Title: Comments on short strings and Wilson loops in AdS/CFT

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Abstract: In the study of the string/gauge theory duality (AdS/CFT), an important role is played by the relation between local operators and Wilson loops. Perhaps the most well known example is the relation between twist two operators and the light-like cusp Wilson loop. On the string side, the twist two operator is represented by a "long" string (GKP). In this talk I use T-duality to argue that such relation is also natural for "short" strings. I discuss some examples and present a map between the shape of a short string crossing the Poincare horizon and the shape of a corresponding Wilson loop.
Based on arXiv:1212.4886 with Arkady Tseytlin (Imperial College).

Comments on short strings and Wilson loops in AdS/CFT

M. Kruczenski

Purdue University

Based on work in collaboration with A. Tseytlin (Imperial College).
arXiv:1212.4886

Perimeter Institute 2013

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Summary

- **Introduction**

String / gauge theory duality (**AdS/CFT**)

- **Relation between local operators and Wilson loops**

Twist two operators and light-like cusp.
Involves far from BPS, “long” strings.

Much was learned from “short” near BPS strings (BMN).

Suggestive facts indicating a relation for “short”, near BPS strings.

- **Map between short strings and Wilson loops.**

Use T-duality to map short strings falling into the Poincare extremal horizon to Wilson loops.

- **Simple Examples**

Point-like string along a light-like geodesic in AdS and on the sphere (BMN vacuum).

- **Fluctuations, LL model, etc.**

One can map more generic states to wavy line Wilson loops.

- **Conclusions**

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- **Conclusions**

Gauge theory ⟷ String theory

λ small $\rightarrow$ gauge th.

λ large $\rightarrow$ string th.

Strings live in curved space, e.g.

$AdS_5 x S^5$

S^5: $Y_1^2+Y_2^2+\ldots+Y_6^2 = 1$

AdS_5: $X_1^2+X_2^2+\ldots-X_0^2-X_{-1}^2 = -1$ (hyperbolic space)

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AdS/CFT correspondence (Maldacena 1997)

Gives a precise example of the relation between strings and gauge theory.

Gauge theory	String theory
$\mathcal{N}=4$ SYM SU(N) on R^4	IIB on $AdS_5 x S^5$
A_μ , Φ^i, Ψ^a	radius R
Operators w/ conf. dim. Δ	String states w/ $E = \frac{\Delta}{R}$

$$g_s = g_{YM}^2; \qquad R / l_s = (g_{YM}^2 N)^{1/4}$$

$N \to \infty, \lambda = g_{YM}^2 N$ **fixed** ➡

λ large → string th.
λ small → field th.

Poincare and global coordinates in AdS.

The boundary of global coordinates is RxS^3 and the boundary of Poincare is $R^{3,1}$.
The field theory lives in the boundary metric, therefore string theory in global AdS is dual to gauge theory on RxS^3 and in Poincare dual to gauge theory on $R^{3,1}$.

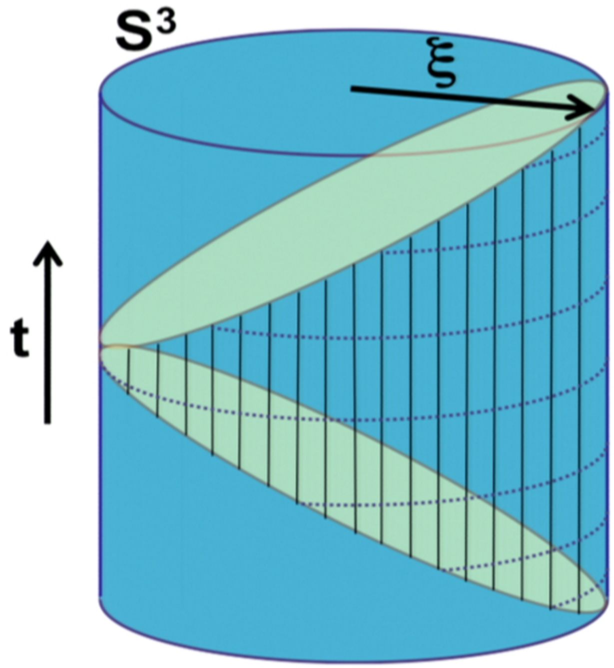
In Minkowski signature, Poincare coordinates have an extremal horizon and, in fact, only cover part of the space.

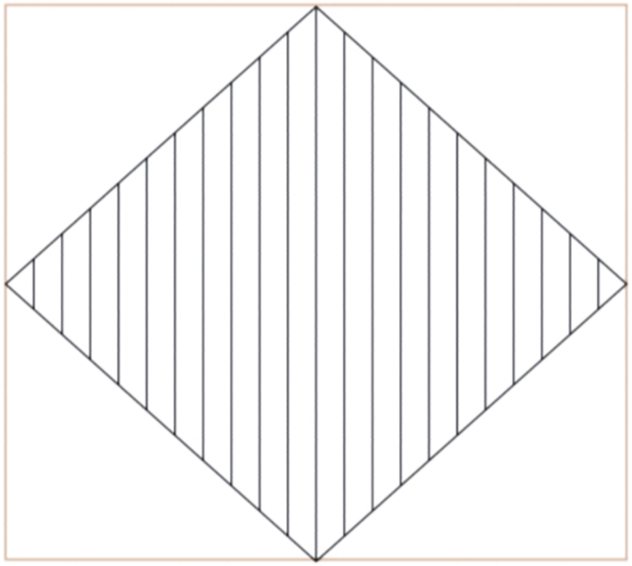
$$ds^2 = -\cosh^2 \rho \; dt^2 + d\rho^2 + \sinh^2 \rho \; d\Omega^2_{[3]}$$

$$ds^2 = \frac{1}{Z^2}(dZ^2 + d\mathcal{X}_\mu d\mathcal{X}^\mu)$$

6

Relation between global and Poincare patches





$$\cosh \rho = 1/\cos \xi$$

$$ds^2 = \frac{-dt^2 + d\xi^2 + \sin^2 \xi \, d\Omega^2_{[3]}}{\cos^2 \xi} \qquad 0 < \xi < \pi/2$$

String states ⟺ Local operators:

Strings moving inside global AdS

⇕ AdS/CFT correspondence

States of field theory on $S^3 x R^1$

⇕ State/op. correspondence (conf. theory)

Local operators for field theory on $R^{(3,1)}$

Example: BMN geodesic (angle ϕ is on the sphere S^5)

$$t = \mathcal{J}\tau, \quad \phi = \mathcal{J}\tau, \quad \rho = 0.$$

⇕

$$\mathcal{O} = \mathrm{Tr} X^J, \quad \mathcal{J} = \frac{1}{\sqrt{\lambda}} J$$

8

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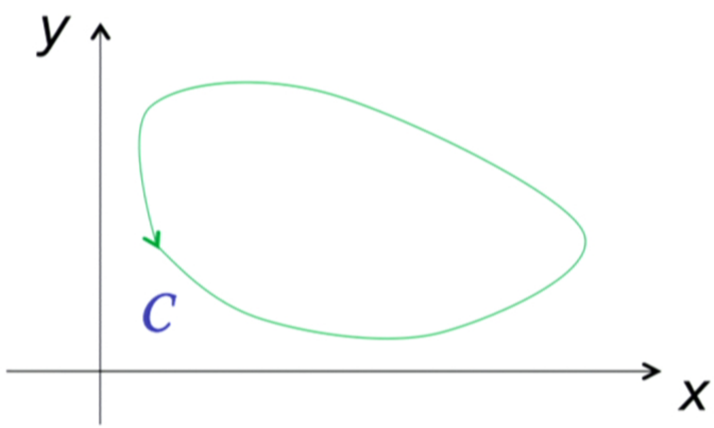
Wilson loops: associated with a closed curve in space. Basic operators in gauge theories. E.g. $q\bar{q}$ potential.

