Title: Gravitational anomaly and topological phases

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Abstract: Since the quantum Hall effect, the notion of topological phases of matter has been extended to those that are well-defined (or:

``protected") in the presence of a certain set of symmetries, and that exist in dimensions higher than two. In the (fractional) quantum Hall effects (and in ``chiral" topological phases in general), Laughlin's thought experiment provides a key insight into their topological characterization; it shows a close connection between topological phases and quantum anomalies.

By taking various examples, I will demonstrate that quantum anomalies serve as a useful tool to diagnose (and even define) topological properties of the systems.

For chiral topological phases in (2+1) dimensions and (3+1) dimensional topological superconductors, I will discuss topological responses of the system which involve a cross correlation between thermal transport, angular momentum, and entropy. We also argue that gravitational anomaly is useful to study symmetry protected topological phases in (2+1) dimensions.

Topological Phases and Gravitational Anomalies

Shinsei Ryu
Univ. of Illinois, Urbana-Champaign

Topological phases:

no analogous phase in classical systems

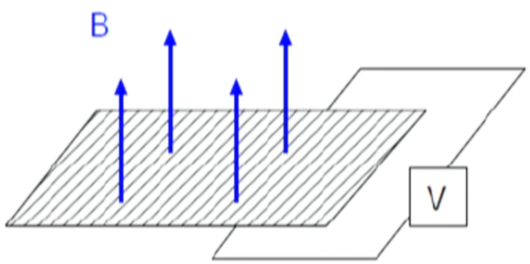
(very quantum state of matter)

Anomalies:

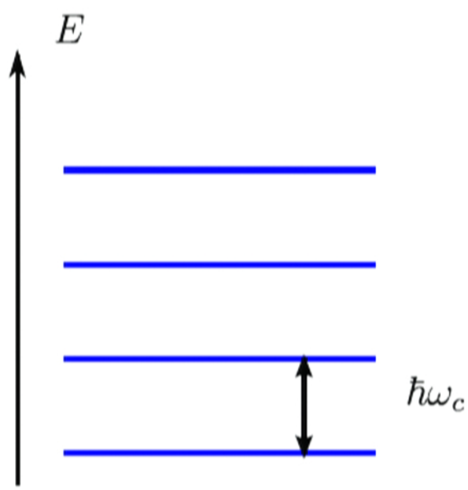
breakdown of a classical symmetry by quantum effect

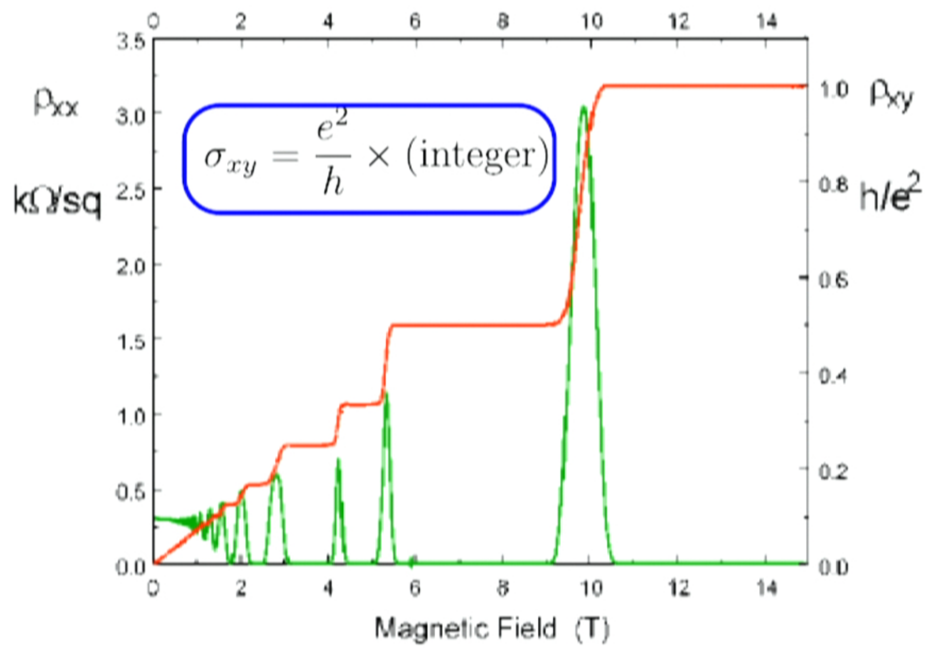
(nothing more quantum than this)

Quantum Hall effect (IQHE)

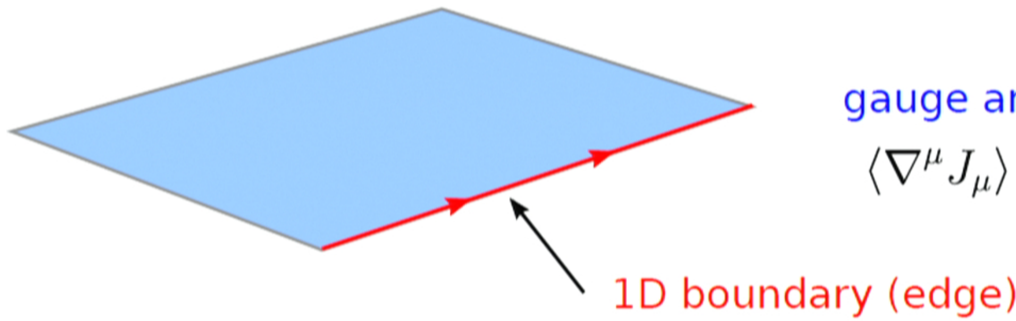


$$\sigma_{xx} = 0 \qquad \rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2} = 0$$



