Title: Wall-Crossing and Quiver Invariants

Date: Feb 19, 2013 02:00 PM

URL: http://www.pirsa.org/13020137

Abstract: We start with a one-slide review of the Kontsevich-Soibelman
(KS) solution to the wall-crossing problem and then proceed to direct and comprehensive physics counting of BPS states that eventually connects to KS. We also asks what input data is needed for either approaches to produce complete BPS spectra, and this naturally leads to the BPS quiver representation of BPS states and the new notion of quiver invariants.
We propose a simple geometrical conjecture that can segregate BPS states in Higgs phases of the BPS quiver dynamics to those that experience wall-crossing and those that do not, and give
proofs for all cyclice Abelian quivers. We close with explanation of how physics distinguishes two such classes of BPS states.

Wall-Crossing & Quiver Invariants

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February 2013

with
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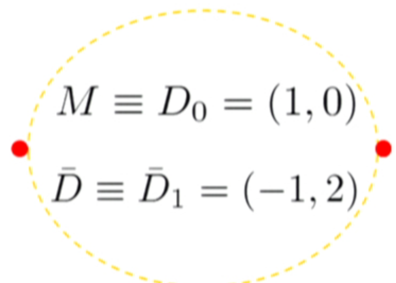
Heeyeon Kim, Jaemo Park, Zhao-Long Wang (1107. 0723),
Seung-Joo Lee, Zhao-Long Wang (1205.6511 / 1207.0821)

prototype : D=4 N=2 SU(2) → U(1) Seiberg-Witten

$$W = (0, 2)$$

$$\bar{D}_n = (-1, 2n)$$



$$M \equiv D_0 = (1, 0)$$

$$\bar{D} \equiv \bar{D}_1 = (-1, 2)$$

$$\Omega(\bar{W}) = \Omega(W) = -2$$

$$\Omega(\bar{D}_n) = \Omega(D_n) = 1$$

or, more generally, the protected spin character

$$SO(4) = SU(2)_{\text{rotation}} \times SU(2)_{\text{R-symmetry}}$$

J $\quad$ I

$$\Omega = -\frac{1}{2}\text{tr}\,(-1)^{2J_3}(2J_3)^2 \quad \underset{y=1}{\Leftarrow} \quad \Omega(y) = -\frac{1}{2}\text{tr}\,(-1)^{2J_3}(2J_3)^2 y^{2I_3+2J_3}$$

2nd helicity trace

protected spin character

Gaiotto, Moore, Neitzke 2010 / Maldacena 2010

$$\to (-1)^{2l} \times (2l+1)$$

on [a spin ½ + two spin 0]
x [angular momentum l multiplet]

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Kontsevich-Soibelman, 2008

(also Gaiotto-Moore-Neitzke, 2008-2009)

Schwinger product

$$[V_\alpha, V_\beta] = (-1)^{\langle \alpha, \beta \rangle} \langle \alpha, \beta \rangle V_{\alpha+\beta} \qquad K_\gamma \equiv \exp\left(\sum_n \frac{V_{n\gamma}}{n^2} \right)$$

a marginal stability wall

+ side

$$\prod_\gamma K_\gamma^{\Omega^+(\gamma)} = \prod_\gamma K_\gamma^{\Omega^-(\gamma)}$$

− side

wall-crossing of BPS states with 4 (or less) supersymmetries

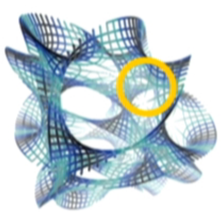
marginal stability wall

γ_1

γ_2

γ_3

$\vdots$



γ_1 $\quad$ $\gamma_1 + \gamma_2$

γ_2 $\quad$ $\gamma_2 + \gamma_3$

γ_3 $\quad$ $\gamma_1 + \gamma_3$

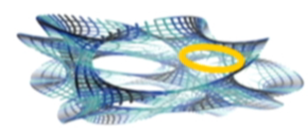
$\vdots$ $\quad$ $\vdots$

$\gamma_1 + \gamma_2 + \gamma_3$

$\vdots$

$\sum k_i \gamma_i$

$\vdots$



true in all physics examples ?

how to see from BPS state building/counting ?

& why rational invariants ?

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

input data ?

$$\Omega^+(\gamma) = \Omega^-(\gamma) \neq 0$$

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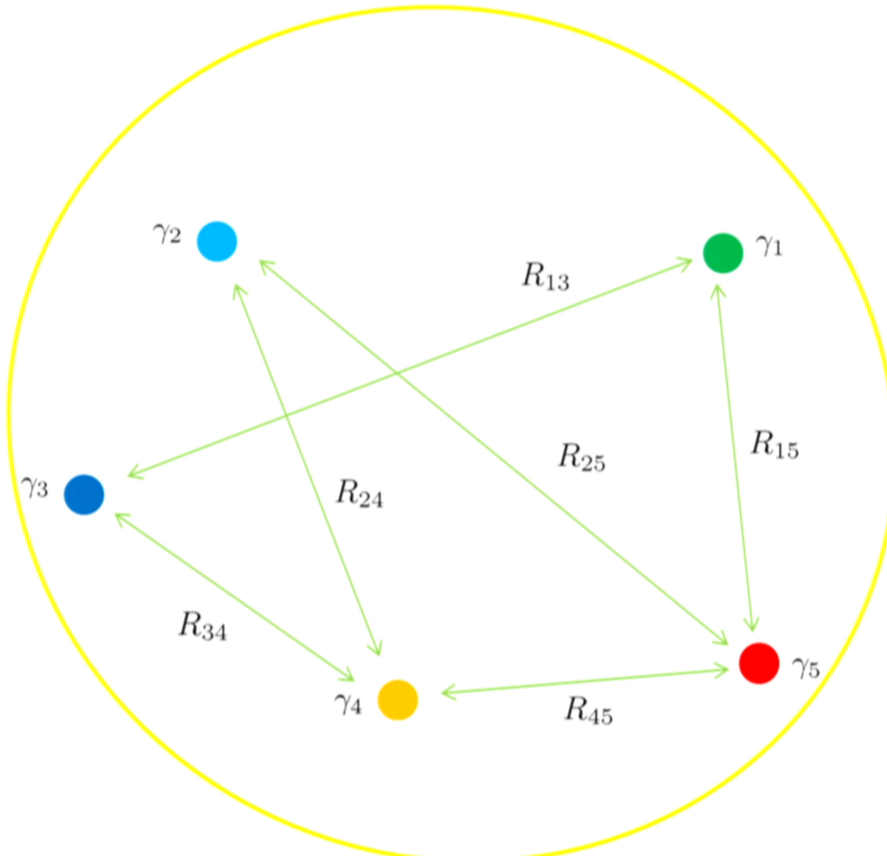
generic BPS "particles" are loose bound states of charge centers

