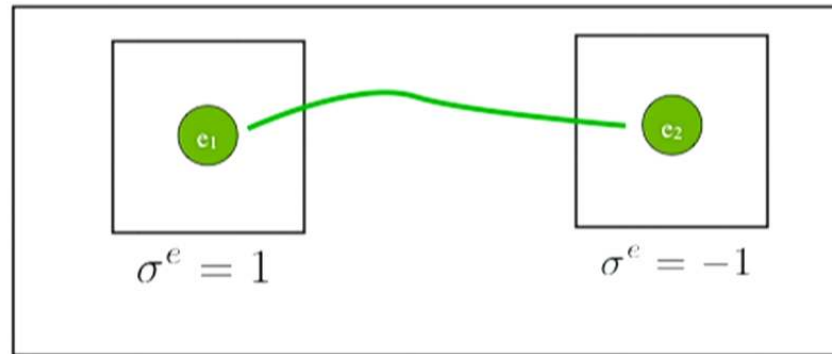
$$T_x^e T_y^e T_x^{e-1} T_y^{e-1} = \sigma^e = \pm 1 \text{ is } \underline{\text{constant}} \text{ on the } e\text{-sector}$$



Translation symmetry

$$T_x^e T_y^e T_x^{e-1} T_y^{e-1} = \sigma^e = \pm 1 \text{ is constant on the } e\text{-sector}$$

Argument: Suppose the contrary...



1-sector region on which $T_x T_y T_x^{-1} T_y^{-1} = -1$



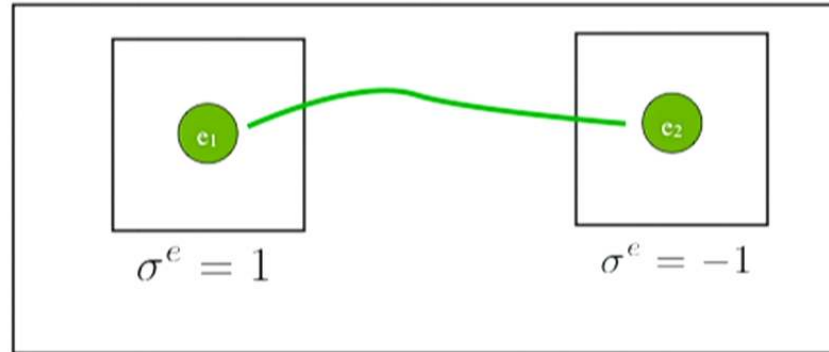
Contradiction!



Translation symmetry

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Contradiction!

This implies that σ^e is a robust property of a Z_2 spin liquid phase, as long as gap remains open and translation symmetry is preserved.



Translation symmetry: classes

	e	m	ε
$T_x T_y T_x^{-1} T_y^{-1}$	-1	1	-1
	-1	-1	1
	1	1	1

- Translation symmetry: 2 fractionalization classes & 3 symmetry classes
- These classes all realized in Kitaev toric code model (vary signs of vertex & plaquette terms)

- This is *not* a classification of irreps, but instead is the coarser classification desired.



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General symmetry group: fractionalization classes

- Some mathematics...
- Consider symmetry group G , elements $g \in G$, projective representation $\Gamma(g)$

$$\Gamma(g_1)\Gamma(g_2) = \omega(g_1, g_2)\Gamma(g_1g_2), \quad \omega(g_1, g_2) \in Z_2$$

“Factor set”

From fusion rules

- Associativity constraint: $\omega(g_1, g_2)\omega(g_1g_2, g_3) = \omega(g_1, g_2g_3)\omega(g_2, g_3)$
- Abelian group structure: $(\omega_A\omega_B)(g_1, g_2) = \omega_A(g_1, g_2)\omega_B(g_1, g_2)$
- “Gauge” transformation:

$$\Gamma(g) \rightarrow \lambda(g)\Gamma(g) \implies \omega(g_1, g_2) \rightarrow \lambda^{-1}(g_1)\lambda^{-1}(g_2)\lambda(g_1g_2)\omega(g_1, g_2)$$

- Classify factor sets up to “gauge” equivalence.
- Factor set classes also form Abelian group: $H^2(G, Z_2)$

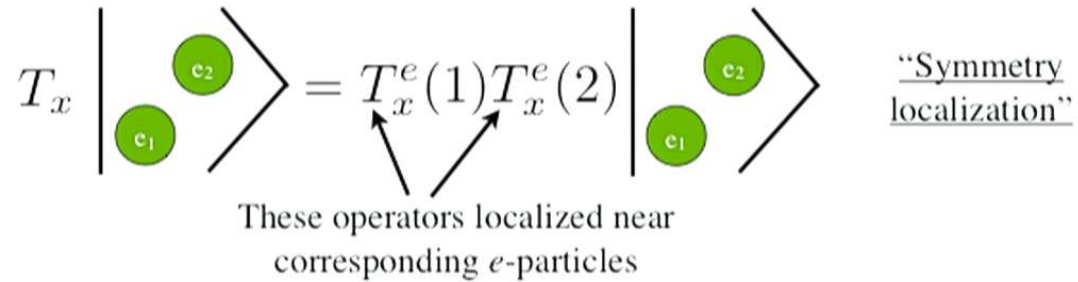
2nd cohomology
group, coefficients
in Z_2