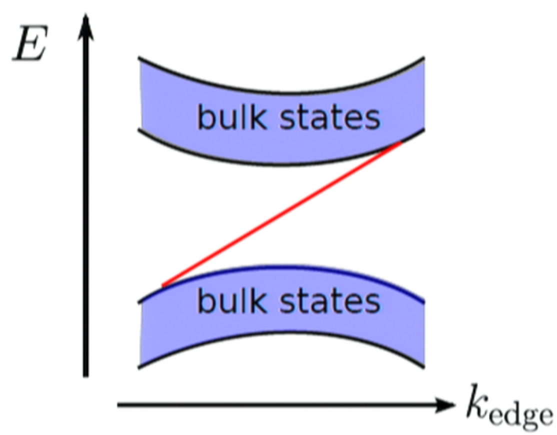


Edge of QHE



gauge anomaly

$\langle \nabla^\mu J_\mu \rangle \neq 0$



Questions and goal

- Diagnose topological phases by anomalies (= "responses")

  (often) observable

  robust against interactions
    e.g., Adler-Bardeen's theorem

- Can *all* topological phases be characterized by
  an anomaly of some kind ?

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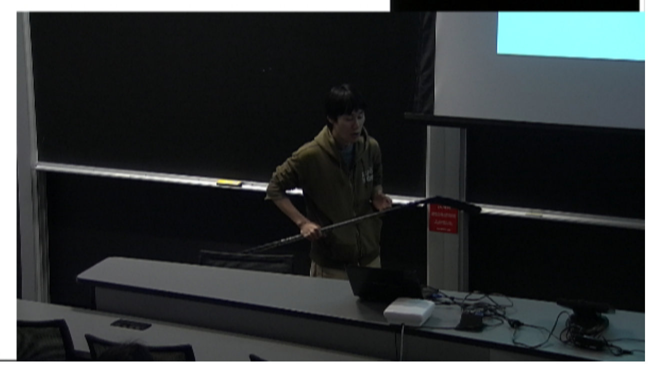

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0: Introduction

1: Thermal/mechanical response in 2d T-breaking topological phases

(infinitesimal gravitational anomaly)

with Ken Nomura (Tohoku), Akira Furusaki (RIKEN), Naoto Nagaosa (U of Tokyo)

2: 2d symmetry-protected topological phases

(global gravitational anomaly)

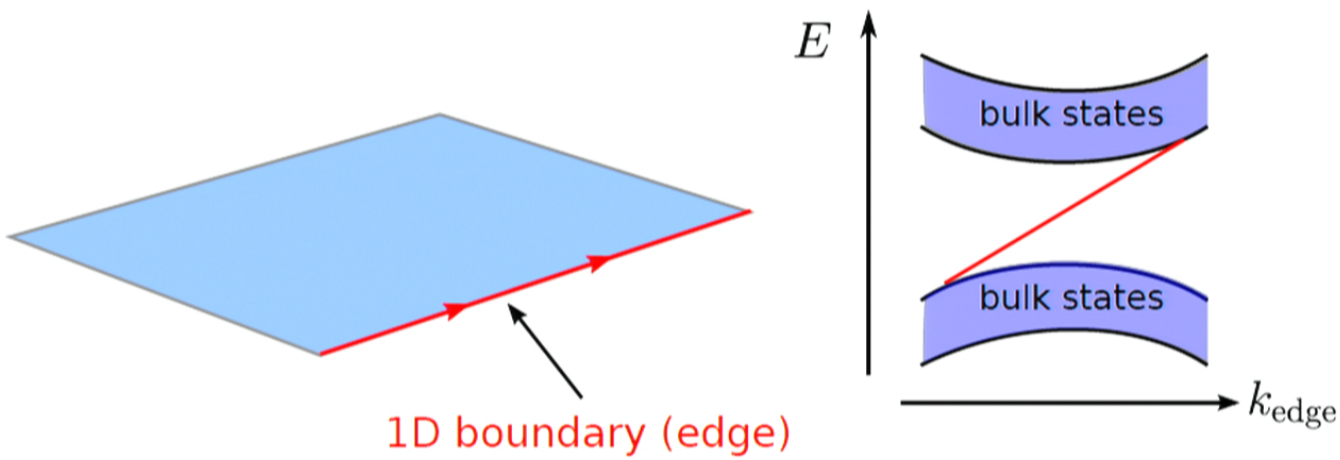
with Shou-cheng Zhang (Stanford) Hong Yao (Tsinghua)

T-breaking topological phases

Setting:
T-breaking topological phases in 2d
with no conserved charge

e.g. 2d topological SC, chiral p-wave SC

What to look for ?



Coupling to gravity

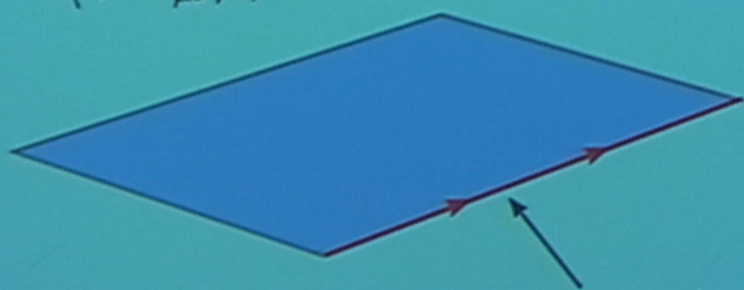
Nothing can escape from gravity

Energy (and momentum) is "always" conserved
--> response to gravity is always well-defined