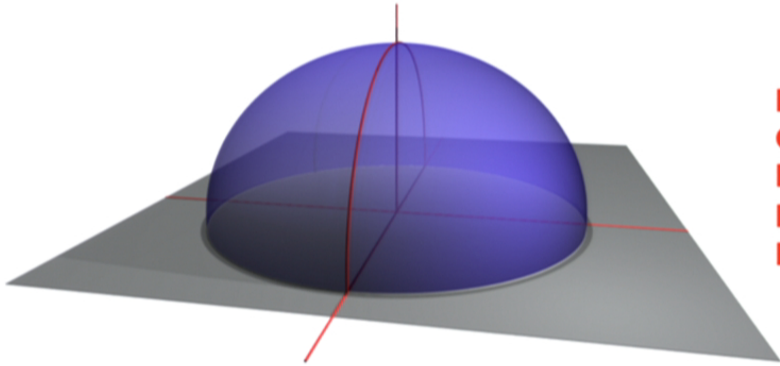


$$W = \frac{1}{N}\mathrm{Tr}\ \hat{\mathrm{P}}\exp\left\{i\oint_{\mathcal{C}}\left(A_\mu\frac{dx^\mu}{ds} + \theta^a(s)\Phi_a\left|\frac{dx^\mu}{ds}\right|\right)\right\}$$

9

Open strings ending on the boundary correspond to Wilson loop operators:

Example: Circular Wilson loop



Berenstein-Corrado-Fischler-Maldacena 1999
Gross-Ooguri 1998,
Erickson-Semenoff-Zarembo 2000
Drukker-Gross 2001
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Equivalent to straight line Wilson loop:

$$T = \tau, \quad Z = \sigma$$

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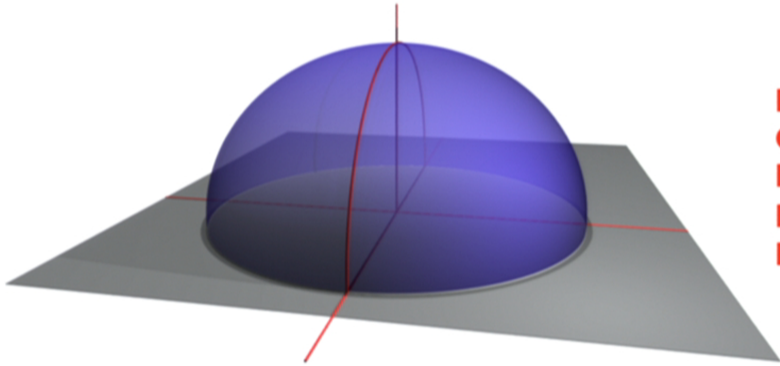

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Is there a relation between the two type of observables, i.e. operators and Wilson loops?

Yes, perhaps the most well known example is the relation between large spin twist two operators and light-like cusp Wilson loop (Korchemsky-Marchesini 1993).

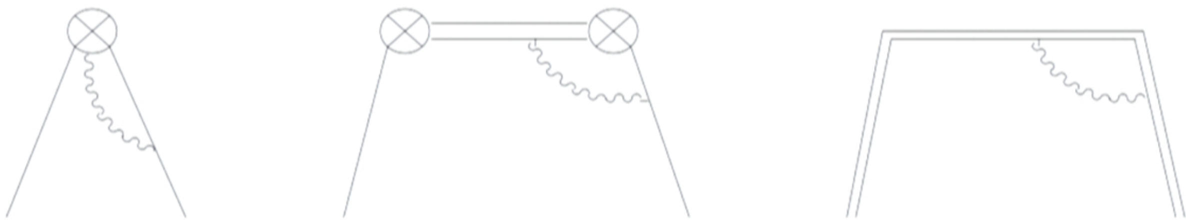
Twist two operators dominate Deep Inelastic Scattering in QCD.

A cusp produces an anomalous dimension (Polyakov 1980).

When the lines forming the cusp become light-like, the angle goes to infinity. The anomalous dimension diverges, the divergence is controlled by the light-like cusp anomaly.

In AdS/CFT it works the same way (MK 2002). In field theory the same argument applies, e.g. in perturbation theory

$$\mathcal{O}_{\{\mu_1\cdots\mu_S\}} = \mathrm{Tr}\,\Phi^I \nabla_{\{\mu_1}\cdots\nabla_{\mu_S\}}\Phi^I$$



$$\mathcal{O}_S(\Delta) = \mathrm{Tr}\left(\Phi^I \nabla_{\mu_1}\cdots\nabla_{\mu_S}\Phi^I\right)\Delta^{\mu_1}\cdots\Delta^{\mu_S}$$

$$\Gamma^{[2]}_{\mathcal{O}_s} = \langle p|\mathcal{O}_s|p\rangle = C_S(ip_\mu\Delta^\mu)^S\left(\frac{\Lambda}{\mu}\right)^{\gamma_S}$$

$$W_{\Delta^\mu} = \mathrm{Tr}\left(\Phi^I(\Delta^\mu)e^{\int_0^\Delta A_\mu(t\Delta^\mu)\Delta^\mu dt}\Phi^I(0)\right) = \sum_{S=0}^{\infty}\frac{1}{S!}\mathcal{O}_S(\Delta)$$

$$\langle p|W(\Delta^\mu)|p\rangle \sim e^{ip.\Delta}\left(\frac{L}{\epsilon}\right)^{-2\Gamma_{\mathrm{cusp}}\ln(p.\Delta)} = e^{ip.\Delta}(p.\Delta)^{-2\bar{\Gamma}_{\mathrm{cusp}}\ln\left(\frac{L}{\epsilon}\right)}$$

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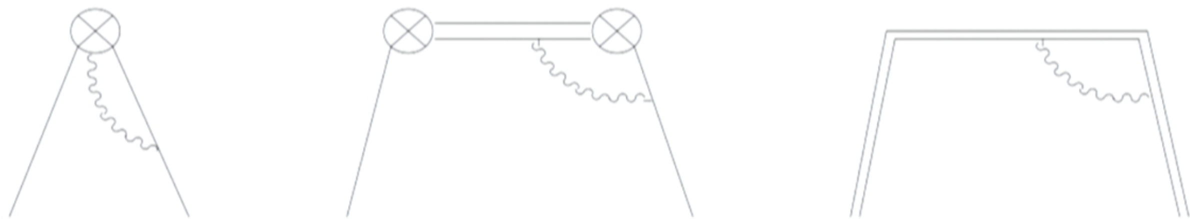

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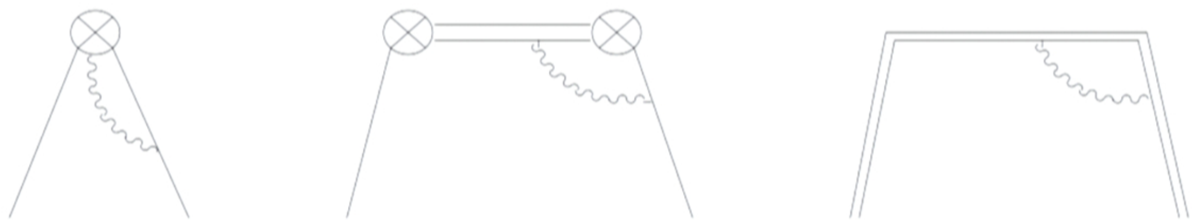

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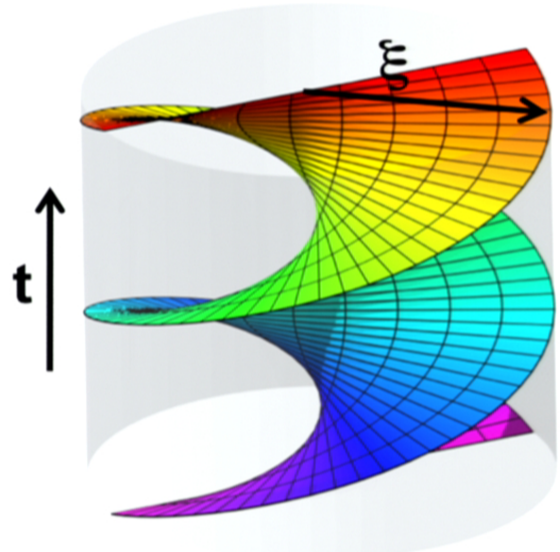
12

In string theory:
Twist two operators can be computed by long rotating strings (Gubser-Klebanov-Polyakov 2002) GKP string.



$$X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 = 1$$

$$X_2 X_{-1} = X_1 X_0$$

$$\rho = \sigma, \quad t = \tau, \quad \theta = \tau$$

13

In fact the two string solutions are related by a double analytic continuation. (Roiban-Tirziu-Tseytlin-MK 2007)

cusp

$$X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 = 1$$

$$X_0^2 - X_1^2 = \frac{1}{2}$$

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Scattering amplitudes:

Field theory:

Bern-Dixon-Smirnov 2005 (BDS) formula, etc.

String theory:

Alday-Maldacena 2007 used T-duality to map the problem of scattering amplitudes to the problem of computing Wilson loops with cusps.

More recently: Basso-Sever-Vieira 2013

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T-duality

Kallosh and Tseytlin 1998 used T-duality to simplify the string action.

The ideas by Alday-Maldacena 2007 suggested an extension of T-duality which was made more precise by Berkovits-Maldacena 2008 and Beisert-Ricci-Tseytlin-Wolf 2008 . It involves an extra (novel) fermionic form of T-duality.

Relates AdS_5xS^5 to itself, equivalently relates $\mathcal{N}=4$ SYM to itself as a "momentum ⟷ position" duality.

