Title: The dynamics of asymptotically anti-de Sitter spacetimes

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Abstract: The majority of work on asymptotically anti-de Sitter spacetimes, much of it motivated by the AdS/CFT correspondence, assumes configurations which are either at or close to equilibrium.

Recently it has been discovered that genuinely dynamical spacetimes with AdS asymptotics can exhibit some surprising behaviour. In particular, numerical studies suggest AdS is dynamically unstable. I will discuss recent analytical (in)stability results for asymptotically AdS spacetimes, in particular how stability can be affected by boundary conditions at the timelike conformal infinity.

Dynamics in asymptotically anti-de Sitter spacetimes

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Perimeter Institute, March 14, 2013

Talk Outline

- Motivation
  - Instability of anti-de Sitter
- The Klein-Gordon equation on asymptotically AdS spaces
  - Well posedness
  - Black hole stability results

Why aAdS?

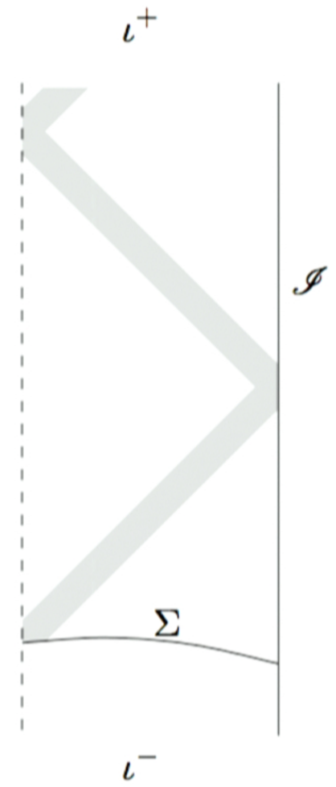
- AdS one of three maximally symmetric ground states for gravitational field far from an isolated system
- Asymptotically AdS spacetimes are of great interest in context of AdS/CFT correspondence
- Many interesting classical effects related to the timelike conformal boundary
  - Choice of boundary conditions for fields
  - Energy 'trapped' inside spacetime
  - New possible sources of instability

Instability of AdS

- anti-de Sitter spacetime described by a metric on $\mathbb{R}^4$:

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{l^2}} + r^2 d\Omega^2$$

- Linear fields remain bounded in time (but do not decay)
- Conjectured by Dafermos, and independently by Anderson (2006) that AdS is *nonlinearly* unstable



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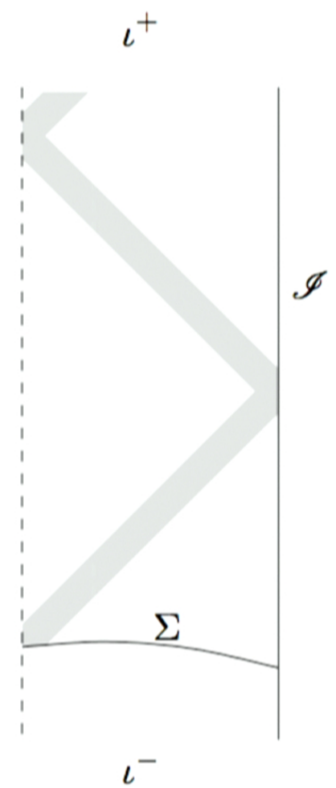