--> thermal response is always well-defined

"gravitational anomaly"

$$\langle \nabla^{\mu} T_{\mu\nu} \rangle \neq 0$$



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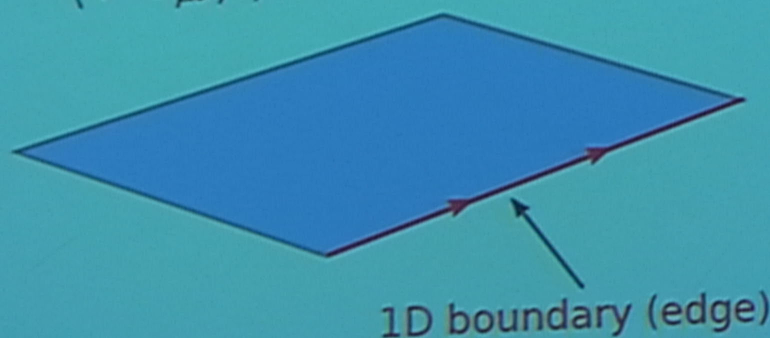

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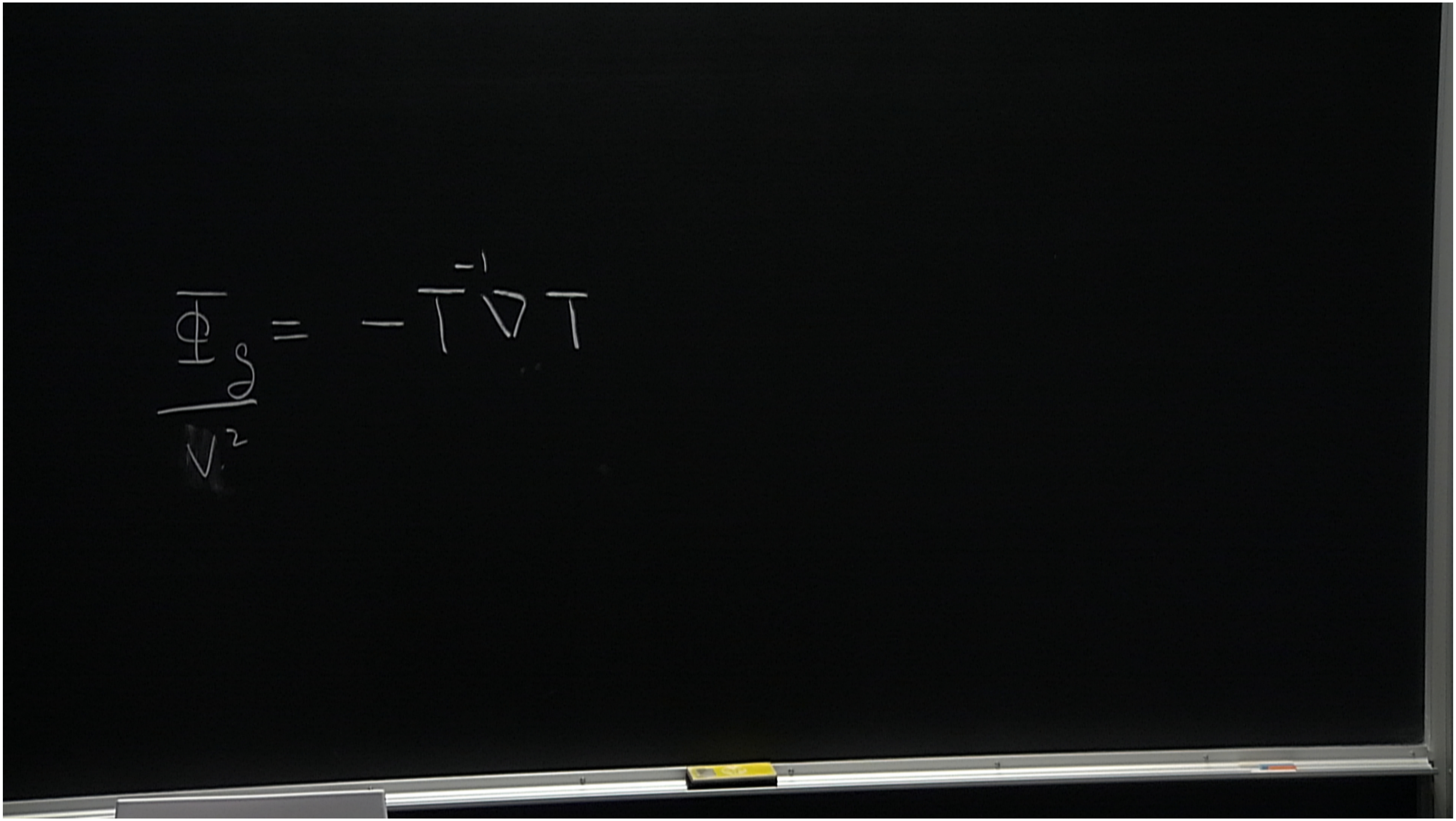

Φ_g = − T⁻¹ ∇ T
∇²

Edge theory and gravity coupling

- Edge theory: $Z = \mathrm{Tr}\left[e^{-\beta(H-\Omega^z L_z)}\right]$

$$H = \frac{v}{R}\left(L_0 - \frac{c}{24}\right) + \frac{v}{R}\left(\bar{L}_0 - \frac{c}{24}\right)$$

edge 1 edge 2

- Can rewrite as:

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}\left[e^{2\pi i\tau(L_0 - c/24) - 2\pi i\bar{\tau}(\bar{L}_0 - c/24)}\right]$$

where

$$2\pi i\tau = -\frac{\beta}{R}(v + i\eta) \quad \eta := i\Omega^z/(\pi R)$$

- Can be expanded in general as:

$$Z(\tau,\bar{\tau}) = \sum_{a,b} N_{a,\bar{b}}\chi_a(\tau)\overline{\chi_b^c(\tau)}$$

focusing on the single edge (and a sector) $Z(\tau,\bar{\tau}) \sim \chi_a(\tau)$