Metric: $$ds^2 = \frac{1}{Z^2}(dZ^2 + d\mathcal{X}_\mu d\mathcal{X}^\mu)$$

Action: $$S = \frac{1}{2}\int \frac{d\sigma d\tau}{Z^2}\left(\partial^a Z \partial_a Z + \partial^a \mathcal{X}_\mu \partial_a \mathcal{X}^\mu\right)$$

EOM:

$$\partial_\sigma\left(\frac{1}{Z^2}\partial_\sigma \mathcal{X}^\mu\right) = \partial_\tau\left(\frac{1}{Z^2}\partial_\tau \mathcal{X}^\mu\right)$$

$$\partial_\sigma \; \partial_\tau \tilde{\mathcal{X}}^\mu = \partial_\tau \; \partial_\sigma \tilde{\mathcal{X}}^\mu$$

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18

Going back to relation between local operators and Wilson loops, the GKP string for large S is a "long" string (large S is the limit when it touches the boundary).

What about "short" strings?. Let's look at small S.

Dimension of the spin S and twist J operator or energy of the dual "short" spinning string has following expansion in the small S limit (Basso 2011, Gromov 2012)

$$E^2 = J^2 + h(\lambda, J)S + \mathcal{O}(S^2)$$

$$h = 2J + 2\sqrt{\lambda}\frac{I_{J+1}(\sqrt{\lambda})}{I_J(\sqrt{\lambda})}$$

And the cusp?. For small angle (Correa-Henn-Maldacena-Sever 2012, Drukker et al. 2012):

$$\Gamma_{\text{cusp}}(\phi,\lambda)\big|_{\phi\to 0} \simeq -\frac{\sqrt{\lambda}}{4\pi^2}\frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}\ \phi^2$$

$$\phi = 0 \quad \Longrightarrow \quad \text{straight line}$$

Recall large S is large angle. Apparently small S may be related to small angle?

Another observation

The string action is the same for closed and open strings.

Short string solutions can be found using Riemann theta functions associated with a hyperelliptic curve.

Closed strings: Dorey-Vicedo 2006, Jevicki-Jin 2009, Dorey-Losi 2008

Open strings: Irrgang-MK 2012.

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Idea:

Use T-duality to map short strings to Wilson loops.

In the spirit of D-branes, instead of considering open strings ending at the horizon, consider closed strings that can fall into the Poincare horizon which is mapped to the boundary. It will therefore give a Wilson loop.

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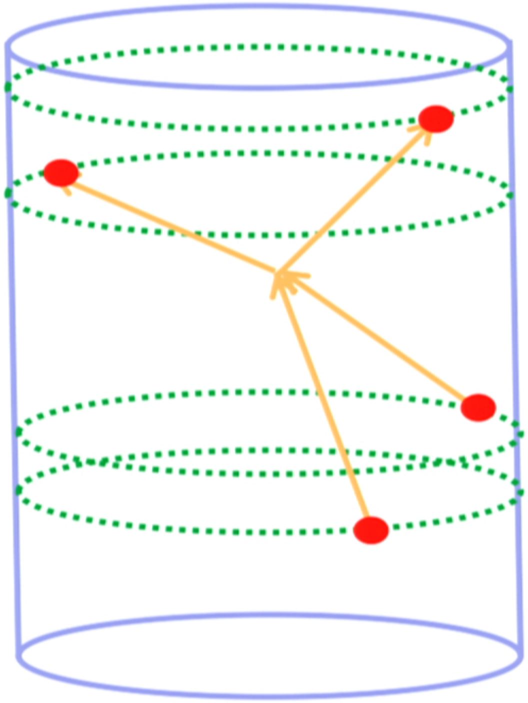
In the spirit of D-branes, instead of considering open strings ending at the horizon, consider closed strings that can fall into the Poincare horizon which is mapped to the boundary. It will therefore give a Wilson loop.

Another way of motivating this idea

Consider the computation of a correlation function for the field theory living on S^3



Should map to a correlation function of the theory in flat space.

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A string can "escape" through the horizon!.

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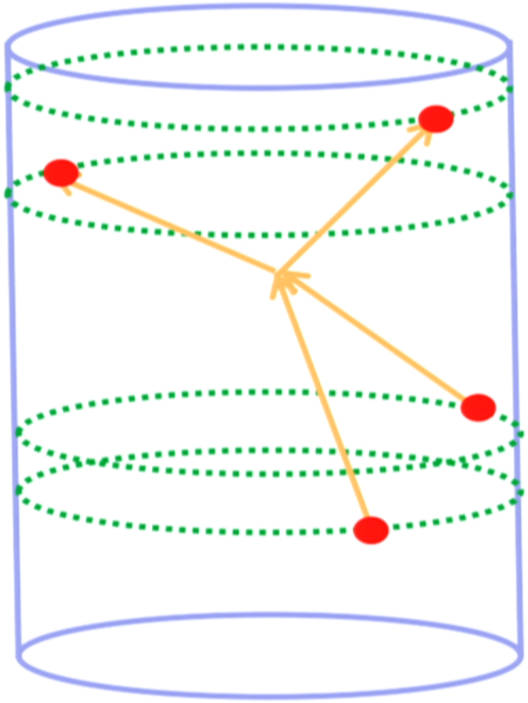

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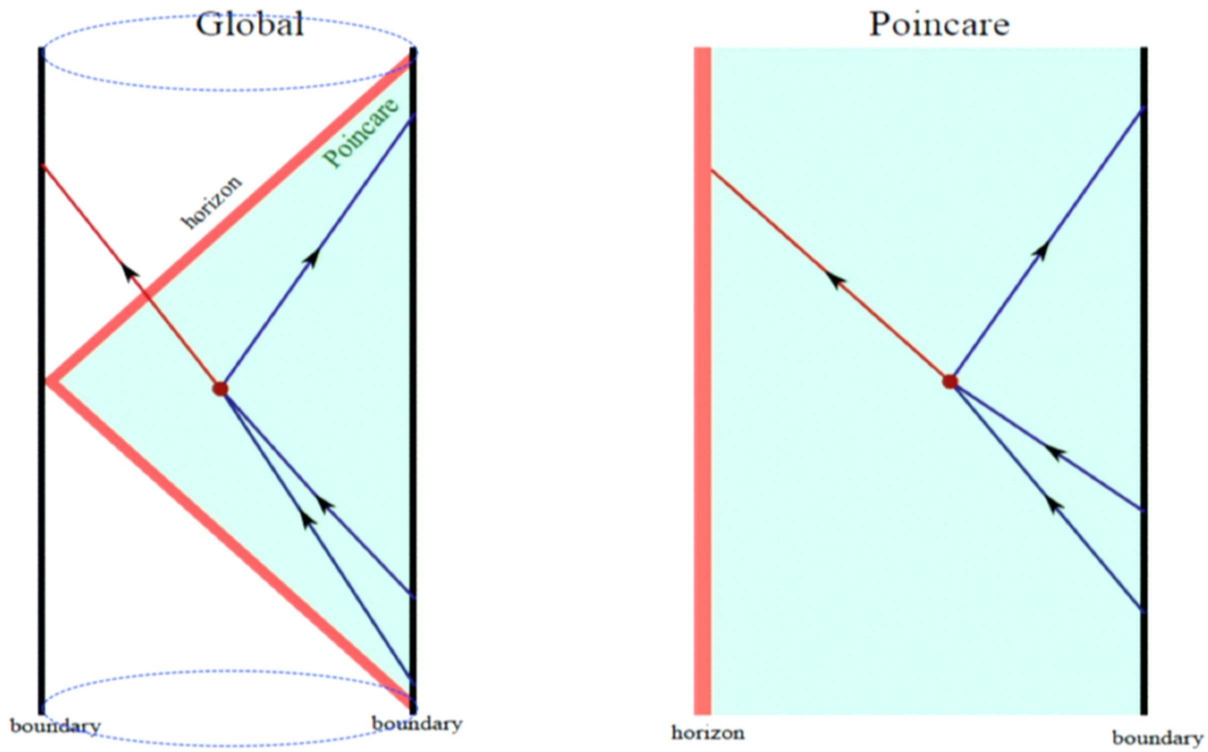
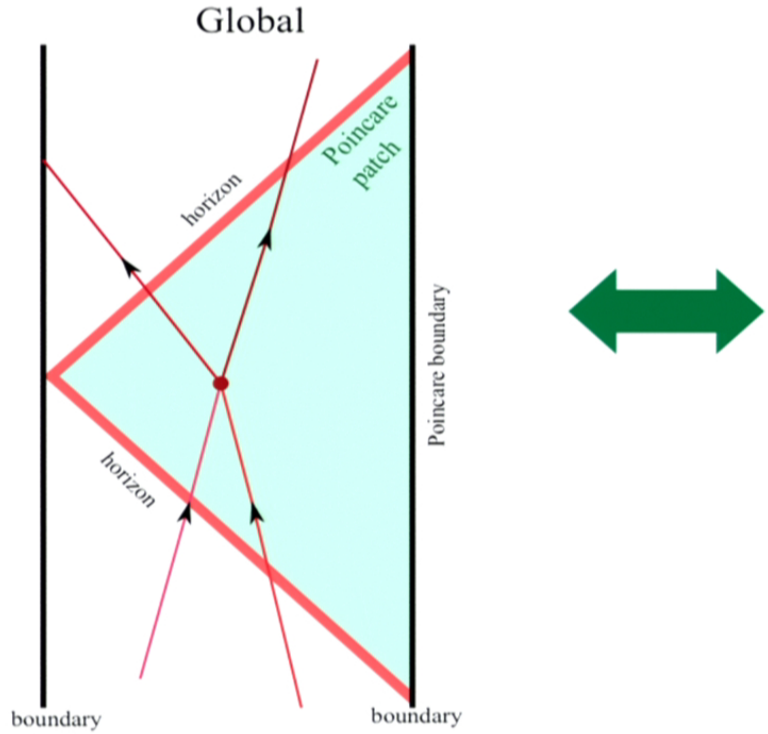