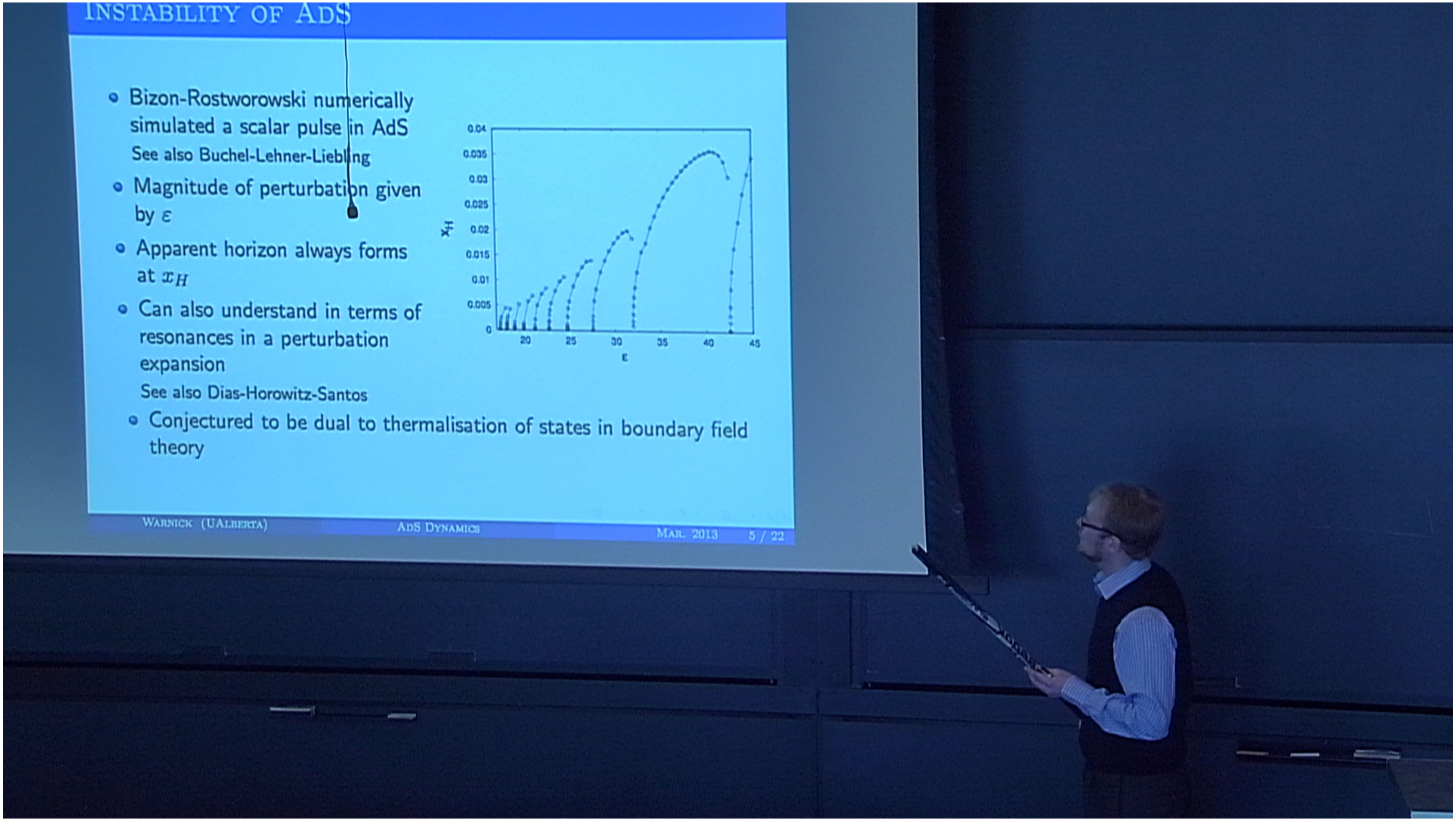
INSTABILITY OF AdS
Bizon-Rostworowski numerically simulated a scalar pulse in AdS
See also Buchel-Lehner-Liebling
Magnitude of perturbation given by ε
Apparent horizon always forms at x_H
Can also understand in terms of resonances in a perturbation expansion