Streda formula

EM Streda formula:

$$\sigma_H = ec\frac{\partial M^z}{\partial \mu} = ec\frac{\partial N}{\partial B^z}$$

[Streda (1982)]

- link to thermodynamics

- can be used as an alternative to Kubo formula

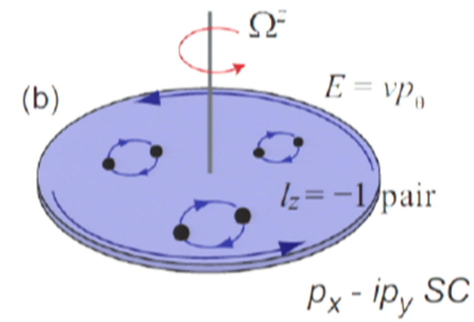
Thermal Streda formula

[Nomura, SR, Furusaki, Nagaosa (11)]

Is there a thermal analogue of Streda formula ?

$$\kappa_H = \frac{v^2}{2}\left(\frac{\partial L^z}{\partial T}\right)_{\Omega^z} = \frac{v^2}{2}\left(\frac{\partial S}{\partial \Omega^z}\right)_T$$

where
- L^z: angular momentum
- Omega^z: external angular velocity
- S: entropy
- v: "velocity"



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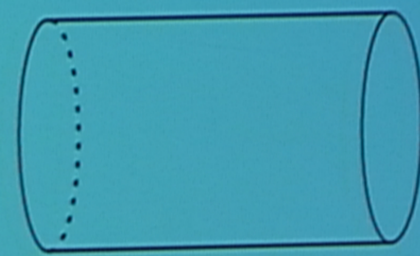

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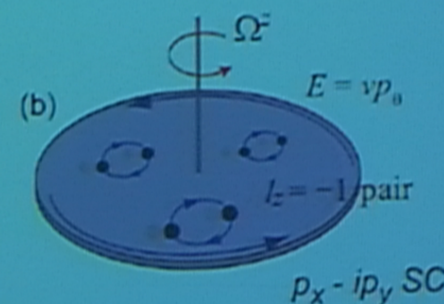

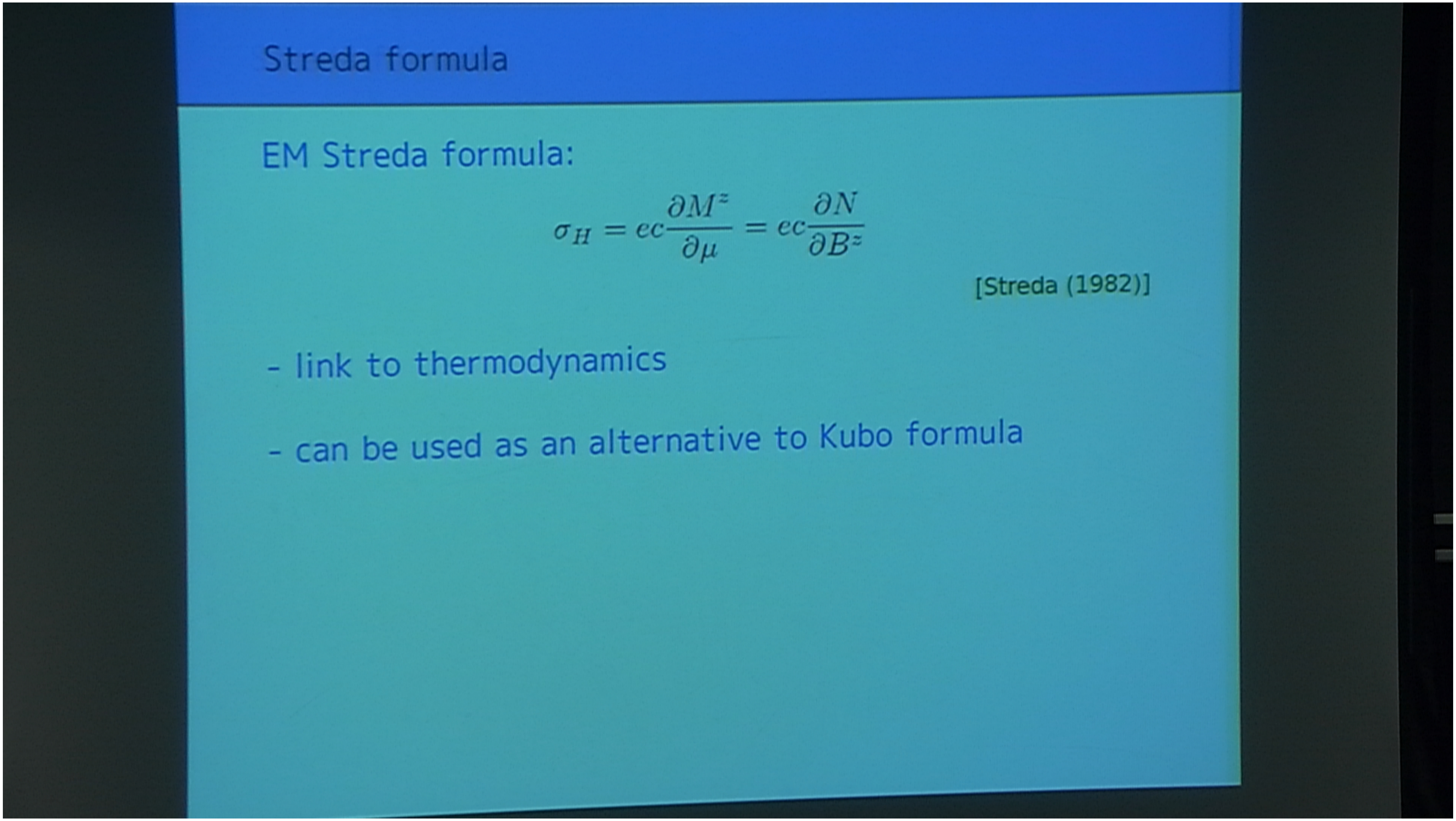
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Derivation from edge theory

