Title: Non-perturbative problems in gauge theories: N=2 quiver QCD

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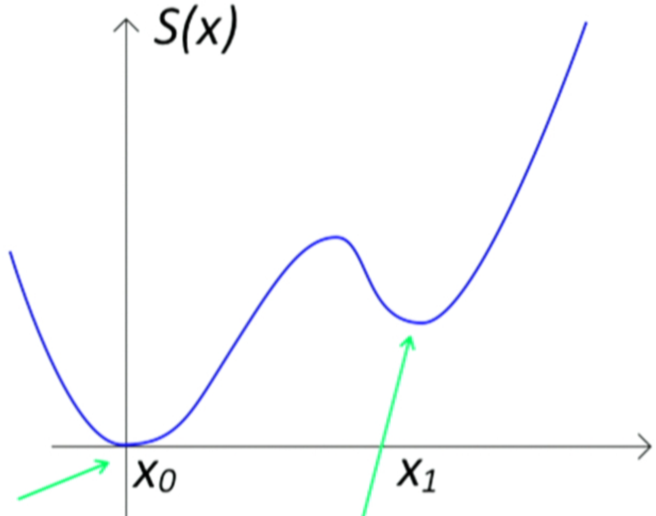
Abstract: Non-perturbative effects are responsible for the essential dynamical features of the four-dimensional gauge theories such as QCD. The N=2 supersymmetric four-dimensional theories are an interesting class of models in which non-perturbative computations can be carried out with arbitrary precision using localization of the path integrals. I will explain the new exact non-perturbative results and the relation to classical and quantum integrable systems for a large class of N=2 supersymmetric QCD.

Non-perturbative QFT

$$\langle \mathcal{O}(X) \rangle = \int DX e^{-\frac{1}{g^2} S(X)} \mathcal{O}(X)$$



Feynman diagrams

Perturbative:

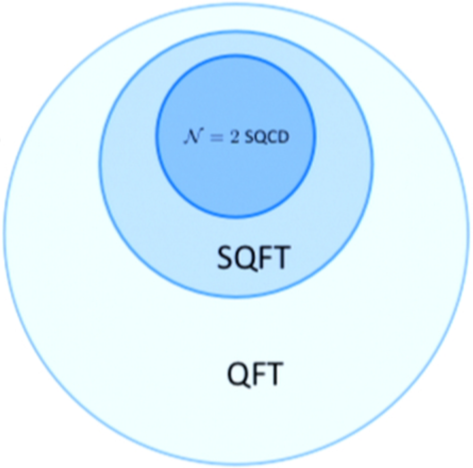
$$S(X) = S(X_0) + \frac{1}{2} S''(X_0)(X - X_0)^2 + \ldots \quad \Longrightarrow \quad \langle \mathcal{O} \rangle = \sum g^k \mathcal{O}_k$$

Non-perturbative: other critical points contribute $\ldots e^{-\frac{S(X_1)}{g^2}}$

Why?

Physics:

- Non-perturbative real QCD:
  in phenomenology we ask about $\mathcal{L}_{UV}$
  but in experiment we measure
  hadron states in IR



- Dynamical supersymmetry breaking:
  might explain the hierarchy problem [Witten]

- May be supersymmetry in UV ?
  May be $\mathcal{N}=2$ / $\mathcal{N}=1$ hybrid model [Antoniadis & others]
  (In a hidden sector and/or in UV)
  (Also motivations in string/brane constructions)

Mathematical physic, statistical mechanics and condensed matter:

- classical and quantum mechanical integrable systems (Toda and others) [ITEP]
- exactly solvable lattice models and spin chains (Yang-Baxter, Bethe ansatz) [Nekrasov-Shatashvili]
- 2d conformal field theory

$$Z_{\mathcal{N}=2}(S^4) \quad = \quad \langle V_1 \dots V_n \rangle_{\text{Liouville}}$$

[V. P.]   [AGT]   [Dorn Otto, Zamolodchikov-Zamolodchikov]

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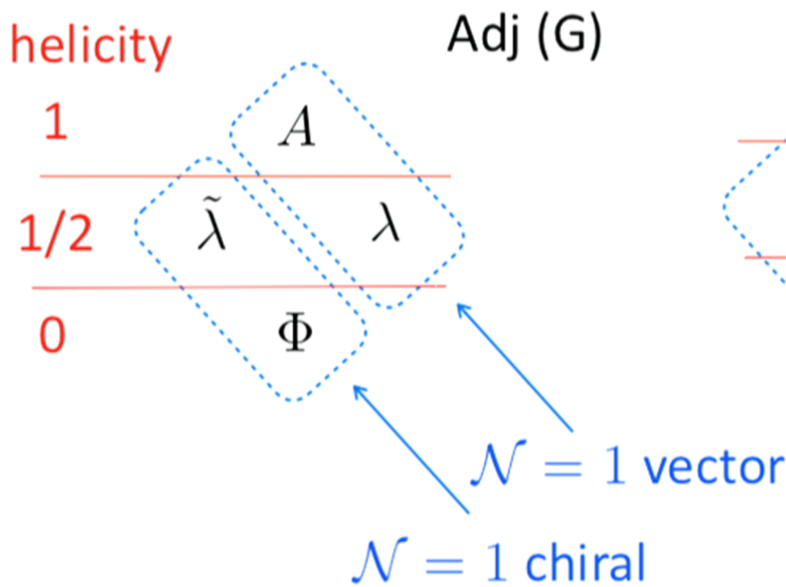
Mathematics:

- Quantum geometry of various moduli spaces: instantons, monopoles, Hitchin systems [Atiyah, Hitchin, Donaldson, Nakajima & others]
- Geometric representation theory
- Mirror symmetry & top strings; GW, SW, GV, DT invariants
- Quantum groups [Faddeev, Drinfeld, Jimbo]
- Wall crossing, BPS states counting [KS, GMN,CV]
- Cluster algebras, Y-systems [Zamolodchikov, Fomin-Zelevisnky]

$$\{Q^i_\alpha, Q^j_{\dot\beta}\} = 2P_\mu \sigma^\mu_{\alpha\dot\beta}\delta^{ij}$$

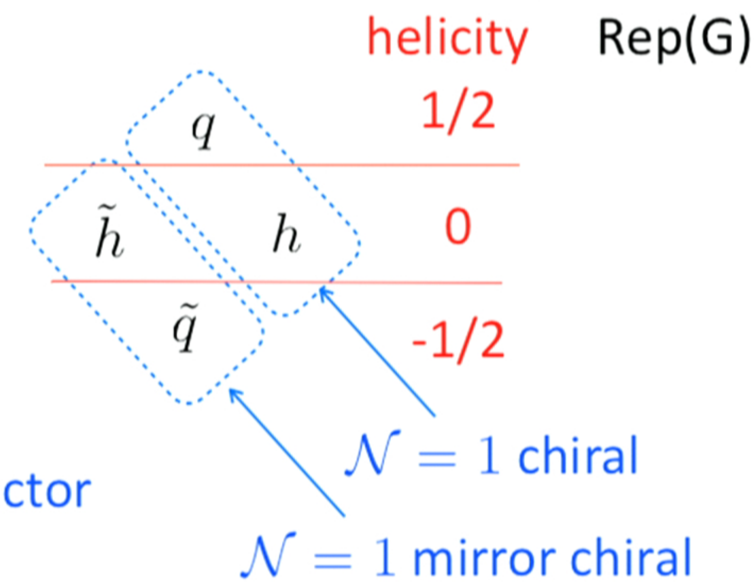
$\mathcal{N} = 2$ SQCD

$\mathcal{N} = 2$ gauge multiplet



$$\frac{1}{2g^2_{\rm YM}} {\rm tr} F \wedge * F$$

$\mathcal{N} = 2$ matter multiplet



$${\rm tr}\psi \not{D} \psi$$

The space of vacua

Supersymmetric ground states, or vacua, are zero energy states $|u\rangle$ such that

$$H|u\rangle = 0$$

In fact: zero energy state <=> supersymmetric state

$$\langle u|H|u\rangle = \langle u|Q^\dagger Q|u\rangle = |Q|u\rangle|^2$$ [Witten]

$\mathcal{N} = 2$ theories have infinitely degenerate ground state !

(without any obvious symmetry relating different ground states, e.g. the spectrum of excitations is different)

Define: $\mathcal{M} =$ { space of inequivalent ground states $|u\rangle$ }

also known as *the space of vacua* [Seiberg-Witten]

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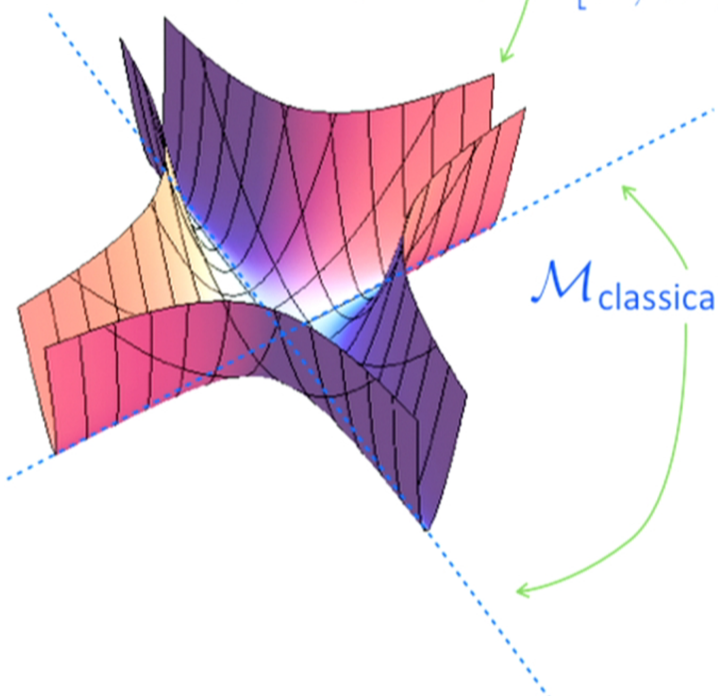
Classical space of vacua

scalar potential: $V = \mathrm{tr}[\Phi, \Phi^\dagger]^2$

$$V = 0 \quad \Rightarrow \quad [\Phi, \Phi^\dagger] = 0 \quad \Rightarrow \quad \Phi \in \mathfrak{t}_G$$



maximal commuting subalgebra of Lie(G)

$$\mathcal{M}_{\text{classical}} = \mathfrak{t}_G / W$$

Weyl group

Non-abelian UV gauge theory

$\mathcal{N}=2$ gauge multiplet $\quad G \quad \mathrm{rk}(G)=r$

$\mathcal{N}=2$ matter multiplet $\quad Rep(G)$

$$\mathcal{L}_{UV}=\frac{1}{2g_{\mathsf{UV}}^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}+i\psi \not{D}\psi+\dots)$$

UV

IR

Abelian IR non-linear sigma model $\quad U(1)^r$

$$\mathrm{Maps}(\mathbb{R}^{3,1},\mathcal{M})$$

metric on $\mathcal{M}$ is *special Kahler*:

$$ds^2=\mathrm{Im}\tau_{ij}da^i d\bar{a}^j \qquad \tau_{ij}(a)=\frac{\partial^2\mathcal{F}(a)}{\partial a^i\partial a^j}$$

[Seiberg-Witten]

special coordinates on $\mathcal{M}$ $\qquad$ *prepotential*

What is $\mathcal{F}(a)$?