The disappearing string should be represented by an operator in the field theory. Which operator? ➡ WL

It should be interesting to compute "pure" correlators, for strings coming out of the past horizon and going into the future horizon never touching the boundary. This should be T-dual to ordinary WL correlators.



Correlator of WL in usual $\mathcal{N}=4$ SYM.

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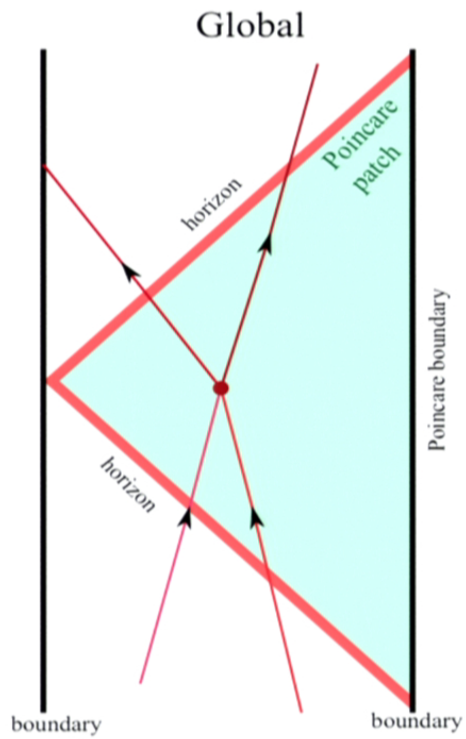

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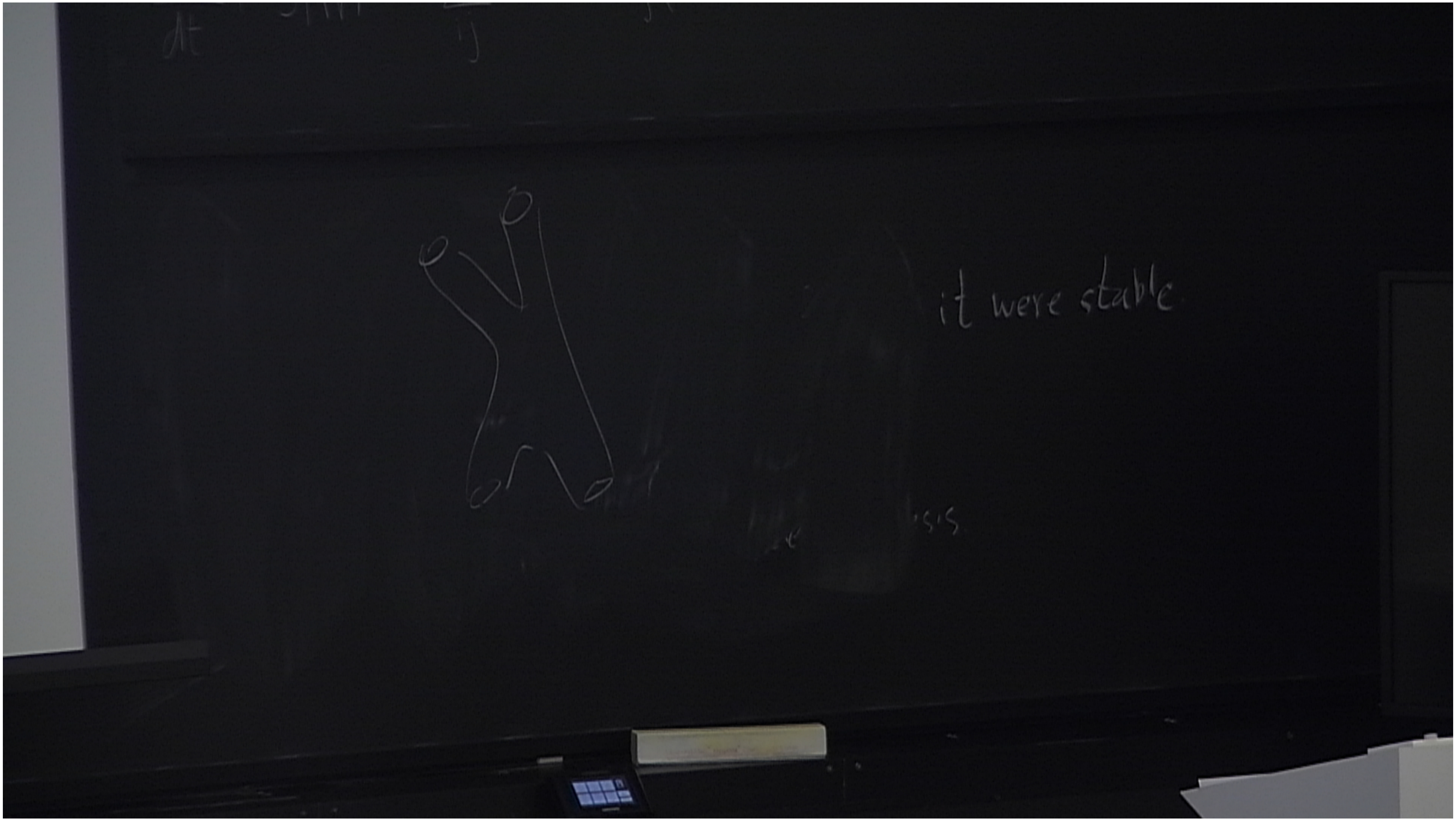
it were stable.

Examples:

Consider the case of a small string (~ massless)
Two possibilities for massless geodesics:

AdS massless

$$\tan t = \kappa\tau, \qquad \sinh\rho = \kappa\tau$$

AdS massive (rotating on S^5)
(BMN geodesic).

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26

First case (Massless AdS geodesic):

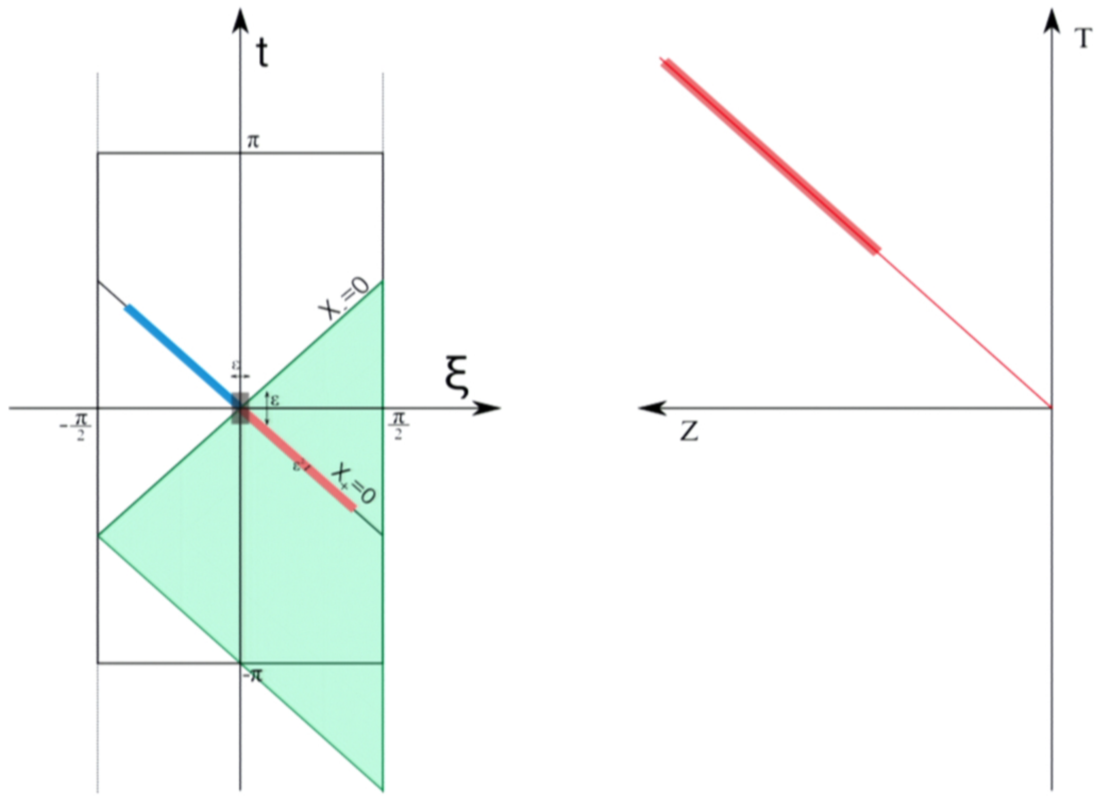
$$T = \frac{1}{\kappa\tau}, \quad Z = \frac{1}{\kappa\tau},$$