$$R^3 = \{\vec{X}\}$$



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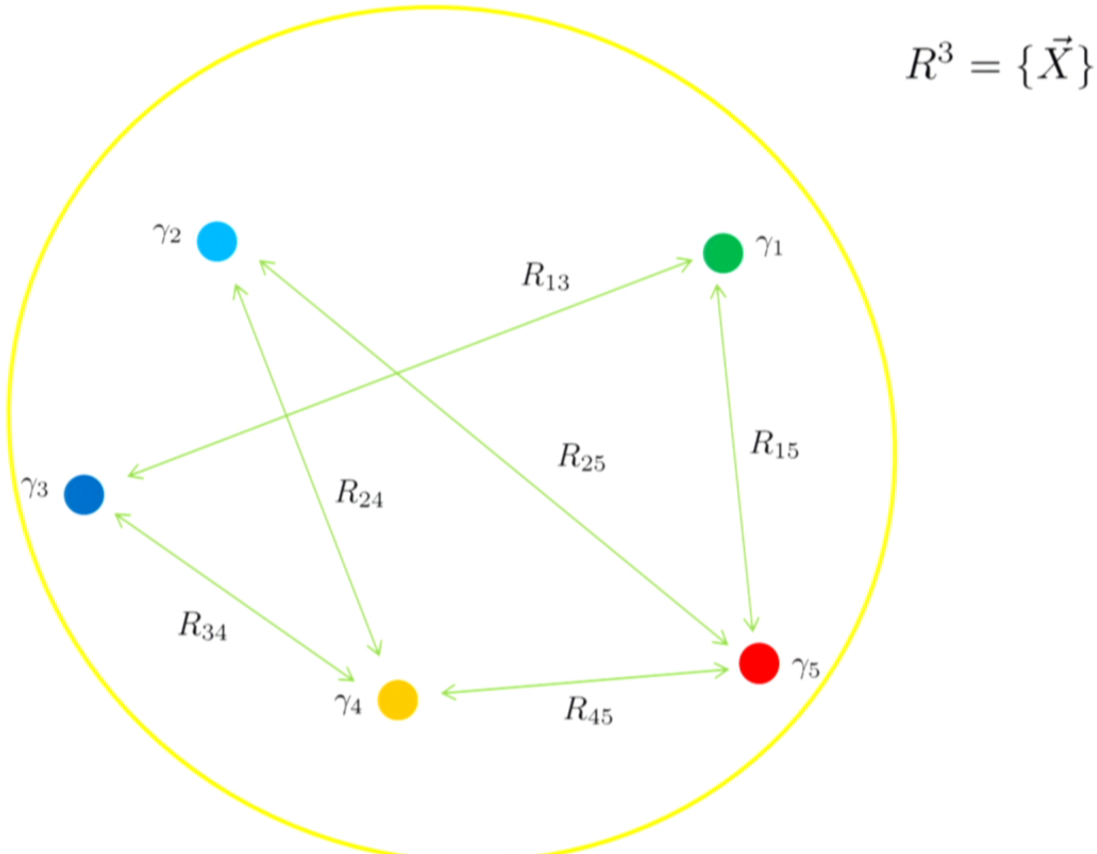


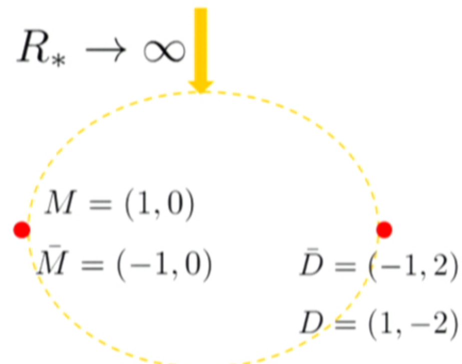
wall-crossing ← dissociation of supersymmetric bound states

$$W^+ = M + \bar{D}$$

$$W^- = \bar{M} + D$$

$$D_n = (n+1)M + n\bar{D}$$

$$\bar{D}_n = (n-1)M + n\bar{D}$$



$R_* \to 0$

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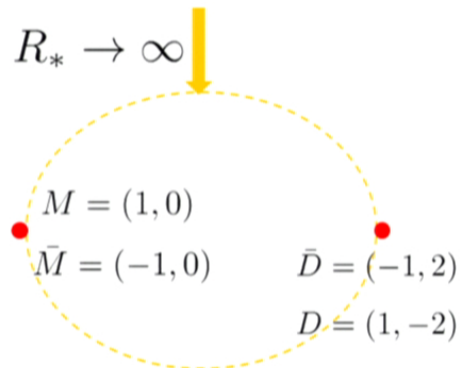

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1998 Lee + P.Y.

N=4 SU(n) ¼ BPS states via multi-center classical dyon solitons

1999 Bak + Lee + Lee + P.Y.

N=4 SU(n) ¼ BPS states via multi-center monopole dynamics

1999-2000 Gauntlett + Kim + Park + P.Y. / Gauntlett + Kim + Lee + P.Y. / Stern + P.Y.

N=2 SU(n) BPS states via multi-center monopole dynamics

2001 Denef

N=2 supergravity via classical multi-center black holes attractor solutions

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{R_{AB}} \qquad \zeta \equiv \frac{\sum_A Z_{\gamma_A}}{|\sum_A Z_{\gamma_A}|}$$

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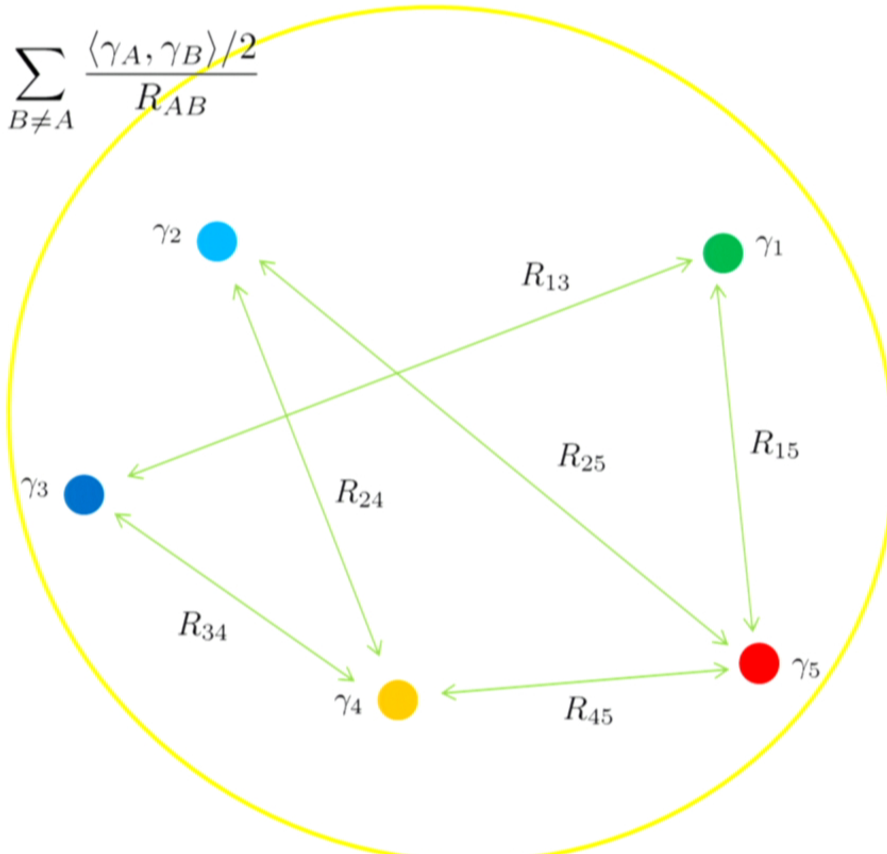

