Title: The Quantum Steering Ellipsoid

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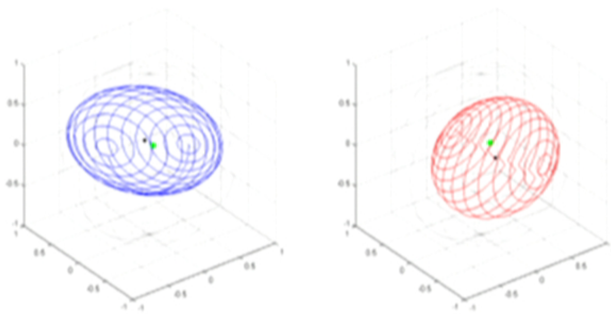
Abstract: A quantum steering ellipsoid may be used to faithfully represent a density matrix ? describing two qubits A and B. The ellipsoid is the geometric set of states that Bob can steer Alice's qubit to when he implements all possible measurements on his qubit. We show how the correlations between qubits A and B manifest themselves in this paradigm, giving simple conditions for when the state is entangled, or has discord. We will also present novel features of the two qubit state that are revealed by the steering ellipsoid formalism, and show that a state corresponding to an ellipsoid with non-zero volume contains a new type of correlation.



Quantum Steering Ellipsoids

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Matt Pusey, David Jennings, Terry Rudolph

arXiv: quant-ph 1303.4724

Imperial College London

Controlled Quantum Dynamics

EPSRC Engineering and Physical Sciences Research Council

Royal Commission Exhibition of 1851

Quantum steering ellipsoid

- Alice and Bob each have a qubit, in state ρ_{AB}

- QSE := set of all states that Bob (Alice) can collapse Alice (Bob) to with finite probability

Why are we interested?

- Allows geometric representation of quantum state
  - Single qubit – Bloch sphere
  - Higher dimensions – difficult

- Discovered most (all?) properties of purely from geometry ρ_{AB}

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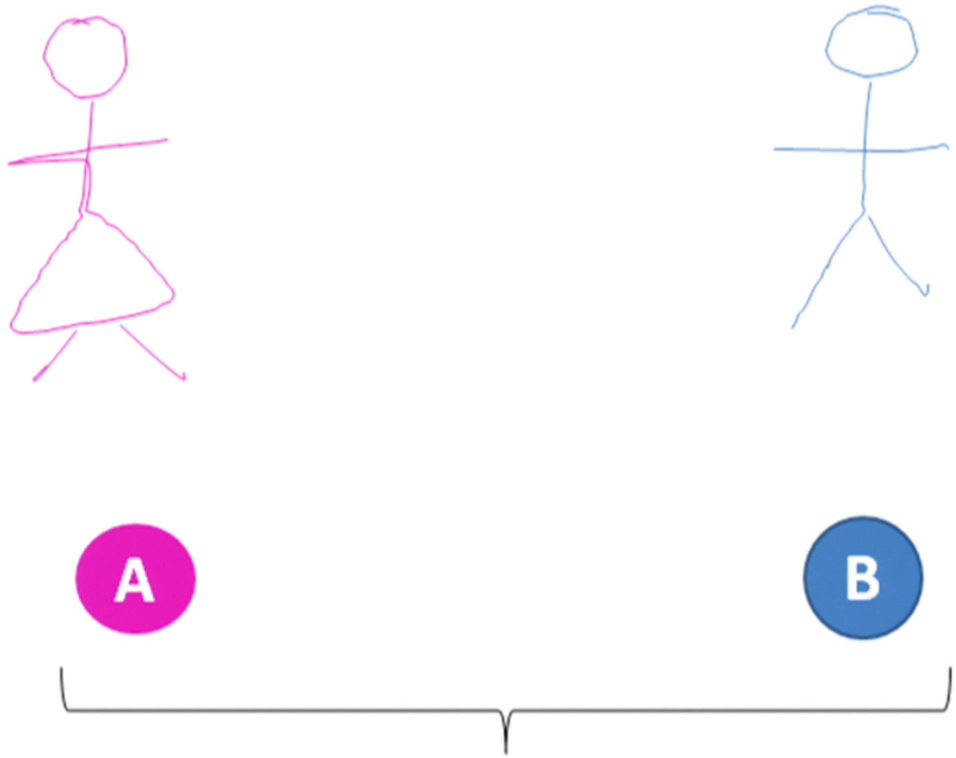
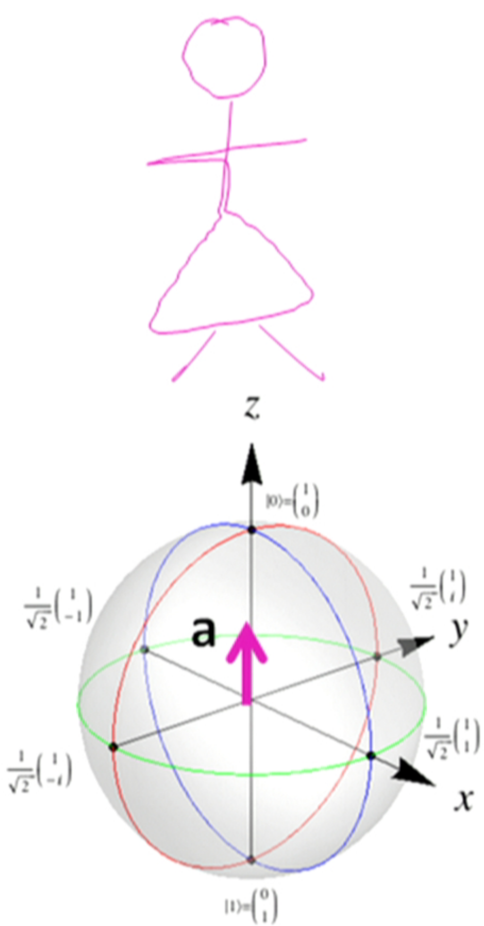
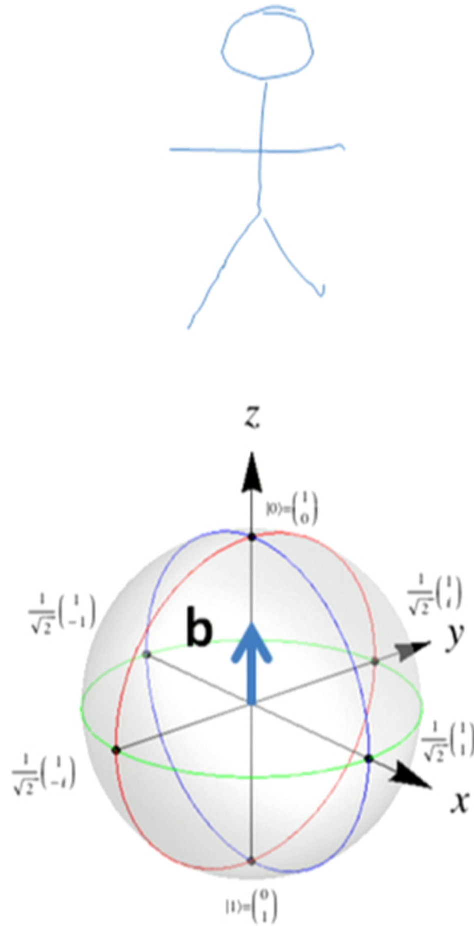
Pure state steering