$$\left.\begin{array}{rcccl} \partial_\sigma \tilde{T} & = & -\frac{1}{Z^2}\partial_\tau T & = & \kappa \\ \partial_\tau \tilde{T} & = & -\frac{1}{Z^2}\partial_\sigma T & = & 0 \end{array}\right\} \Rightarrow \tilde{T} = \kappa\sigma$$

$$\tilde{Z} = \frac{1}{Z} = \kappa\tau$$

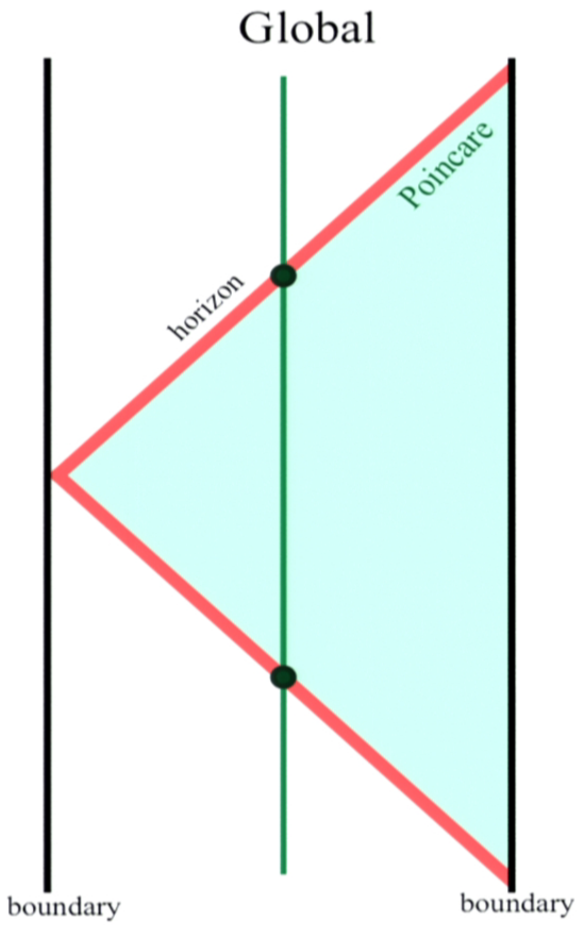
$$\sigma \leftrightarrow \tau \quad : \qquad \tilde{T} = \kappa\tau, \ \tilde{Z} = \kappa\sigma$$

Straight-line Wilson loop!



A point-like string becomes a straight WL of the T-dual theory.

Second case (Massless BMN geodesic):



$$Z = \frac{1}{\sin \mathcal{J}\tau}, \quad T = \text{cotan}\mathcal{J}\tau$$

$$\tau : 0 \to \frac{\pi}{\mathcal{J}}$$

$$\partial_\sigma \tilde{T} = -\frac{1}{Z^2}\partial_\tau T = \mathcal{J}$$

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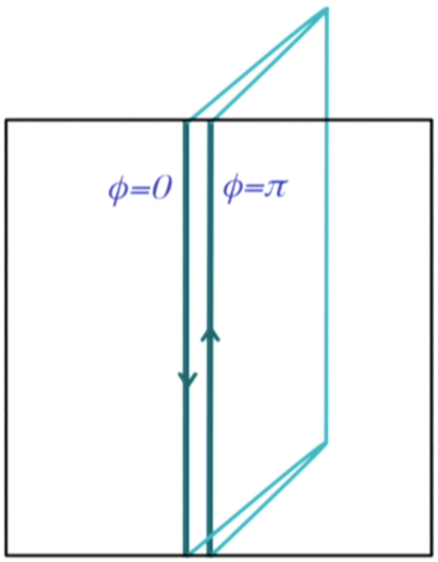
$$\sigma : 0 \to \frac{\pi}{\mathcal{J}}$$

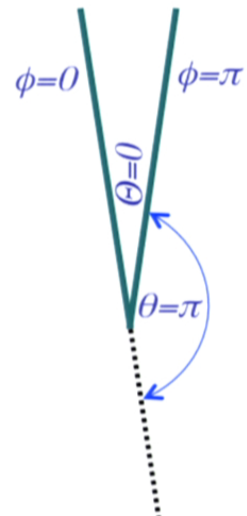
29

Therefore the dual to the BMN geodesic is two parallel lines running on opposite directions and on top of each other.

Each line is at opposite poles on the S^5.

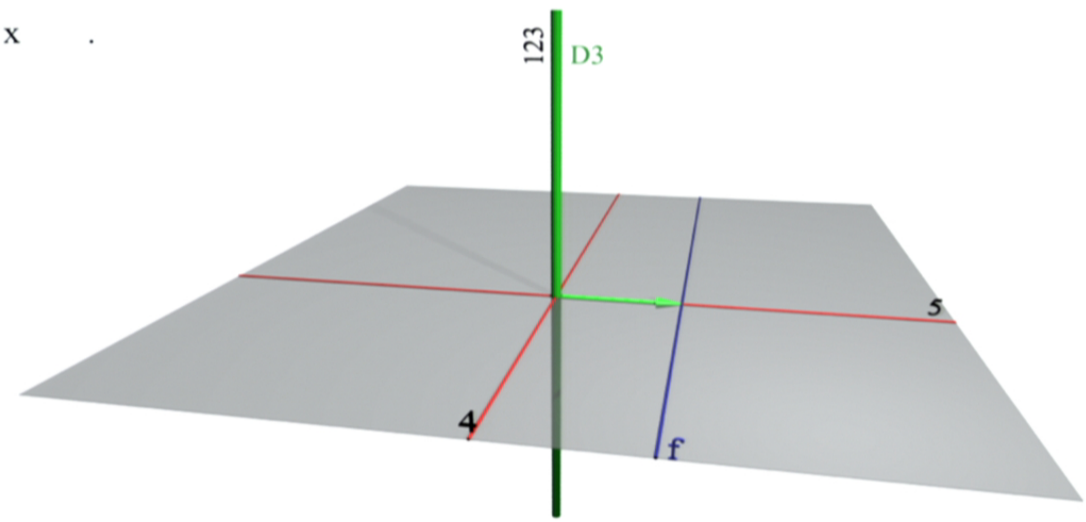
In Euclidean signature this is equivalent to a BPS cusp with angle π (or 0 depending on convention) in space and π on the sphere.





Another simple way to see that this Wilson loop is BPS is to look at the D3-brane picture. The string is just perpendicular to the brane, there is a minimum radius and on the sphere goes from pole to pole.

	0	1	2	3	4	5	6	7	8	9
D3	x	x	x	x		.				
f	x				x	.				



31

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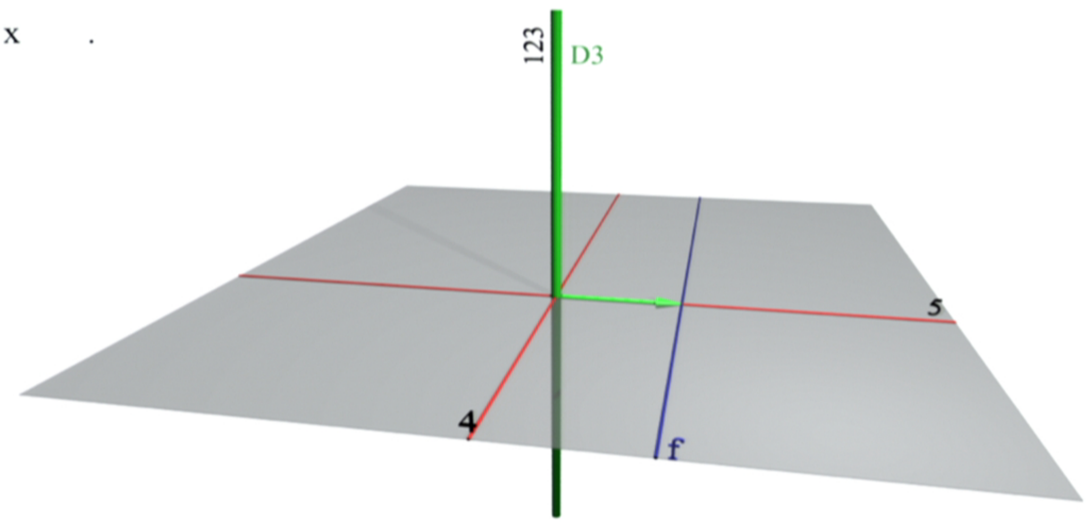

31

Consider now small fluctuations of the (first) string.