from Seiberg-Witten BPS dyons as semiclassical & asymptotic solutions

with electric charges n^i and magnetic charges m^i

$$\mathcal{E} = |Z| \qquad Z \equiv \langle m^i \phi_D^i + n^i \phi^i \rangle = |Z|\zeta$$

$$F_a^i = i\zeta^{-1}\partial_a \phi^i \qquad F_a^i \equiv B_a^i + iE_a^i \qquad \mathrm{Re}\int_{S^2} F^i = 4\pi m^i$$

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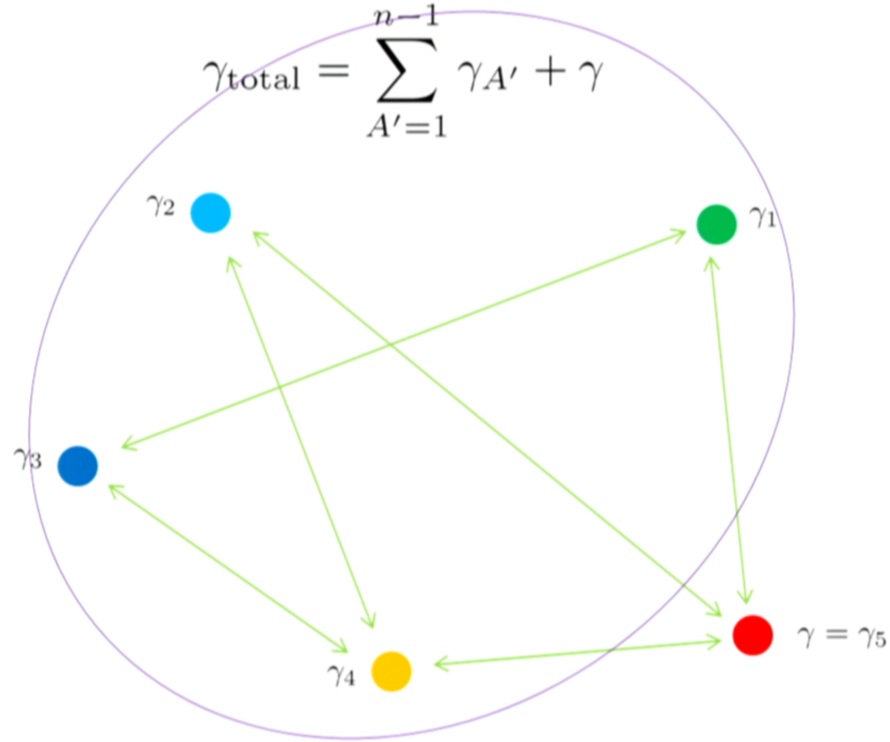
as a preliminary step, treat one dyon dynamical at a time

$$\gamma_{\text{total}} = \sum_{A'=1}^{n-1} \gamma_{A'} + \gamma$$



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$$\gamma = (p, 2q) \qquad \sum_{A'} \gamma_{A'} = (m_{A'}, 2n_{A'})$$

$\mathcal{Z}_\gamma$: the central charge function of the probe dyon in the background of other dyons

$$-\partial_a \mathrm{Im}[\zeta^{-1}\partial_a \mathcal{Z}_{\gamma=(p,2q)}]$$

$$= \mathrm{Re}\left(q^i \partial_a F_a^i + p^i \partial_a (F_D)_a^i\right) = \sum_{A'} (q^i m_{A'}^i - p^i n_{A'}^i) 4\pi \delta^3(\vec{x} - \vec{x}_{A'})$$

$$= \langle \gamma, \gamma_{A'} \rangle / 2$$

$$\mathcal{K}_\gamma \equiv \mathrm{Im}[\zeta^{-1}\mathcal{Z}_\gamma] = \mathrm{Im}[\zeta^{-1} Z_\gamma] - \sum_{A'} \frac{\langle \gamma, \gamma_{A'} \rangle /2}{|\vec{x} - \vec{x}_{A'}|}$$

→ Lorentz Force !!!

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a probe charge to a system of background "core" dyons

Sungjay Lee+P.Y. 2011

$$\mathcal{L}_{probe} = -|\mathcal{Z}_\gamma|\sqrt{1-\dot{\vec{x}}^2} + \mathrm{Re}[\zeta^{-1}\mathcal{Z}_\gamma] - \dot{\vec{x}}\cdot\vec{W}$$

$$\simeq \frac{1}{2}\,|\mathcal{Z}_\gamma|\dot{\vec{x}}^2 - (|\mathcal{Z}_\gamma| - \mathrm{Re}[\zeta^{-1}\mathcal{Z}_\gamma]) - \dot{\vec{x}}\cdot\vec{W}$$

$$\simeq \frac{1}{2}\,|\mathcal{Z}_\gamma|\dot{\vec{x}}^2 - \frac{(\mathrm{Im}[\zeta^{-1}\mathcal{Z}_\gamma])^2}{2|\mathcal{Z}_\gamma|} - \dot{\vec{x}}\cdot\vec{W}$$

$$\vec{\partial}\times\vec{W} \equiv \vec{\partial}\,\mathrm{Im}\,[\zeta^{-1}\mathcal{Z}_\gamma]$$

$$\zeta^{-1}\mathcal{Z}_\gamma = |\mathcal{Z}_\gamma|e^{i\alpha}, \quad |\alpha| \ll 1$$

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(1) ab initio, real space N=4 susy quantum mechanics for n dyons

Lee+P.Y. 2011
Kim+Park+P.Y.+Wang 2011

$$\int dt\ \mathcal{L}_{kinetic} = \int dt \int d\theta^2 d\bar{\theta}^2 F(\hat{\Phi}^{Aa})$$

