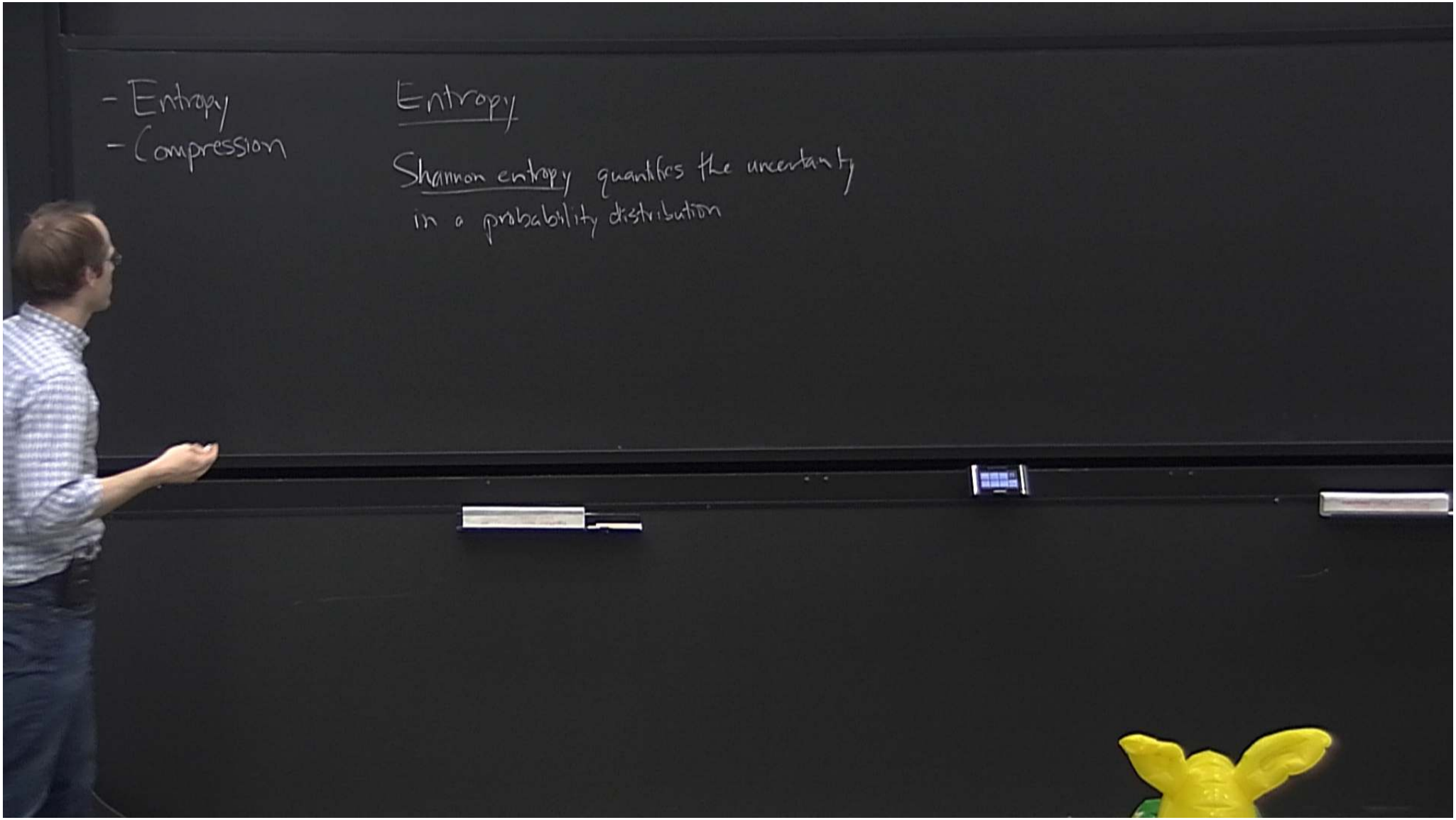
Title: 12/13 PSI - Quantum Information Review Lecture 11

Date: Mar 05, 2013 10:15 AM

URL: http://www.pirsa.org/13030050

Abstract:

- Entropy
- Compression
Entropy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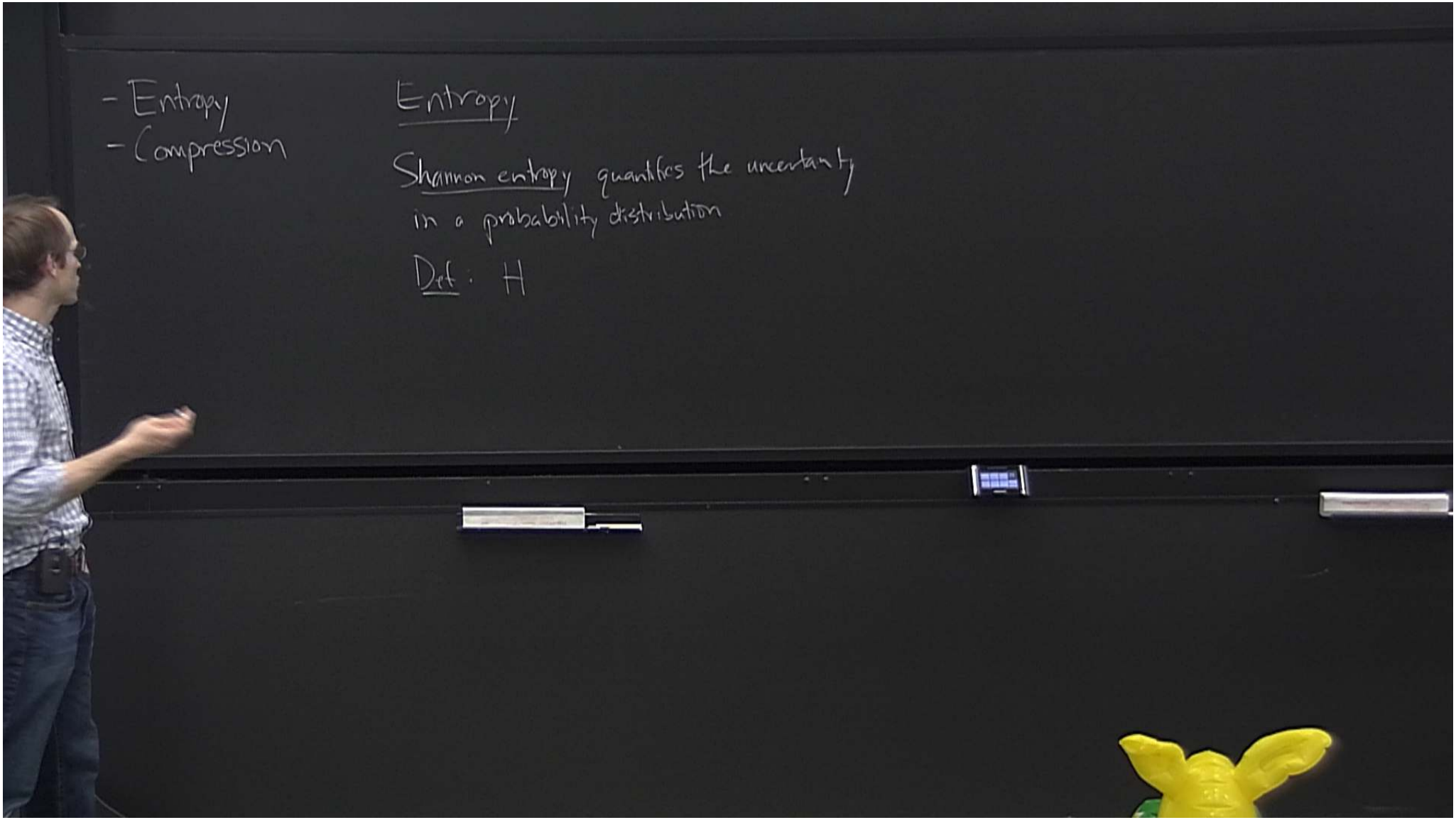
Entropy
Shannon entropy quantifies the uncertainty
in a probability distribution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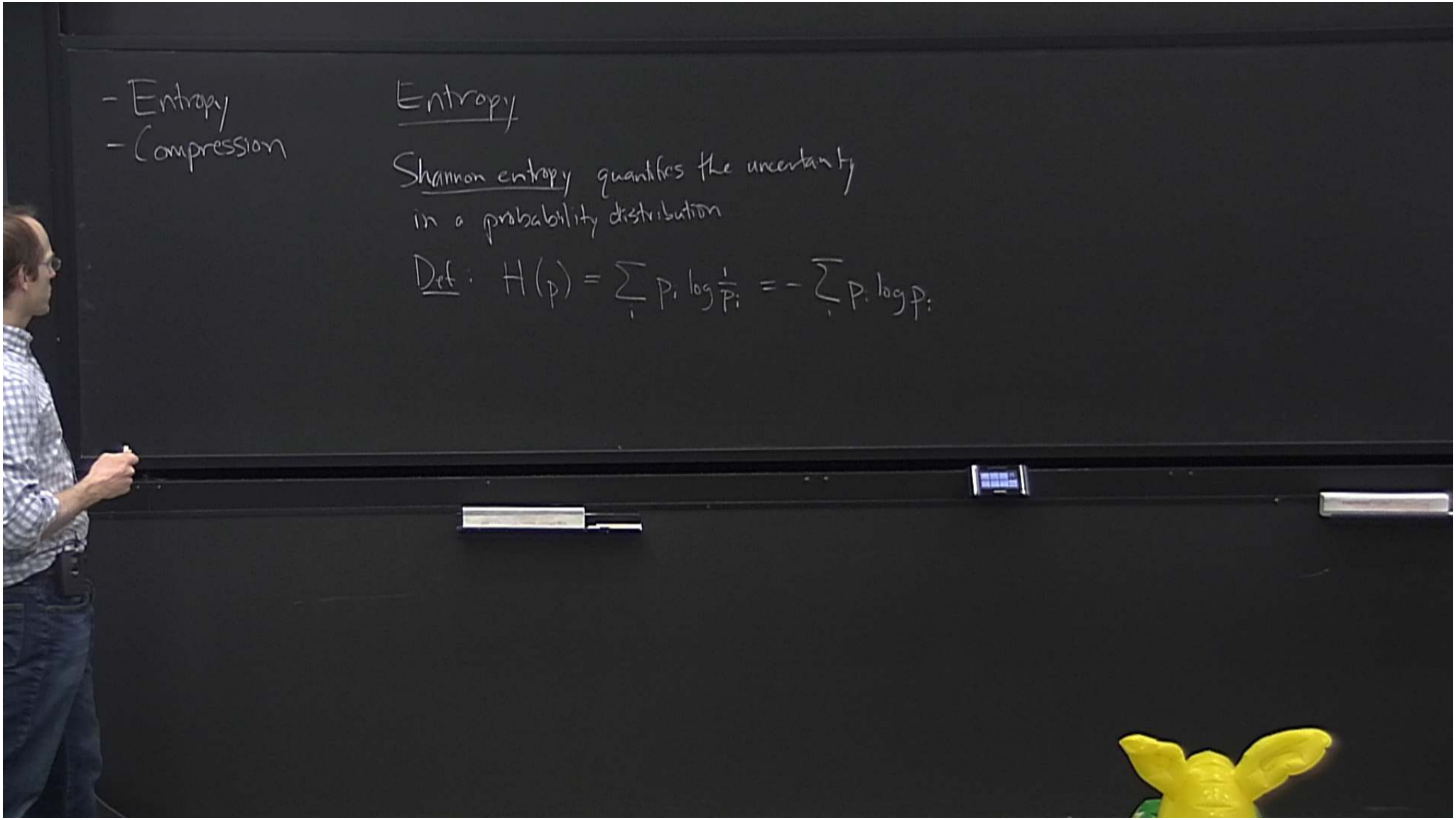
Def: H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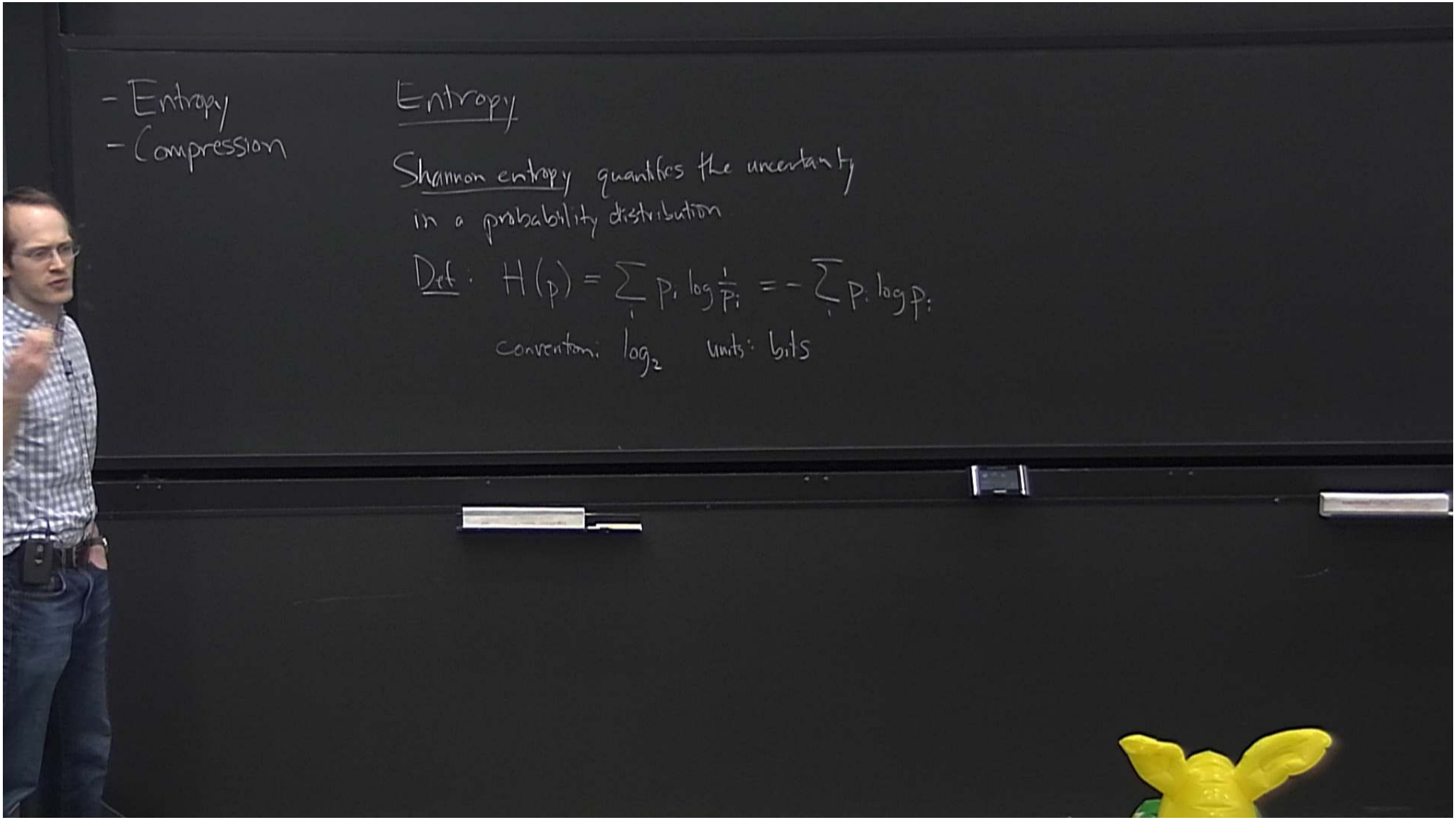
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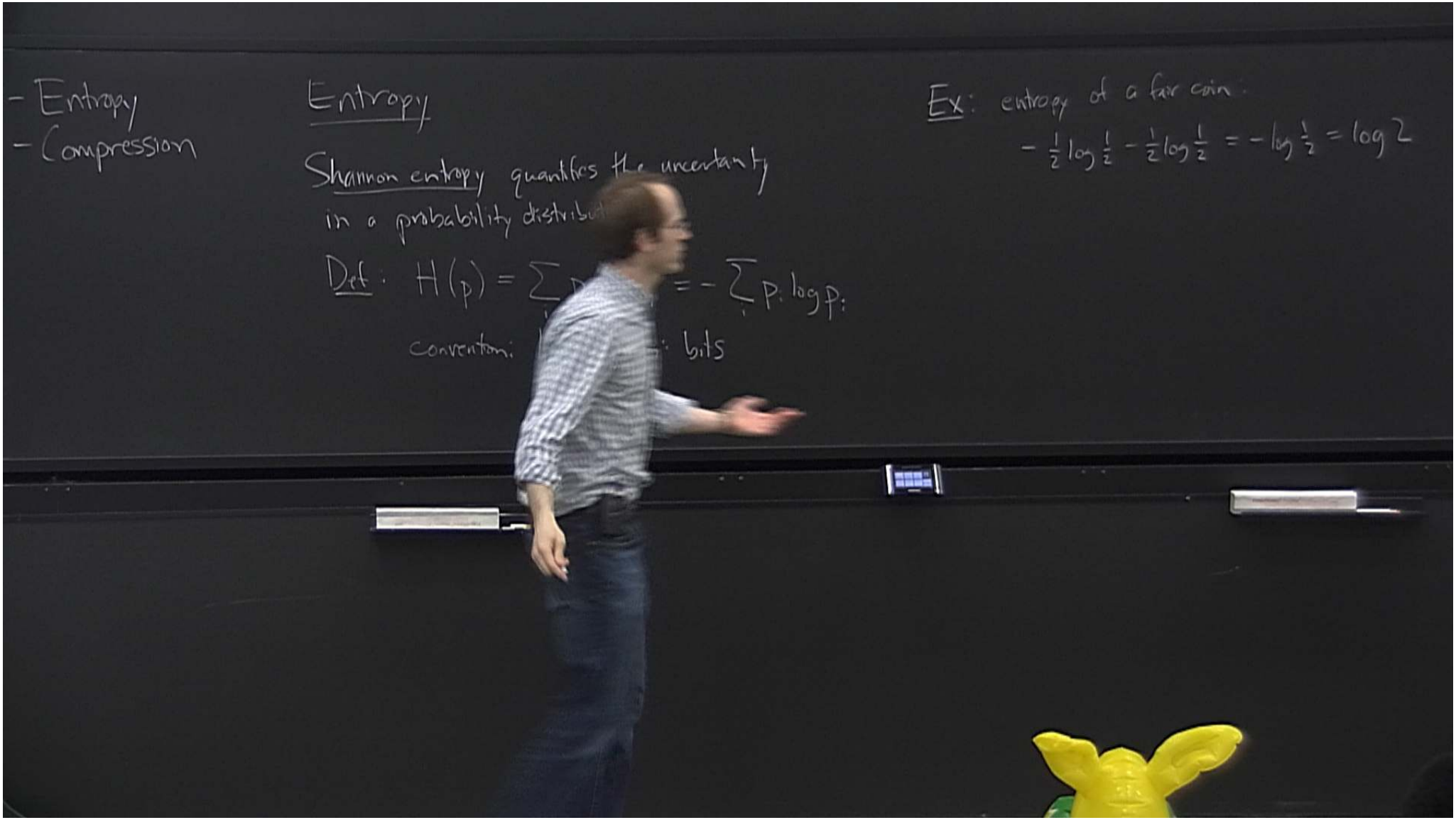
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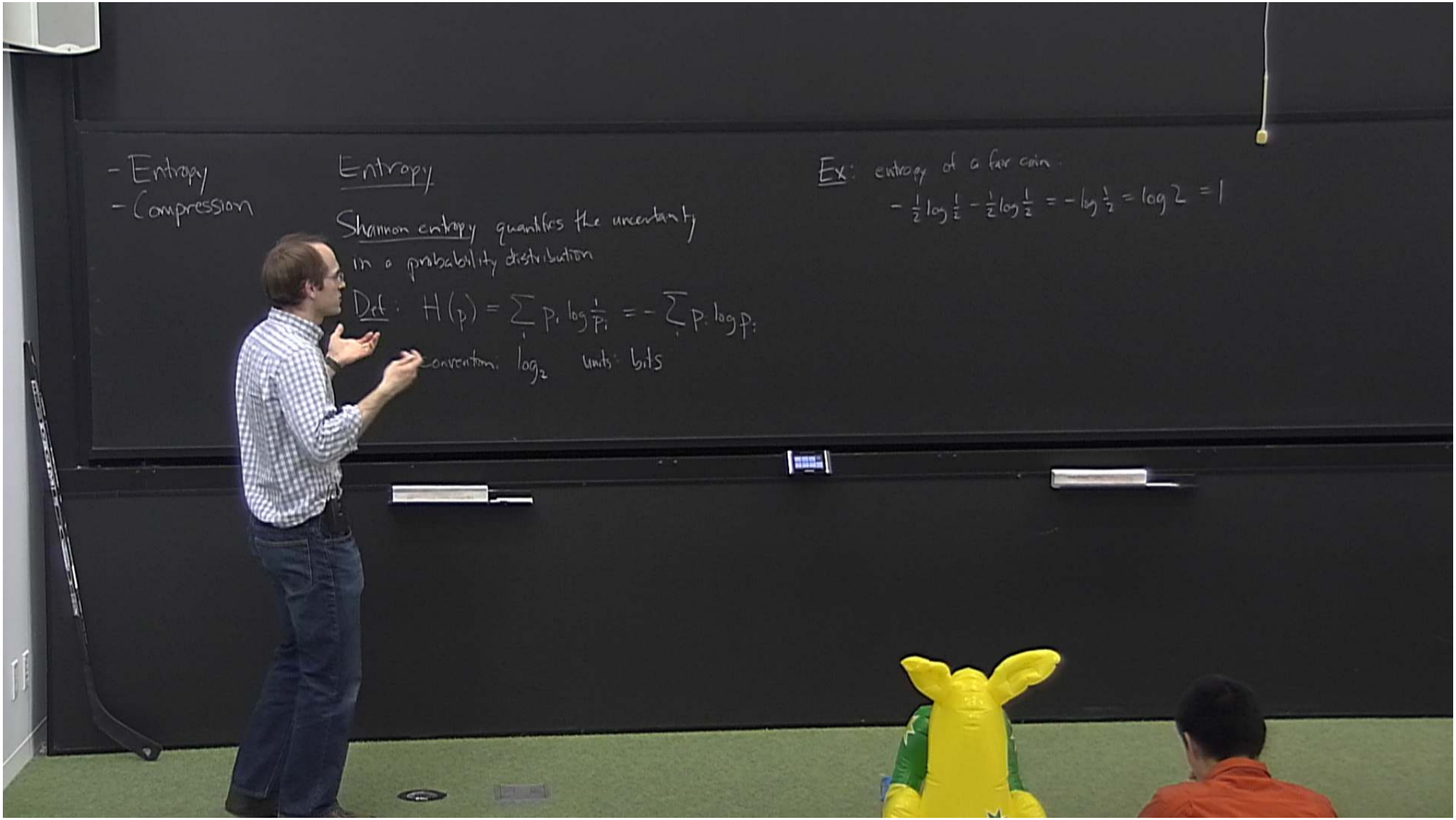
Ex: entropy of a fair coin:
- 1/2 log 1/2 - 1/2 log 1/2 = - log 1/2 = log 2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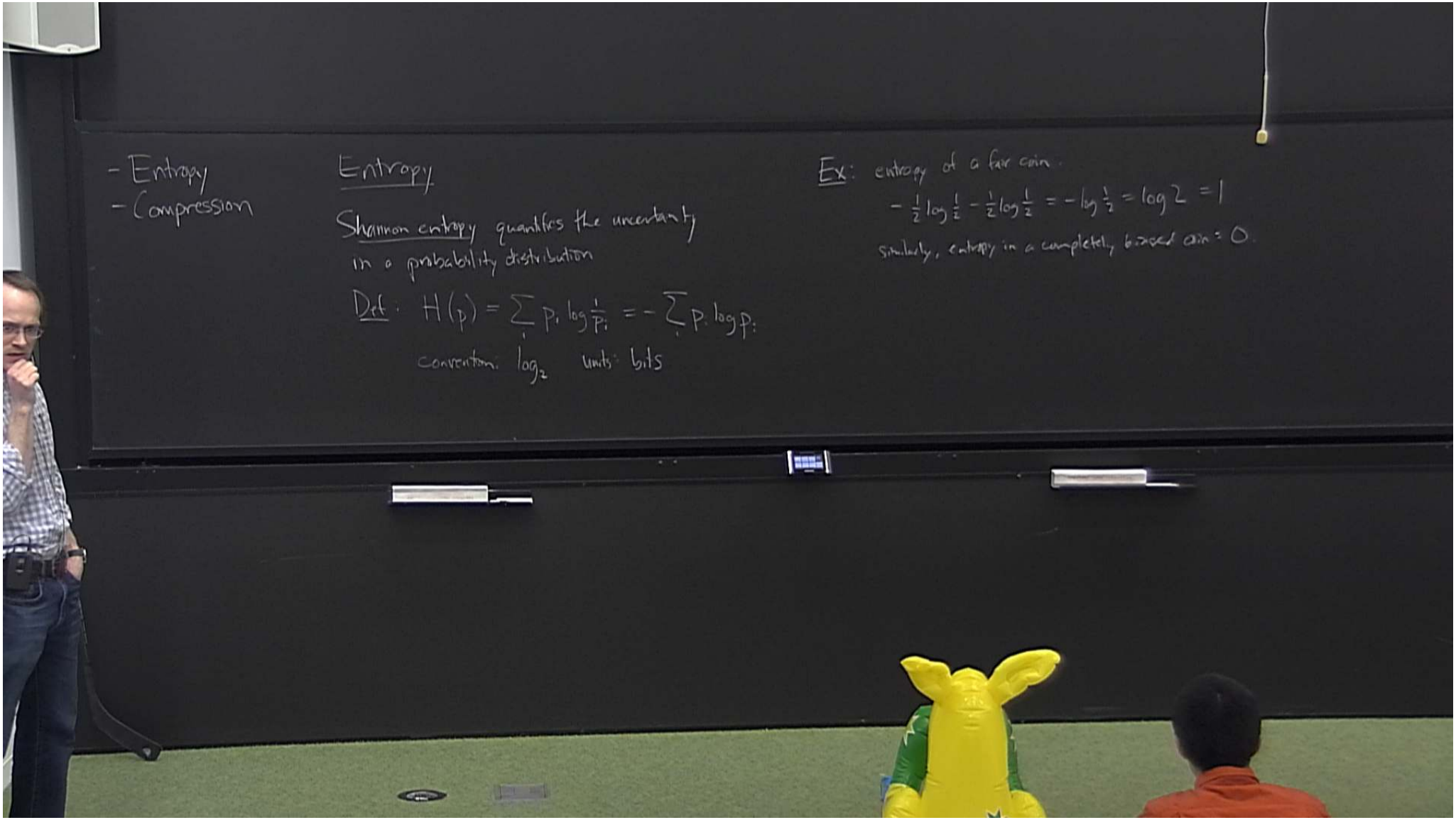
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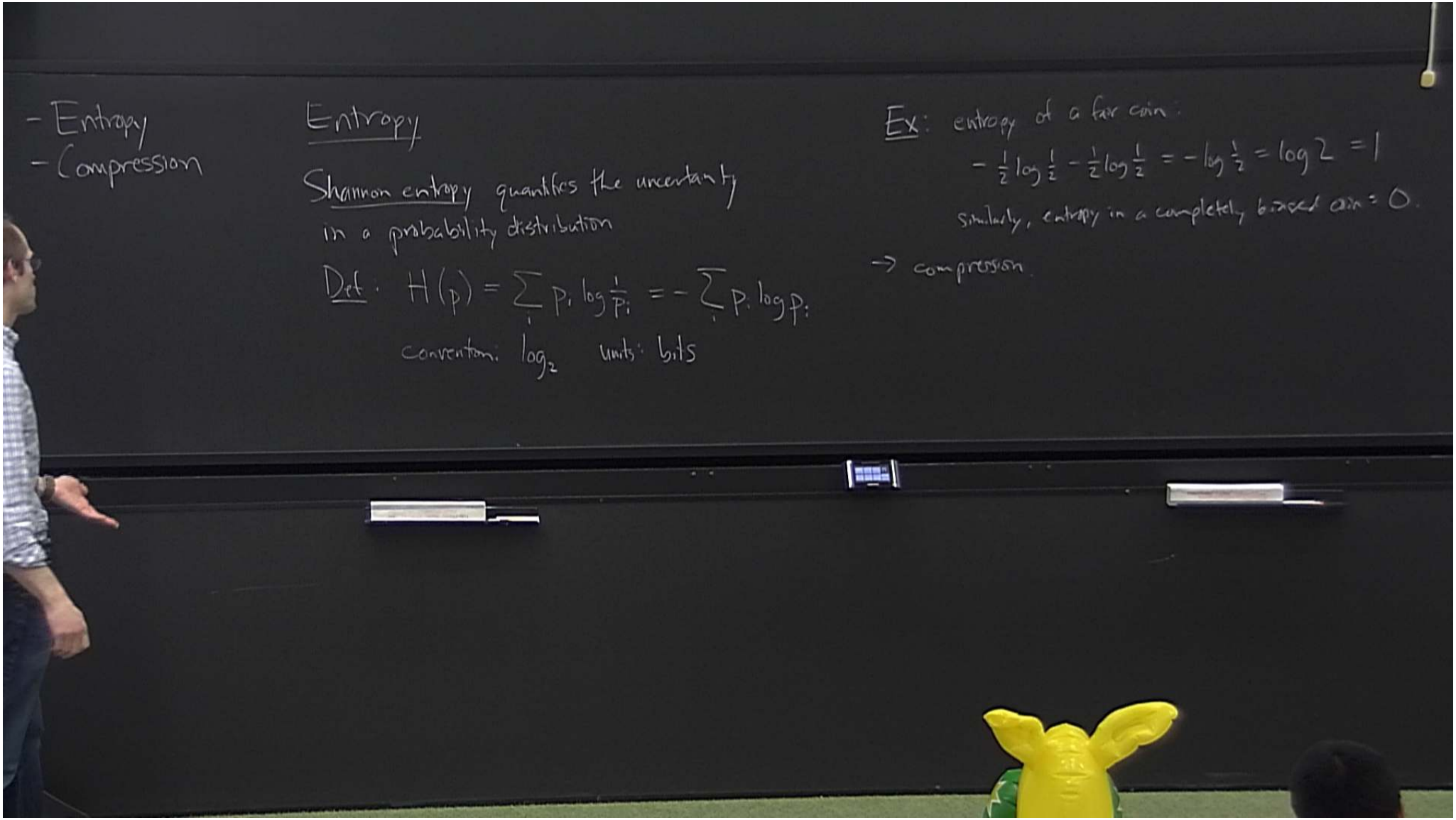
Similarly, entropy in a completely biased coin = 0.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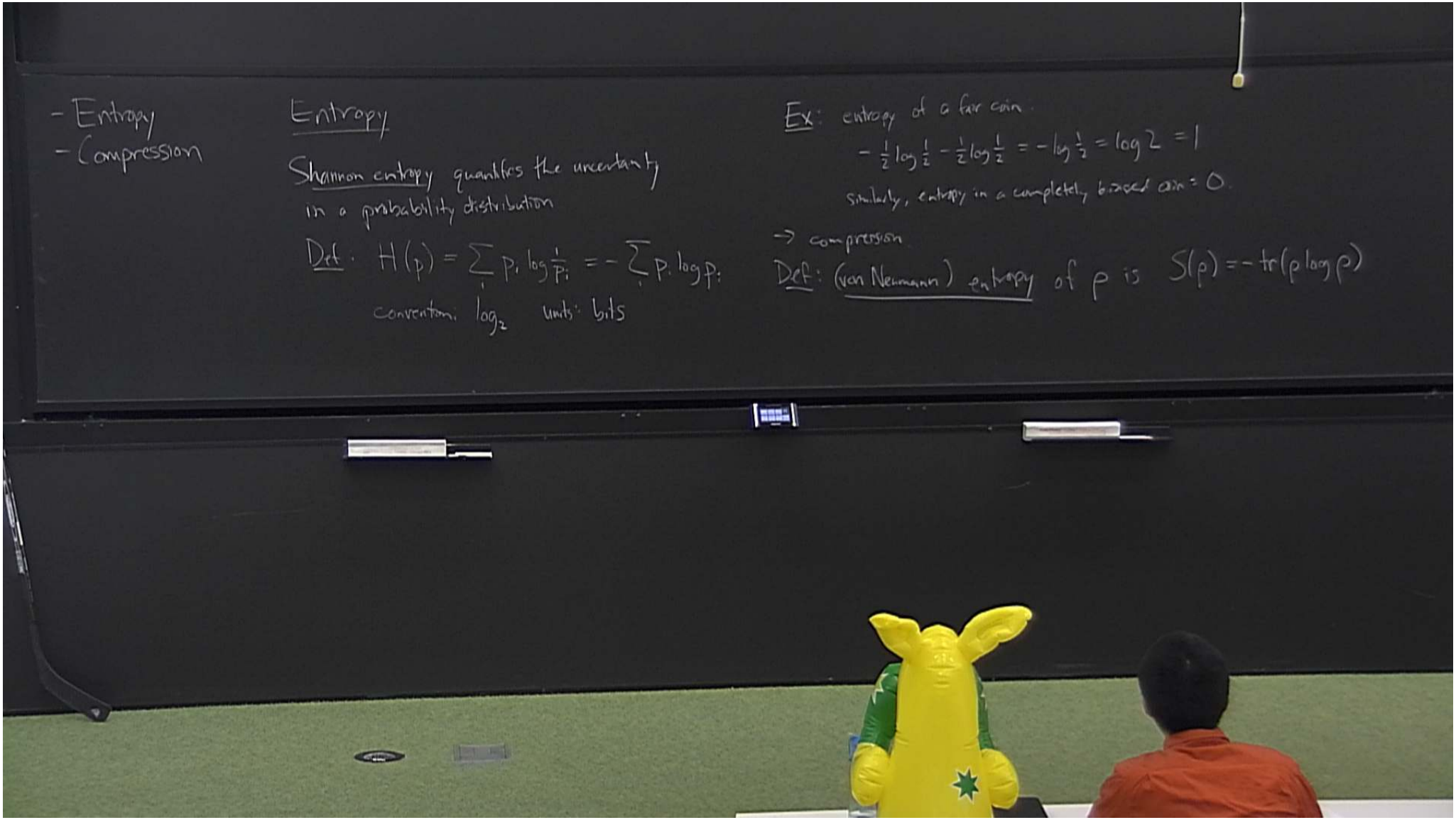
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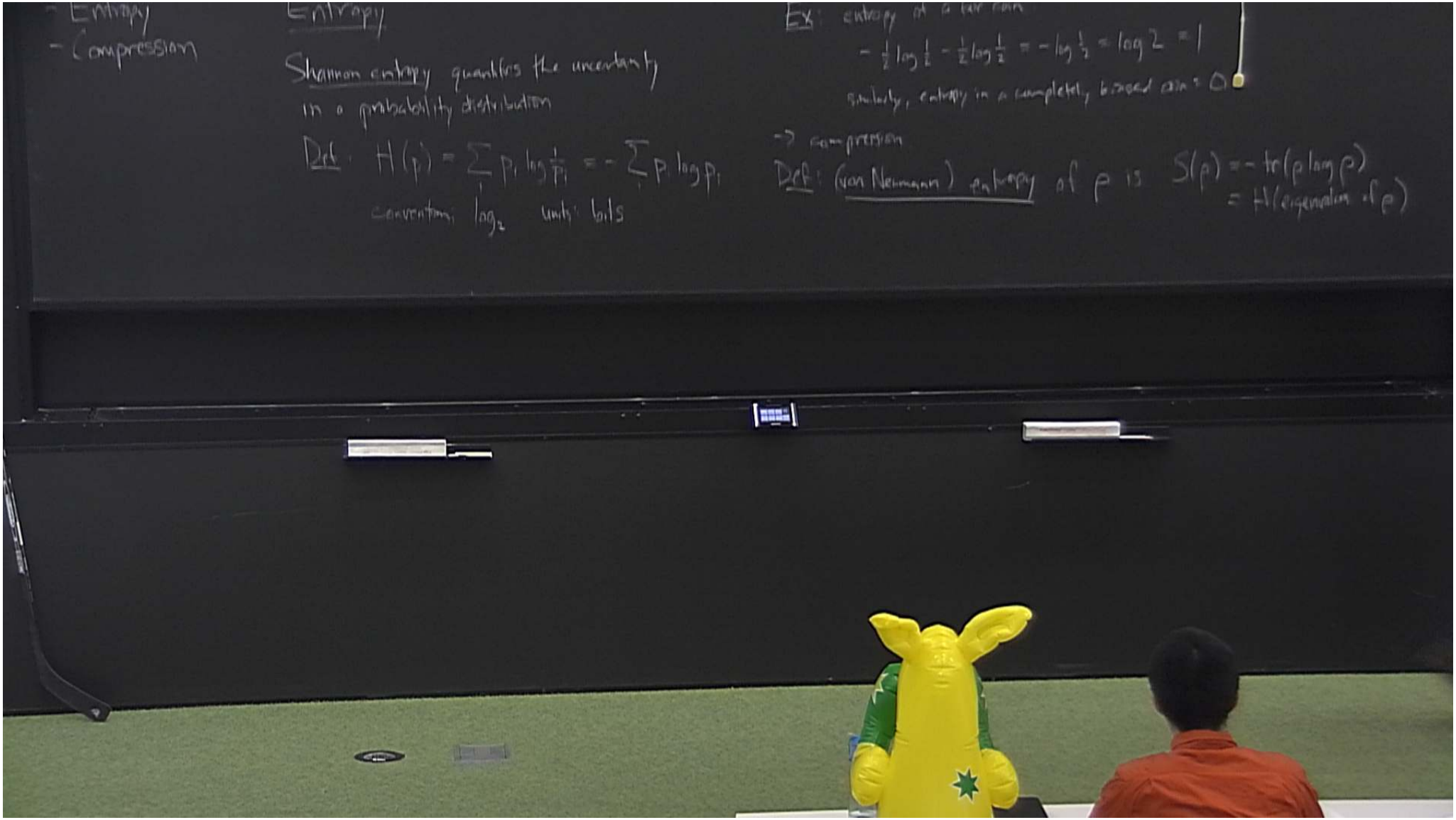
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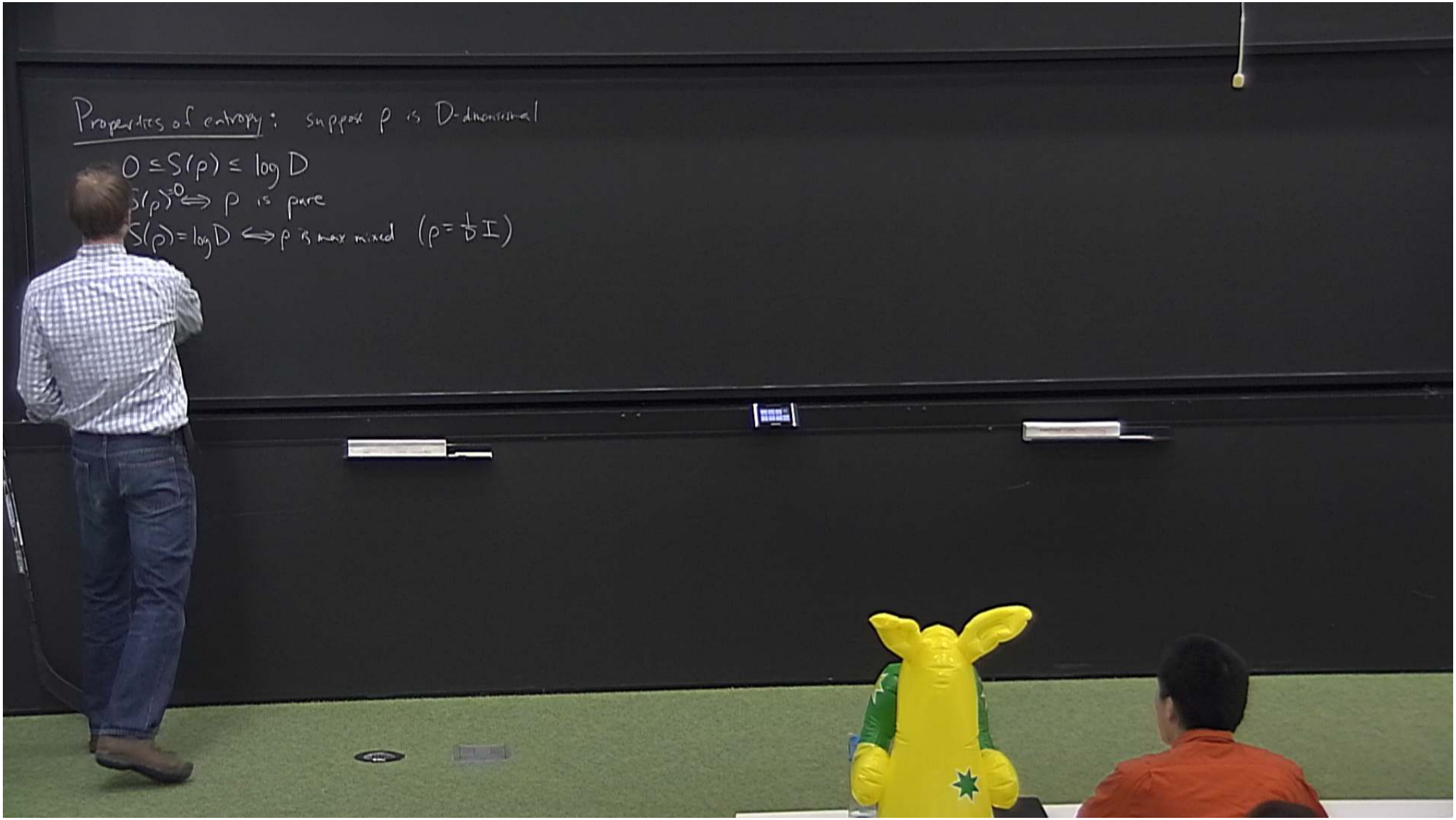
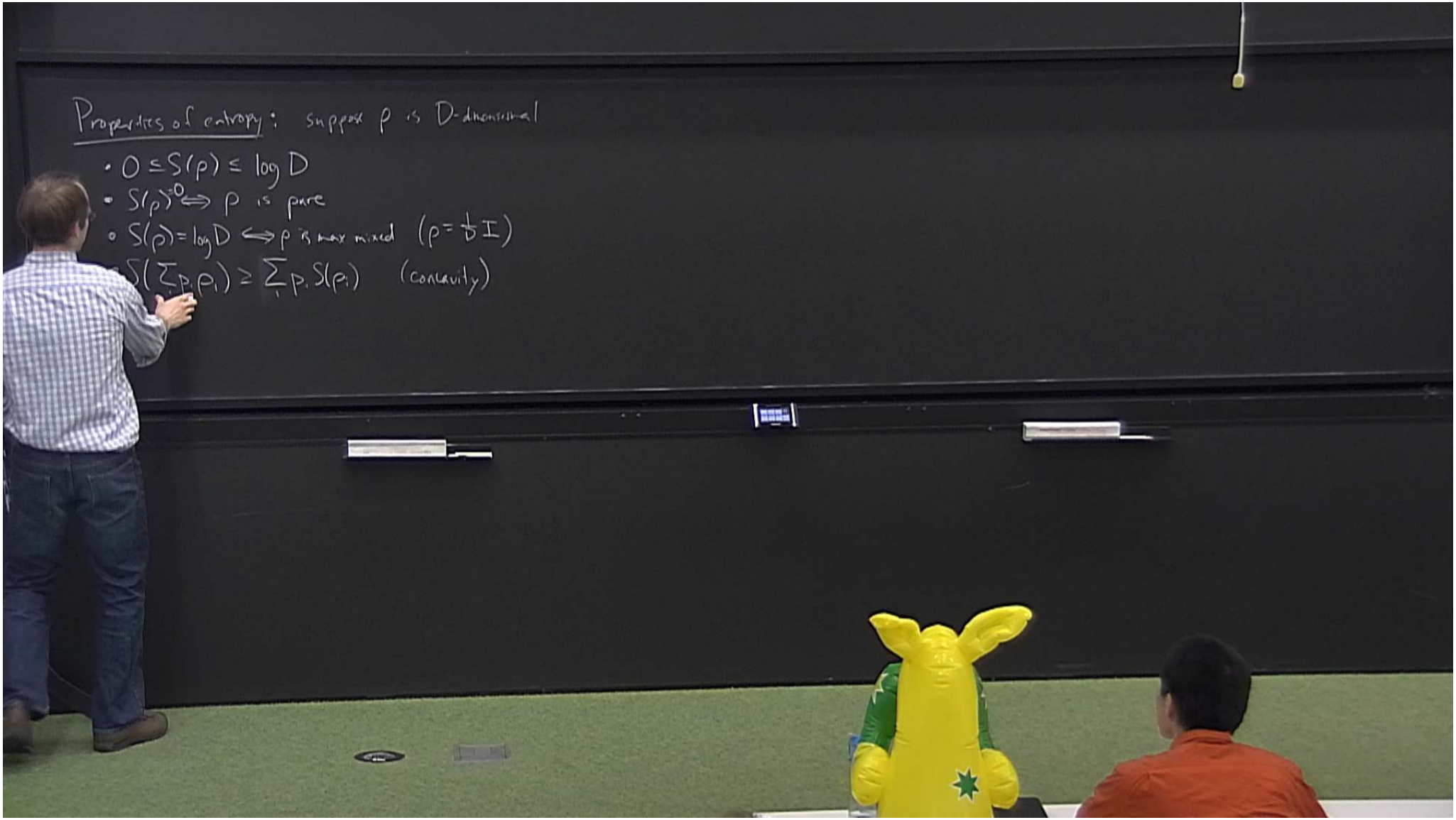
$= H(\text{eigenvalues of } \rho)$