Since the string is a small fluctuation of the point-like string we can just study the string in flat space. Moreover, we can use light-cone gauge (since in flat space it is a type of conformal gauge) and the equations reduce to a set of harmonic oscillators.

More formally, start from embedding coordinates:

$$X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = 1$$

and expand around $X_0=1$:

$$X_0 \sim 1 + \epsilon^2 \ , \qquad\qquad X_{M\neq 0} \sim \epsilon$$

32

The metric around this (any) point is approximately:

$$ds^2 \simeq \quad \epsilon^2 \left(-dy_0^2 + dy_r dy_r\right)$$

Light-cone gauge: $r = 1..4$

$$y_- \equiv y_0 - y_4 = \tau$$

Most general solution for the other coordinates:

$$y_i(\sigma, \tau) = \mathrm{y}_i(\sigma, \tau) \equiv \mathrm{y}_i^+(\sigma + \tau) + \mathrm{y}_i^-(\sigma - \tau)$$

τ=0 corresponds to the horizon. Classically we need to specify the shape and velocity of the string on that surface.

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Examples:

Consider the case of a small string (~ massless)
Two possibilities for massless geodesics:

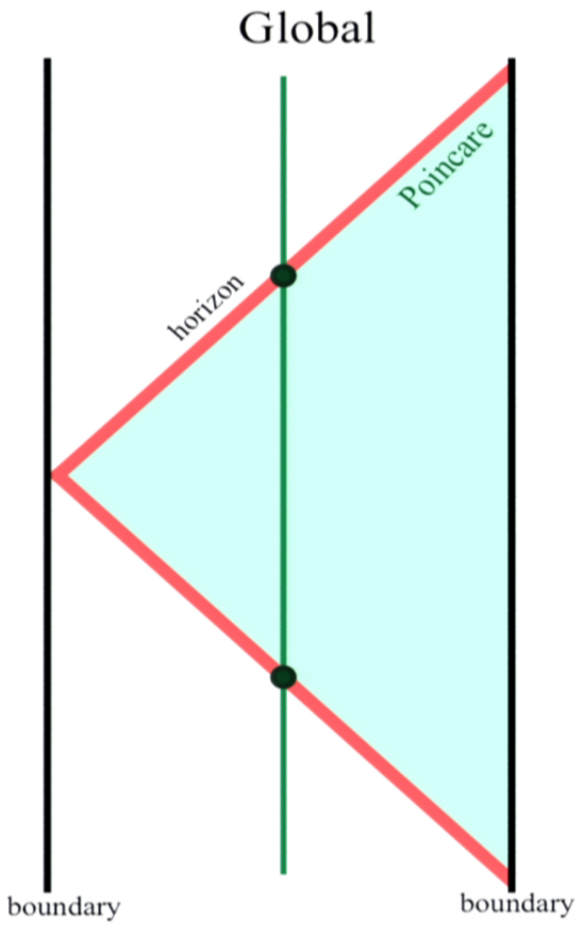
AdS massless

$$\tan t = \kappa\tau, \quad \sinh\rho = \kappa\tau$$

AdS massive (rotating on S^5)
(BMN geodesic).

$$t = \mathcal{J}\tau, \quad \phi = \mathcal{J}\tau, \quad \rho = 0.$$

Second case (Massless BMN geodesic):



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$$\tau : 0 \to \frac{\pi}{\mathcal{J}}$$

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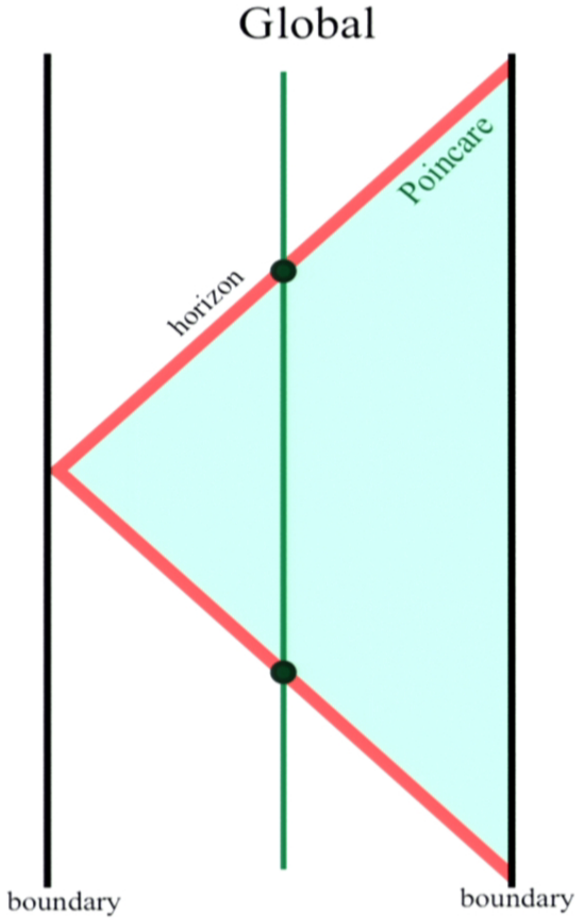

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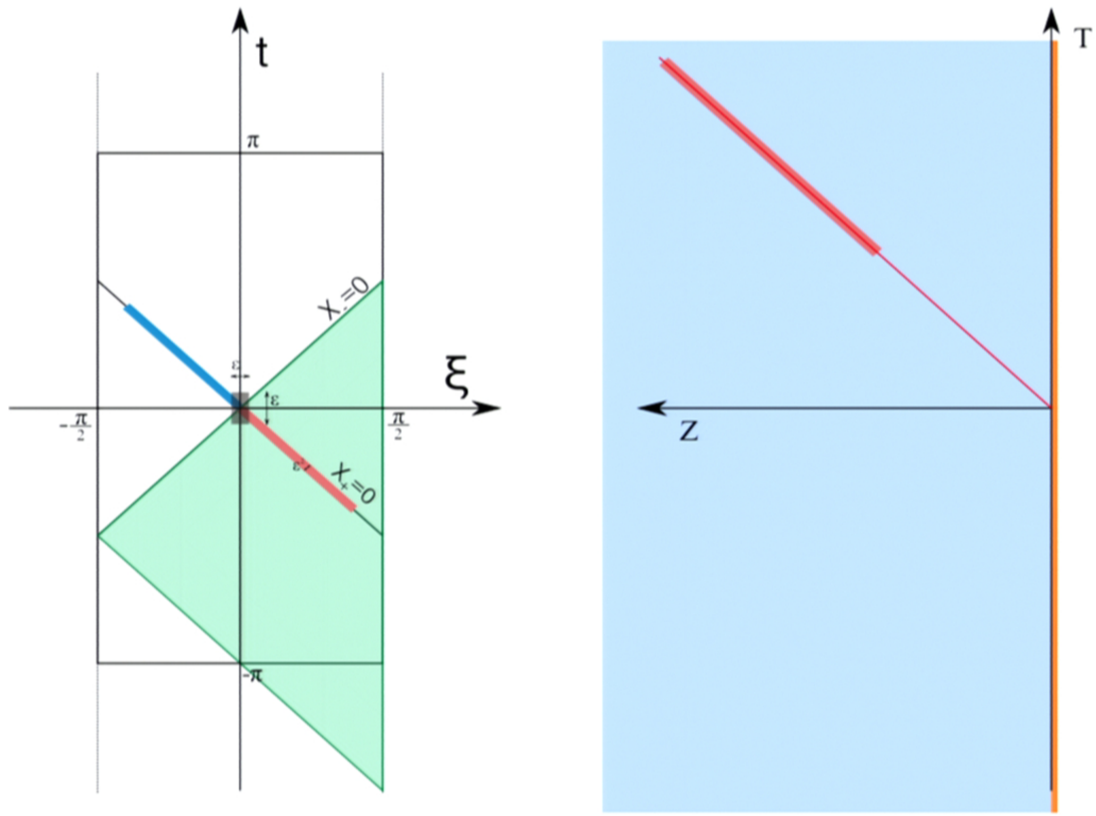
33



A point-like string becomes a straight WL of the T-dual theory.