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N=4 Q.M. +
D=4 N=2 Central Charge

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3n bosons + 4n fermions in two different superspaces

Kim+Park+P.Y.+Wang 2011

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is manifestly N=4 supersymmetric

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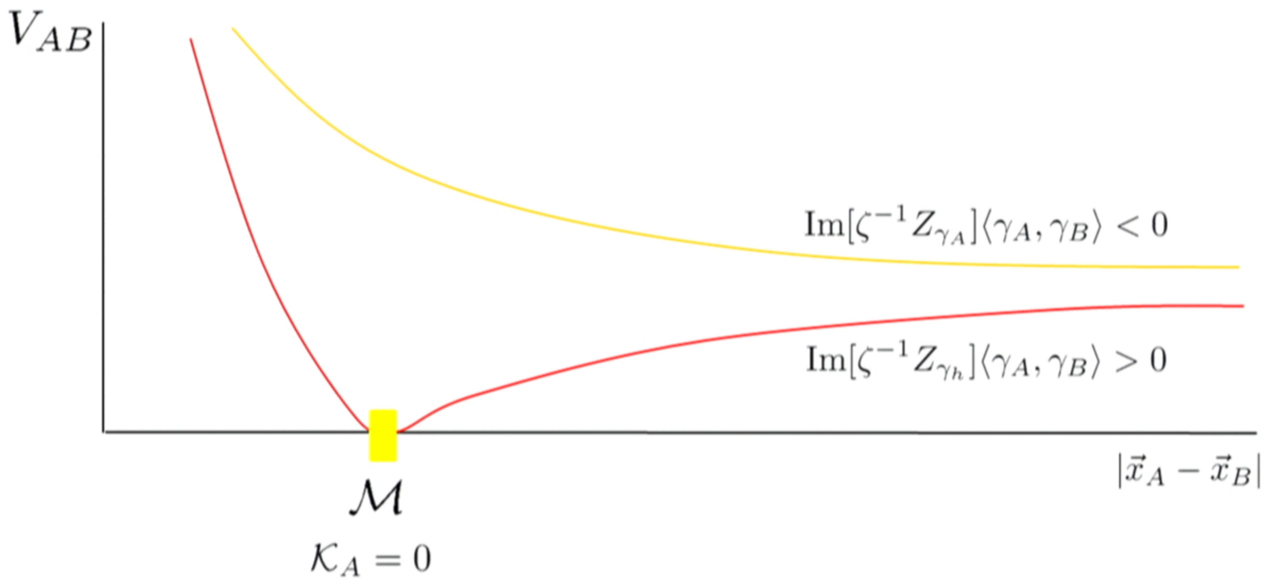
$$V(\{\vec{x}_A\}) \sim \sum_A \mathcal{K}_A^2 = \sum_A \left(\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] - \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{|\vec{x}_A - \vec{x}_B|} \right)^2$$



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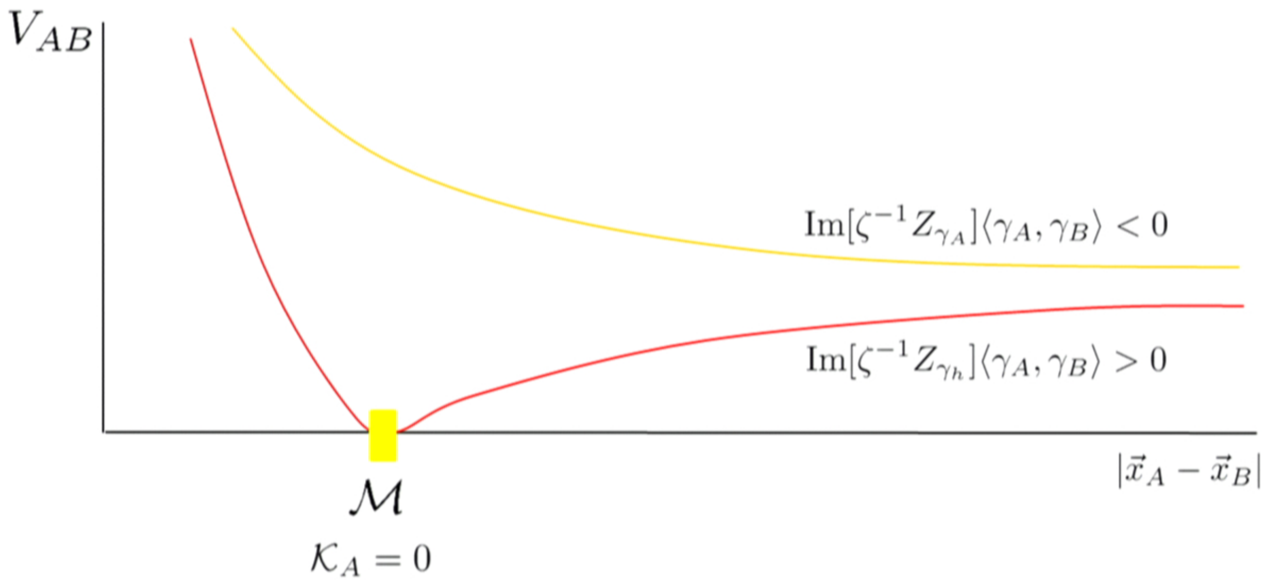


3n bosons + 4n fermions in two different superspaces

4n N=1 supermultiplets

$$\Phi^{Aa} = x^{Aa} - i\theta\psi^{Aa} \qquad \Lambda^A = i\lambda^A + i\theta b^A \qquad A = 1, 2, \ldots, n$$