Properties of entropy: suppose ρ is D-dimensional
0 ≤ S(ρ) ≤ log D
S(ρ) = 0 ⇔ ρ is pure
S(ρ) = log D ⇔ ρ is max mixed (ρ = 1/D I)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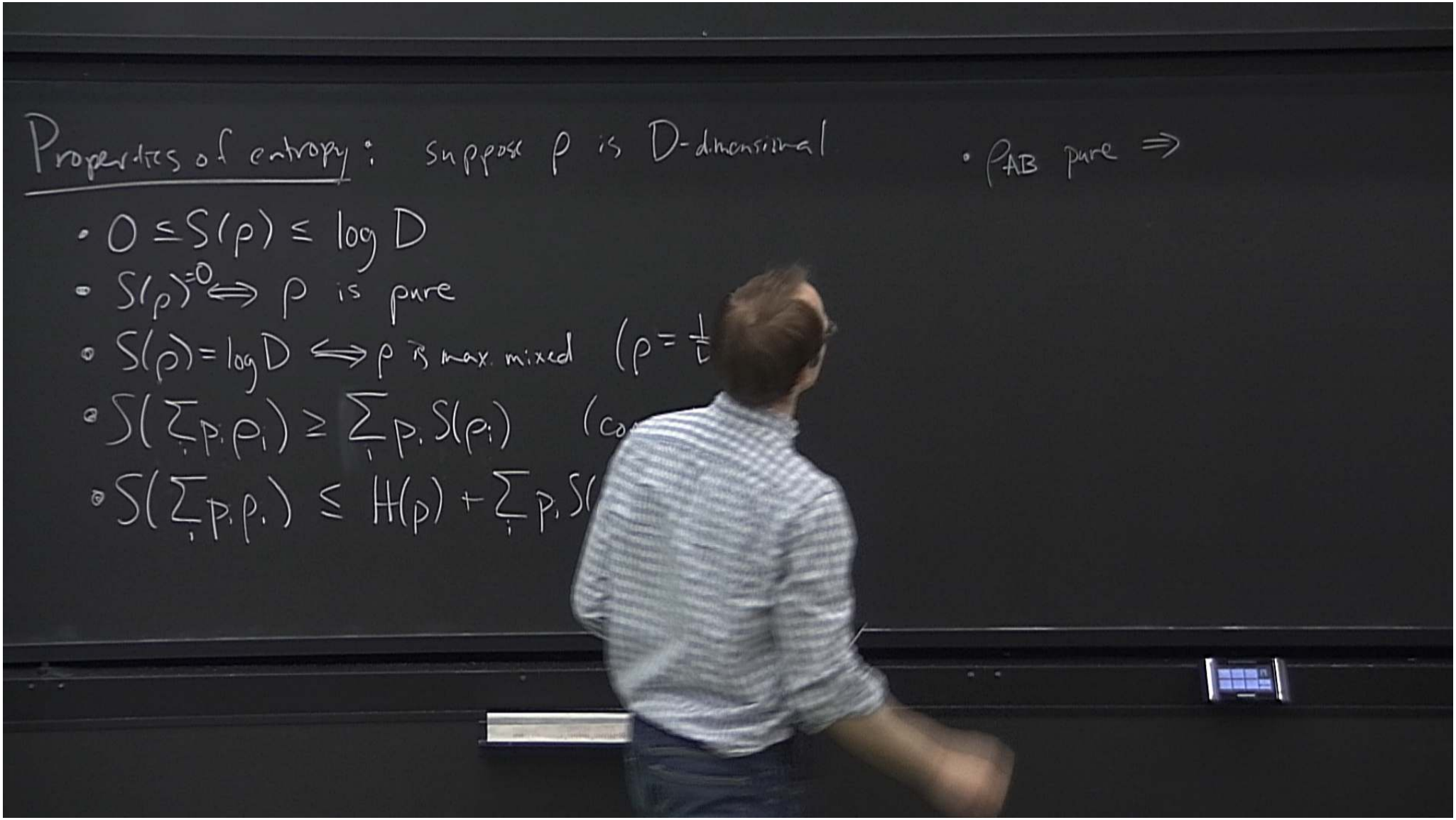
S(Σ p_i ρ_i) ≥ Σ p_i S(ρ_i) (concavity)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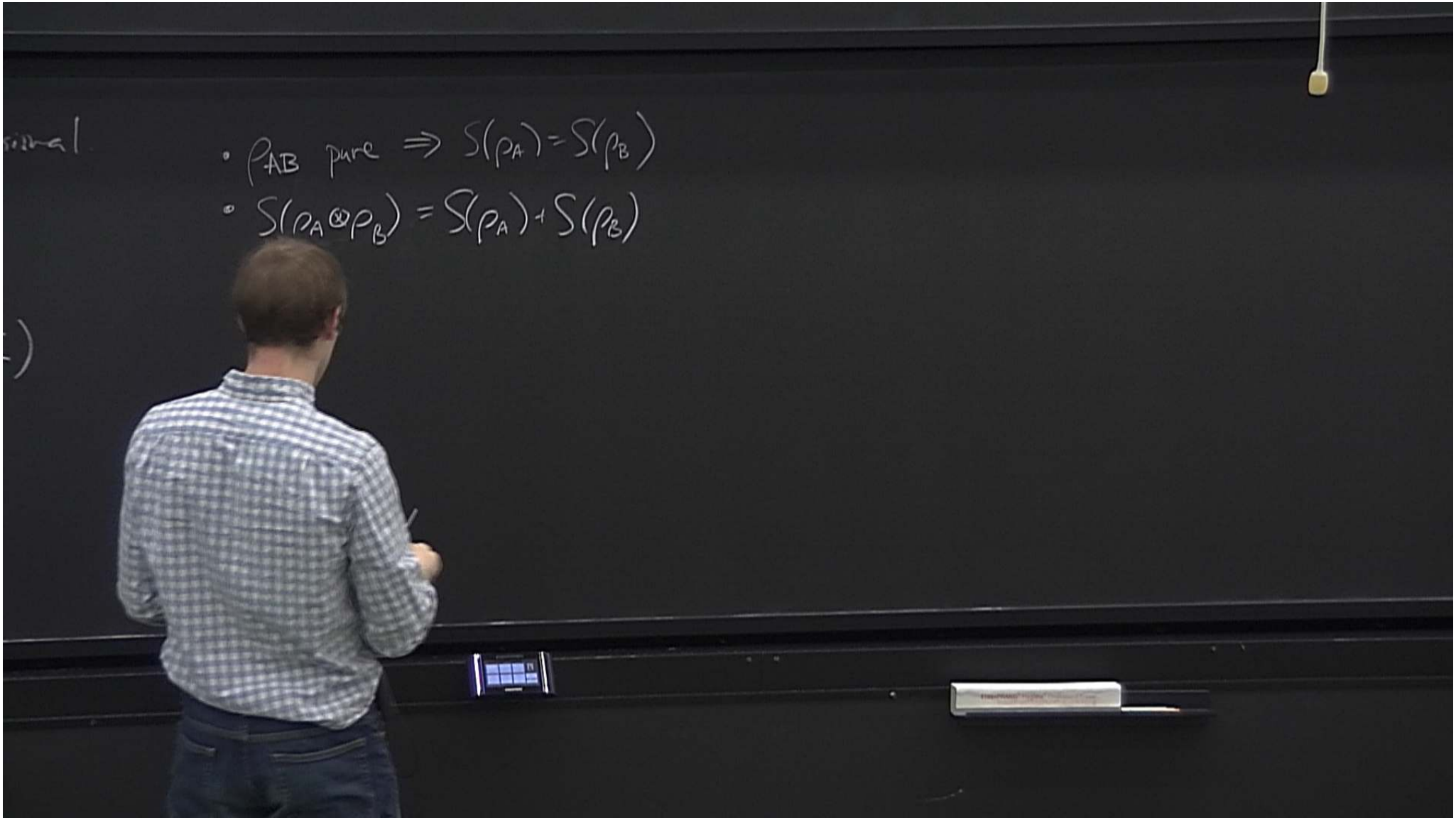
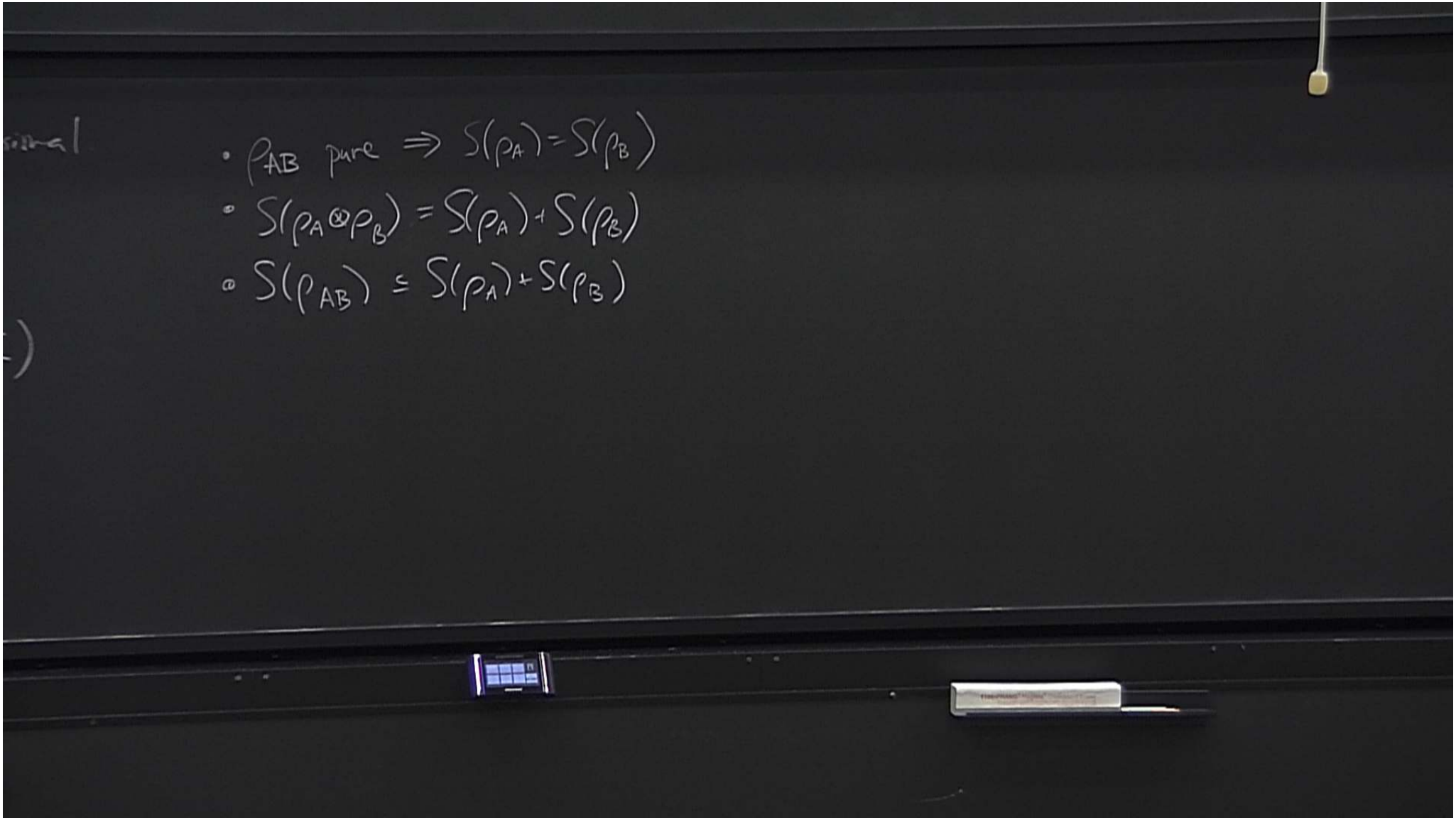