Typically:

$$\mathcal{F}(a) = \frac{1}{2}\tau a^2 + a^2 \log a + \sum_{k=0}^{\infty} \mathcal{F}_k(a) q^k$$

classical — one-loop — instanton (non-perturbative)

instanton expansion parameter $q = \exp(2\pi i \tau) = e^{-\frac{8\pi^2}{g^2} + i\theta}$

complexified gauge coupling constant $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$

contribution of charge k instanton $\mathcal{F}_k(a)$

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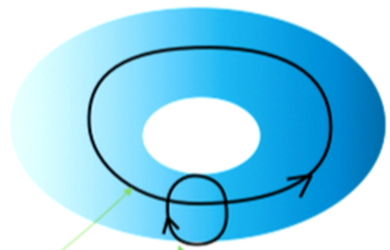
Algebraic integrable system: hyperKahler fibration

[Donagi-Witten]

fibers: complex tori
(abelian variety)



$\rightarrow \mathcal{P}$

$\dim_{\mathbb{C}}\mathcal{P} = 2r$

$\dim_{\mathbb{C}}(\text{fiber}) = r$

$$a_i = \oint_{A^i} d^{-1}\omega$$

$$a_i^D = \oint_{B^i} d^{-1}\omega$$

$\dim_{\mathbb{C}} \mathcal{M} = r$

base $\mathcal{M}$: quantum space of vacua

a_i are coordinates on $\mathcal{M}$

a_i^D are dual coordinates on $\mathcal{M}$

$$a_i^D = \frac{\partial \mathcal{F}}{\partial a_i}$$

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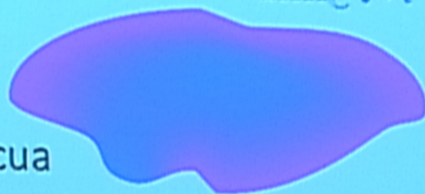

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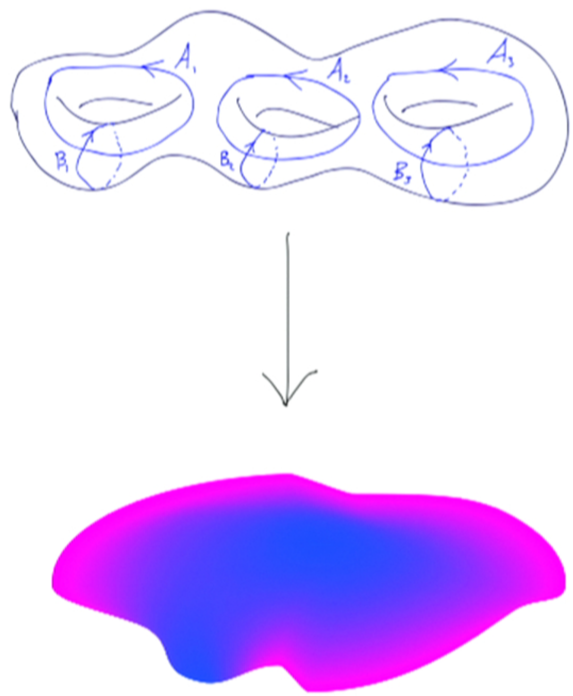
Seiberg-Witten curve

[Seiberg-Witten]



quantum space of vacua $\mathcal{M}$

a and a^D are periods
of *Seiberg-Witten differential*

$$a_i = \oint_{A_i} \lambda_{SW}$$

$$a_i^D = \oint_{B_i} \lambda_{SW}$$

Goal:

Given the $\mathcal{N}=2$ QCD (G,R) in UV, compute (sum up all instantons):

- The prepotential $\mathcal{F}(a)$
- The SW curve and the differential λ_{SW}
- The algebraic integrable system $\mathcal{P} \to \mathcal{M}$

The following is based on a joint work with Nikita Nekrasov (2012)