$$|\psi_{AB}\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$$

Pure state steering





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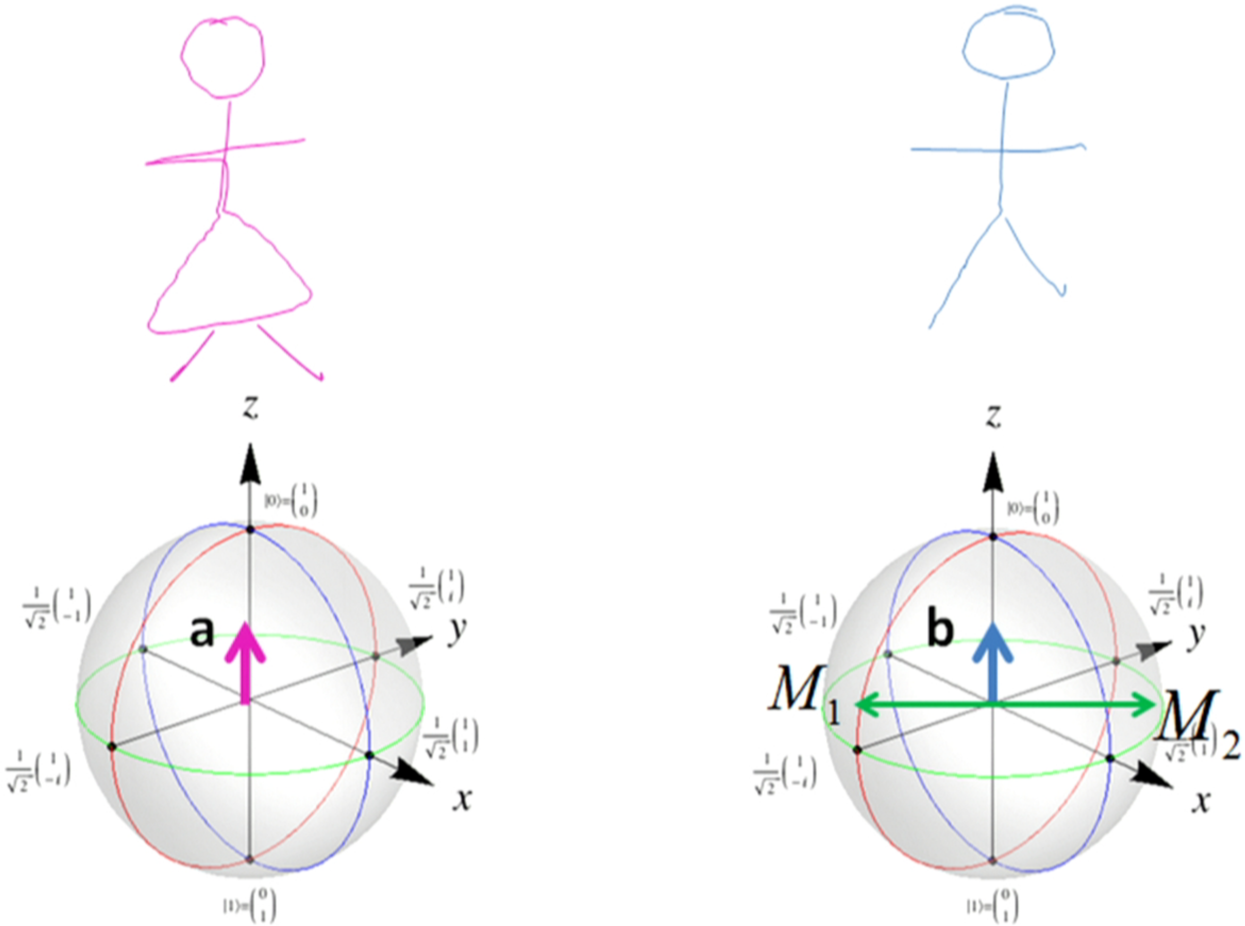

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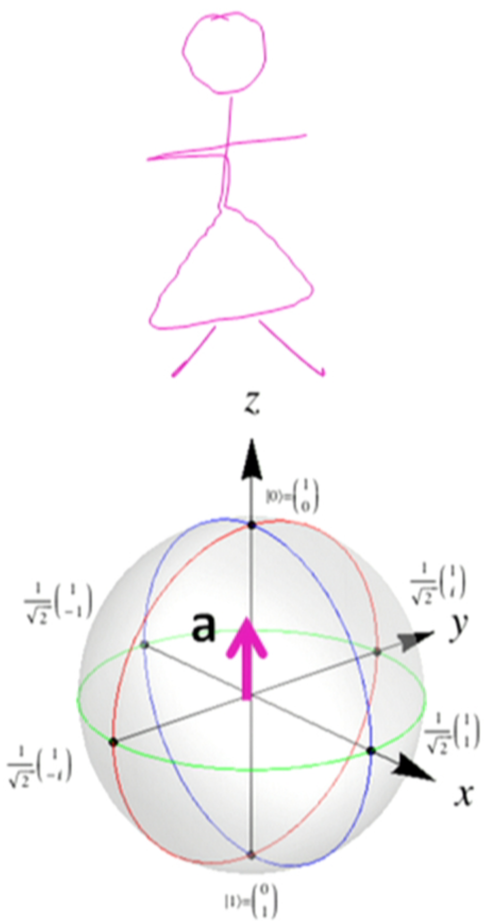