- All bulk excitations are gapped
  --> partition function for the edge theory

- Thermal current and thermal Hall conductance

$$\kappa_H = \partial J_E / \partial T \qquad J_E = v\varepsilon$$

[Cappelli et al]

$$\varepsilon \sim v\frac{\partial}{\partial \eta}\ln \chi_a \qquad \kappa_H \sim v\frac{\partial}{\partial T}\frac{\partial}{\partial \eta}\ln \chi_a$$

- Entropy:

$$S = \frac{\partial}{\partial T}\frac{1}{\beta}\ln \chi_a$$

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- Valid for interacting topological phases
  v = velocity at the edge.

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- Topological phases characterized by infinitesimal anomalies are by now "familiar".

- New setting:
symmetry protected topological phases in 2d;

topological phases with no conserved charge
no chiral edge mode
--> no infinitesimal anomaly

- What to look for ?

A simple $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric model

- Topological phases with Z2xZ2 symmetry
  "spin up" and "down" are separately conserved mod 2

- Possible edge theory:

$$\mathcal{L} = \frac{1}{4\pi} \sum_{a=1}^{N_f} \left[\psi_L^a (\partial_\tau + iv\partial_x) \psi_L^a + \psi_R^a (\partial_\tau - iv\partial_x) \psi_R^a \right]$$

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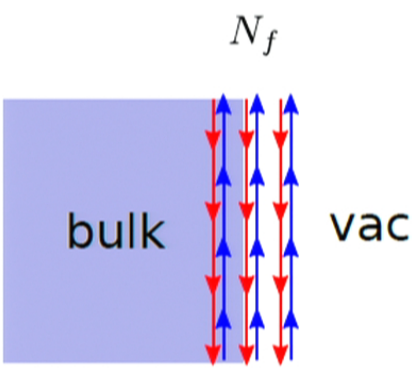

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[Similar (same) model: Gu-Wen, Qi]

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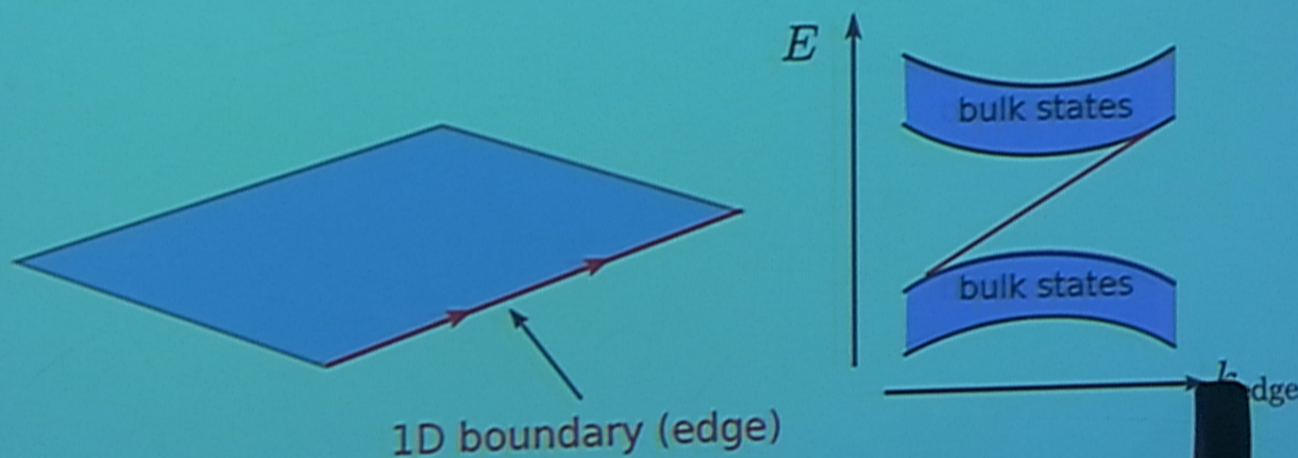

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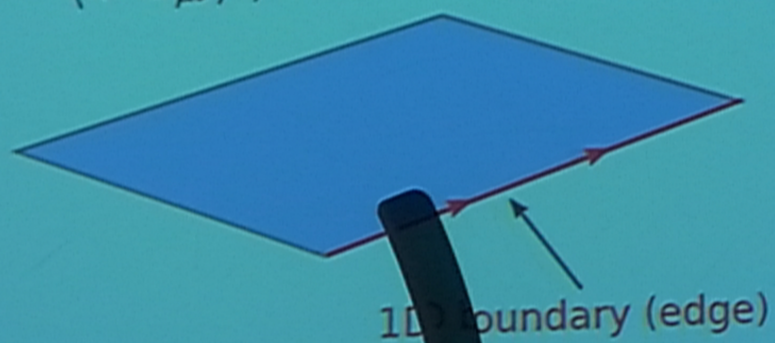

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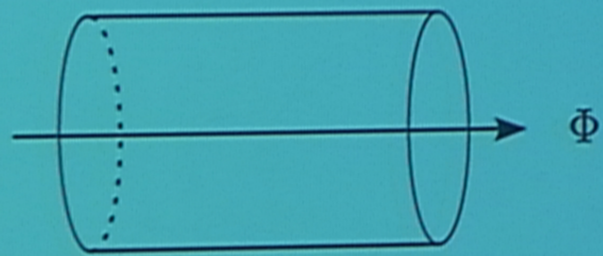
Laughlin's argument -- "large gauge transformation"