• ρ_{AB} pure $\Rightarrow S(\rho_A) = S(\rho_B)$
• $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$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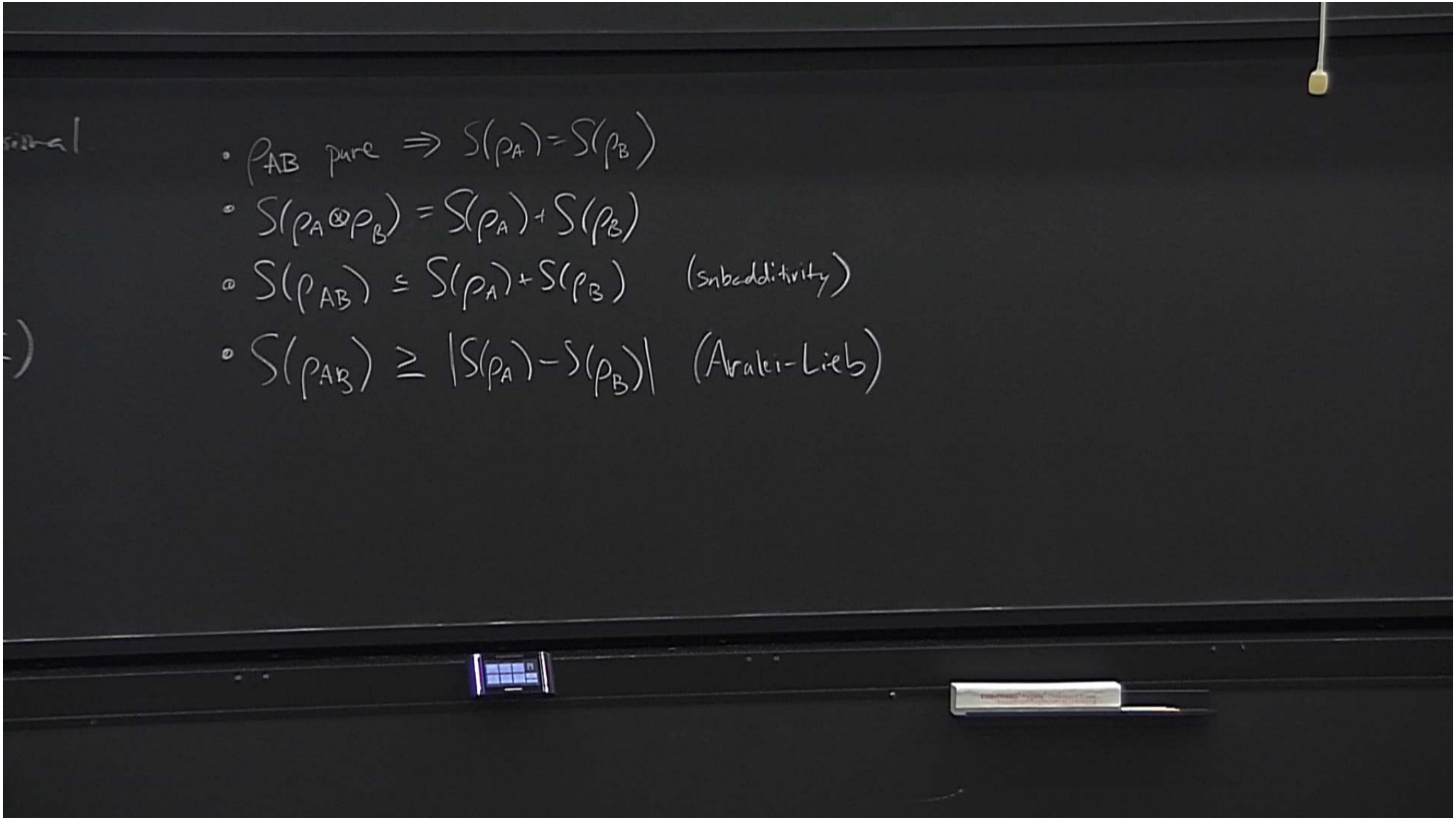
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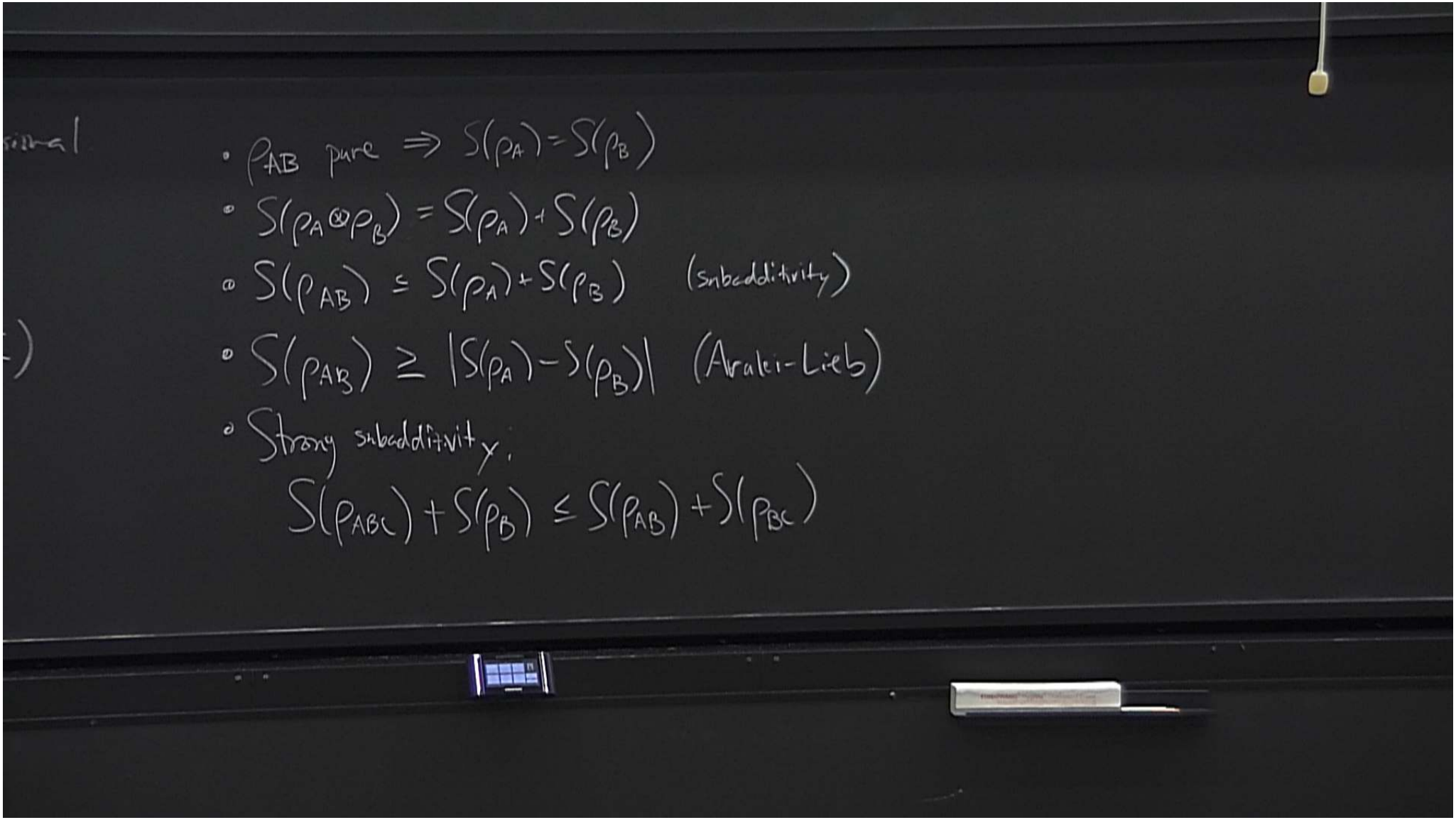
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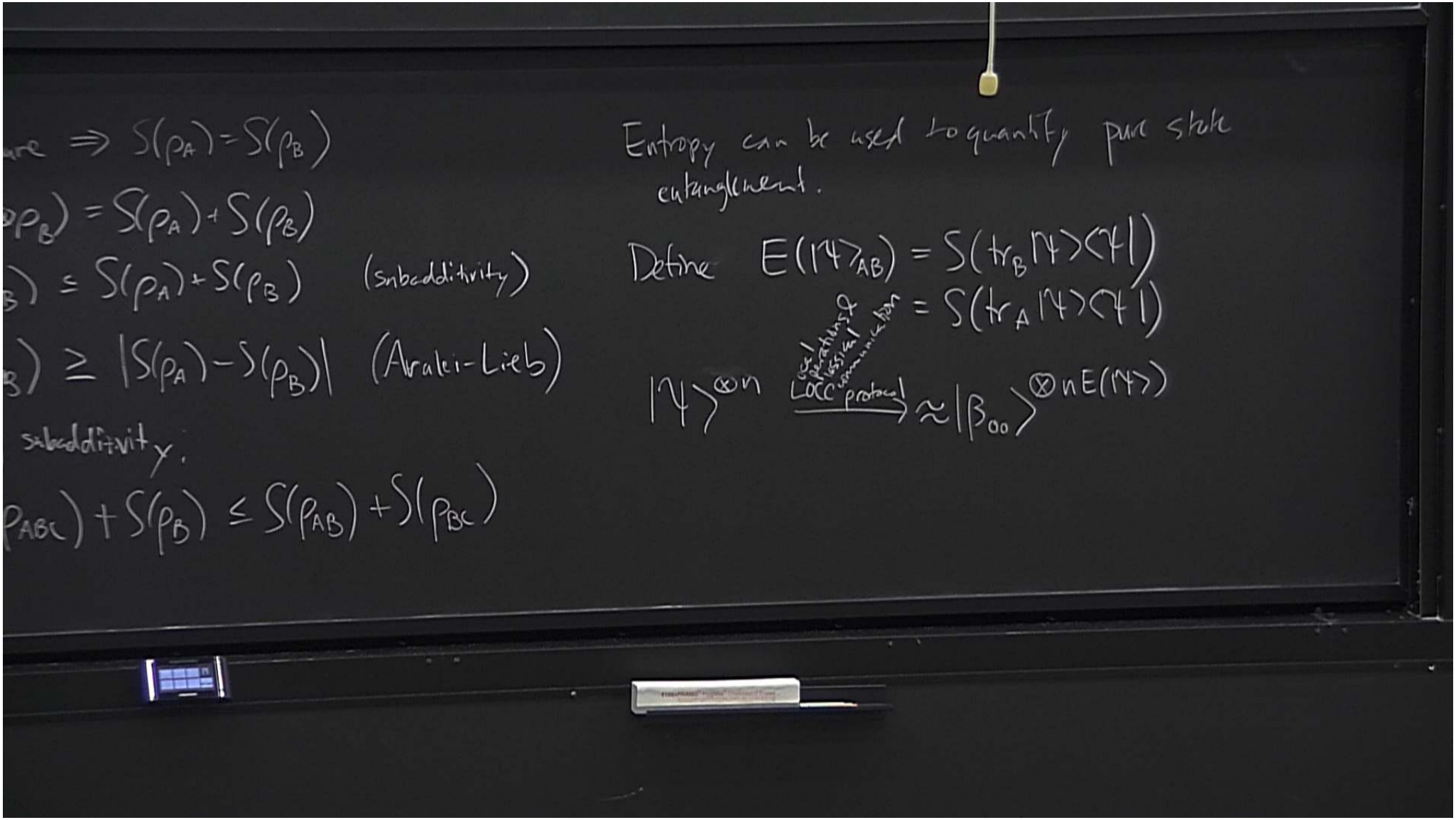
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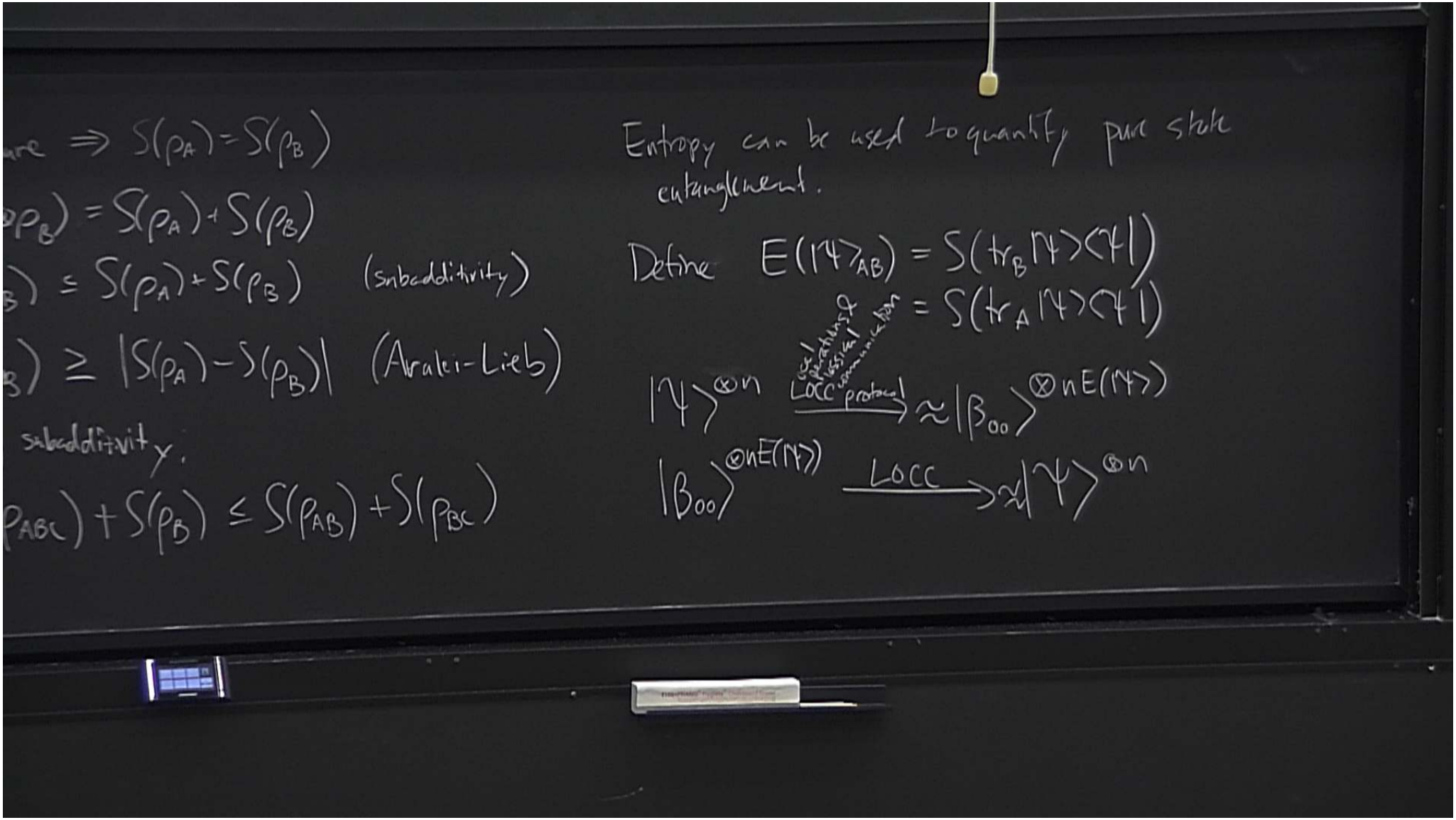
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Entropy can be used to quantify pure state entanglement.
Define $E(|\psi\rangle_{AB}) = S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$
$= S(\mathrm{tr}_A |\psi\rangle\langle\psi|)$
$|\psi\rangle^{\otimes n}$
local quantum & classical communication
LOCC protocol
$\approx |\beta_{00}\rangle^{\otimes nE(|\psi\rangle)}$
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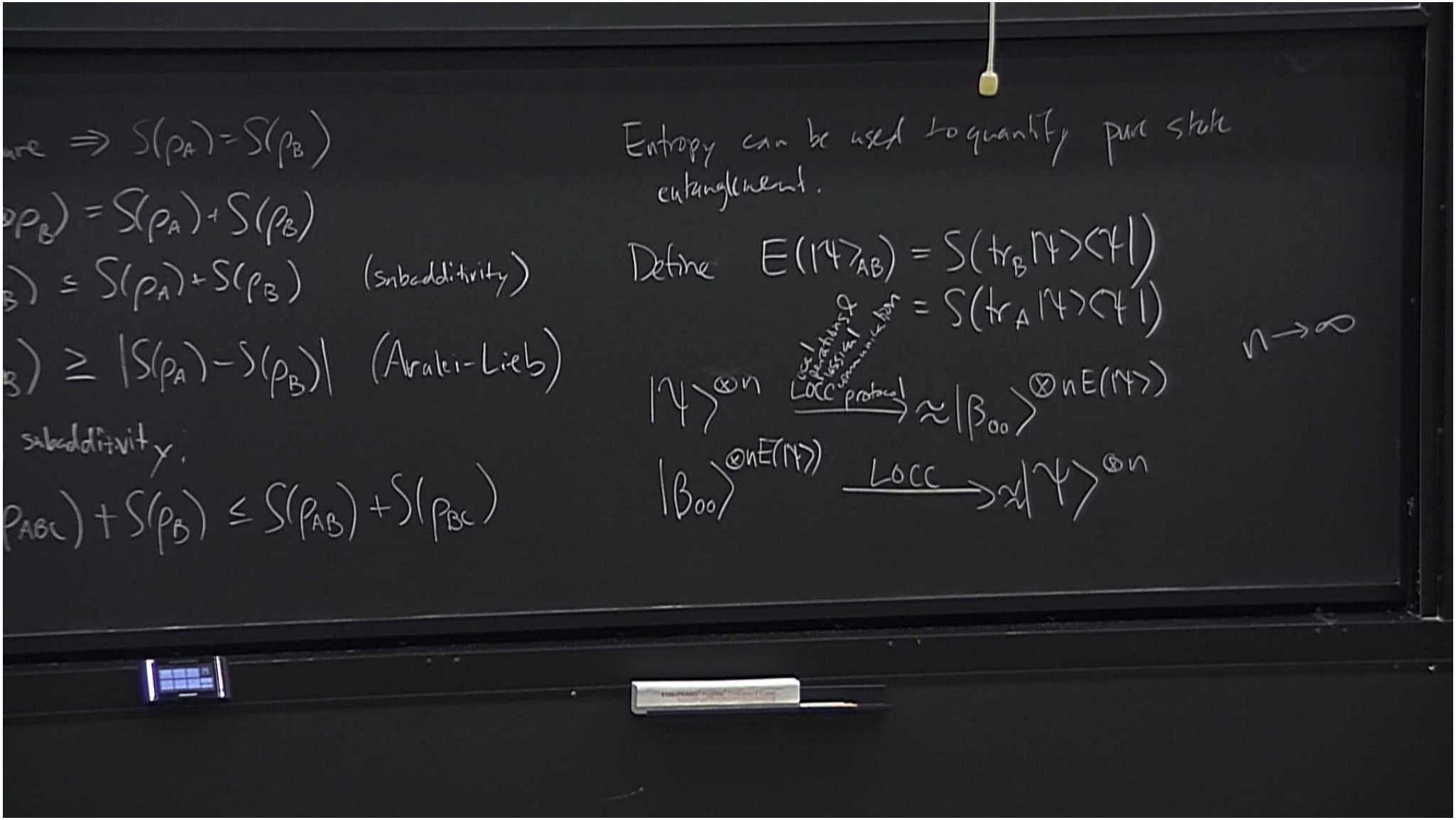
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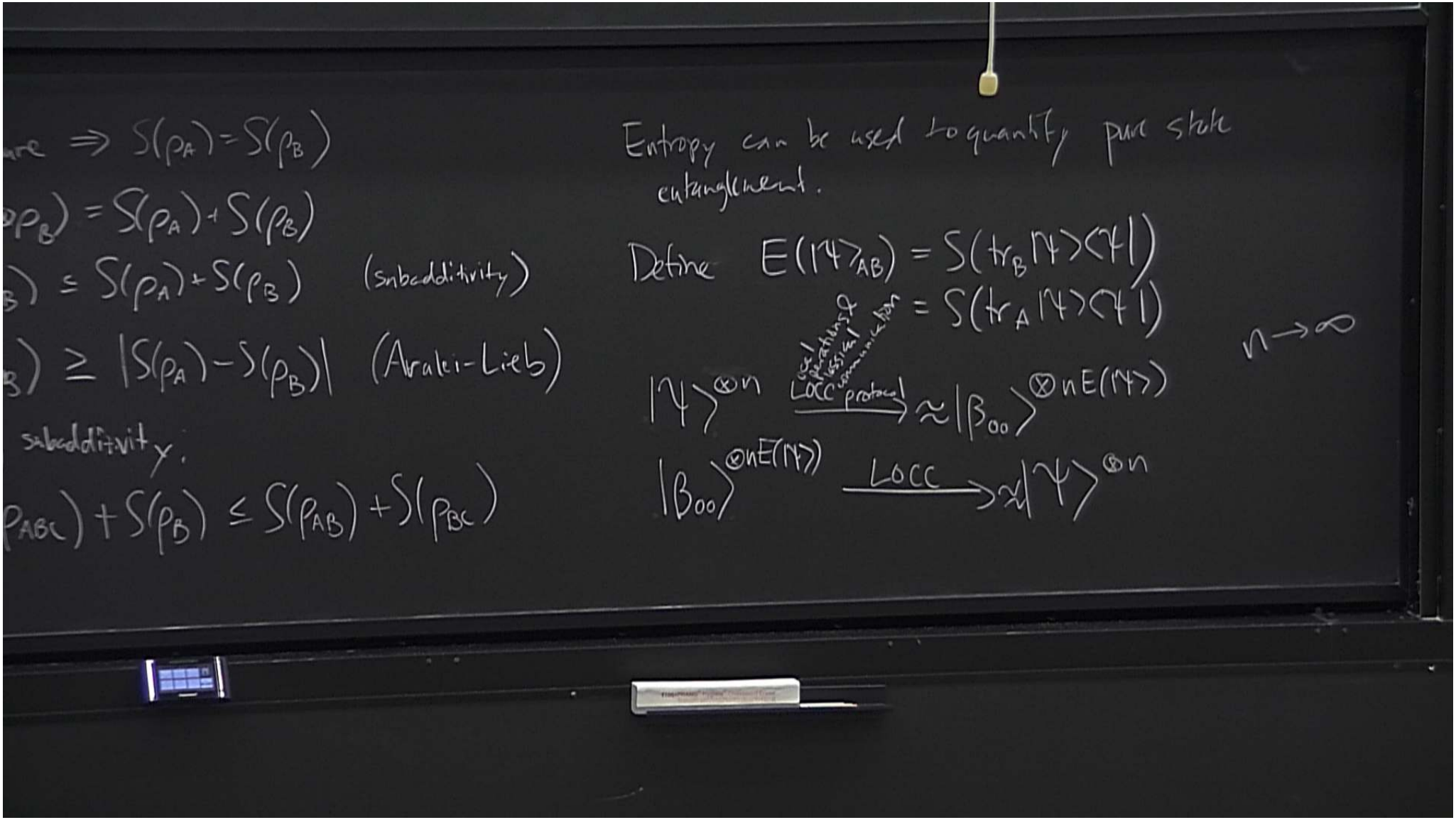
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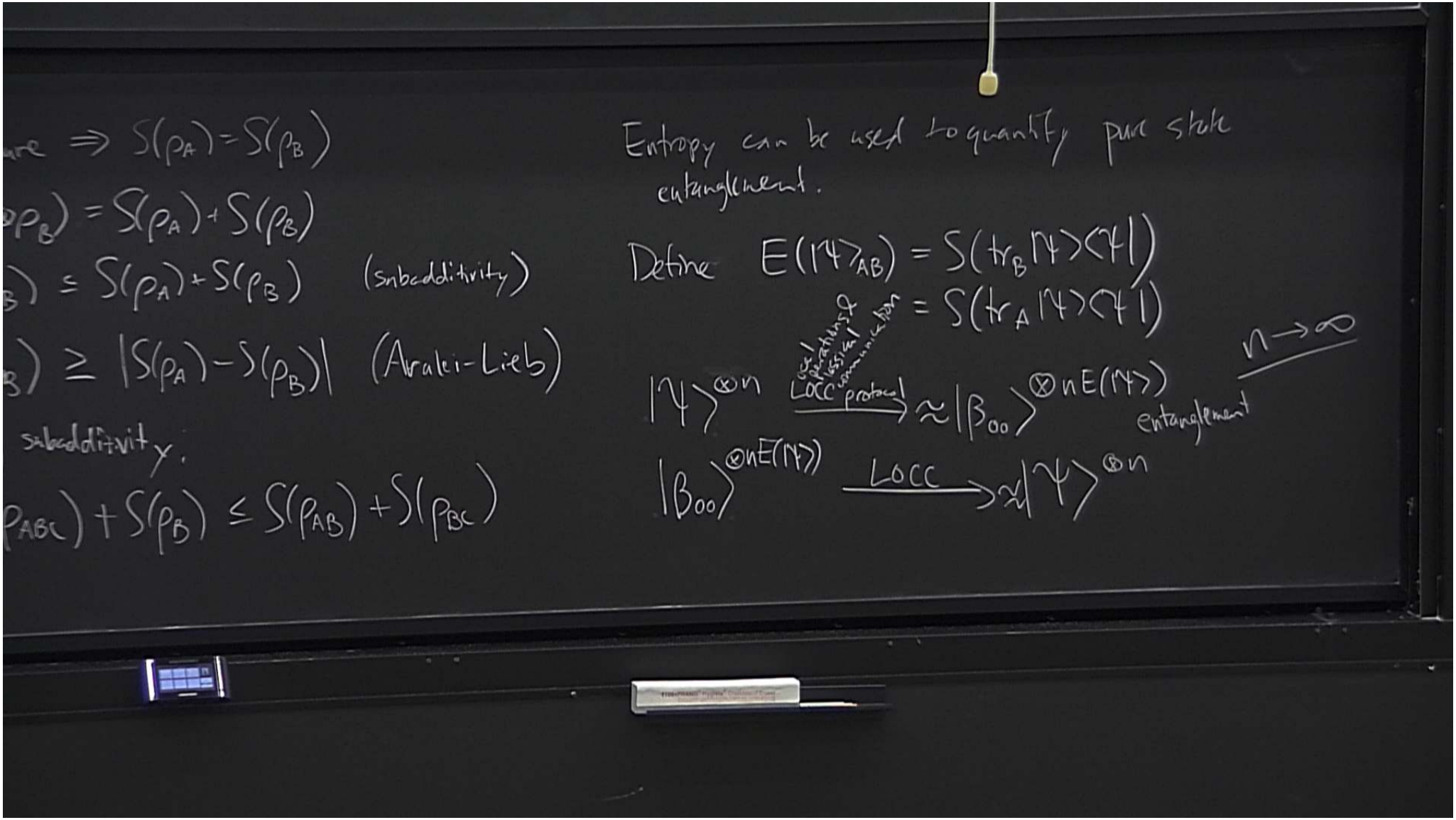
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$|\psi\rangle^{\otimes n}$ LOCC protocol $\approx |\beta_{00}\rangle^{\otimes nE(|\psi\rangle)}$
$n \to \infty$
entanglement
$|\beta_{00}\rangle^{\otimes nE(|\psi\rangle)}$ LOCC $\to \approx |\psi\rangle^{\otimes n}$