arXiv:1211.2240

Chiral ring operators

$$\mathcal{O}_k = \mathrm{tr}\Phi^k$$

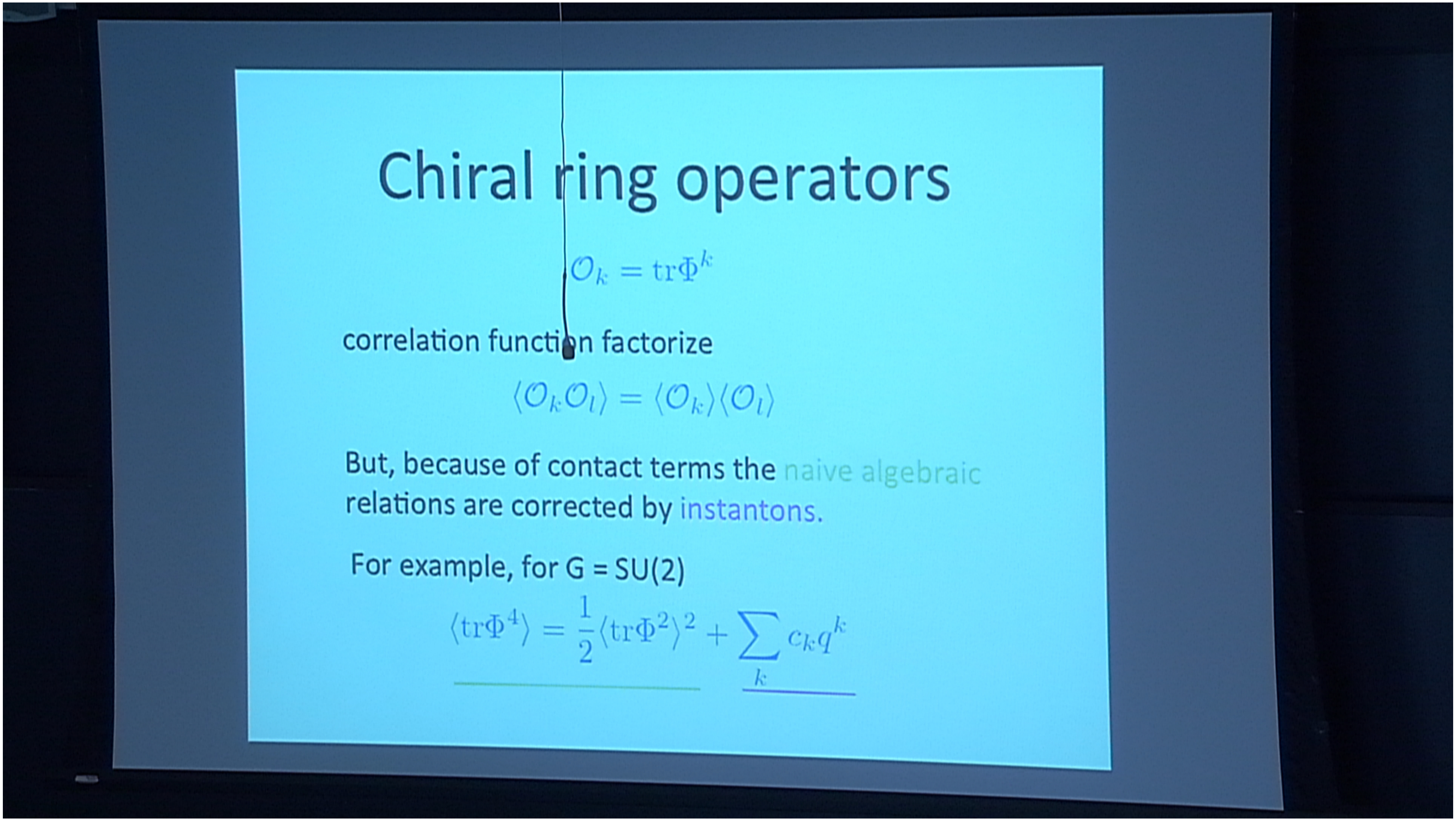
correlation function factorize

$$\langle \mathcal{O}_k \mathcal{O}_l \rangle = \langle \mathcal{O}_k \rangle \langle \mathcal{O}_l \rangle$$

But, because of contact terms the naive algebraic relations are corrected by instantons.

For example, for G = SU(2)