- Consider an adiabatic process $\Phi \to \Phi + \Delta\Phi$

- When $\Delta\Phi = \text{integer} \times \Phi_0$ system goes back to itself ("large gauge equivalent")

$$H(\Phi) = H(\Phi + n\Phi_0) \qquad Z(\Phi) = Z(\Phi + n\Phi_0)$$

- However, by this adiabatic process, an integer multiple of charge is transported from the left (right) to right (left) edge

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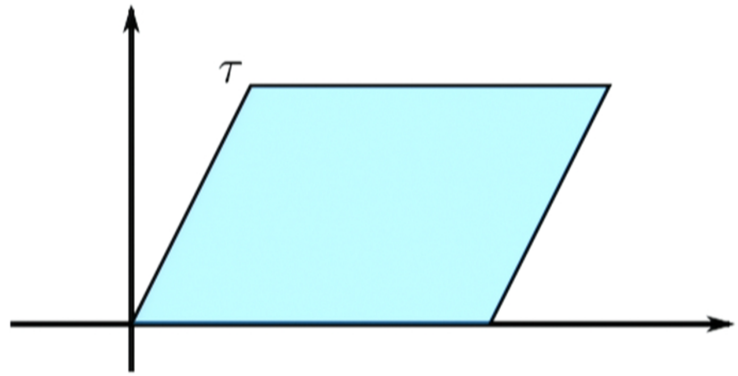
large coordinate transformations -- modular transformations



- Space-time manifold of the edge theory = 2d torus

- "Shape" of torus is parameterized by a single complex number

$$z \equiv z + 2\pi(m + \tau n)$$

- $\tau' = \tau + 1$ and $\tau' = -1/\tau$ represents the same torus since

$$(m,n) \to (m-n,n) \qquad (m,n) \to (n,-m)$$

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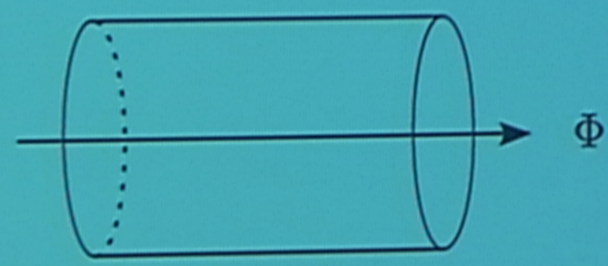

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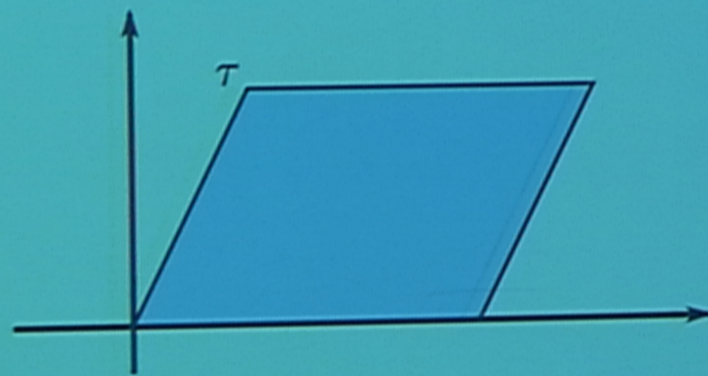

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Modular invariance/non-invariance in CFTs

- Any CFT derived as a continuum limit of a lattice system is expected to be modular invariant (anomaly free).

[Cardy, Cappelli-Itzykson-Zuber, Kato]

- Chiral CFTs are often modular non-invariant (anomalous).

- "Gluing" left-moving and right-moving parts properly, non-chiral CFTs can usually be made modular invariant.

- However, demanding a CFT to be invariant under some symmetry (e.g. Z2xZ2 symmetry) might conflict with modular invariance

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simple example

- free complex left-moving fermion $\mathcal{L}_L = \frac{1}{2\pi}\Psi_L^\dagger\left(\partial_\tau + v\partial_x\right)\Psi_L$

- partition function with a given boundary condition

spatial bc

$$Z^\alpha{}_\beta(\tau) = \mathrm{Tr}_\alpha\left[e^{\pi i \beta N_L} q^{H_L}\right], \quad q = e^{2\pi i \tau} \qquad N_L := \int dx\, \Psi_L^\dagger \Psi_L$$

time bc

alpha, beta = A, P (anti PBC or PBC)

- modular trsf:

$$S : \begin{cases} Z^0{}_0 \to Z^0{}_0, \\ Z^1{}_1 \to Z^1{}_1, \\ Z^1{}_0 \to Z^0{}_1, \\ Z^0{}_1 \to Z^1{}_0, \end{cases} \qquad T : \begin{cases} Z^0{}_0 \to e^{i\pi/12} Z^0{}_1, \\ Z^1{}_1 \to e^{-i\pi/6} Z^1{}_1, \\ Z^1{}_0 \to e^{-i\pi/6} Z^1{}_0, \\ Z^0{}_1 \to e^{i\pi/12} Z^0{}_0, \end{cases}$$

- combining left- and right-moving sectors, we acheive modular inv.:

$$Z = |Z^0{}_0|^2 + |Z^0{}_1|^2 + |Z^1{}_0|^2 + |Z^1{}_1|^2$$

Enforcing symmetries by projection

- Is the modular invariance "consistent" with the Z2xZ2 symmetry?