There seems to be a problem. These fluctuations should be mapped to fluctuations of the Wilson loop which is in AdS and not in flat space ?!.

Indeed, the action for the fluctuations around Z=σ, T=τ is (notice interchange of σ and τ):

$$\bar{S} = \int \frac{d\tau d\sigma}{2\sigma^2} \left[(\partial_\tau x_i)^2 - (\partial_\sigma x_i)^2 \right]$$

which gives the equations:

$$\partial_\tau^2 x_i - \partial_\sigma^2 x_i + \frac{2}{\sigma} \partial_\sigma x_i = 0 \qquad ?!$$

35

But, in fact, it was already shown by A. Mikhailov that these equations are solved by

$$\begin{aligned} x_i(\tau,\sigma) &= \mathrm{x}_i(\tau,\sigma) - \sigma\partial_\sigma \mathrm{x}_i(\tau,\sigma) \\ &= \mathrm{x}_i^+(\tau+\sigma) + \mathrm{x}_i^-(\tau-\sigma) - \sigma\Big[\dot{\mathrm{x}}_i^+(\tau+\sigma) - \dot{\mathrm{x}}_i^-(\tau-\sigma)\Big] \end{aligned}$$

$$x_i(\tau,0) \equiv \mathrm{x}_i(\tau) = \mathrm{x}_i^+(\tau) + \mathrm{x}_i^-(\tau)$$

$$\partial_\sigma^3 x_i(\tau,0) = -2\partial_\tau^3[\mathrm{x}_i^+(\tau) - \mathrm{x}_i^-(\tau)]$$

So, indeed the equations **are** like in flat space. In fact using the rules of T-duality we find

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36

First change to Poincare coordinates

$$Z = \frac{1}{\epsilon\,(y_0 - y_4)}\ , \qquad \mathcal{X}_0 \equiv \mathcal{T} = \frac{1}{\epsilon\,(y_0 - y_4)}\ , \qquad \mathcal{X}_i = \frac{y_i}{y_0 - y_4}$$

and perform a rescaling (boost) by ε, the solution is

$$Z = \mathcal{T} = \frac{1}{\tau}\ , \qquad \mathcal{X}_i = \frac{y_i(\sigma,\tau)}{\tau}\ , \qquad y_i = \mathrm{y}_i(\sigma,\tau)$$

T-duality gives:

$$\begin{aligned} \tilde{Z} &= \frac{1}{Z} = \tau\ , \\ \partial_\sigma \tilde{\mathcal{T}} &= -\frac{1}{Z^2}\partial_\tau \mathcal{T} = 1\ , \\ \partial_\sigma \tilde{\mathcal{X}}_i &= -\frac{1}{Z^2}\partial_\tau \mathcal{X}_i = -\tau^2 \partial_\tau \frac{y_i}{\tau} = y_i - \tau \partial_\tau y_i \end{aligned}$$

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The expectation value of the Wilson loop is related to the area of the dual world-sheet

$$\bar{S}_{\rm fin} = -\frac{1}{4}\int d\tau\, x_i(\tau,0)(\partial_\sigma^3 x_i)(\tau,0)$$

$$\bar{S}_{\rm fin} = \frac{1}{2}\int d\tau\,(\mathrm{x}_i^+ + \mathrm{x}_i^-)(\partial_\tau^3 \mathrm{x}_i^+ - \partial_\tau^3 \mathrm{x}_i^-) = \frac{1}{2}\int d\tau\; v_i(a_i^- - a_i^+)$$

$$v_i = \partial_\tau \mathrm{x}_i(\tau, \sigma = 0), \qquad a_i^\pm = \partial_\tau^2 \mathrm{x}_i^\pm(\tau)$$

38

The energy of the open string ending on the boundary is not conserved (we need do work on the quark to move it along a specified trajectory). The total energy change, however, for *these* solutions is:

$$\Delta\bar{E} = \bar{E}(+\infty) - \bar{E}(-\infty) = \int_{-\infty}^{+\infty} d\tau \ (a^{-2} - a^{+2})$$

$$= \int_{-\infty}^{+\infty} d\tau \ \left[(\partial_\tau \mathrm{y}_i^-)^2 - (\partial_\tau \mathrm{y}_i^+)^2\right] = 0$$

It vanishes from the level matching condition for the closed string. However if we add the left and right movers we get the energy of the closed string:

$$\bar{E}^{\mathrm{cl}} = \int_{-\infty}^{+\infty} d\tau \ (a^{-2} + a^{+2})$$

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39

The spin is also not conserved.

$$\partial_\tau \bar{\mathrm{S}}_{ij} = (v_i^+ a_j^+ - v_j^+ a_i^+) - (v_i^- a_j^- - v_j^- a_i^-)$$

The velocity and acceleration are evaluated at the boundary. The reason is that the spin defines a conserved current, any increase in spin is due to a flux from the boundary. The total spin absorbed by the open string is precisely equal to the spin of the original closed string.

$$\Delta \bar{\mathrm{S}}_{ij} = \int d\tau \left[(v_i^+ a_j^+ - v_j^+ a_i^+) - (v_i^- a_j^- - v_j^- a_i^-) \right] = \mathrm{S}_{ij}^+ - \mathrm{S}_{ij}^-$$

Second case: BMN geodesic.

An interesting way to study the motion around the BMN vacuum is to use a LL model which contains the leading terms in the action in the case where the motion is slow compared with the fast motion of the center of mass. In the field theory side it appear also as a way to study the operators in term of spin chains.

$$t = \mathcal{J}\tau, \quad \phi = \mathcal{J}\tau, \quad \rho = 0.$$

$$\Updownarrow$$

$$\mathcal{O} = \mathrm{Tr} X^J, \quad \mathcal{J} = \frac{1}{\sqrt{\lambda}} J$$

Ground state (s)

$$| \uparrow\uparrow \ldots \uparrow\uparrow\uparrow\uparrow\rangle \quad \Longleftrightarrow \quad \mathrm{Tr}\,(XX \ldots XXXX)$$

$$| \downarrow\downarrow \ldots \downarrow\downarrow\downarrow\downarrow\rangle \quad \Longleftrightarrow \quad \mathrm{Tr}\,(YY \ldots YYYY)$$

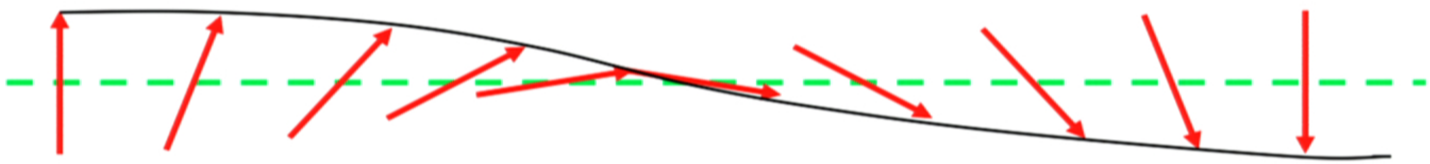
First excited states

$$|k\rangle = \sum e^{ikl} | \uparrow \uparrow \ldots \underset{l}{\downarrow} \ldots \uparrow \uparrow \rangle, \quad k = \frac{2\pi n}{J}; \, (J = J_1 + J_2)$$

$$\varepsilon(k) = \frac{\lambda}{J^2}\left(-1 + \cos k\right) \xrightarrow[k \to 0]{} \frac{\lambda n^2}{2J^2} \qquad \textbf{(BMN)}$$

More generic (low energy) states: Spin waves

Other states, e.g. with $J_1=J_2$



Spin waves of long wave-length have low energy and are described by an effective action in terms of two angles θ, φ: direction in which the spin points.

$$S_{eff.} = J\left\{ -\frac{1}{2}\int d\sigma\, d\tau \cos\theta\, \partial_\tau \phi - \right.$$

$$\left. -\frac{\lambda}{32\pi J^2}\int d\sigma\, d\tau \left[(\partial_\sigma\theta)^2 + \sin^2\theta (\partial_\sigma\phi)^2 \right] \right\}$$

The same action describes the motion of a string.