$$\langle \mathrm{tr}\Phi^4 \rangle = \frac{1}{2} \langle \mathrm{tr}\Phi^2 \rangle^2 + \sum_k c_k q^k$$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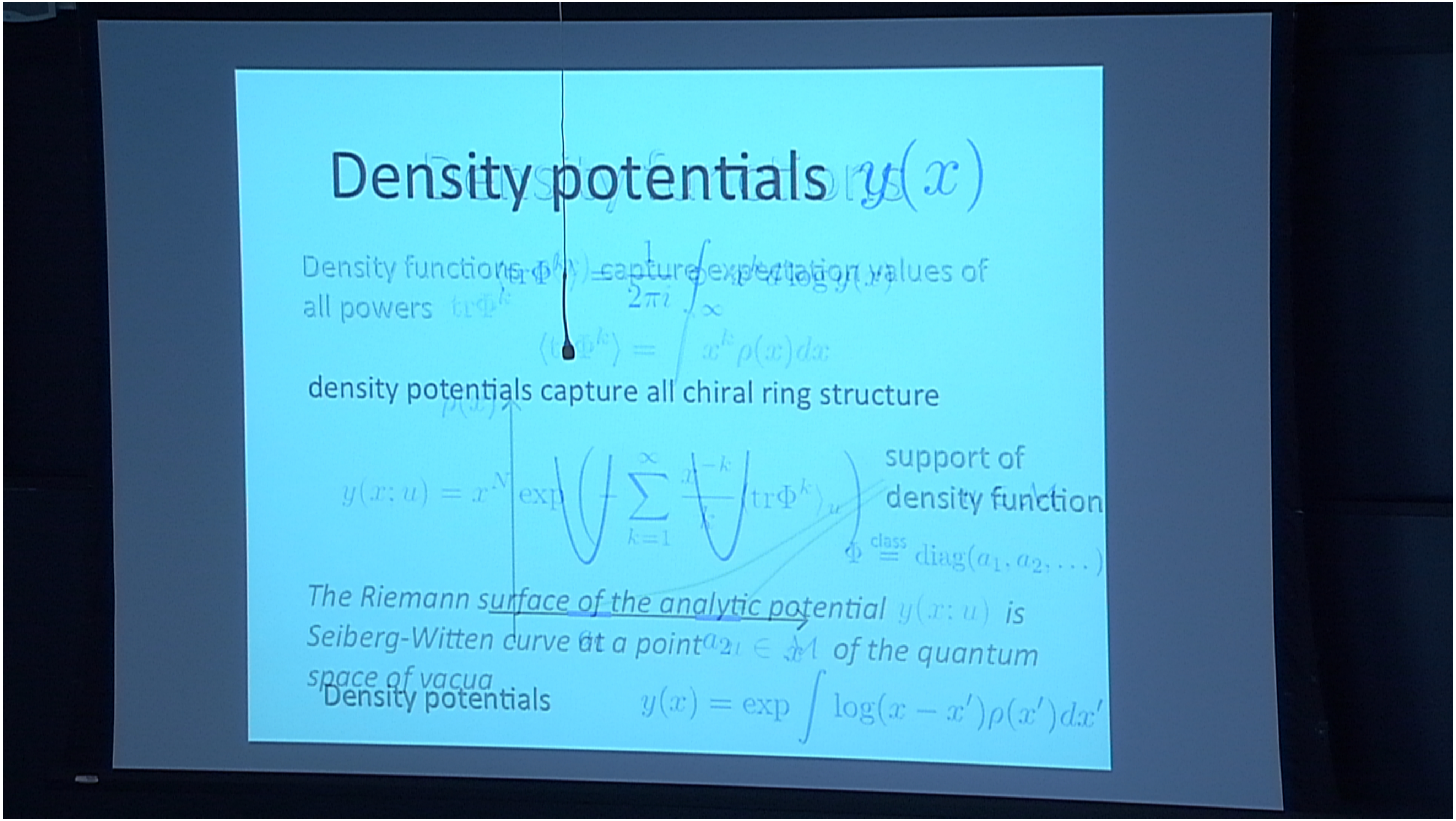
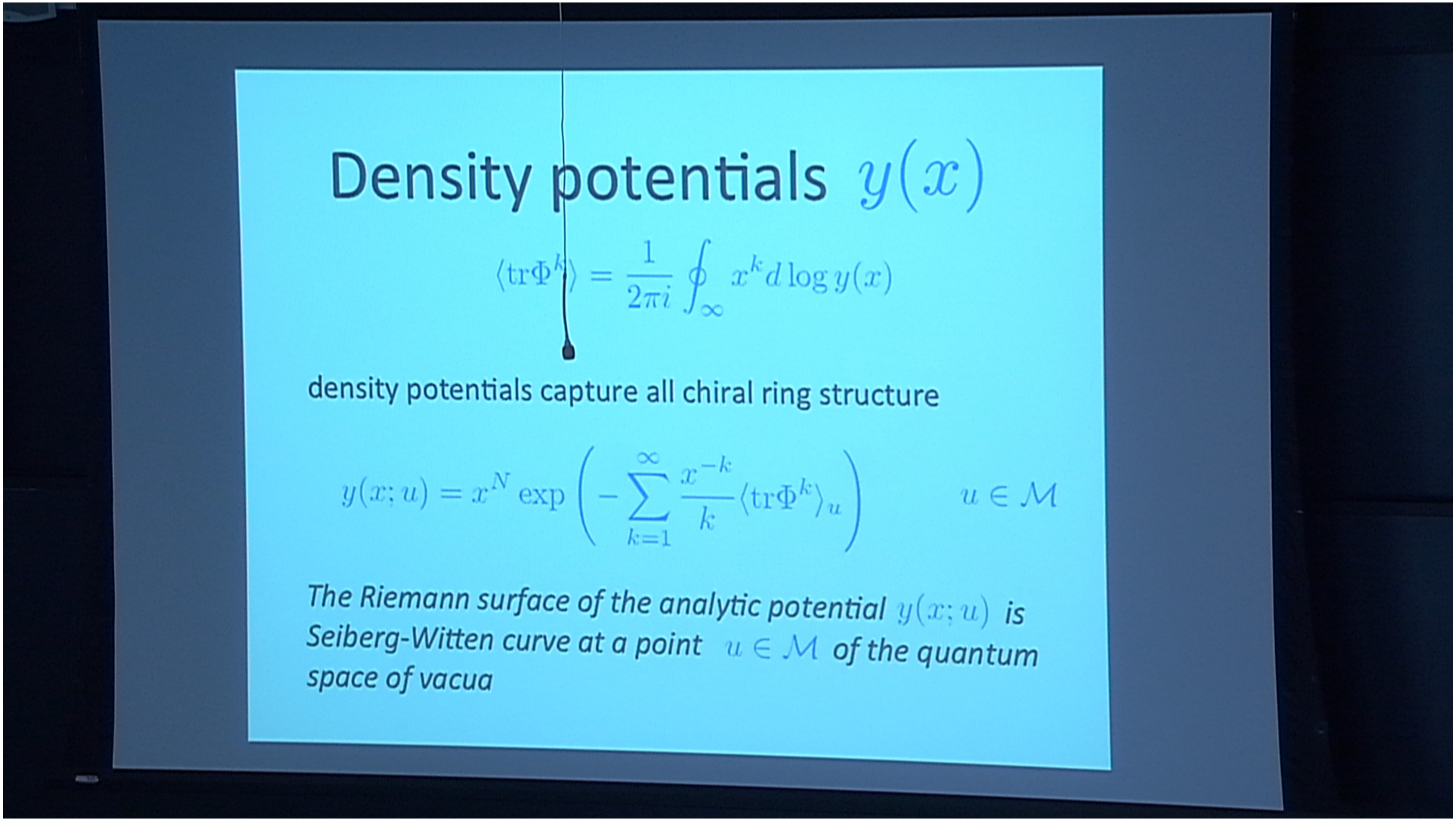