- Let's enforce Z2xZ2 symmetry by considering *projection*

$$P = \frac{1+(-1)^{N_L}}{2}\frac{1+(-1)^{N_R}}{2}$$

$$N_L = \sum_{i=1}^{N} N_L^i$$

$$N_R = \sum_{i=1}^{N} N_R^i$$

- Projected partition function:

$$Z_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{A}}\left[Pq^{H_L}\right] = \frac{1}{2}\left[{Z^0}_0(\tau)^N \pm {Z^0}_1(\tau)^N\right]$$

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  --> the bulk topolgoical phase

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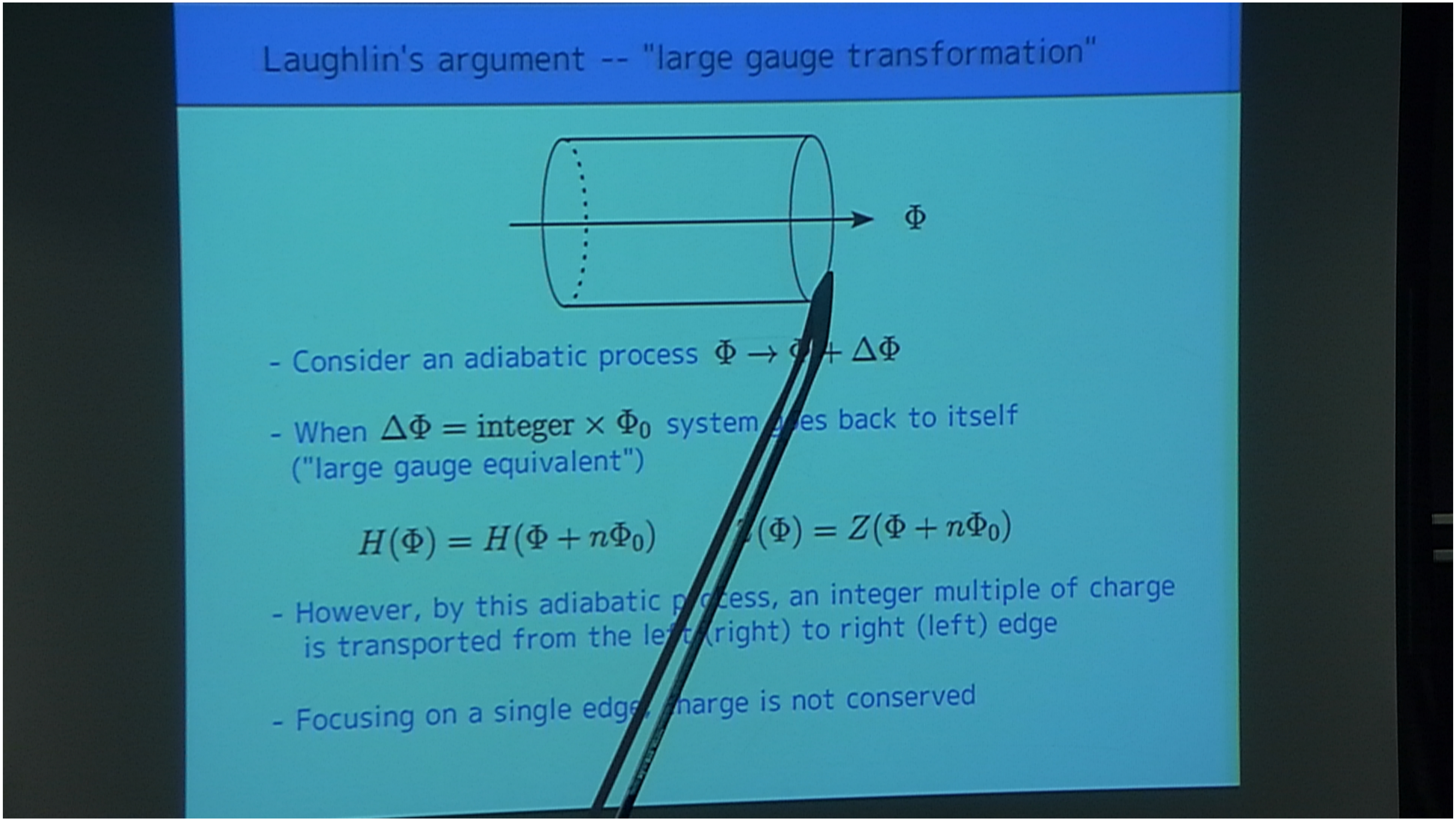
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- When N=4 (8 flavors of Majorana fermions), however, we acheive modular invariance !

$$Z(\tau) = \frac{1}{2}\left[{Z^0}_0(\tau)^4 - {Z^0}_1(\tau)^4 - {Z^1}_0(\tau)^4 \pm {Z^1}_1(\tau)^4\right]$$

- Suggesting there is no bulk topological phase when N=4.

- In fact, we can find interactions which destabilize the edge theory when N = 4.

- Interactions in terms of "spinors":

$$\Psi_L^i \simeq e^{i\varphi_L^i}$$

$$e^{\frac{i}{2}\left[\pm\varphi_L^1 \pm\varphi_L^2 \pm\varphi_L^3 \pm\varphi_L^4\right]}$$

Haldane phase,
two-leg ladder Hubbard model
[Tsvelik, Lin-Balents-Fisher, ...]