See also Dias-Horowitz-Santos
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x_H
ε
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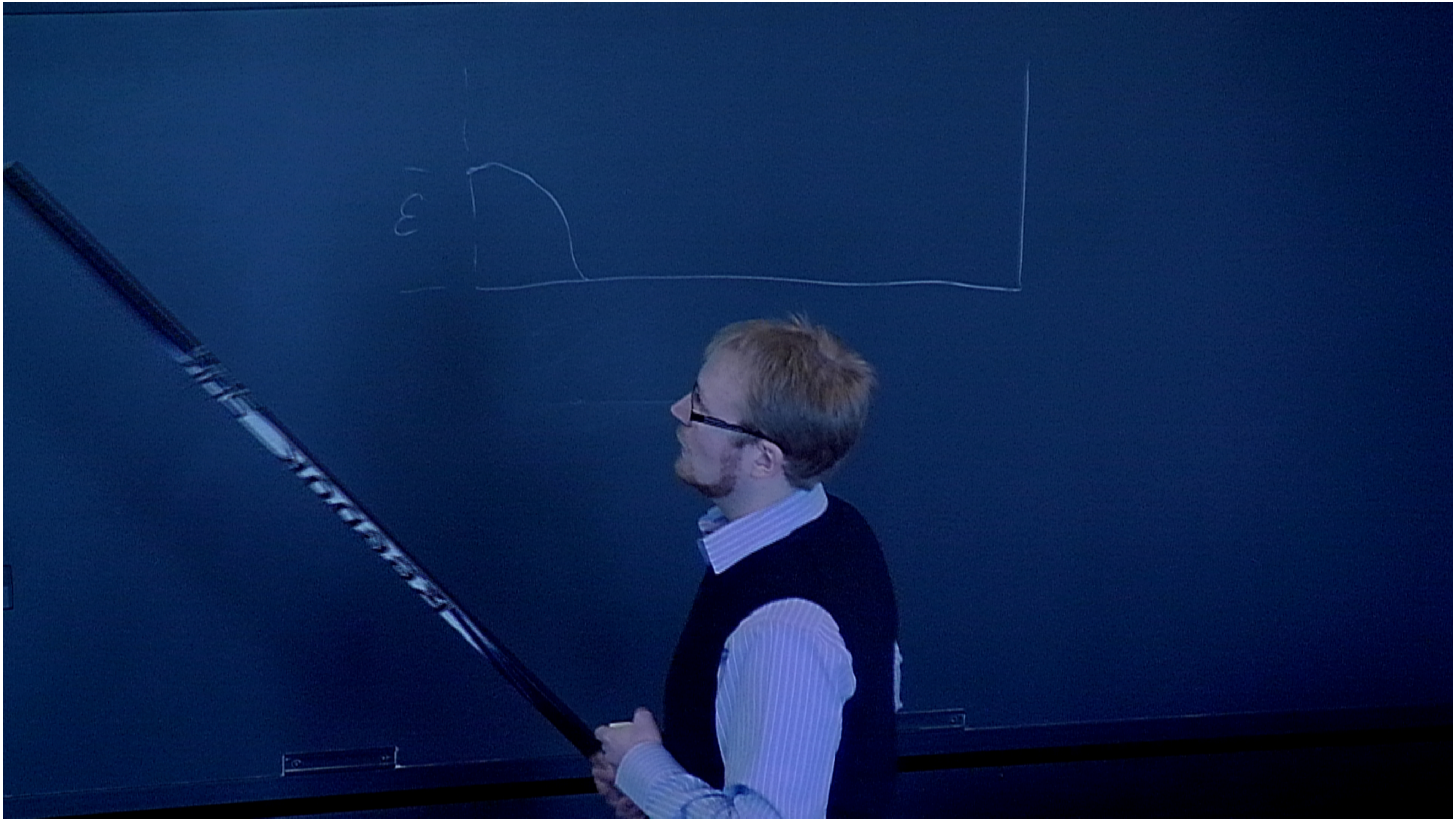
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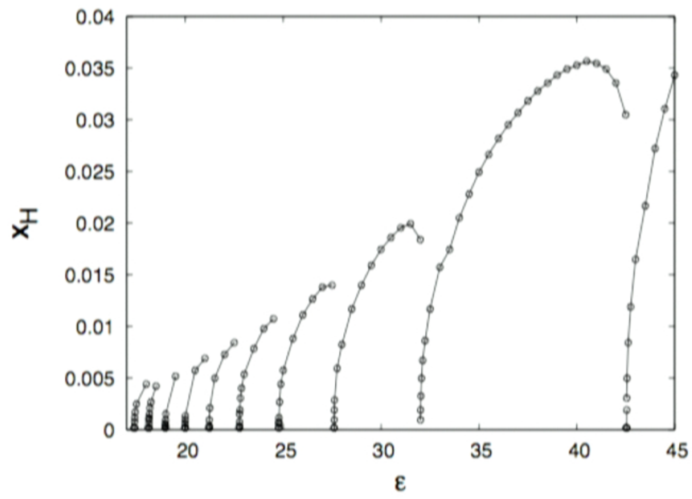