Fractionalization class (for one sector) \longleftrightarrow Element of $H^2(G, Z_2)$

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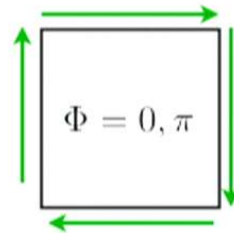
- Acting on state with two e -particles:



$$T_x T_y T_x^{-1} T_y^{-1} = 1$$

\Downarrow

$$T_x^e T_y^e (T_x^e)^{-1} (T_y^e)^{-1} = \pm 1$$



Interpretation: e -particle feels 0 or π flux per plaquette

General symmetry group: symmetry classes

Fractionalization class (for one sector) \longleftrightarrow Element of $H^2(G, Z_2)$

- Symmetry class given by specifying fractionalization class for each non-trivial superselection sector: $\omega_e, \omega_m, \omega_\epsilon \in H^2(G, Z_2)$
- But only two are independent: $\omega_\epsilon = \omega_e \omega_m$ (From $\epsilon = e \times m$ fusion rule.)
- Pair (ω_e, ω_m) can also be viewed as element of $H^2(G, Z_2 \times Z_2)$, since $H^2(G, Z_2 \times Z_2) \simeq H^2(G, Z_2) \times H^2(G, Z_2)$
- Symmetry classes are elements of $H^2(G, Z_2 \times Z_2)$, up to $e \leftrightarrow m$ relabeling

General symmetry group: symmetry classes

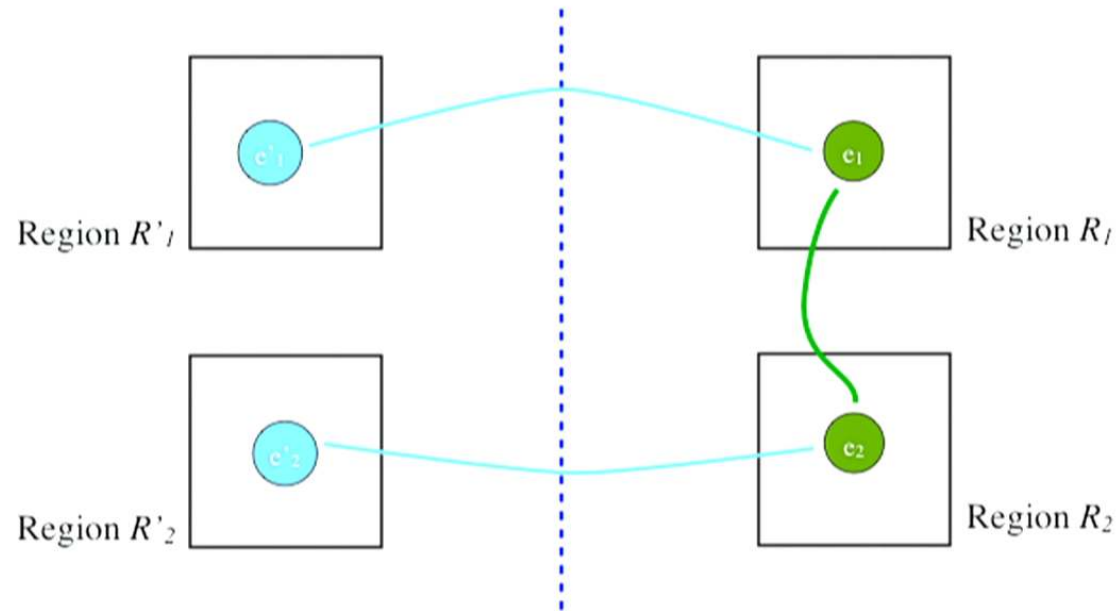
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Space group fractionalization classes