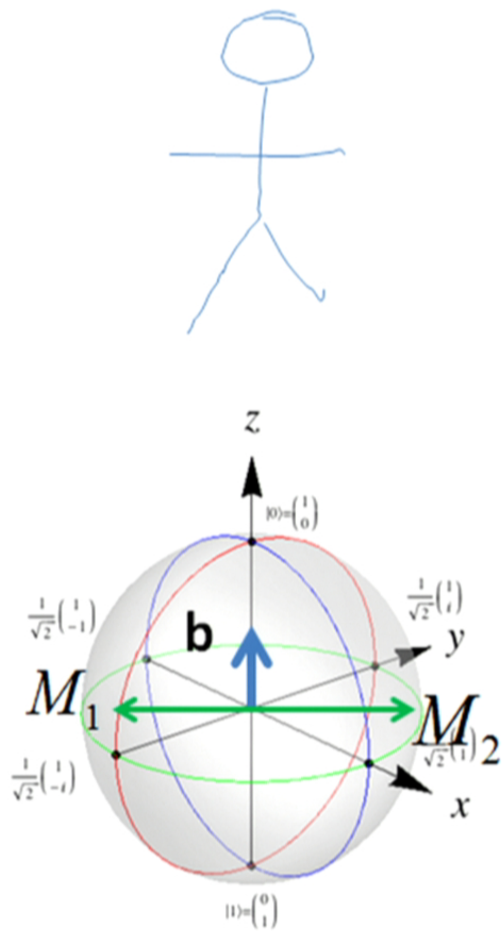

Pure state steering

- (1) Bob can collapse Alice to any state in her Bloch sphere

$$\left(I \otimes \langle M|\right)|\psi_{AB}\rangle = \cos\theta \langle M|0\rangle|0\rangle + \sin\theta \langle M|1\rangle|1\rangle$$
$$\propto \cos\theta' |0\rangle + e^{i\phi} \sin\theta' |1\rangle$$

- So Alice's steering ellipsoid is the *full Bloch sphere*

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Pure state steering

- (2) Given a convex decomposition of a, there is a POVM that Bob can do to steer Alice to it
  - Call this *complete steering of qubit A*

Pure state steering

- (1) Bob can collapse Alice to any state in her Bloch sphere

- (2) Given a convex decomposition of a, there is a POVM that Bob can do to steer Alice to it

- (1) + (2) = pure state steering theorem
  - Schrodinger $|\psi_{AB}\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$

General steering ellipsoid

$$\left|\psi_{AB}\right\rangle \to \rho_{AB}$$

- Now the set of states Bob can steer Alice to is a subset (ellipsoid) of the full Bloch sphere
  - (1) always breaks down

General steering ellipsoid

$$\rho_{AB} = \frac{1}{4} \sum_{\mu\nu=0}^{3} \Theta_{\mu\nu} \sigma_{\mu} \otimes \sigma_{\nu} \qquad \sigma_{\mu} = \{I, X, Y, Z\}$$

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$$E = \frac{1}{2}\sum_{\mu=0}^{3} X_\mu \sigma_\mu \qquad E \geq 0 \qquad X_0 \geq 0$$

$$X_0^2 \geq \sum_{i=1}^{3} X_i^2$$

General steering ellipsoid

$$\rho_A^{steer} \propto tr_B\left(\rho_{AB} I \otimes E\right)$$

$$= tr_B\left[\frac{1}{4}\sum_{\mu\nu=0}^{3}\Theta_{\mu\nu}\sigma_\mu \otimes \sigma_\nu . I \otimes \frac{1}{2}\sum_{\gamma=0}^{3} X_\gamma \sigma_\gamma\right]$$

General steering ellipsoid

- **Full set of steered states at A is invariant under *local filters* at B**

$$\rho_{AB} \to \frac{I \otimes F_B \rho_{AB} (I \otimes F_B)^+}{tr\left(I \otimes F_B \rho_{AB} (I \otimes F_B)^+\right)}$$

- **Local filters on density matrices are Lorentz boosts on Pauli basis matrices** $\Theta' = \Theta \Lambda_B$

$$Y' = \frac{1}{2}\Theta' X = \frac{1}{2}\Theta X' \qquad X' = \Lambda_B X$$

- **Same steering ellipsoid!**

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General steering ellipsoid

- With probability = ½, Alice is steered to

$$y = a' + T' x$$

- Set of Alice's collapsed states is unit sphere of possible x, $|x| \leq 1$ shrunk and rotated by T', translated by a', i.e. an ellipsoid
  - Semiaxes and orientation: SVD of $T' = O_1 D O_2^T$
  - Centre: a'

General steering ellipsoid

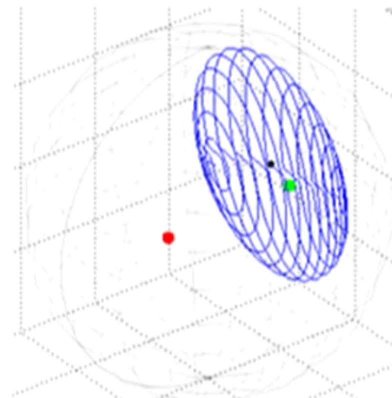
- Undoing the filter to get back to original state does not change the steering ellipsoid at A, but gets us the form of the ellipsoid in terms of the original components of Θ

$$Q_A = \frac{1}{1-b^2}\left(T - ab^T\right)\left(I + \frac{bb^T}{1-b^2}\right)\left(T^T - ba^T\right)$$

$$c_A = \frac{a - Tb}{1-b^2}$$

General steering ellipsoid – not new

- Frank Verstraete: for many ρ_{AB}, $\left(Q_A, c_A, a, b\right)$ give faithful representation