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Dynamics

- Much (but not all) analytical work on asymptotically AdS spacetimes considers equilibrium situations:
  - Static or stationary fields: often explicit metrics
  - Can often 'Wick rotate': Riemannian metrics; elliptic PDEs
- What sort of methods do we need to study dynamics?
  - No Wick rotation: equations are hyperbolic
  - Methods based on separation of variables not as useful
  - Avoid symmetry assumptions
  - Understand role of boundary conditions

DYNAMICS

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Asymptotically AdS spacetimes

- X an $(n+1)-$dimensional manifold with boundary ∂X, and g a smooth Lorentzian metric on $\mathring{X}$.

Definition

∂X is an *asymptotically anti-de Sitter end of* (X, g) *with radius* l if:

I) There exists a smooth r such that $r^{-1} = 0$, $\nabla r^{-1} \neq 0$ on $\partial X = \mathscr{I}$

II) If x^α are coordinates on the slices $r = \text{const.}$, we have locally

$$g_{rr} = \frac{l^2}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right), \qquad g_{r\alpha} = \mathcal{O}\left(\frac{1}{r^2}\right), \qquad g_{\alpha\beta} = r^2 \mathfrak{g}_{\alpha\beta} + \mathcal{O}(1),$$

where $\mathfrak{g}_{\alpha\beta} dx^\alpha dx^\beta$ is a Lorentzian metric on $\mathscr{I}$.

III) $r^{-2} g$ extends as a smooth metric on a neighbourhood of $\mathscr{I}$.

- Examples include: AdS, AdS-Schwarzschild, AdS-Kerr, ...

Instability of AdS

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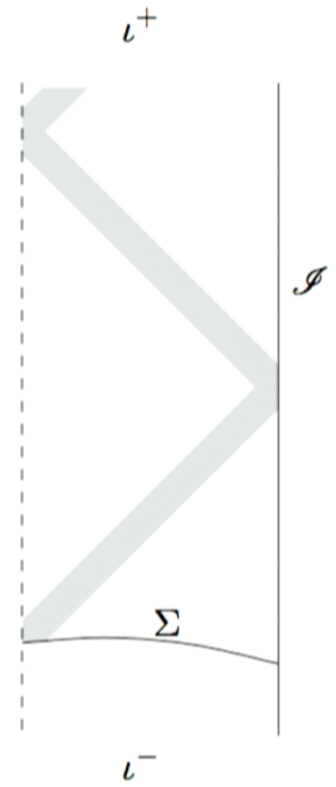

Klein-Gordon Equation

- Simplest equation to consider is the Klein-Gordon equation

$$\Box_g u - \frac{\mu}{l^2} u = 0, \qquad \mu > -\frac{n^2}{4} \tag{1}$$

- On AdS^{n+1}, find that near infinity

$$u \sim a_+ r^{-\frac{n}{2}+\sqrt{\frac{n^2}{4}+\mu}} + a_- r^{-\frac{n}{2}-\sqrt{\frac{n^2}{4}+\mu}}$$

- Can solve (1) with initial conditions $u = u_0, \partial_t u = u_1$ at $t = 0$ and choice of boundary conditions

$a_+ = 0$	$\mu > -\frac{n^2}{4}$	Dirichlet
$a_- = 0$	$-\frac{n^2}{4} + 1 > \mu > -\frac{n^2}{4}$	Neumann
$a_- - \beta a_+ = 0$	$-\frac{n^2}{4} + 1 > \mu > -\frac{n^2}{4}$	Robin

(Breitenlohner-Freedman; Ishibashi-Wald)

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Klein-Gordon Equation

Theorem (CMW 2012)

Let Σ be a spacelike, n-dimensional surface in (X^{n+1}, g), an asymptotically AdS spacetime. Choose Dirichlet, Neumann or Robin boundary conditions on $\mathscr{I}$, appropriate to μ. Then for suitable $u_0, u_1 : \Sigma \to \mathbb{R}$ there exists a unique u solving