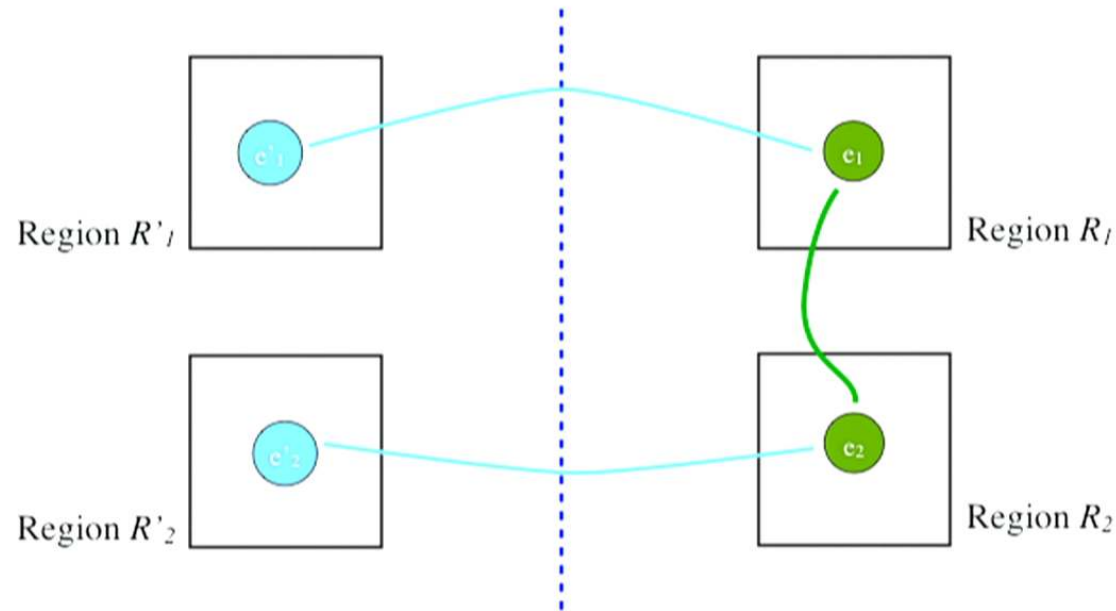
- General space group operations move some points large distances. Notion of symmetry localization needs to be modified.
- Example: P_x ($x \rightarrow -x$ reflection symmetry)
- Still have $P_x|\psi\rangle = P_x^e(1)P_x^e(2)|\psi\rangle$



- But now $P_x^e(i)$ has support on the union of regions R_i and R'_i and a linear region connecting the two. On this linear region, $P_x^e(i)$ is an e -string.

Space group fractionalization classes

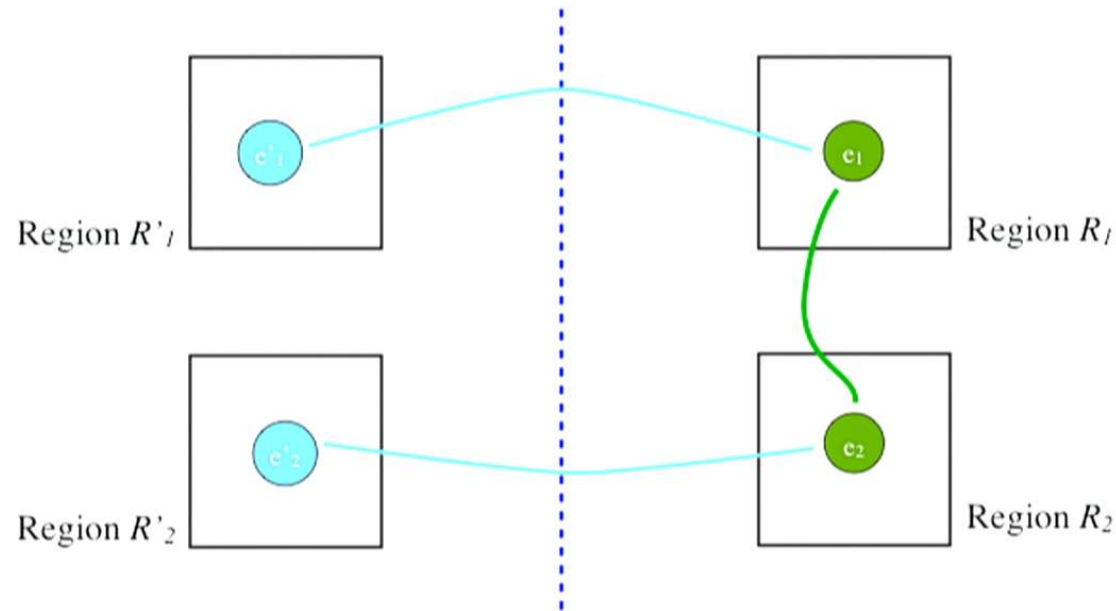
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Square lattice example

- $G = \text{Square lattice space group} \times \text{time reversal} \times \text{spin rotation}$.

- Square lattice space group generators: T_x, P_x, P_{xy}

- Note that: $T_y = P_{xy} T_x P_{xy}^{-1}$

- Time reversal \mathcal{T}

- Spin rotation (by θ about \hat{n} -axis): $R(\theta\hat{n})$

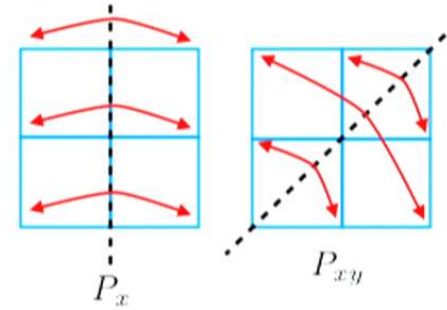
- Generators + relations specify the symmetry class in each non-trivial sector:

$$\begin{aligned}
 P_x^2 &= \sigma_{px} & \mathcal{T}T_x\mathcal{T}^{-1}T_x^{-1} &= \sigma_{Ttx} \\
 P_{xy}^2 &= \sigma_{pxy} & \mathcal{T}P_x\mathcal{T}^{-1}P_x &= \sigma_{Tpx} \\
 (P_x P_{xy})^4 &= \sigma_{pxpxy} & \mathcal{T}P_{xy}\mathcal{T}^{-1}P_{xy} &= \sigma_{Tpxy} \\
 T_x T_y T_x^{-1} T_y^{-1} &= \sigma_{txty} & R(2\pi\hat{n}) &= \sigma_R \\
 T_x P_x T_x P_x^{-1} &= \sigma_{txpx} & R(\theta\hat{n})\mathcal{T} &= \mathcal{T}R(\theta\hat{n}) \\
 T_y P_x T_y^{-1} P_x^{-1} &= \sigma_{typx} & R(\theta\hat{n})P_x &= P_x R(\theta\hat{n}) \\
 \mathcal{T}^2 &= \sigma_{\mathcal{T}} & R(\theta\hat{n})P_{xy} &= P_{xy} R(\theta\hat{n}) \\
 & & R(\theta\hat{n})T_x &= T_x R(\theta\hat{n})
 \end{aligned}$$

(+ Lie algebra of spin rotations)

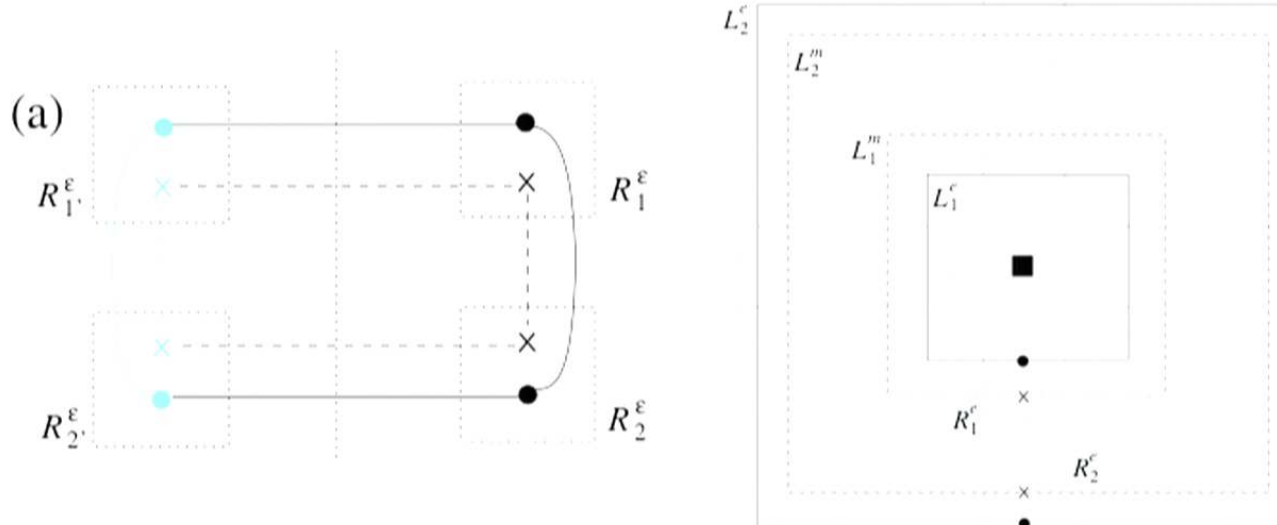
- Here the σ 's = ± 1

- 11 independent Z_2 parameters $\rightarrow H^2(G, Z_2) = (Z_2)^{11}$



Space group symmetry classes and braiding

- How to determine ε fractionalization class from e and m classes?
- It turns out that the H^2 product is “twisted”: $\omega_\varepsilon = \omega_l \omega_e \omega_m$
- Basic idea: view ε as a composite of e and m , work out symmetry-localized group relations keeping track of new statistical phase factors.



Space group symmetry classes and braiding

- Result for ω_t factor set (specify in terms of group relations):

$$(P_x P_{xy})^4 = -1 \quad \text{Other relations trivial.}$$

$$(P_x^e P_{xy}^e)^4 = \sigma_{r_x r_{xy}}^e \quad (P_x^m P_{xy}^m)^4 = \sigma_{r_x r_{xy}}^m \quad (P_x^\epsilon P_{xy}^\epsilon)^4 = -\sigma_{r_x r_{xy}}^e \sigma_{r_x r_{xy}}^m$$

- Only the “rotation” relation is modified ... consequence of braiding statistics.

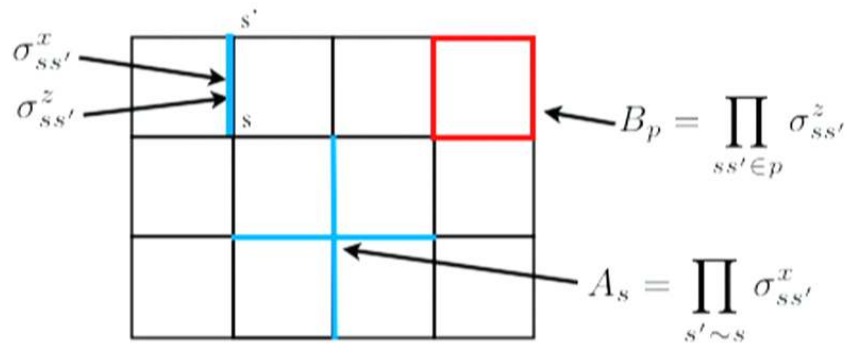
Outline

1. Motivation/background
2. Symmetry classification for non-point group symmetry
3. General symmetry classification
4. Realization of (some) symmetry classes in microscopic models