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$|\beta_{00}\rangle^{\otimes nE(|\psi\rangle)}$ —LOCC→ $\approx |\psi\rangle^{\otimes n}$ (entanglement dilution)

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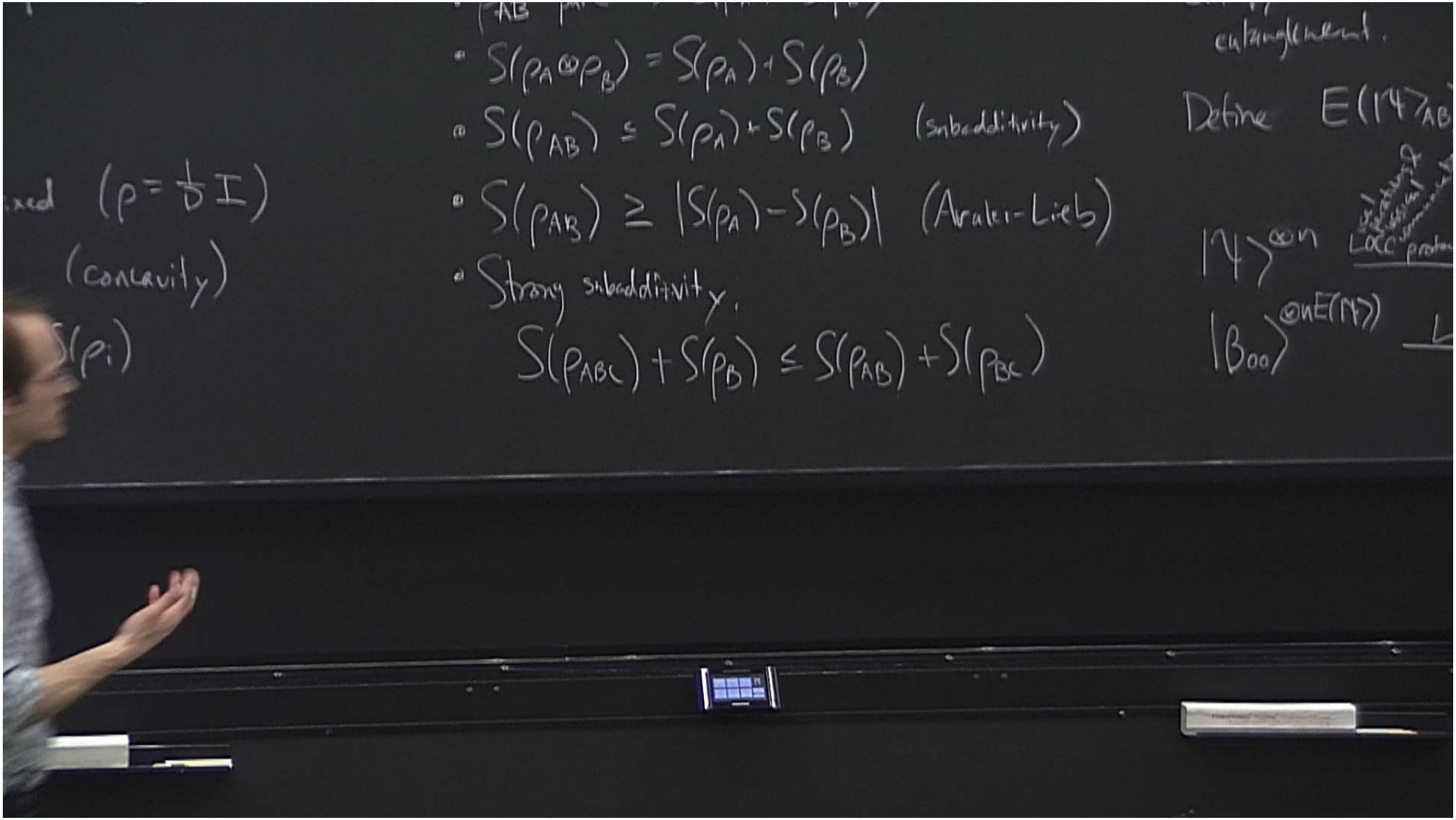
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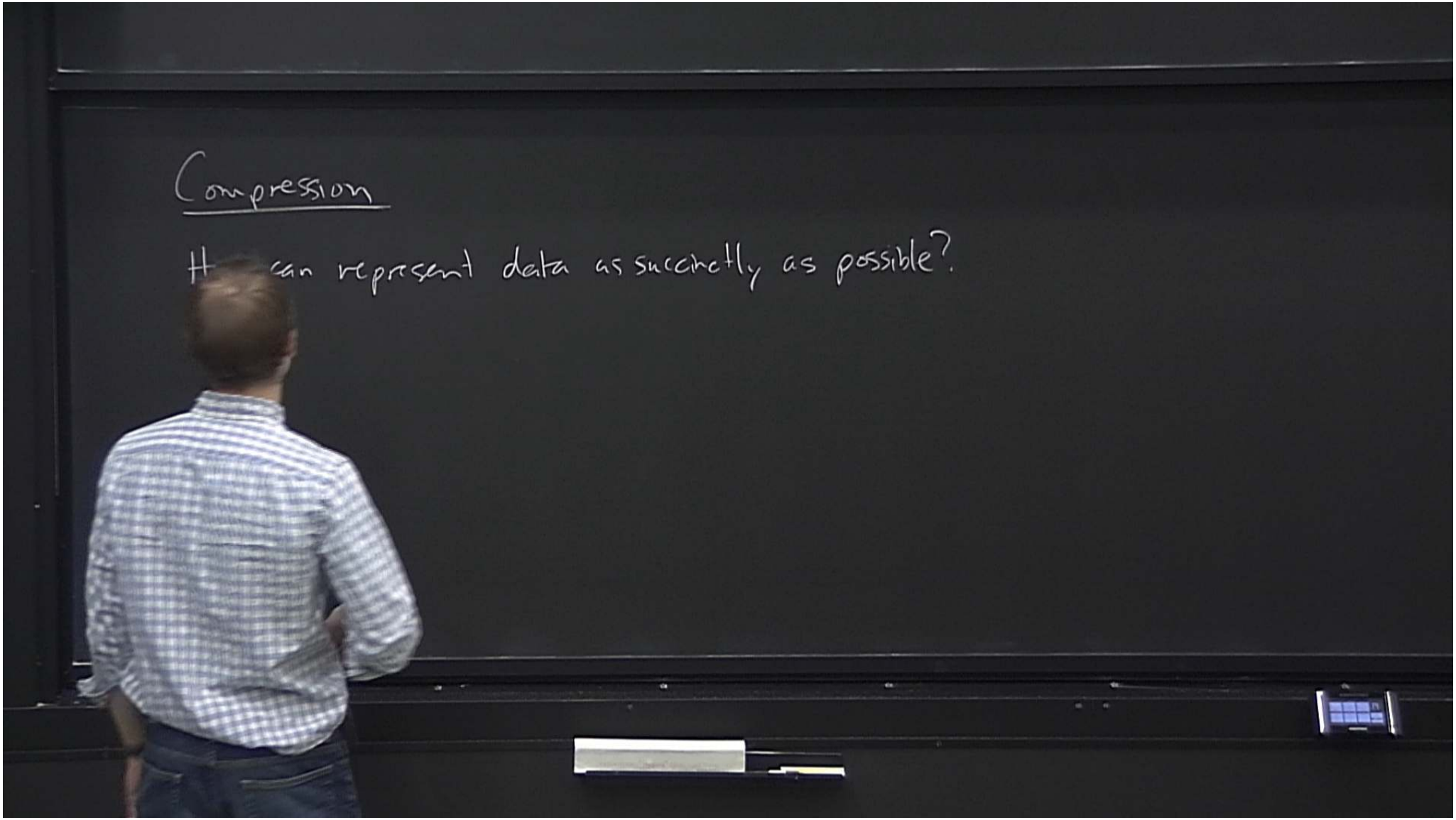
$(\rho = \frac{1}{D} I)$
(concavity)
$S(\rho_i)$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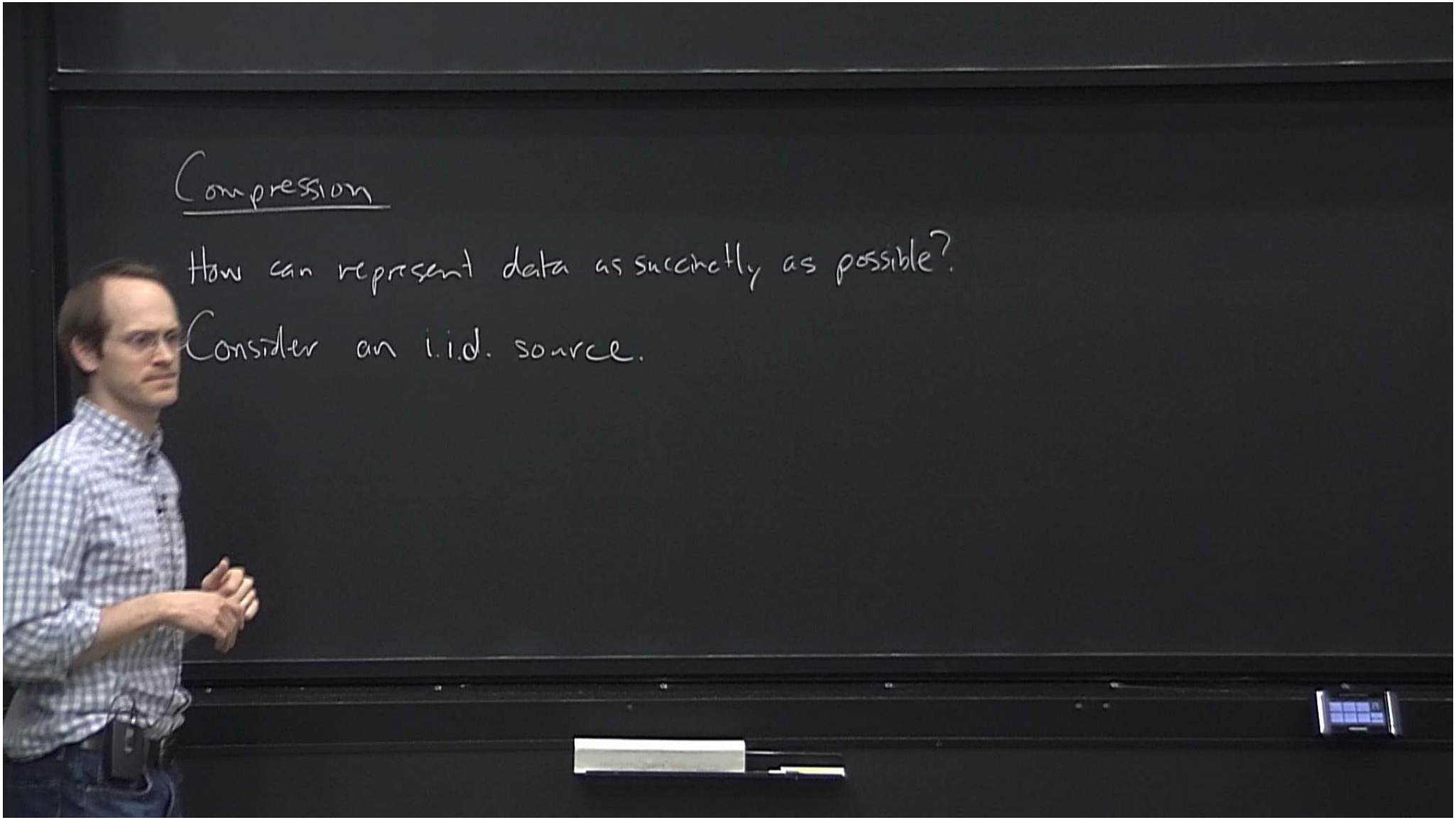
How can represent data as succinctly as possible?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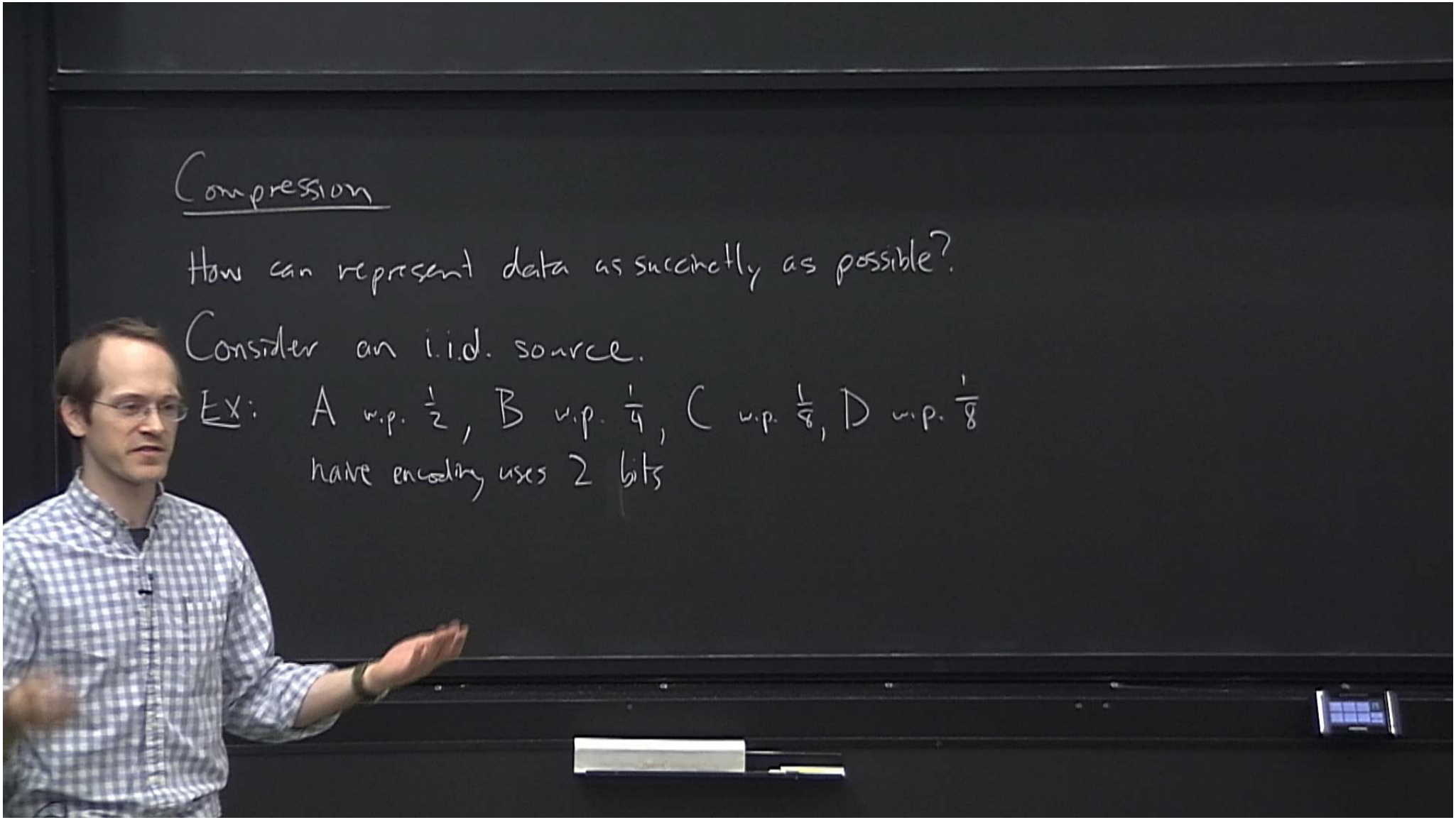