Density potentials $y(x)$
Density functions $\rho(x)$ capture expectation values of all powers $\mathrm{tr}\Phi^k$
$\langle \mathrm{tr}\Phi^k \rangle = \frac{1}{2\pi i}\oint_\infty x^k d\log y(x)$
$\langle \mathrm{tr}\Phi^k \rangle = \int x^k \rho(x) dx$
density potentials capture all chiral ring structure
$\rho(x)$
$y(x:u) = x^N \exp\left(-\sum_{k=1}^{\infty} \frac{x^{-k}}{k} \langle \mathrm{tr}\Phi^k \rangle_u\right)$
support of density function
$\Phi \overset{\text{class}}{=} \mathrm{diag}(a_1, a_2, \ldots)$
The Riemann surface of the analytic potential $y(x:u)$ is Seiberg-Witten curve at a point $u \in \mathcal{M}$ of the quantum space of vacua
Density potentials
$y(x) = \exp\int \log(x - x')\rho(x')dx'$


Density potentials $y(x)$
$\langle \mathrm{tr}\Phi^k \rangle = \frac{1}{2\pi i} \oint_\infty x^k d \log y(x)$
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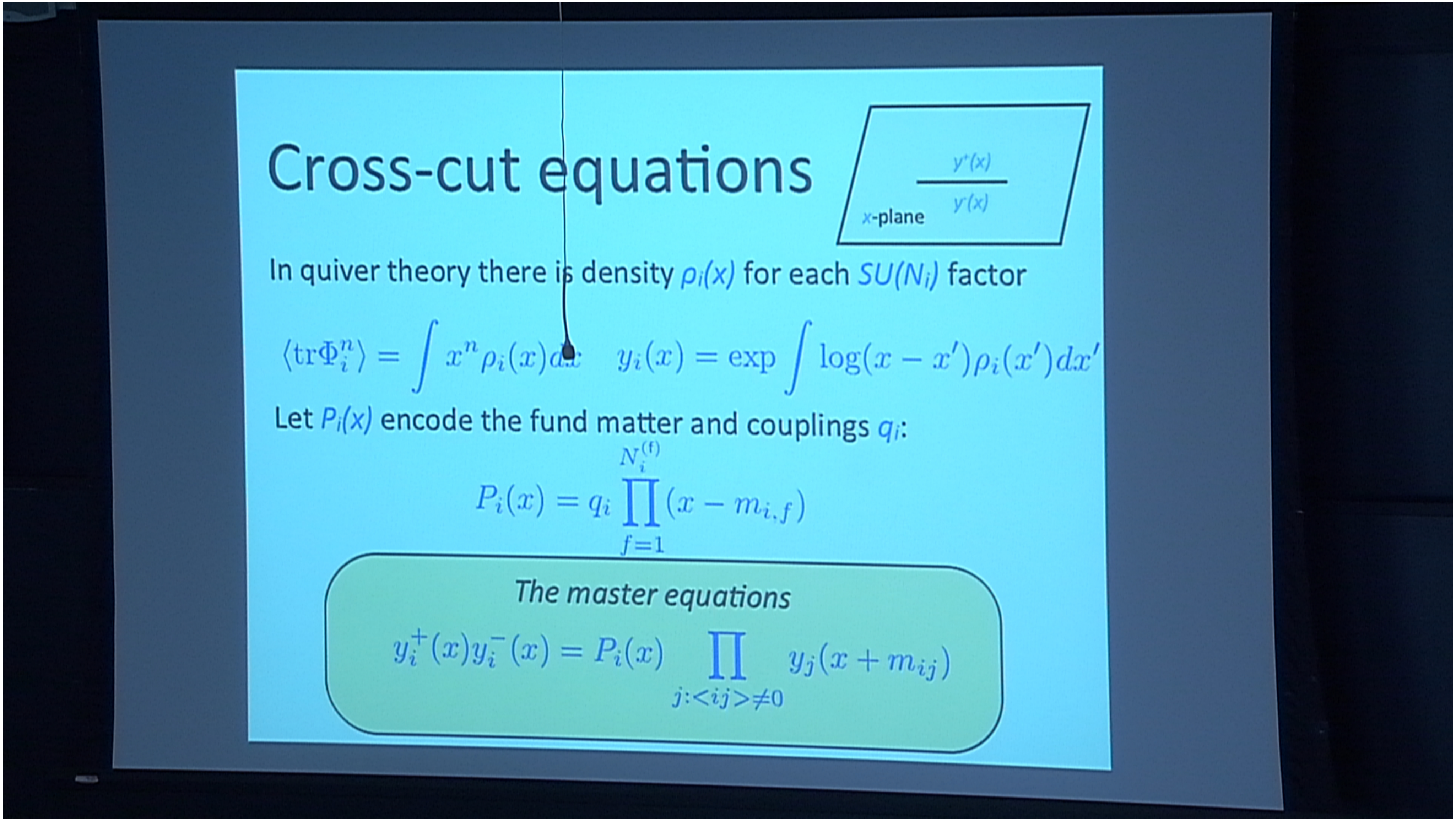
$y(x;u) = x^N \exp\left(- \sum_{k=1}^{\infty} \frac{x^{-k}}{k} \langle \mathrm{tr}\Phi^k \rangle_u \right) \qquad u \in \mathcal{M}$
The Riemann surface of the analytic potential $y(x;u)$ is Seiberg-Witten curve at a point $u \in \mathcal{M}$ of the quantum space of vacua