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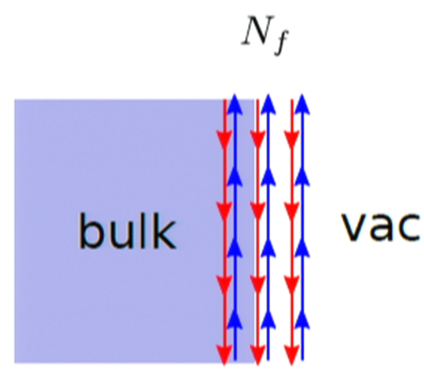

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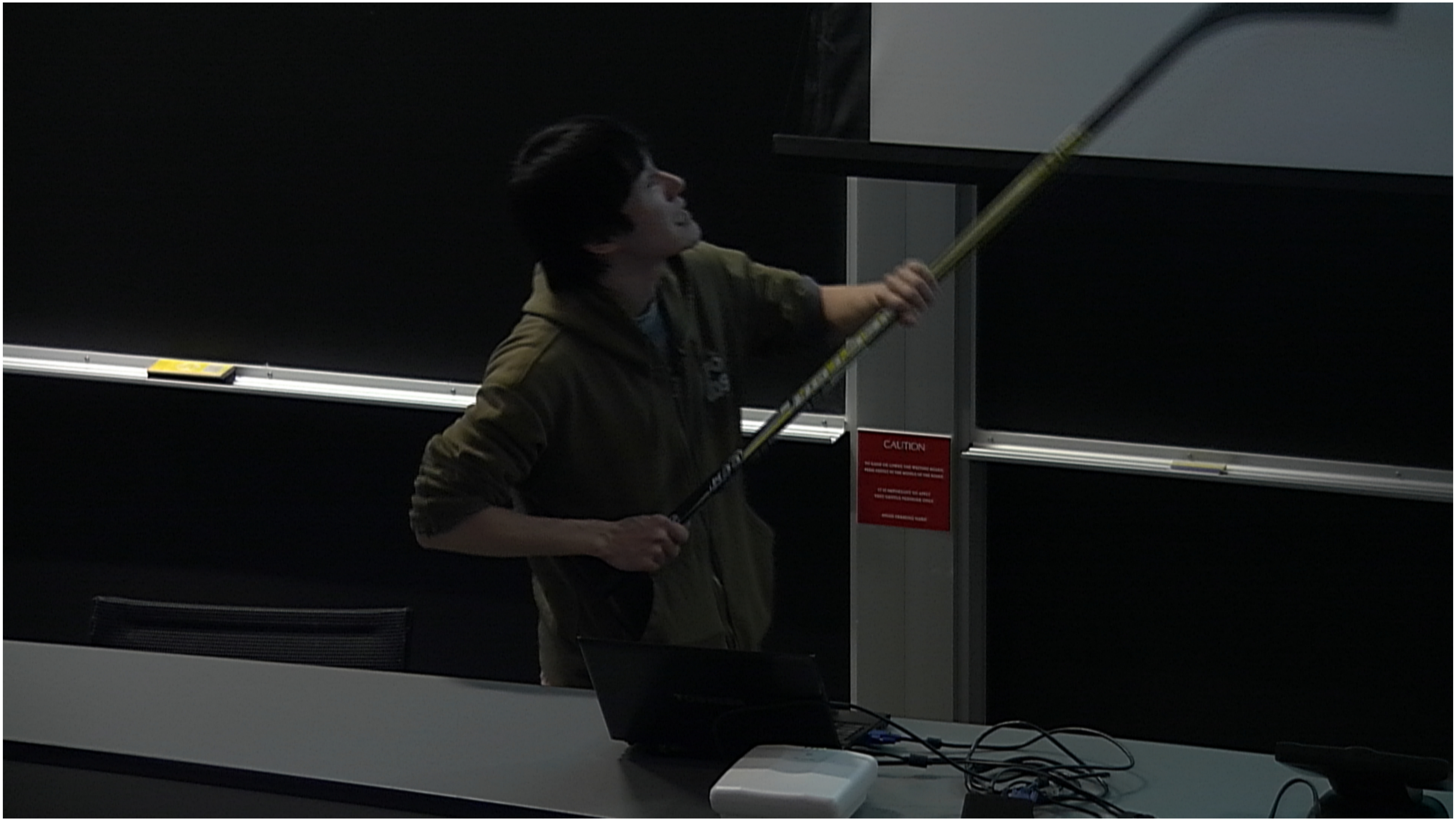
two-leg ladder Hubbard model
[Tsvelik, Lin-Balents-Fisher, ...]


CAUTION

Summary and outlook

- Thermal/mechanical response theory for topological SC in 2d.

- Can also be applied to 3d topological SCs.
  quasi-quantized thermal-mechanical responses.

	TI	TSC
2d	$\sigma_H = ec\frac{\partial M^z}{\partial \mu} = ec\frac{\partial N}{\partial B^z}$	$\kappa_H = \frac{v^2}{2}\frac{\partial L^z}{\partial T} = \frac{v^2}{2}\frac{\partial S}{\partial \Omega^z}$
3d	$\chi_\theta^{ab} = \frac{\partial M^a}{\partial E^b} = \frac{\partial P^a}{\partial B^b}$	$\chi_{\theta,g}^{ab} = \frac{\partial L^a}{\partial E_g^b} = \frac{\partial P_E^a}{\partial \Omega^b}$

- Symmetry protected topological phases and (global) anomalies. Proposed strategy;

SPTs = asymmetric "orbifolds"

(i) Projection by symmetry group G --> orbifold part. function

$$Z^{\mathrm{orb}} = \frac{1}{|G|}\sum_{g_1, g_0 \in G} \epsilon(g_1, g_0) Z^{g_0}{}_{g_1}$$

(ii) Look for anomaly

[See also: Levin-Gu (2012)]

- Tested for a wider range of systems.

[Sule-Ryu (unpublished) Levin arXiv:1301.7355]

- Connection with other approaches ?

[Chen-Liu-Wen (11), Chen-Gu-Liu-Wen (11) Lu-Vishwanath (12-13), Gu-Wen (11), Hung-Wan(13) ....]

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Outline

1. Acceptance of Dark Energy Today
2. Newton's Methodology: Acceptance empirically guided by theory mediated measurements
3. Example: Mercury's perihelion before and after Einstein
4. Determining Cosmological Parameters and Supporting Dark Energy