Concrete model: toric code

A. Kitaev



$$H = -u \sum_s A_s - K \sum_p B_p$$

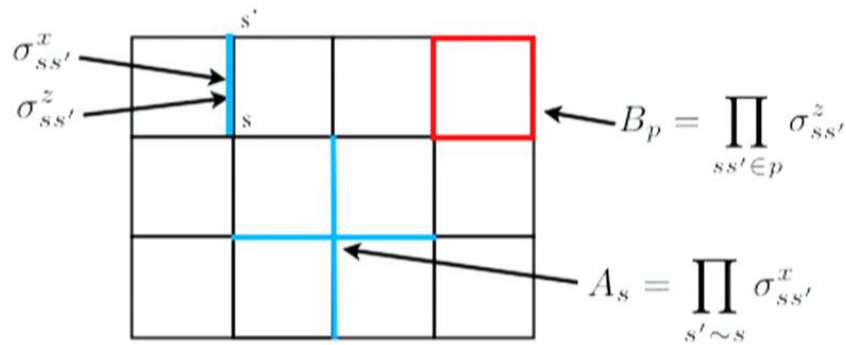
Choose $u, K = \pm 1$

- Ground state has: $A_s = u, B_p = K$
- e -particles live at vertices s where $A_s = -u$
- m -particles live at plaquettes p where $B_p = -K$
- Focus on square lattice space group symmetry. Can show *four* symmetry classes realized depending on u, K :

$$\begin{aligned} (P_x^e)^2 &= 1 & (P_x^m)^2 &= 1 \\ (P_{xy}^e)^2 &= 1 & (P_{xy}^m)^2 &= 1 \\ (P_x^e P_{xy}^e)^4 &= 1 & (P_x^m P_{xy}^m)^4 &= u \\ T_x^e T_y^e T_x^{e-1} T_y^{e-1} &= K & T_x^m T_y^m T_x^{m-1} T_y^{m-1} &= u \\ T_x^e P_x^e T_x^e P_x^{e-1} &= 1 & T_x^m P_x^m T_x^m P_x^{m-1} &= 1 \\ T_y^e P_x^e T_y^{e-1} P_x^{e-1} &= 1 & T_y^m P_x^m T_y^{m-1} P_x^{m-1} &= u. \end{aligned}$$

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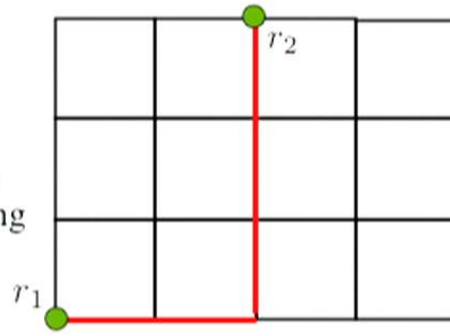
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$T_x^e T_y^e T_x^{e-1} T_y^{e-1} = K$	$T_x^m T_y^m T_x^{m-1} T_y^{m-1} = u$
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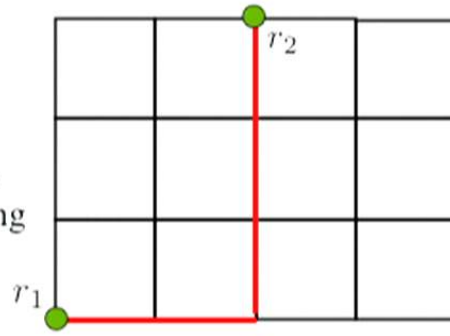
Explicit construction of symmetry localization

Initial state:
two e-particles,
connected by string

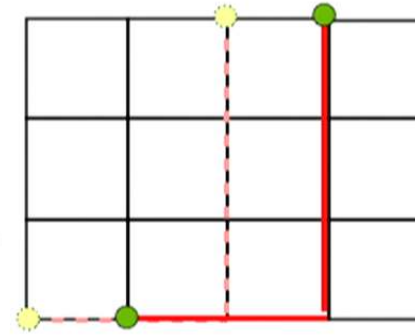


Explicit construction of symmetry localization

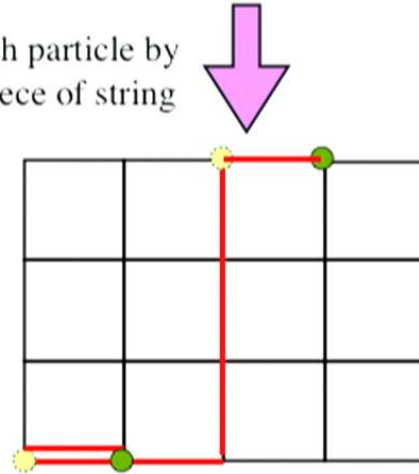
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Translation