Cross-cut equations
$y^+(x)$
$y^-(x)$
x-plane
In quiver theory there is density $\rho_i(x)$ for each $SU(N_i)$ factor
$\langle \mathrm{tr}\Phi_i^n \rangle = \int x^n \rho_i(x) dx \quad y_i(x) = \exp \int \log(x - x')\rho_i(x') dx'$
Let $P_i(x)$ encode the fund matter and couplings q_i:
$P_i(x) = q_i \prod_{f=1}^{N_i^{(\mathrm{f})}} (x - m_{i,f})$
The master equations
$y_i^+(x) y_i^-(x) = P_i(x) \prod_{j:<ij>\neq 0} y_j(x + m_{ij})$

Cross-cut transformations

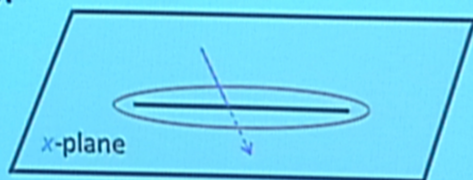
If quiver graph is acyclic, represent bi-fund masses as

$$m_{ij} = m_i - m_j$$

and redefine

$$y_i(x) \to y(x + m_i)$$

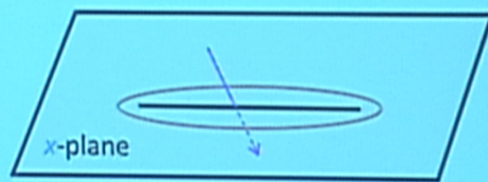
The cross-cut equations can be thought as cross-cut transformations:



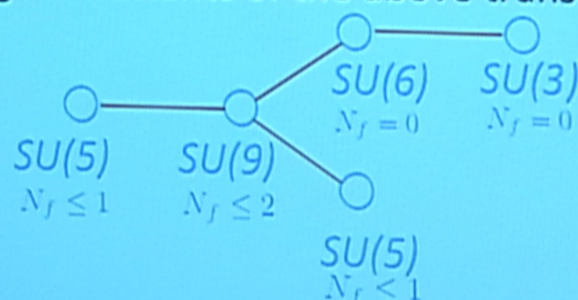
$$y_i(x) \mapsto y_i^{-1}(x) P_i(x) \prod_{j:<ji>\neq 0} y_j(x)$$

Cross-cut invariants for quiver theories

$$y_i(x) \mapsto y_i^{-1}(x) P_i(x) \prod_{j:<ji>\neq 0} y_j(x)$$



How we get invariants of the above transform for quivers?



We shall use the deep fact: the $\mathcal{N}$=2 quiver graphs fall in the famous ADE classification

Cross cut invariants for $\mathcal{N} = 2$ quivers are ADE Weyl invairiants

the transformation

$$y_i(x) \mapsto y_i^{-1}(x) P_i(x) \prod_{j:<ji>\neq 0} y_j(x)$$

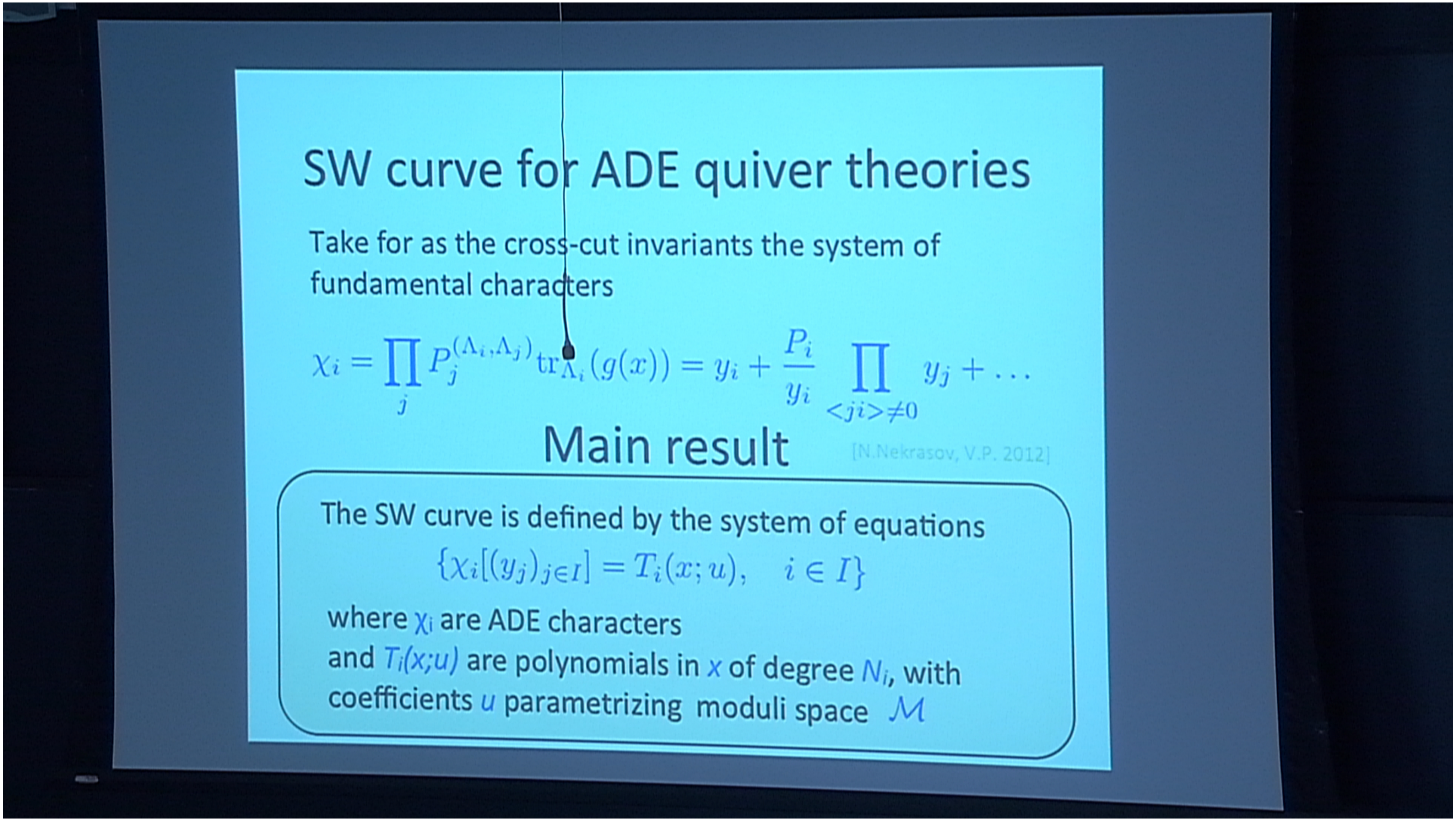
is equivalent to the *i*-th Weyl reflection on the ADE quiver group element:

$$g(x) = \prod_{i \in I} \frac{y_i(x)^{\alpha_i^\vee}}{P_i(x)^{\Lambda_i^\vee}}$$

A_1 example:

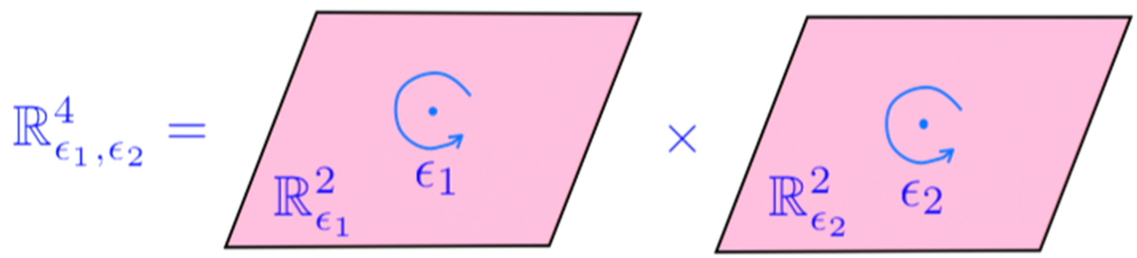
$$g_{A_1}(x) = \begin{pmatrix} y(x)/\sqrt{P(x)} & 0 \\ 0 & \sqrt{P(x)}/y(x) \end{pmatrix}$$

here $\alpha_i^\vee, \Lambda_i^\vee : \mathbb{C}^\times \to T_{G_{\text{ADE quiver}}}$ are coroots (coweights) of the ADE quiver group


SW curve for ADE quiver theories
Take for as the cross-cut invariants the system of fundamental characters
$\chi_i = \prod_j P_j^{(\Lambda_i,\Lambda_j)} \mathrm{tr}_{\Lambda_i}(g(x)) = y_i + \frac{P_i}{y_i} \prod_{<ji>\neq 0} y_j + \dots$
Main result
[N.Nekrasov, V.P. 2012]
The SW curve is defined by the system of equations
$\{\chi_i[(y_j)_{j\in I}] = T_i(x;u), \quad i \in I\}$
where χ_i are ADE characters
and $T_i(x;u)$ are polynomials in x of degree N_i, with
coefficients u parametrizing moduli space $\mathcal{M}$

Method

To derive the cross-cut equations on the density potentials $y_i(x)$ we compute the partition function of $\mathcal{N} = 2$ theories in a twisted space-time $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$



then we take the limit $\epsilon_1, \epsilon_2 \to 0$ [LMNS, Nekrasov, Nekrasov-Okounkov]

$$\mathbb{R}^4_{1/r,1/r} \leftrightarrow S^4_r$$ [V.P.]

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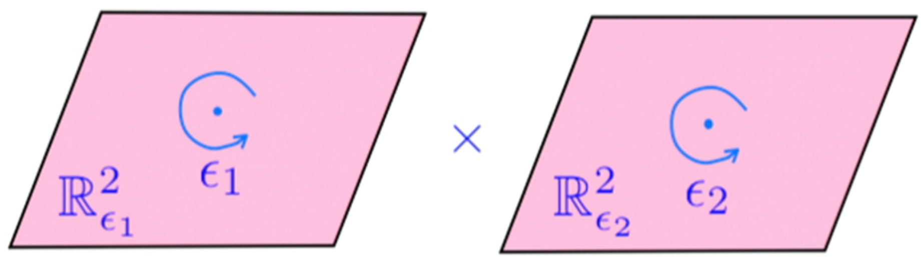

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