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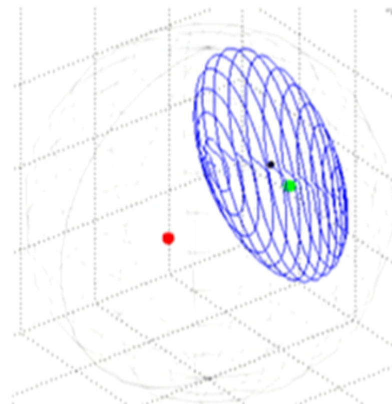


Our work

- We've shown that for *all* states, $\left(Q_A, c_A, a, b\right)$ give faithful representation

- All interesting properties deducible from geometry

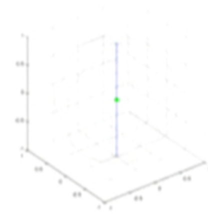
- Frank only considered "pure-like" mixed states – steer to any convex decomposition, but there exist non-completely steerable states

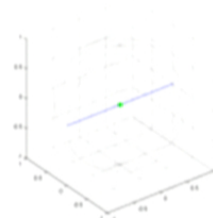
Volume related to quantum correlations

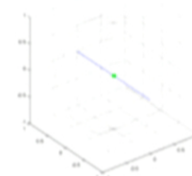
$$V_A = \frac{64\pi}{3} \frac{\left|\det\rho - \det\rho^{T_B}\right|}{\left(1-b^2\right)^2}$$

Volume related to quantum correlations

$$\frac{1}{3}\left[\frac{1}{2}\left(|00\rangle\langle 00|+|11\rangle\langle 11|\right)\right]+\frac{1}{3}\left[\frac{1}{2}\left(|++\rangle\langle ++|+|--\rangle\langle --|\right)\right]+\frac{1}{3}\left[\frac{1}{2}\left(|+i+i\rangle\langle +i+i|+|-i-i\rangle\langle -i-i|\right)\right]$$







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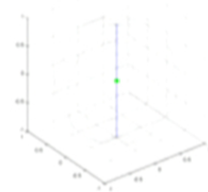

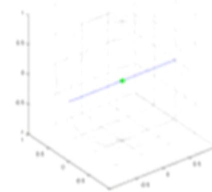

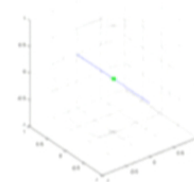

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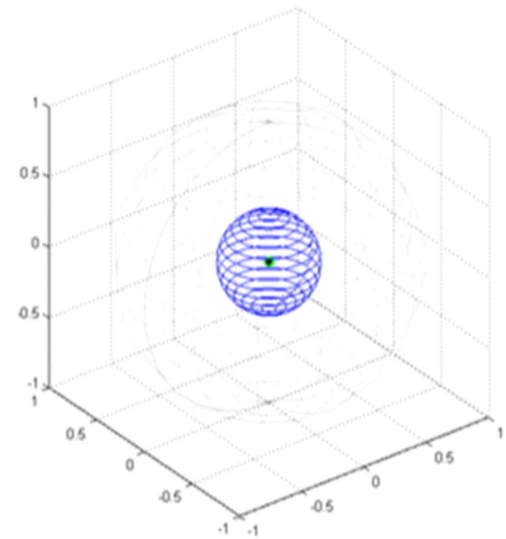

Volume related to quantum correlations

$$V_A = \frac{64\pi}{3} \frac{\left|\det \rho - \det \rho^{T_B}\right|}{\left(1 - b^2\right)^2}$$

Can deduce that entangled states *always* have finite volume

Separable states $V \geq 0$

Volume acts as an entanglement witness:
If $V \geq V_* > 0$ then entangled

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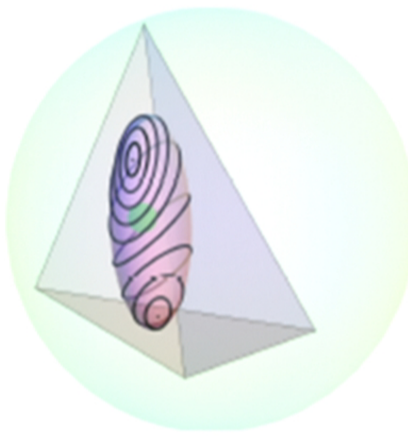
Necessary and sufficient condition for entanglement

ρ_{AB} **separable iff steering ellipsoid fits in tetrahedron inside Bloch sphere**



Necessary and sufficient condition for entanglement

IF: Any separable state

$$\rho_{AB} = \sum_{i=1}^{n=4} p_i \alpha_i \otimes \beta_i$$

A. Sanpera, R. Tarrach, and G. Vidal, Phys. Rev. A 58, 2 (1998)

ONLY IF: Harder

Decomposition of separable states

The minimal number of product states a separable state can be decomposed into:

$$\rho_{AB} = \sum_{i=1}^{n} p_i \alpha_i \otimes \beta_i \quad \textbf{where} \quad \min n = \dim \varepsilon_A + 1$$