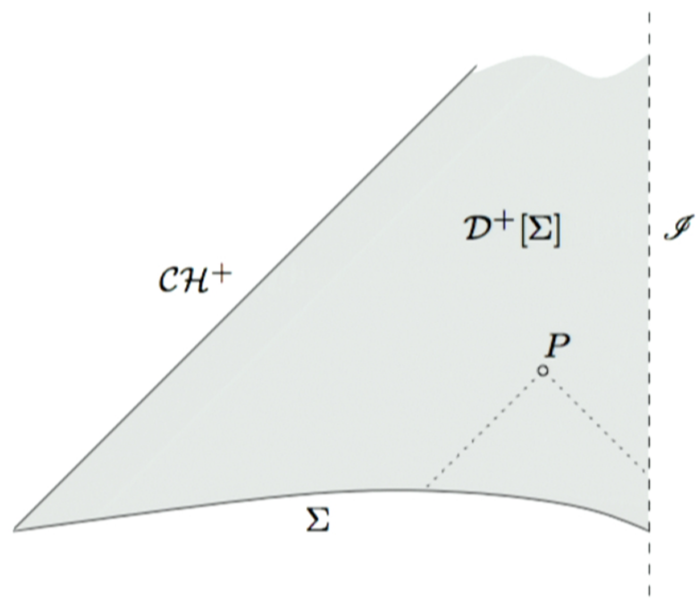
$$\Box_g u - \frac{\mu}{l^2} u = 0 \quad in \quad D^+[\Sigma \cup (\mathscr{I} \cap I^+(\Sigma))] := \mathcal{D}^+[\Sigma]$$

with

$$u|_\Sigma = u_0, \quad n_\Sigma u|_\Sigma = u_1.$$

Furthermore u has an expansion near infinity as in the AdS case.

Klein-Gordon Equation



Klein-Gordon Equation

- No assumption of symmetry for (X, g)
- Do not require that the Klein-Gordon equation separates
- Note initial conditions necessary: boundary data alone does *not* determine field everywhere
- Theorem applies for *inhomogeneous* Dirichlet, Neumann, Robin conditions also (for suitable μ)
- Proof uses energy space methods based on a *renormalized* energy
- In general, energy on a spacelike surface Σ' to the future of Σ will be larger than initial energy, but still finite
- Need more structure to say more about behaviour in time

Stationary Black Holes

Definition

$(\mathcal{R}, g)$ is the exterior region of a stationary, asymptotically anti-de Sitter, black hole space time if the following holds

I) $(\mathcal{R}$, g) is asymptotically AdS with null infinity $\mathscr{I}$

II) $\mathcal{R} = \mathcal{D}^+[\Sigma]$ for some spacelike Σ

III) g admits a Killing field T such that:

1. T is timelike in the interior of $\mathcal{R}$
2. T is tangent to $\mathscr{I}$ and normal to $\mathcal{H}^+ = \mathcal{CH}^+$
3. $\mathcal{H}^+$ is a Killing horizon of T, with surface gravity $\kappa > 0$.
4. $r^{-1}T$ is uniformly bounded on $\mathcal{R}$ (w.r.t g)