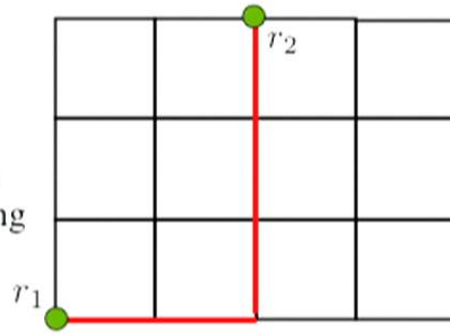


Move each particle by
adding piece of string

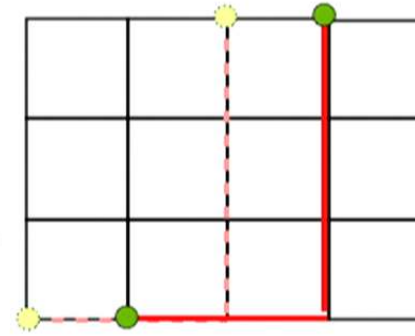


Explicit construction of symmetry localization

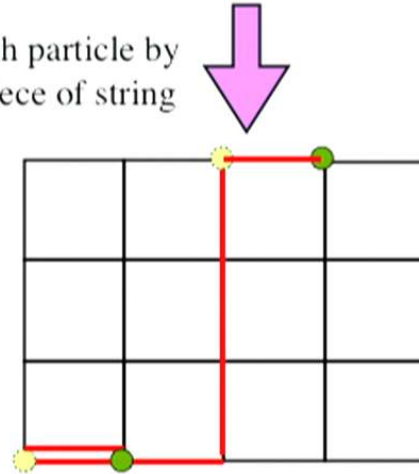
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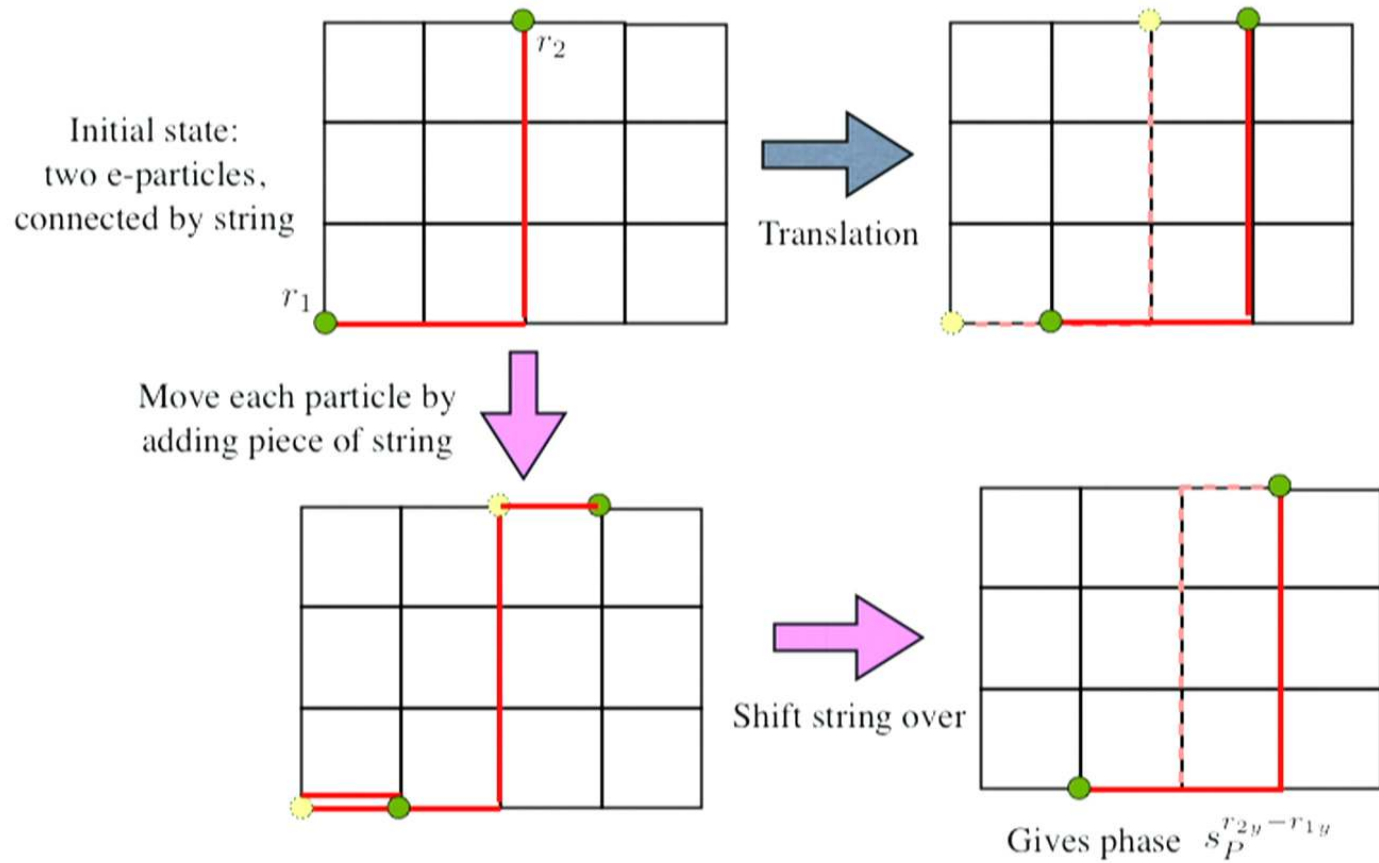
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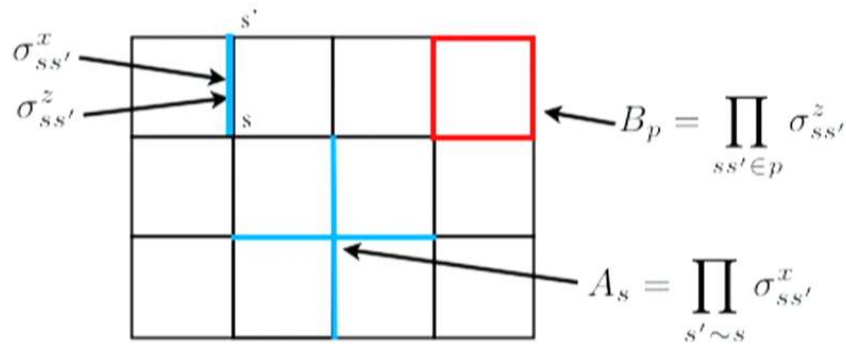
Explicit construction of symmetry localization