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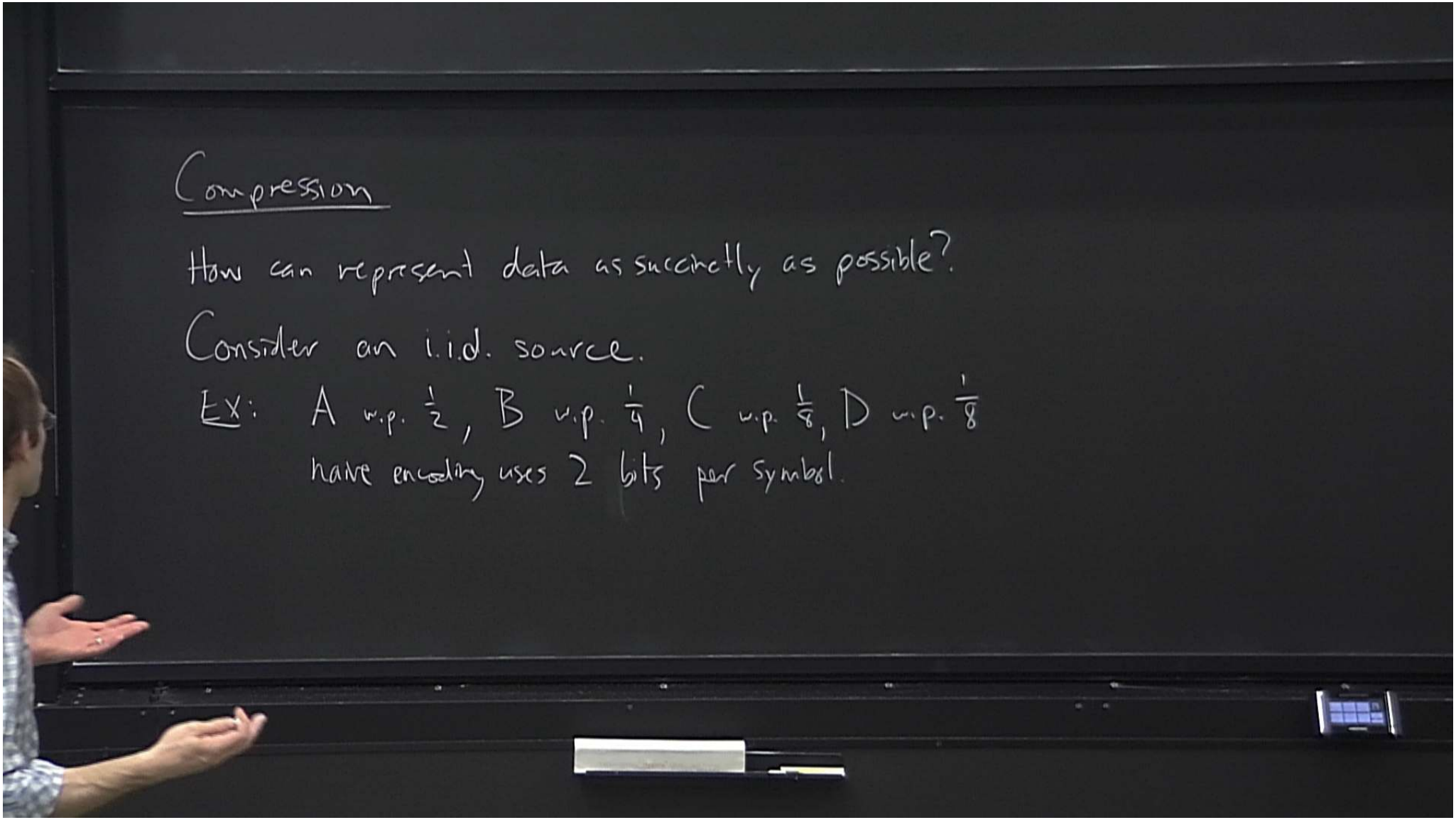
Ex: A w.p. 1/2, B w.p. 1/4, C w.p. 1/8, D w.p. 1/8
have encoding uses 2 bits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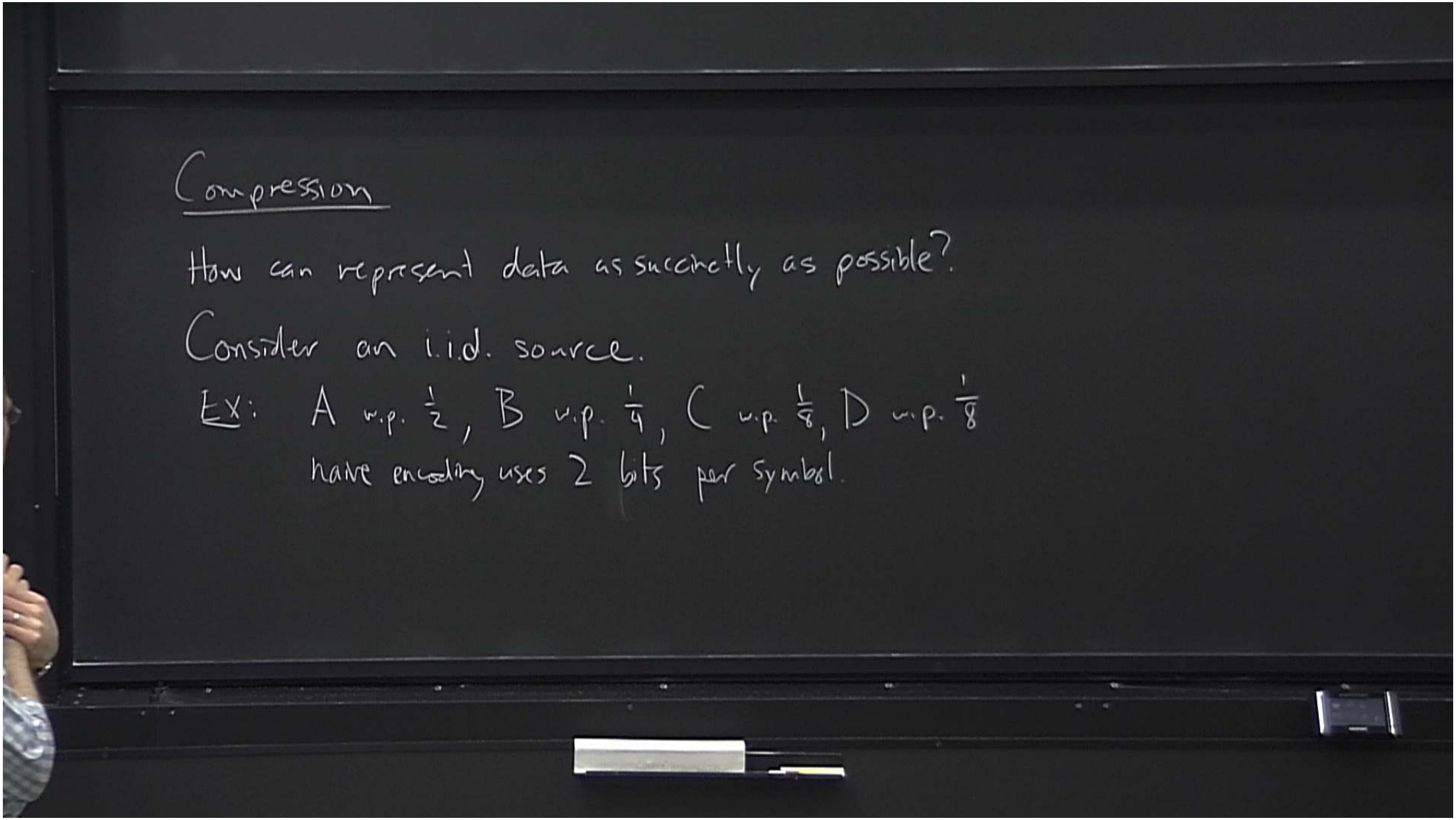
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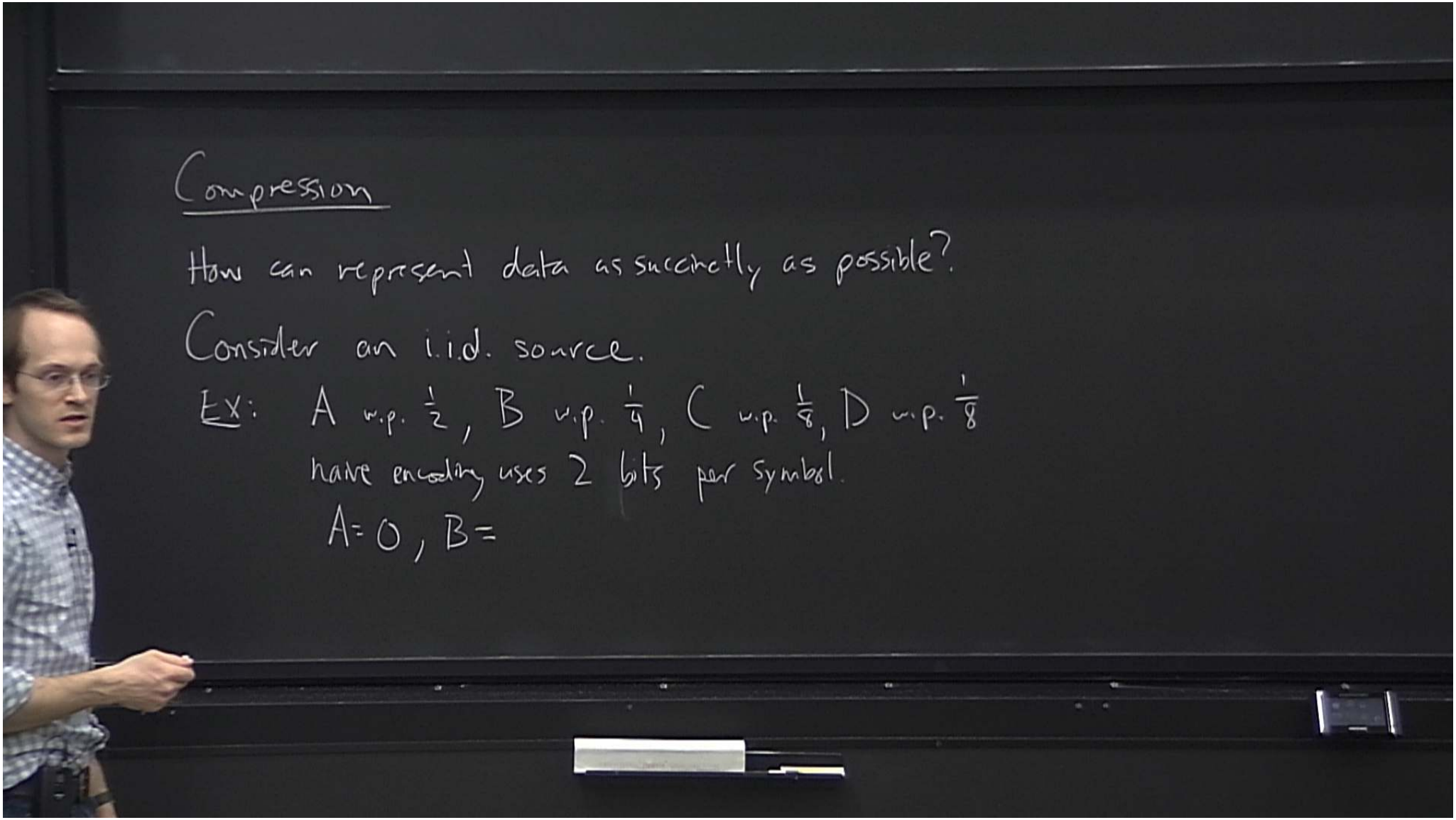
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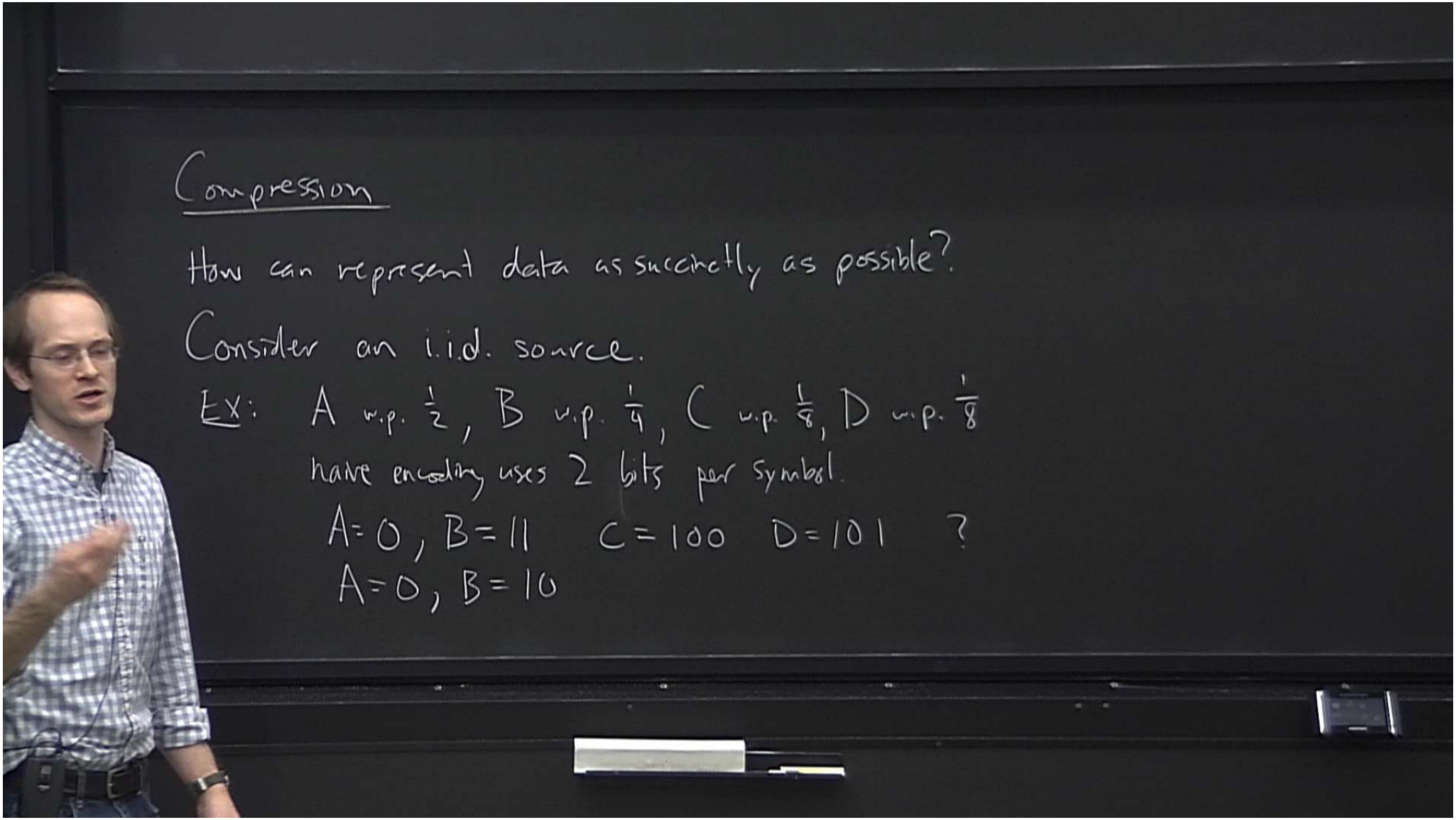
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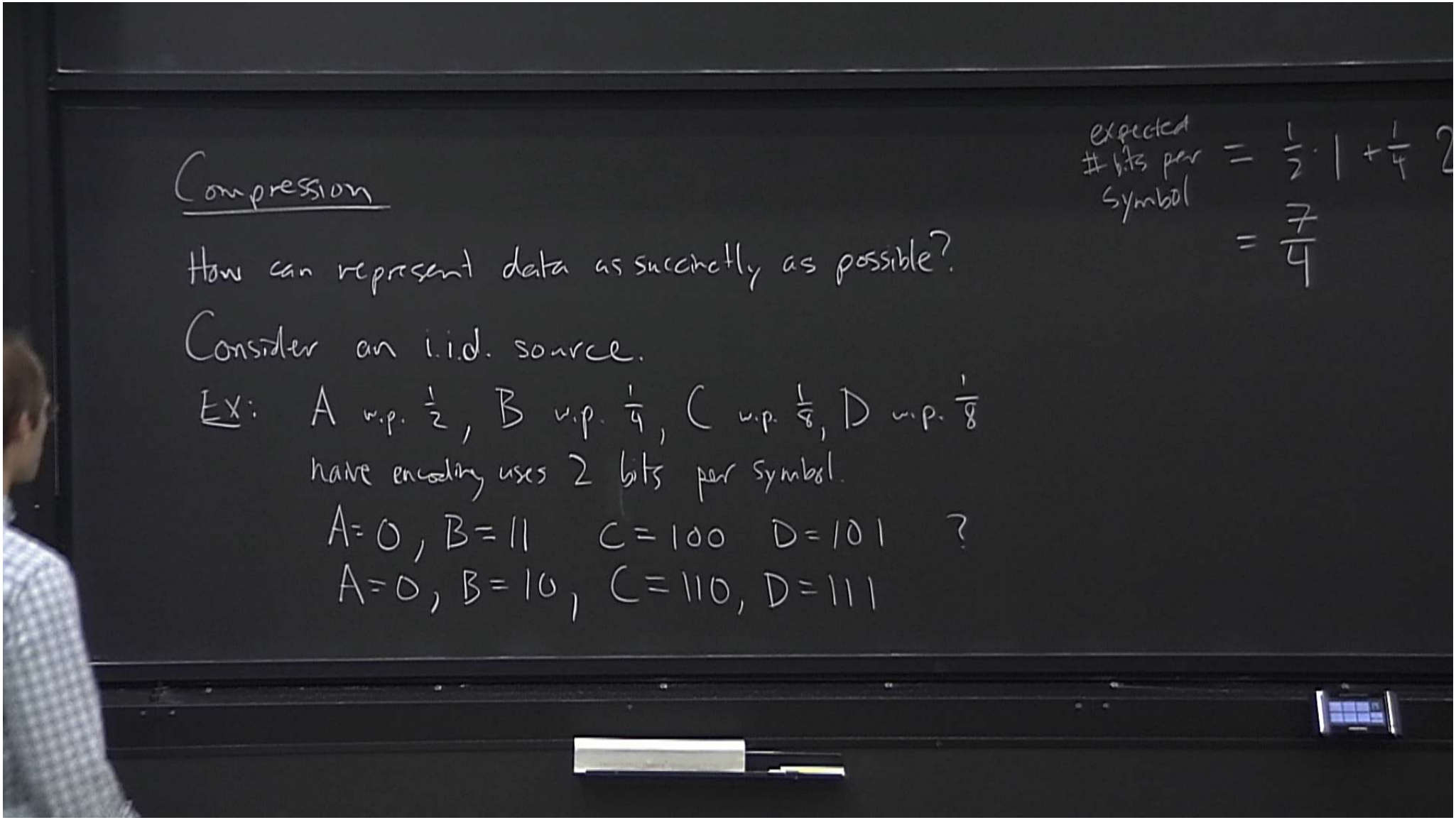
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expected # bits per symbol = 1/2 · 1 + 1/4 2
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Compression