IV) Σ, $\mathcal{H}^+ \cap \Sigma$ and $\mathscr{I} \cap \Sigma$ are compact

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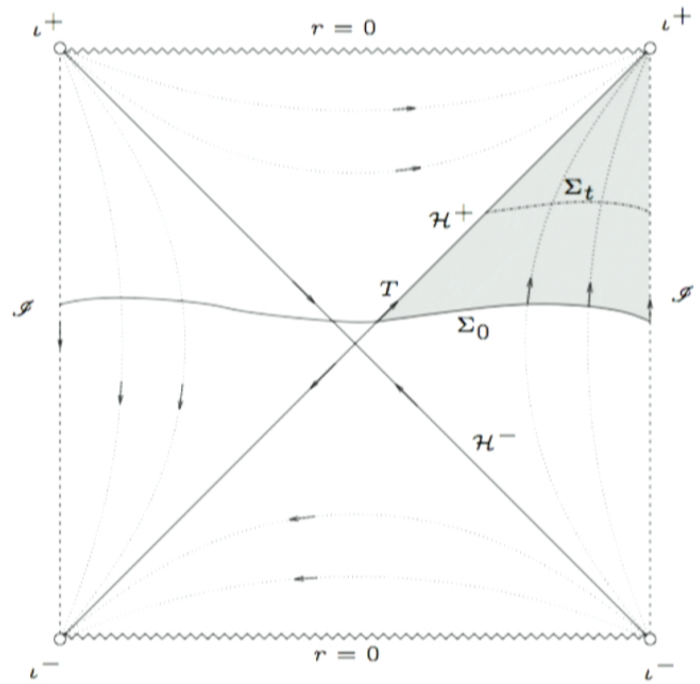
Example: AdS Schwarzschild



$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{r^2}{l^2}} + r^2 d\Omega^2$$

Other examples

- AdS-Schwarzschild with hyperbolic / flat symmetry orbits
- AdS-Kerr below Hawking-Reall bound
- AdS-Reissner-Nördstrom
- various 'hairy' black holes, supported by non-trivial fields
- Higher dimensional generalisations

Linear fields outside black holes

- What can we say about linear fields on black holes?
- Re-write the wave equation

$$\Box_g u - \frac{\mu}{l^2} u = 0, \qquad \mu > -\frac{n^2}{4}$$

as

$$\ddot{u} - B\dot{u} + Lu = 0, \qquad {}^{\cdot} \equiv \frac{\partial}{\partial t} = T.$$

- B and L are spatial operators
- Large t behaviour controlled by L

Linear fields outside black holes

Theorem (G. H. Holzegel, CMW 2012)

Fix homogeneous Dirichlet, Neumann or stationary Robin boundary conditions at $\mathscr{I}$, as appropriate to μ. The eigenvalue problem:

$$Lu = \lambda u, \qquad u \text{ regular at } \mathcal{H}^+, \text{ obeys b.c.s at } \mathscr{I}.$$

is exactly analogous to the eigenvalue problem for the Laplacian in a finite domain.

Let λ_0 be the smallest eigenvalue

- *If $\lambda_0 > 0$, solutions of the wave equation are bounded in time*
- *If $\lambda_0 < 0$, there exist solutions which grow exponentially in time*