↓

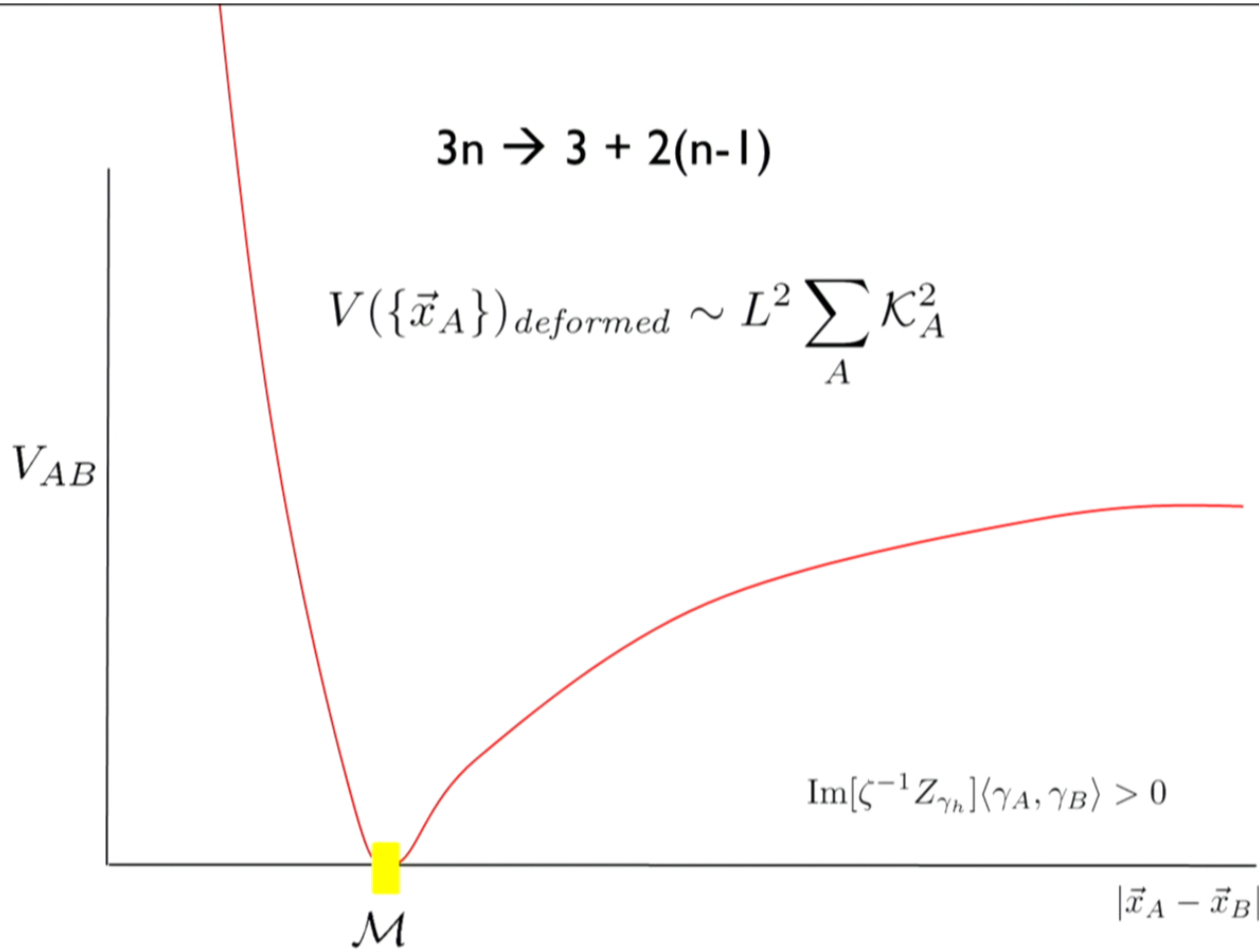
position of A-th dyon

n N=4 supermultiplet ~ n D=4 N=1 vector multiplets

$$\hat{\Phi}^{Aa} = -\frac{i}{4}(\epsilon\sigma^a)^{\alpha\beta}\Phi^A_{\alpha\beta} \; ; \qquad \Phi^A_{\alpha\beta} = (D_\alpha \bar{D}_\beta + \bar{D}_\beta D_\alpha)V^A$$

3n → 3 + 2(n-1)

$$V(\{\vec{x}_A\})_{deformed} \sim L^2 \sum_A \mathcal{K}_A^2$$



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the counting problem reduces to a N=1 Dirac index of a nonlinear sigma model on the manifold $\mathcal{K}_A = 0$

Kim+Park+P.Y.+Wang 2011

3n bosons + 4n fermions → 2(n-1) bosons + 2(n-1) fermions

$$\mathcal{L}^{for\ index\ only}_{deformed}\Big|_{L\to\infty} \quad \to \quad \mathcal{L}_{index}$$

$$\mathcal{L}_{index} \simeq \frac{1}{2} g_{\mu\nu}\dot{z}^\mu \dot{z}^\nu - \dot{x}^\mu \cdot \mathcal{A}_\mu + \frac{i}{2} g_{\mu\nu}\psi^\mu \left(\dot{\psi}^\nu + \dot{z}^\alpha \Gamma^\nu_{\alpha\beta}\psi^\beta\right) + i\mathcal{F}_{\mu\nu}\psi^\mu\psi^\mu$$

$$\mathcal{F} = d\mathcal{A} \equiv \sum_A dW_A\Big|_{\mathcal{K}_A=0}$$

(2) basic state counting index

Manschot, Pioline, Sen 2010/2011

Kim+Park+P.Y.+Wang 2011

$$I_n(\{\gamma_A\}) = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] = \mathrm{tr}\left[(-1)^F e^{-\beta Q^2}\right]$$

$$= \int_{\mathcal{M}=\{\vec{x}_A \mid \mathcal{K}_A=0\}/R^3} ch(\mathcal{F}) \wedge \boxed{\mathbf{A}(\mathcal{M})}$$

trivial for a complete intersection in flat ambient space

$$= \int_{\mathcal{M}=\{\vec{x}_A \mid \mathcal{K}_A=0\}/R^3} ch(\mathcal{F})$$

$$= \frac{1}{(2\pi)^{n-1}(n-1)!} \int_{\mathcal{M}_n} \mathcal{F}^{n-1}$$

$$\mathcal{F} \equiv \sum_{A>B} \frac{\langle \gamma_A, \gamma_B \rangle}{2} \, \mathcal{F}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)\Big|_{\mathcal{K}_A=0}$$

(2) basic state counting index

Manschot, Pioline, Sen 2010/2011

Kim+Park+P.Y.+Wang 2011

$$I_n(\{\gamma_A\}) = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] = \mathrm{tr}\left[(-1)^F e^{-\beta Q^2}\right]$$

$$= \int_{\mathcal{M}=\{\vec{x}_A \mid \mathcal{K}_A=0\}/R^3} ch(\mathcal{F}) \wedge \mathbf{A}(\mathcal{M})$$

trivial for a complete intersection in flat ambient space

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(3) protected spin character = equivariant index on $\mathcal{M}$

Kim+Park+P.Y.+Wang 2011

$$\Omega = -\frac{1}{2}\mathrm{tr}\left[(-1)^{2J_3}(2J_3)^2 y^{2(J_3+I_3)}\right]$$