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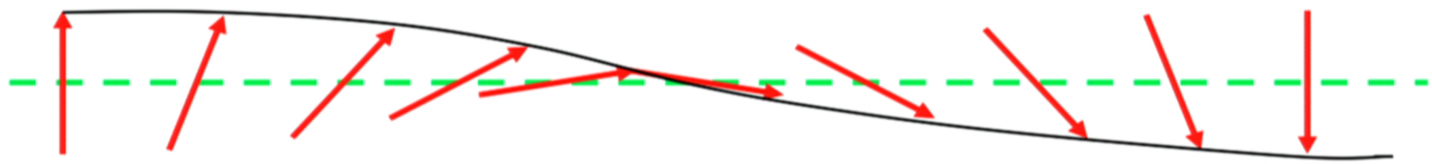

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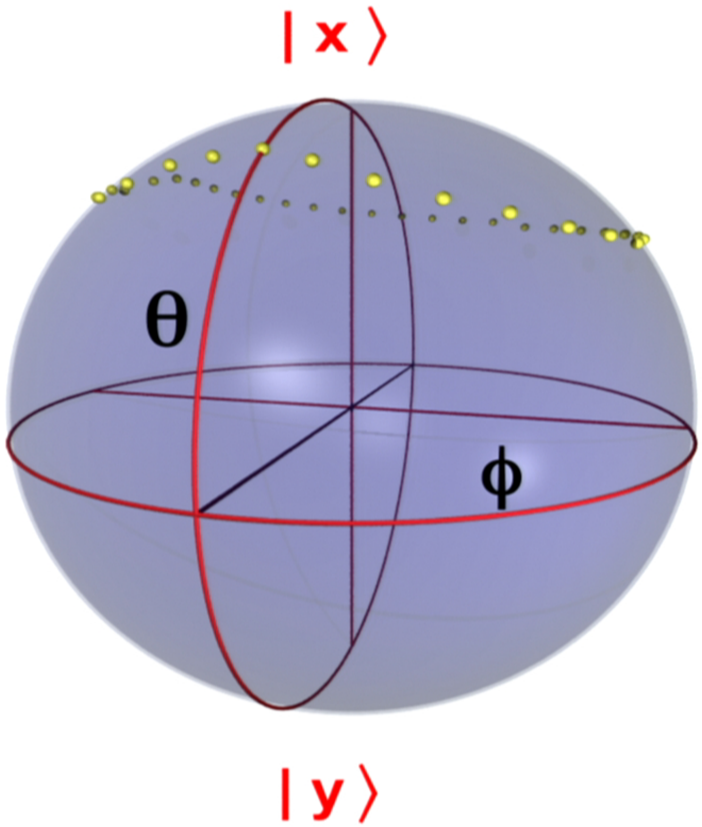
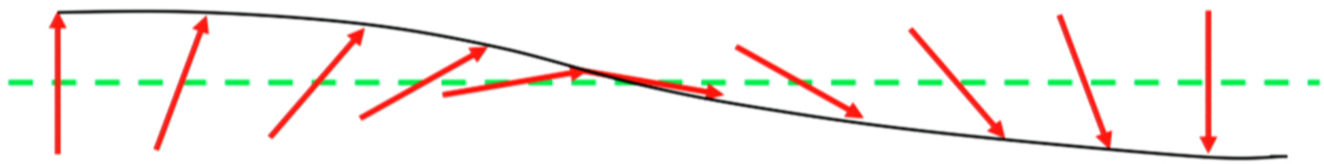


Strings are useful to describe states of a large number of particles.

Other states, e.g. with $J_1=J_2$



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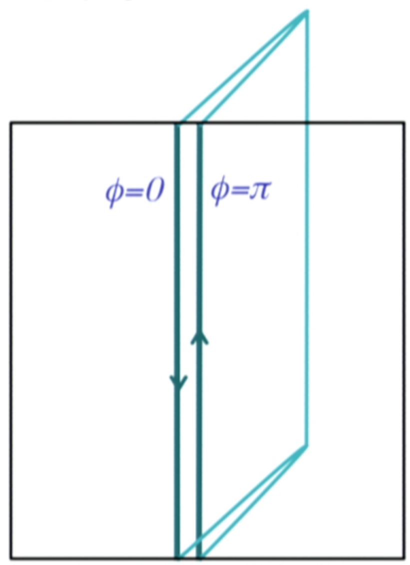
The same action describes the motion of a string.

LL model also works for WL at least on string side.

Consider the Wilson loop we had as T-dual of BMN

$$\tilde{T} = \mathcal{J}\tau,\ \tilde{Z} = \sin \mathcal{J}\sigma,\ \tilde{\phi} = \mathcal{J}\sigma \qquad \sigma : 0 \to \frac{\pi}{\mathcal{J}}$$

and do fluctuations that are near BPS. What we need to do is to keep, at all times, the quark and anti-quark approximately on opposite poles of the S^5



$$Y_1 = \sin\psi\,\sin(\phi_1 - \phi_2) \quad Y_3 = \cos\psi\,\sin(\phi_1 + \phi_2)$$
$$Y_2 = \sin\psi\,\cos(\phi_1 - \phi_2) \quad Y_4 = \cos\psi\,\cos(\phi_1 + \phi_2)$$

$$ds^2 = \frac{-dt^2 + dz^2}{z^2} + d\psi^2 + d\phi_1^2 + d\phi_2^2 + 2\cos 2\psi\, d\phi_1 d\phi_2$$

$$t = \tau, \quad z = \cos\sigma \qquad -\frac{\pi}{2} < \sigma < \frac{\pi}{2}, \quad -\infty < \tau < \infty$$

$$\phi_1 = \sigma + \chi_1(\sigma, \tau), \quad \psi = \psi(\sigma, \tau), \quad \phi_2 = \phi_2(\sigma, \tau),$$

$$\psi(\pm\frac{\pi}{2}, \tau) = \psi_\pm(\tau), \quad \phi_2(\pm\frac{\pi}{2}, \tau) = \phi_\pm(\tau),$$

$$\Delta\psi(\tau) = \psi_+(\tau) - \psi_-(\tau) \ll 1$$

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$\partial_\sigma\psi(\sigma,\tau)$ and $\partial_\sigma\phi_2(\sigma,\tau)$ are small

$$\begin{aligned}\partial_\tau\chi_1 &= -\cos 2\psi\ \partial_\tau\phi_2\\ \partial_\sigma\chi_1 &= -\cos 2\psi\ \partial_\sigma\phi_2 - \tfrac{1}{2}(\partial_\tau\psi)^2 - \tfrac{1}{2}\sin^2 2\psi\ (\partial_\tau\phi_2)^2\end{aligned}$$

$$\begin{aligned}\partial_\tau^2\psi - 2\sin 2\psi\cos 2\psi\ (\partial_\tau\phi_2)^2 - 2\sin 2\psi\ \partial_\sigma\phi_2 &= 0\\ \partial_t(\sin^2 2\psi\ \partial_t\phi_2) - \partial_\sigma(\cos 2\psi) &= 0\end{aligned}$$

$$\begin{aligned}S_{\rm eff} &= \tfrac{1}{2}\int\left[(\partial_\tau\psi)^2 + \sin^2 2\,\psi(\partial_\tau\phi_2)^2\right] - \int \cos 2\psi\,\partial_\sigma\phi_2\\ &= \frac{1}{8}\int\left[(\partial_\tau\theta)^2 + \sin^2\theta\ (\partial_\tau\varphi)^2\right] - \tfrac{1}{2}\int\cos\theta\,\partial_\sigma\varphi\end{aligned}$$

$$\theta = 2\psi,\ \varphi = 2\phi_2$$

Conclusions

We review the relation between twist two operators and the light-like cusp and discussed suggestive facts for the case of short, near BPS strings.

We proposed that a simple way to relate small strings and Wilson loops is to apply to closed strings the same T-duality used by Alday and Maldacena on open strings.

We discussed two simple examples that showed some interesting relations between the fluctuations of a small closed string and the open string dual to a Wilson loop.

This should open new possibilities to study AdS/CFT.

Field theory side? [--Work in progress--]

On the field theory side one can consider perturbation theory. The only observation is that, for the near BPS Wilson loop $\theta \sim \phi$ the cusp anomaly is (Correa-Henn-Maldacena-Sever 2012, Drukker et al. 2012):

$$\Gamma_{\text{cusp}}(\phi, \theta) = -\pi^2(\phi^2 - \theta^2)\frac{B(\tilde{\lambda})}{\pi^2 - \phi^2} + \mathcal{O}((\phi^2 - \theta^2)^2)$$

$$\tilde{\lambda} = \frac{\lambda}{\pi^2}(\pi^2 - \phi^2)$$

Notice that here we are interested in $\theta \sim \phi \sim \pi$ and therefore it seems that one can have $\tilde{\lambda}$ small also at strong coupling. (reminiscent of BMN)

49

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Notice that here we are interested in θ~φ~π and therefore it seems that one can have $\tilde{\lambda}$ small also at strong coupling. (reminiscent of BMN)

49

And the cusp?. For small angle (Correa-Henn-Maldacena-Sever 2012, Drukker et al. 2012):

$$\Gamma_{\rm cusp}(\phi,\lambda)\big|_{\phi\to 0} \simeq -\frac{\sqrt{\lambda}}{4\pi^2}\frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}\;\phi^2$$

$$\phi = 0 \quad\Longrightarrow\quad \text{straight line}$$

Recall large S is large angle. Apparently small S may be related to small angle?