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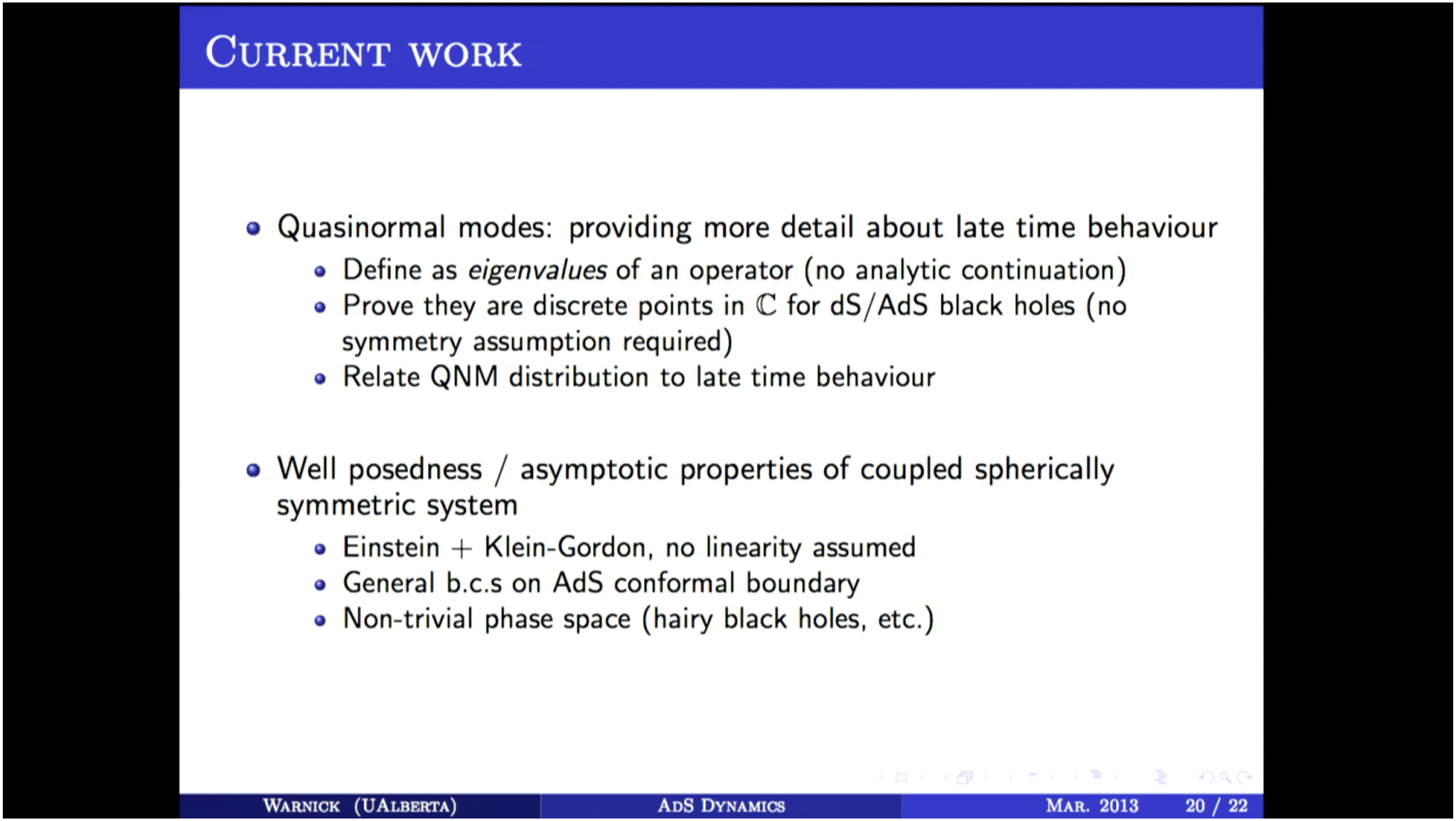
Linear fields outside black holes

- Proof again uses renormalised energy spaces
- λ_0 depends continuously on g
  - Metrics admitting stationary solutions of the wave equation are marginally stable
- Gives a numerically computable criterion for linear stability
- Can apply to known four dimensional spacetimes:

Are solutions to the Klein-Gordon equation bounded in time?

	Dirichlet	Neumann	Robin
AdS-Schwarzschild	Yes	Yes	$\beta > 0$ $\beta > \beta_c(M, \mu)$
AdS-Kerr	Yes	Small a all a	Small a, $\beta > 0$ all a, $\beta > 0$

(numerical results: Holzegel-CMW; Dias-Monteiro-Reall-Santos)


Current work
Quasinormal modes: providing more detail about late time behaviour
Define as eigenvalues of an operator (no analytic continuation)
Prove they are discrete points in ℂ for dS/AdS black holes (no symmetry assumption required)
Relate QNM distribution to late time behaviour
Well posedness / asymptotic properties of coupled spherically symmetric system
Einstein + Klein-Gordon, no linearity assumed
General b.c.s on AdS conformal boundary
Non-trivial phase space (hairy black holes, etc.)
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Current work