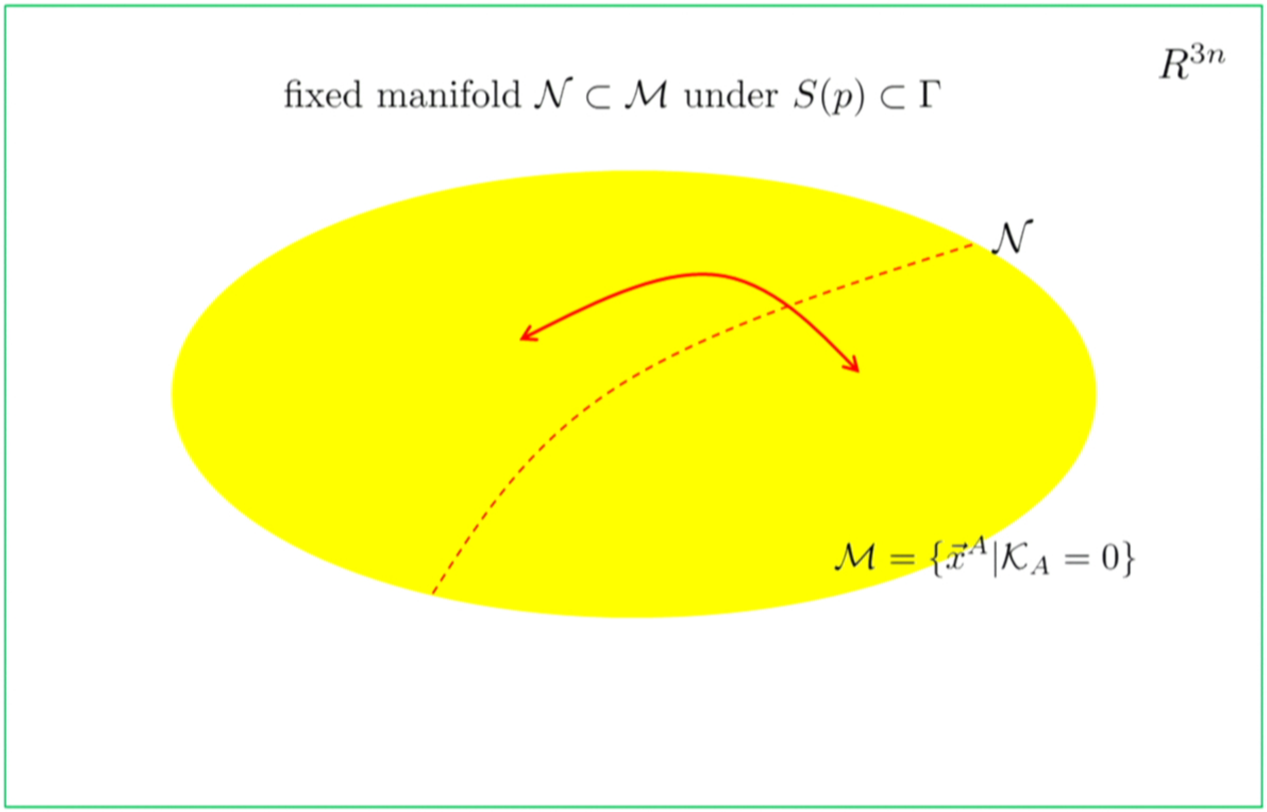
$$H = H_{\text{center of mass}} \otimes H_{\text{reduced}}$$

$$\Omega = \mathrm{tr}_{H_{\text{reduced}}}\left[(-1)^{2L_3+2(S_3-I_3)}(-1)^{2I_3}y^{2(J_3+I_3)}\right]$$

reduction to $\mathcal{M}$

$$\Omega = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\mathrm{tr}((-1)^F y^{2\mathcal{J}_3})$$

with quantum statistics taken into account



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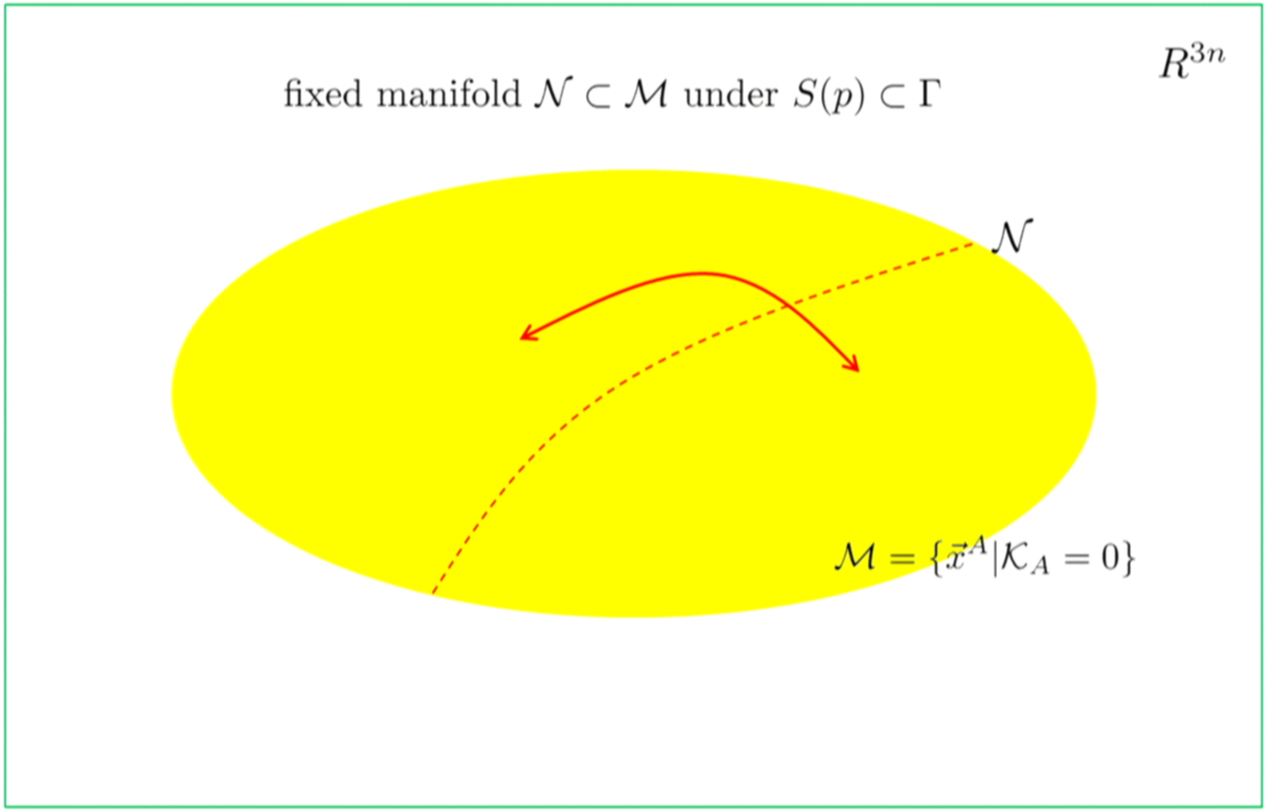