Define: $T_x^e(r) = (-1)^{r_y} \sigma_{r, r+x}^z \longrightarrow T_x = T_x^e(r_1) T_x^e(r_2)$

Concrete model: toric code

A. Kitaev



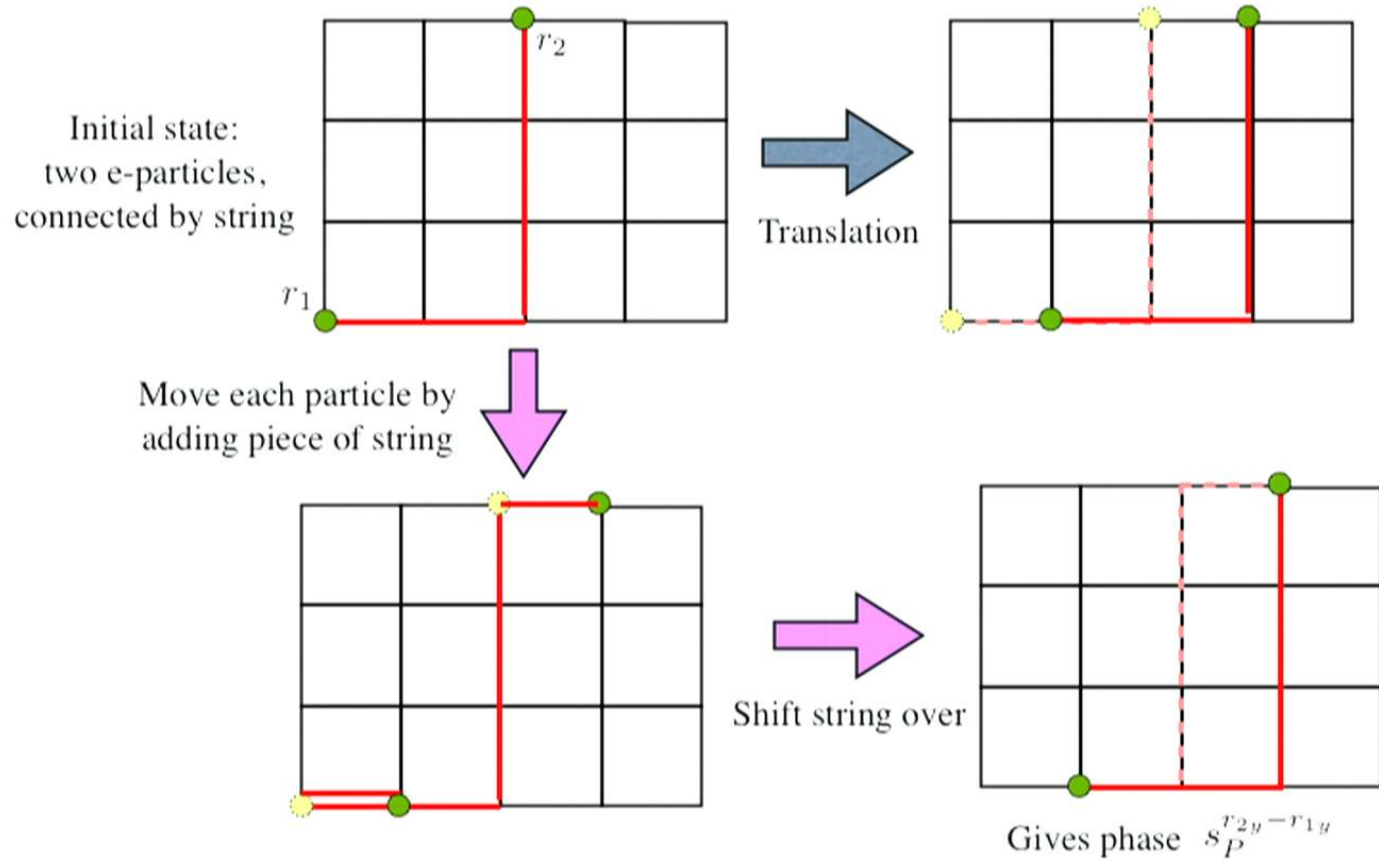
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Explicit construction of symmetry localization