- Quasinormal modes: providing more detail about late time behaviour
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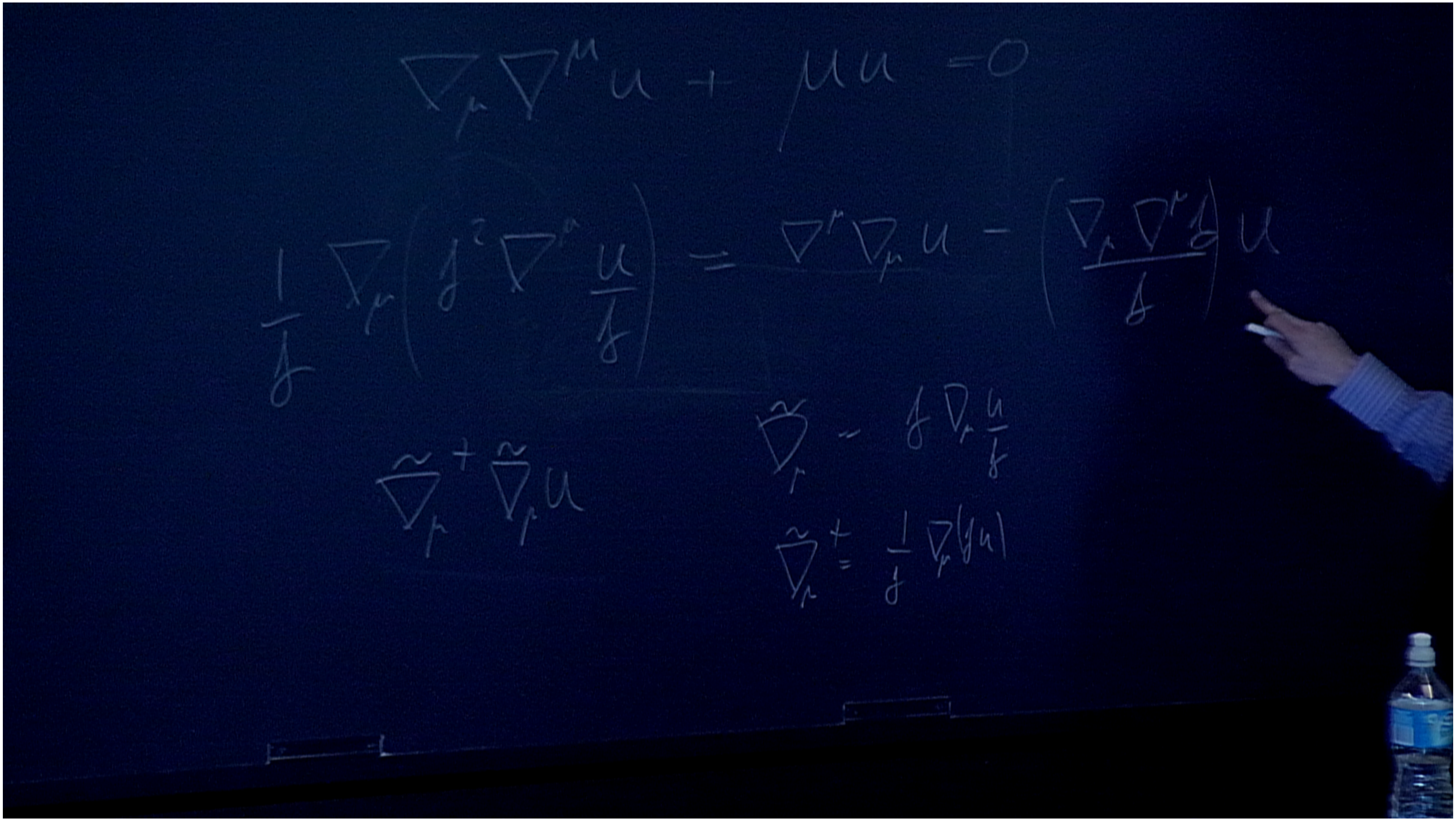
- Well posedness / asymptotic properties of coupled spherically symmetric system
  - Einstein + Klein-Gordon, no linearity assumed
  - General b.c.s on AdS conformal boundary
  - Non-trivial phase space (hairy black holes, etc.)

Conclusion

- Asymptotically AdS spacetimes can have interesting dynamics, owing to the timelike conformal boundary
- The Klein-Gordon equation is well posed with (in)homogeneous Dirichlet, Neumann or Robin boundary conditions
- Boundary data alone is *insufficient* to determine the field in the interior uniquely
- Boundedness of the K-G equation for black holes can be reduced to an eigenvalue problem


∇_μ ∇^μ u + μ u = 0
(1/g) ∇_μ (g² ∇^μ u/g) = ∇^μ ∇_μ u − (∇_μ ∇^μ g / g) u
∇̃_μ^† ∇̃_μ u
∇̃_μ = g ∇_μ u/g
∇̃_μ^† = (1/g) ∇_μ (g u)