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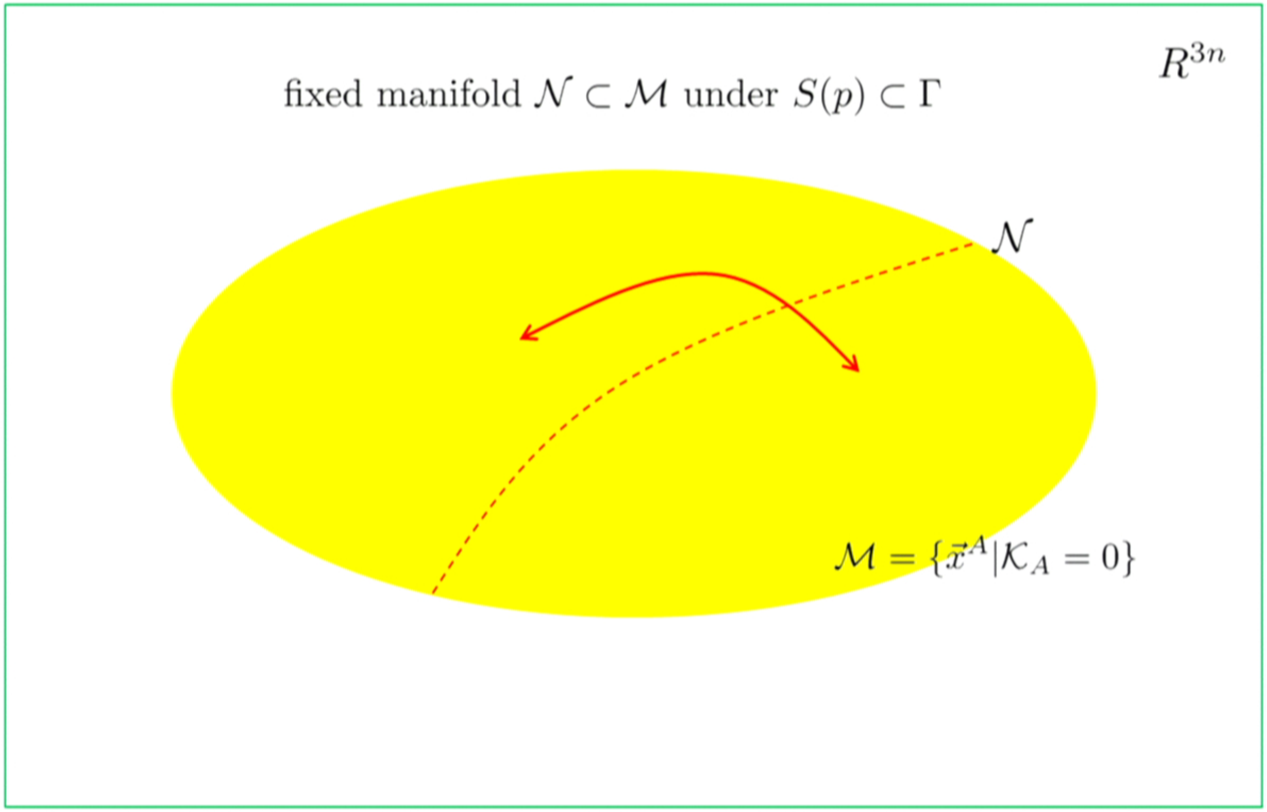

with quantum statistics taken into account

fixed manifold $\mathcal{N} \subset \mathcal{M}$ under $S(p) \subset \Gamma$

$$\Gamma' = \Gamma / S(p) \qquad \mathcal{P}' = \sum_{\sigma \in \Gamma'} (\pm 1)^{|\sigma|} \sigma$$

$$\mathrm{tr}\,(-1)^F e^{-\beta H} \mathcal{P}$$

$$= \mathrm{tr}_{\mathcal{M}/\Gamma - \mathcal{N}}\,(-1)^F e^{-\beta H} \mathcal{P} + \boxed{\Delta_{\mathcal{N}}} \mathrm{tr}_{\mathcal{N}/\Gamma' - \mathcal{N}'}\,(-1)^F e^{-\beta H} \mathcal{P}' + \cdots$$

with quantum statistics taken into account for a pair

$$\mathcal{P}_2^{(\pm)} \;:\; x \to -x,\; \psi \to -\psi$$

P.Y. 1997

$$\Delta_{\mathcal{N}}^{(\pm)}\Big|_{p=2} \leftarrow \lim_{\beta\to 0} \mathrm{tr}_{R^d;n_f}\left[(-1)^{F^{\perp}} e^{\beta\partial^2/2}\mathcal{P}_2^{(\pm)}\right]$$

$$= \lim_{\beta\to 0}\int_{R^d} d^dx\,\langle -x|e^{\beta\partial^2/2}|x\rangle \times (\pm 2^{n_{fermion}/2-1})$$

$$= \lim_{\beta\to 0}\frac{\pm 2^{n_{fermion}/2-1}}{(2\pi\beta)^{d/2}}\int_{R^d} d^dx\; e^{-(x+x)^2/2\beta}$$

$$= \lim_{\beta\to 0}\frac{\pm 2^{n_{fermion}/2-1}}{2^d}$$

$$\to \frac{\pm 1}{2^2}$$

n_f	=	2	4	8	16
d	=	2	3	5	9

(4) wall-crossing formula from real space dynamics

Manschot, Pioline, Sen 2010/2011
Kim+Park+P.Y.+Wang 2011

$$\Omega^-\left(\sum \gamma_A\right) - \Omega^+\left(\sum \gamma_A\right) = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\}+n-1} \frac{\prod_A \bar{\Omega}^-(\gamma_A)}{|\Gamma|} \int_{\mathcal{M}} ch(\mathcal{F})$$

$$\vdots$$

$$+(-1)^{\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\}+n'-1} \frac{\prod_{A'} \bar{\Omega}(\gamma'_{A'})}{|\Gamma'|} \int_{\mathcal{M}'} ch(\mathcal{F}')$$

$$\vdots$$

$$+(-1)^{\sum_{A''>B''}\langle\gamma''_{A''},\gamma''_{B''}\}+n''-1} \frac{\prod_{A''} \bar{\Omega}(\gamma''_{A''})}{|\Gamma''|} \int_{\mathcal{M}''} ch(\mathcal{F}'')$$

$$\vdots$$

$$\sum_{A=1}^{n} \gamma_A = \sum_{A'=1}^{n'} \gamma'_{A'} = \sum_{A''=1}^{n''} \gamma''_{A''} = \cdots$$