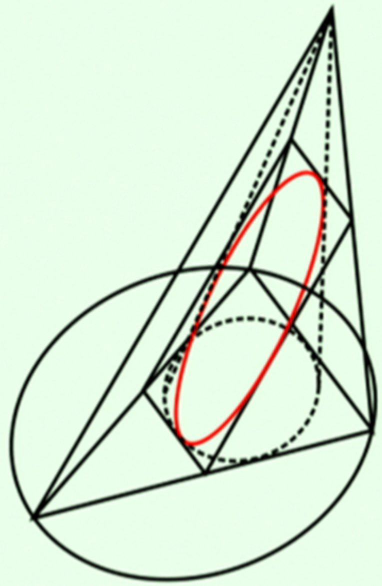
Decomposition of separable states

Came down to purely geometrical question:

Does every ellipse inside a tetrahedron inside a ball fit in a triangle inside the ball?

For simplicity, assume that the setup is "tight" in that E meets every face of T and the vertices of T are on the boundary of B. The generalisation should be straightforward.

If any vertices of T lie in the plane of E then the cross-section is already a triangle, so assume otherwise. Pick an arbitrary vertex of T and draw a cone staring from there with E (shown in red) as a base. This cone meets the opposite face of T in a new ellipse E' (shown as a dotted black curve):



E' lies inside a triangle (the face of T), which itself lies inside a circle (the cross-section of B). By our simplifying assumption, E' touches the edges of the triangle and the triangle's vertices are on the circle. Therefore Poncelet's Porism applies. Hence we can find a different triangle around E' with one edge in the plane of E.

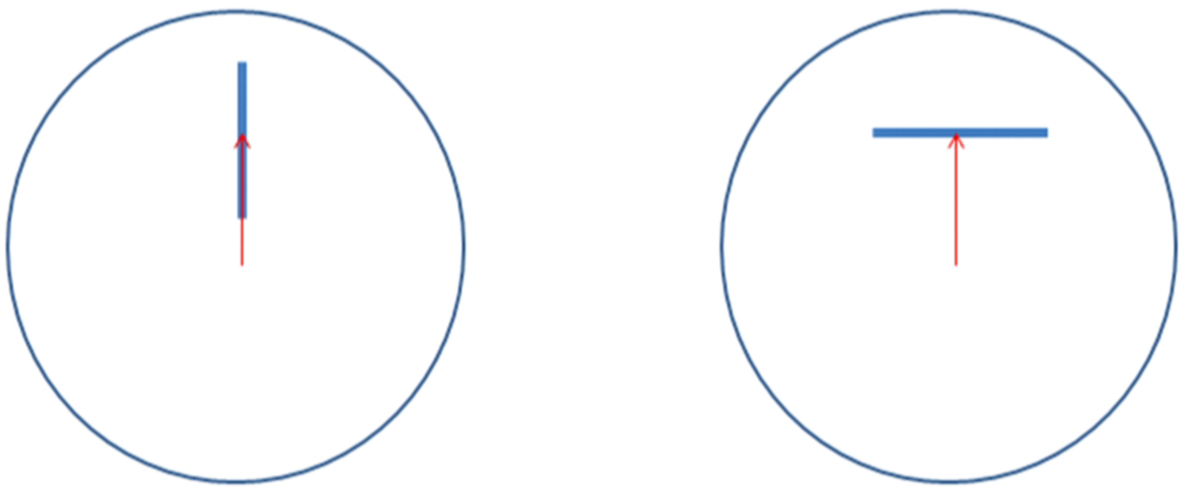
http://mathoverflow.net/questions/108331/does-every-ellipse-inside-a-tetrahedron-inside-a-ball-fit-in-a-triangle-inside-th