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Explicit realization of (more) symmetry classes

with Hao Song

- Can *all* symmetry classes be realized? Probably not, in strictly 2d models. For some on-site symmetries, we know some classes can only be realized as the boundary theory of a d=3 SPT phase! (Vishwanath & Senthil; C. Wang & Senthil).
- Then, which/how many classes can be realized in 2d models?
- Approach: we study a class of generalized toric code models. Within this class, we prove that most of the 2080 symmetry classes are impossible, and find explicit realizations for the 82 others.
- Class of models: toric code defined on (almost) arbitrary 2d lattice. (Links may cross, but vertices may not stack.)
- No “spin-orbit coupling.” Label links by ℓ , then:

$$\text{Symmetry } S : \ell \rightarrow S(\ell)$$

$$S : \sigma_{\ell}^z \rightarrow \sigma_{S(\ell)}^z$$

$$S : \sigma_{\ell}^x \rightarrow \sigma_{S(\ell)}^x$$



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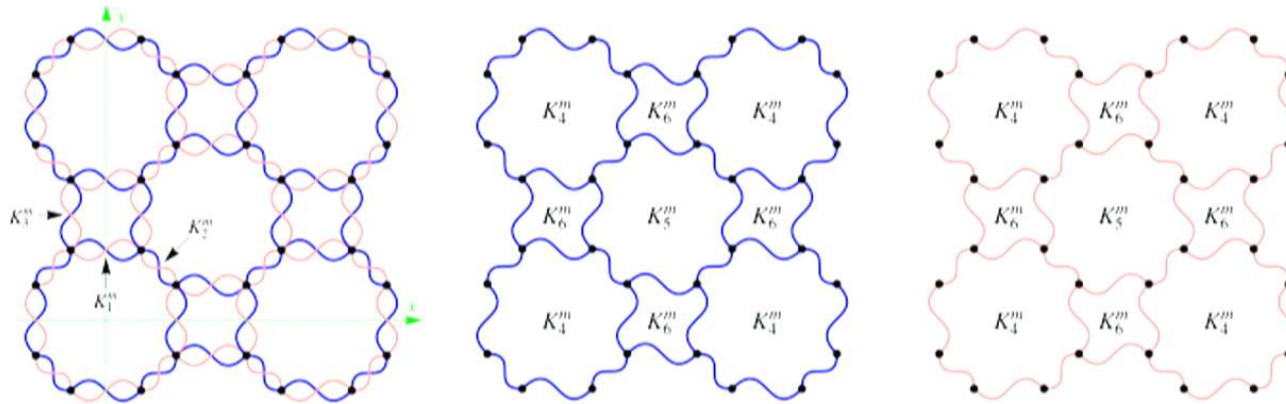
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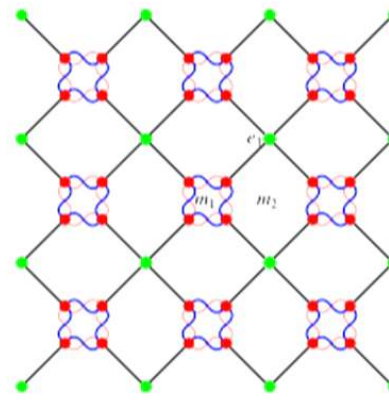
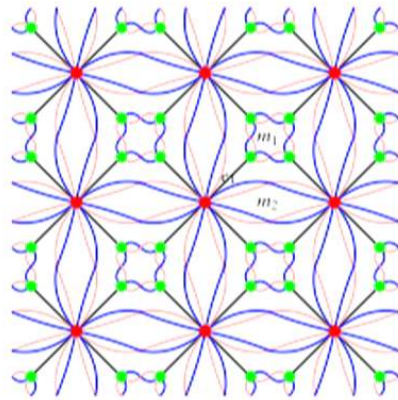
All e -fractionalization classes can be realized



- Model on this lattice has six independent Z_2 parameters controlling the sign of fluxes in the ground state.
- All 2^6 e -fractionalization classes realized by varying these parameters
- m -fractionalization class is trivial

Other classes

- General arguments show that only 82 symmetry classes are possible (so only 18 where both e and m classes are non-trivial)



Open issues

- General understanding of which symmetry classes possible in strictly 2d? Can all classes be realized at boundary of 3d SPT phases?
- Generalization to other topological orders (this is trivial for any Abelian topological order with non-point group symmetry)
- Full symmetry classification, including “beyond fractionalization” $e \leftrightarrow m$ interchange
- Three dimensions ... connection to edge states of 2d SPT phases?
- How can symmetry class be determined given ground state wavefunction, excited states? Application to numerics on kagome & J_1 - J_2 Heisenberg models?
- Experimental signatures?