How can represent data as succinctly as possible?

Consider an i.i.d. source.

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expected # bits per symbol $= \frac{1}{2} \cdot 1 + \frac{1}{4} 2$

$= \frac{7}{4}$

$H(\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\}) = \frac{7}{4}$

as possible?

$\frac{1}{8}$, D w.p. $\frac{1}{8}$

ymbol.

D = 101 ?

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expected # bits per symbol $= \frac{1}{2} \cdot 1 + \frac{1}{4} 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} 3$

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A string from this source has

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A n-bit string from this source has αn 0s with prob

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We expect $\approx pn$ 0s. The prob. of such a string is

$$p^{pn}(1-p)^{(1-p)n}$$

$$\log\left(p^{pn}(1-p)^{(1-p)n}\right) = pn\log p + (1-p)n\log(1-p)$$

$$= -nH_2(p)$$

where $H_2(p) = -p\log p - (1-p)\log(1-p)$

("binary entropy")

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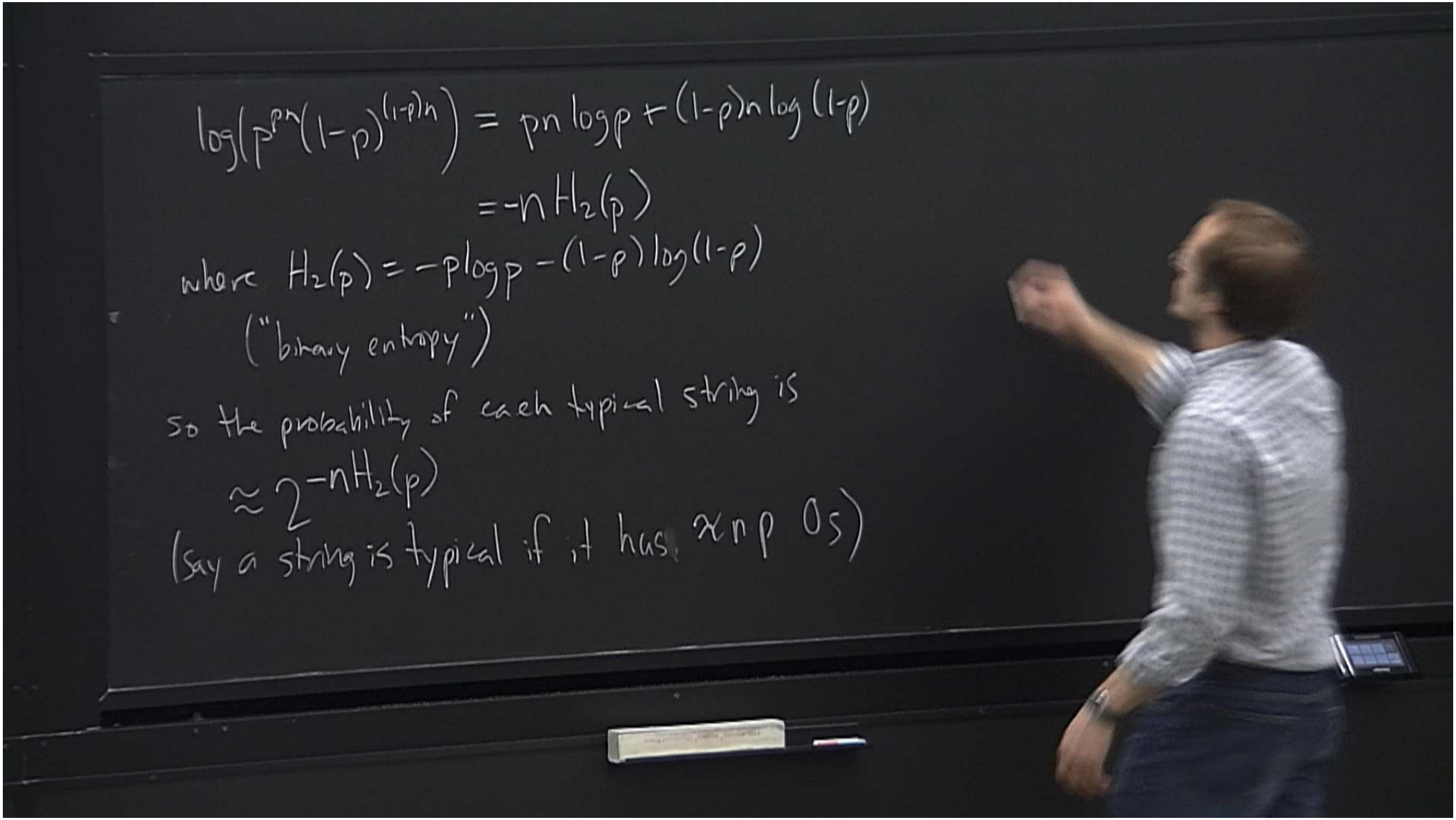

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of each typical string is

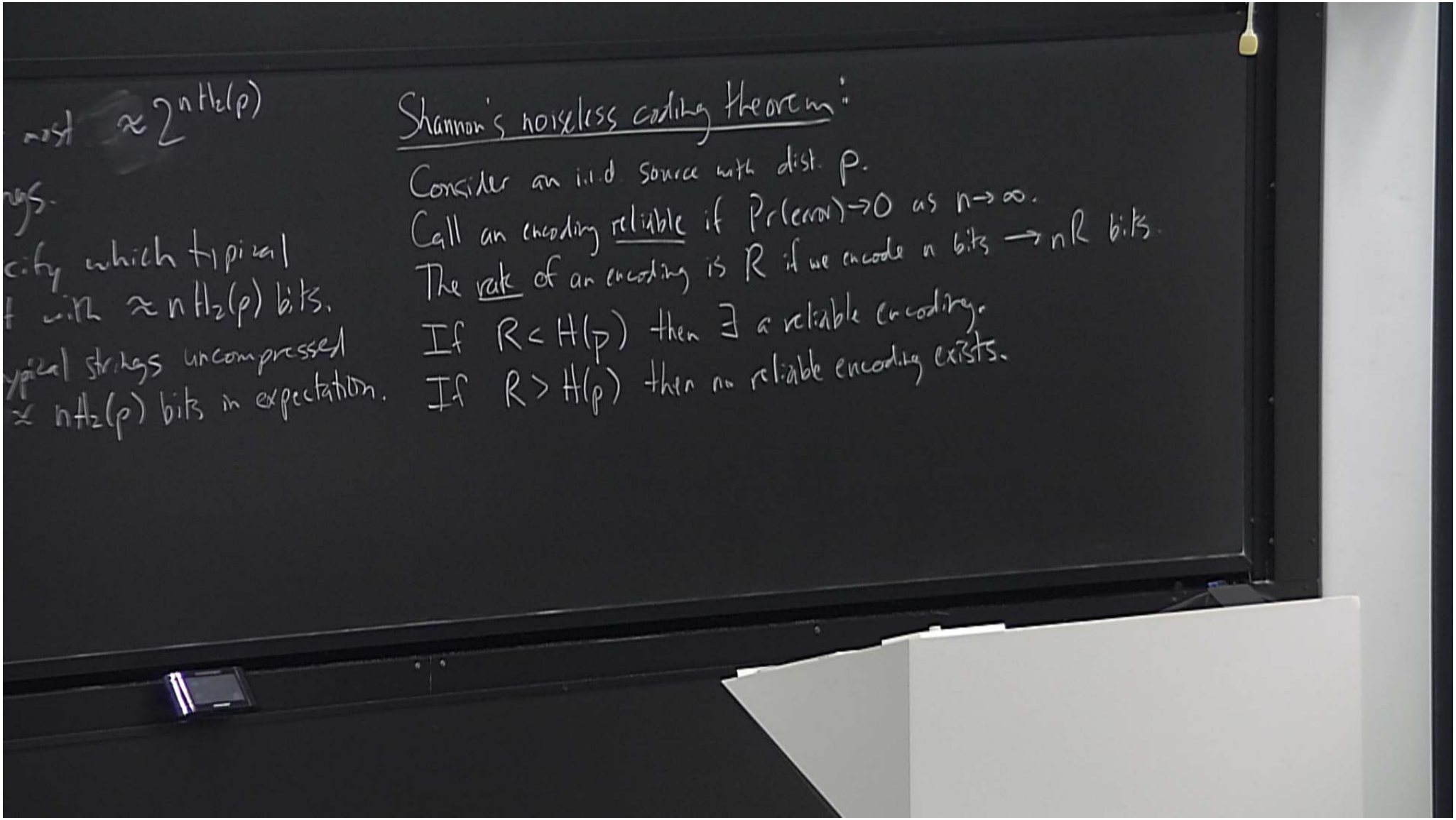
(p)

typical if it has $\approx np$ 0s)

There can be at most $\approx 2^{nH_2(p)}$ typical strings.

So we can specify which typical string we got with $\approx nH_2(p)$ bits.

We can encode atypical strings uncompressed and still use $\approx nH_2(p)$ bits in expectation.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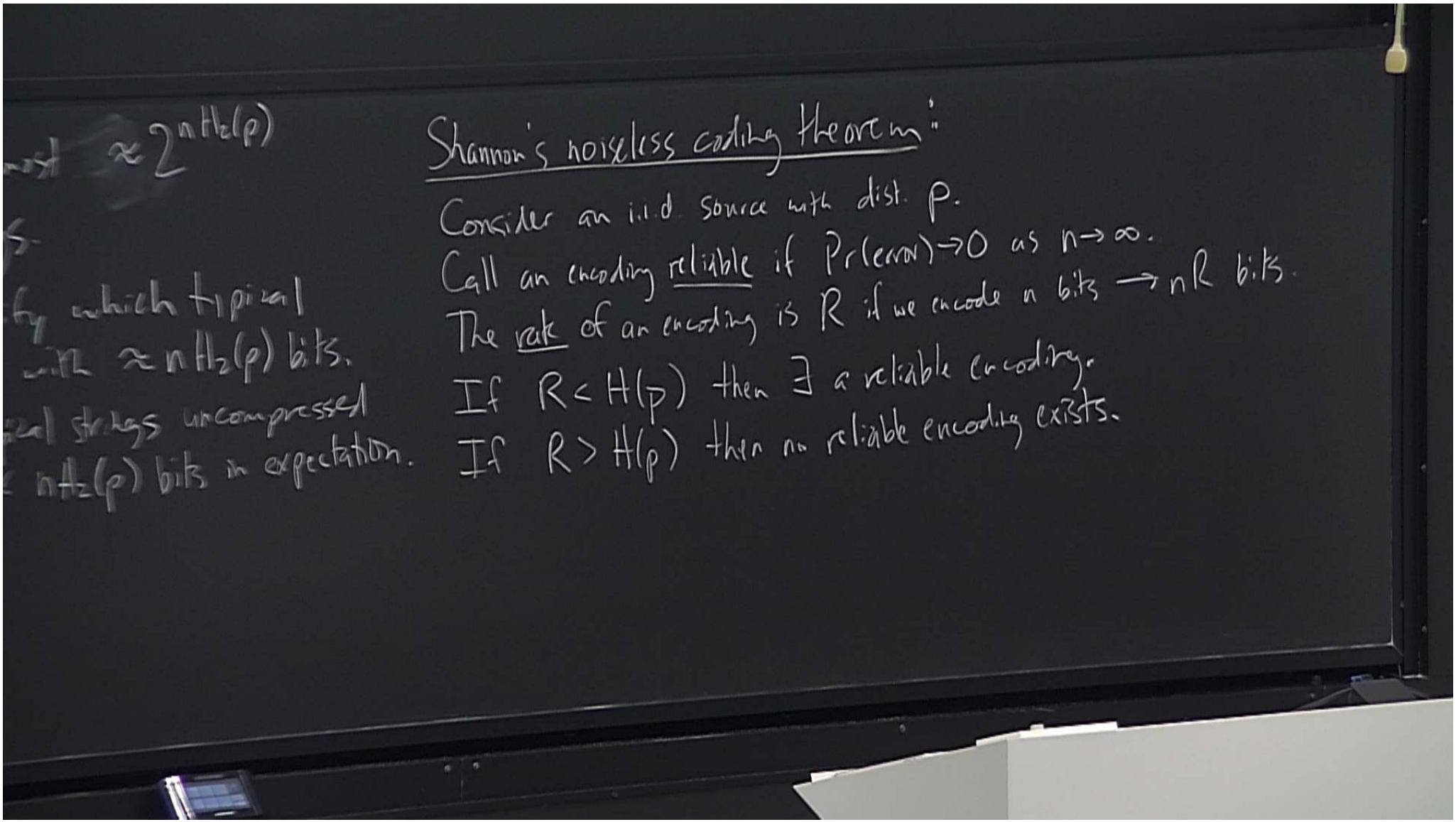
Consider an i.i.d source with dist. p.
Call an encoding reliable if $Pr(\text{error}) \to 0$ as $n \to \infty$.
The rate of an encoding is R if we encode n bits $\to nR$ bits.
If $R < H(p)$ then $\exists$ a reliable encoding.
If $R > H(p)$ then no reliable encoding exists.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