Strength of correlations depends on three key ellipsoid features

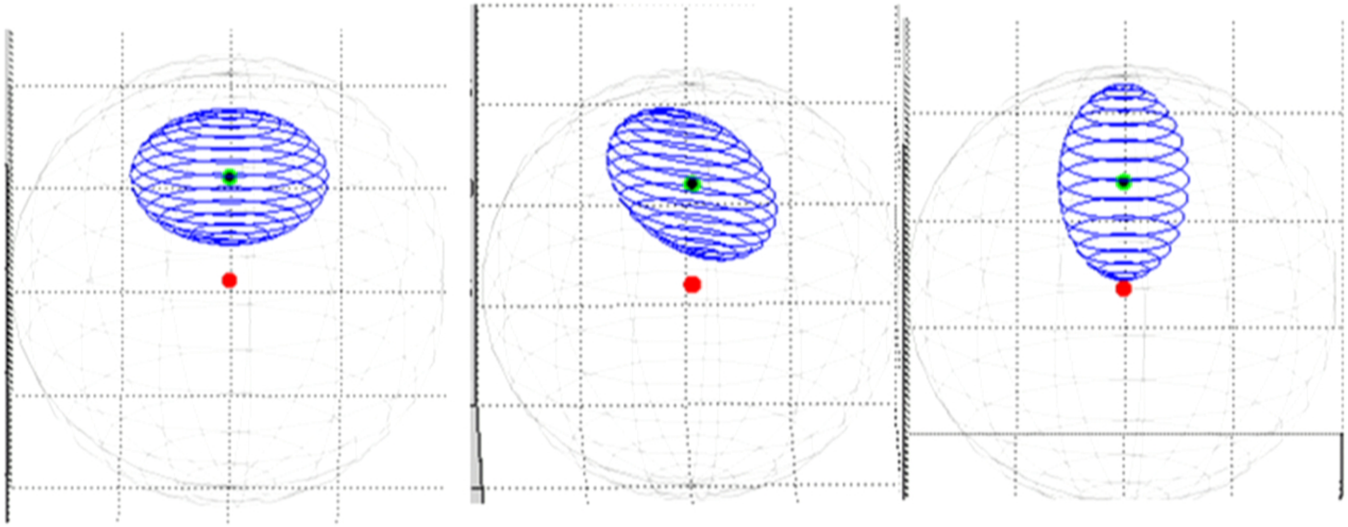
- Volume
- Distance centre from origin
- Spatial orientation – "skew"

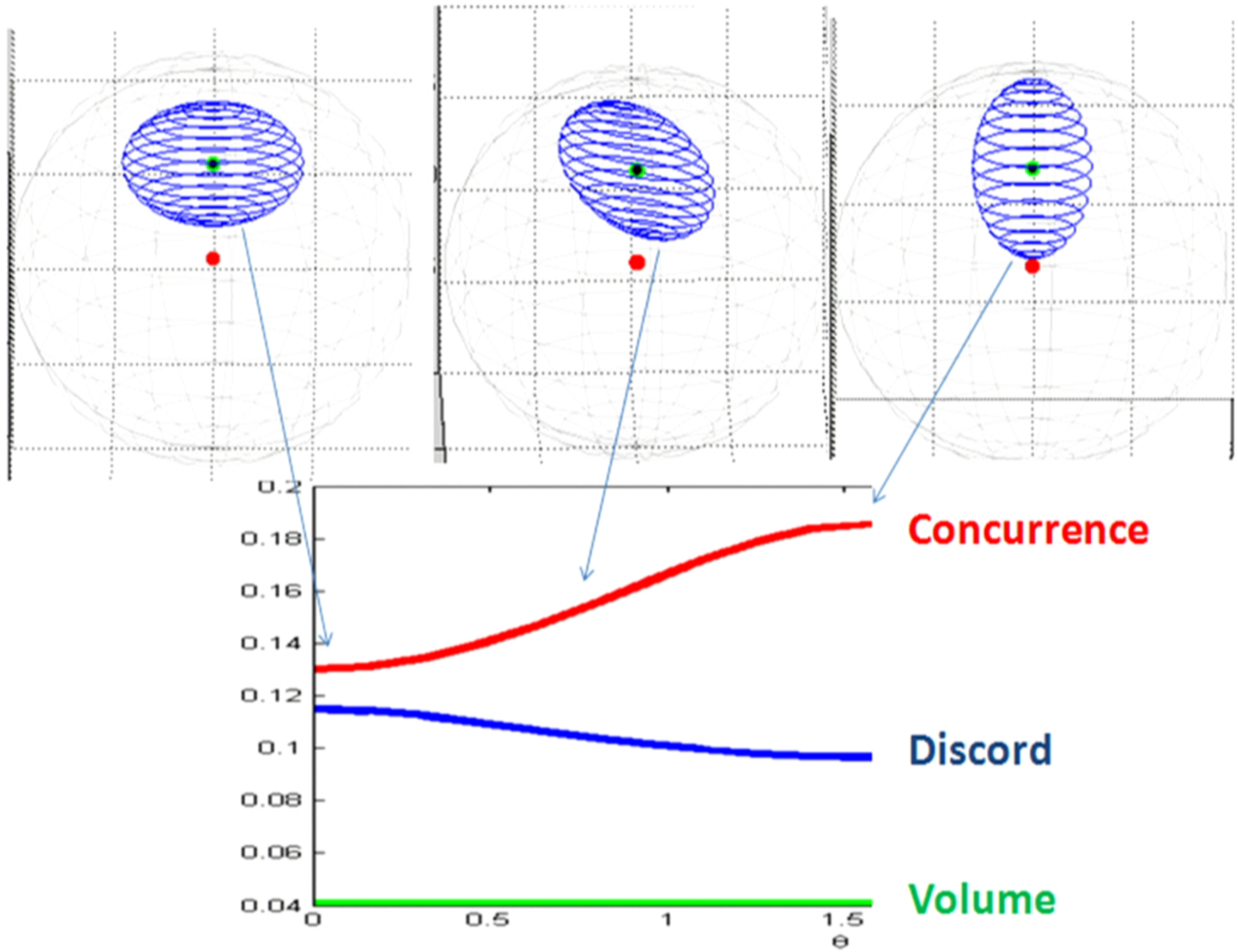
Strength of correlations depends on three key ellipsoid features

Example: skew and discord



Zero discord | Finite discord

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θ
Concurrence
Discord
Volume

Obesity

$$V > 0$$

Obesity

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Obesity

- New type of correlation:
  - A state is obese if $V > 0$
- Non trivial:
  - All entangled states are obese
  - Some separable states are obese
  - All obese states have finite discord
- Obese states are a resource but at the moment our only example is fairly contrived teleportation type scenarios

Complete steering of A

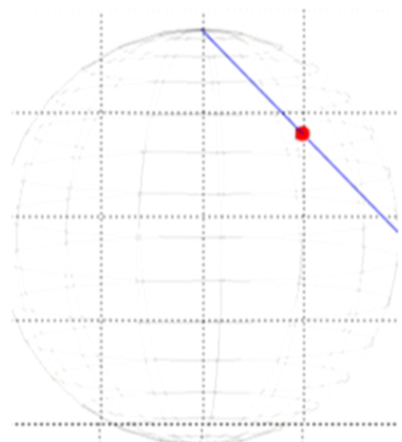
- Qubit A can be steered to any state in its steering ellipsoid
- For any convex decomposition in A's ellipsoid, there exists a POVM that will steer to it

Holds for: all obese states, states with b=0,
some others

Incomplete steering of qubit A

- Consider the state

$$\rho_{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |++\rangle\langle ++|\right)$$



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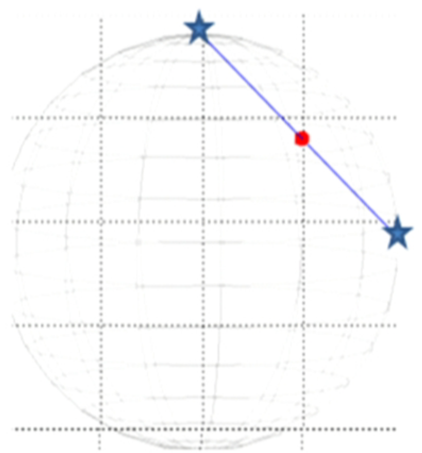

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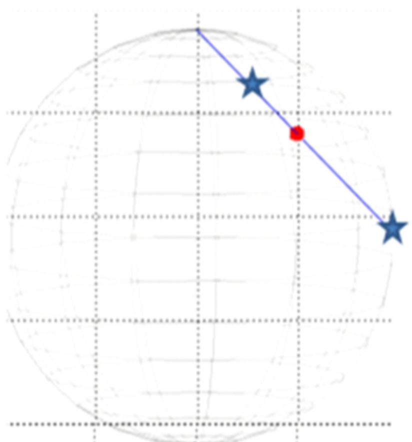

$$\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$$

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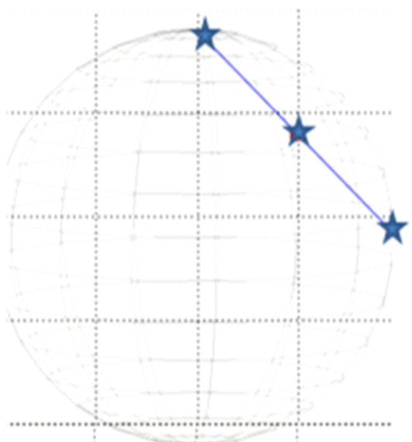

$$\left\{\begin{matrix} |1\rangle\langle 1|, |-\rangle\langle -|, \\ I - |1\rangle\langle 1| - |-\rangle\langle -| \end{matrix}\right\}$$

Conclusion & Further Work

- Steering ellipsoids are great because:
  - Can faithfully represent ρ_{AB} in a single Bloch sphere
  - Visualise properties geometrically
  - Leads to new features of two qubit states

- Calculating Wiseman steerability

- Easier way to find best separable approximation?

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