$$\bar{\Omega}(\gamma) = \sum_{p|\gamma} \Omega(\gamma/p)/p^2$$

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BPS quivers

calibrated geometry : special Lagrange submanifolds in Calabi-Yau

type II string theories on CY3 : wrapped D-branes

D=4 N=2 supergravity : BPS black holes

D=4 N=2 Seiberg-Witten theory : BPS dyons

⋮

D=4 N=4 …… ¼ BPS ……

⋮

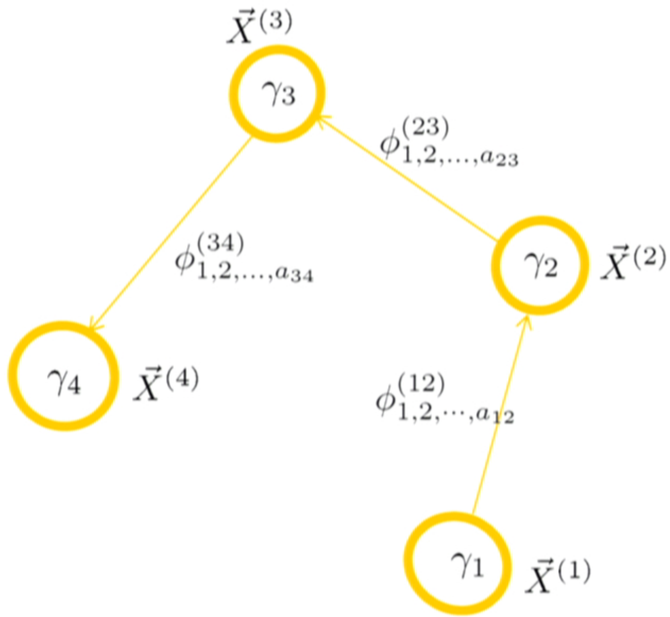
D=2 N=2 Landau-Ginzburg : BPS kinks

D3 on SL 3-cycles in CY3 → BPS quiver quantum mechanics

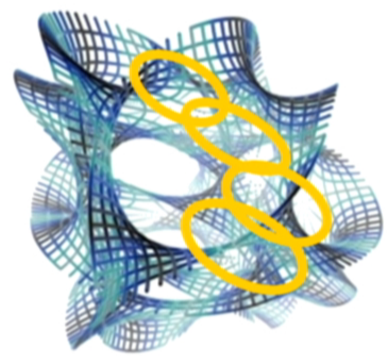
Denef 2002

$$\begin{array}{ccccccc} \vec{X}^{(1)} & & \vec{X}^{(2)} & & \vec{X}^{(3)} & & \vec{X}^{(4)} \\ U(1) & \times & U(1) & \times & U(1) & \times & U(1) \end{array}$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \quad \phi^{(23)}_{1,2,\ldots,a_{23}} \quad \phi^{(34)}_{1,2,\ldots,a_{34}}$$



$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



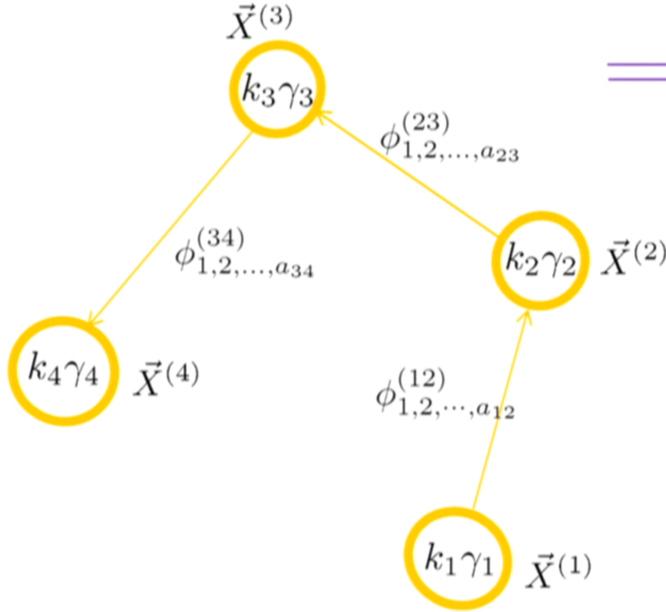
Coulomb "phase"

Denef 2002

$$\vec{X}^{(1)} \quad \vec{X}^{(2)} \quad \vec{X}^{(3)} \quad \vec{X}^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

~~$\phi^{(12)}_{1,2,\cdots,a_{12}}$ $\phi^{(23)}_{1,2,\ldots,a_{23}}$ $\phi^{(34)}_{1,2,\ldots,a_{34}}$~~



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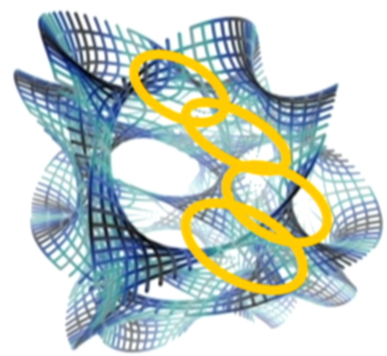
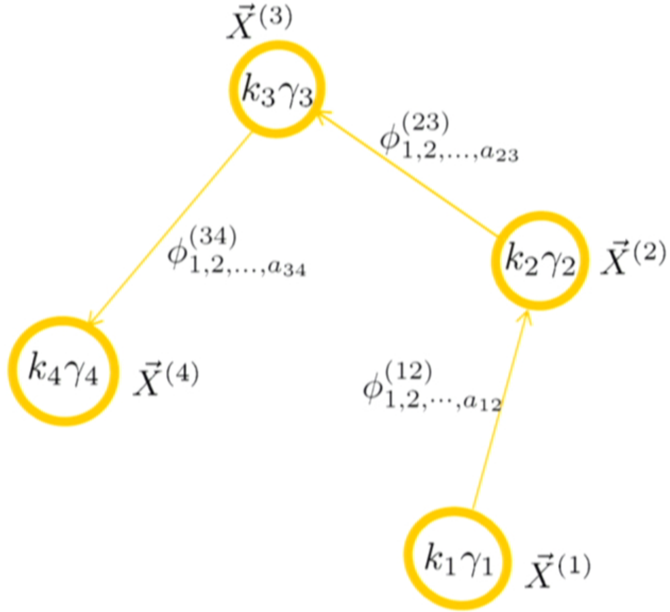


Higgs "phase"



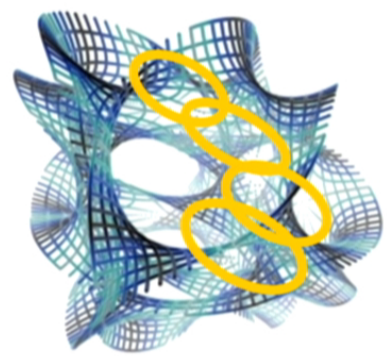
$\vec{X}^{(1)}$ $\vec{X}^{(2)}$ $\vec{X}^{(3)}$ $\vec{X}^{(4)}$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \quad \phi^{(23)}_{1,2,\ldots,a_{23}} \quad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$\xi^i \sim \mathrm{Im}[\zeta^{-1} Z_{\gamma_i}]$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



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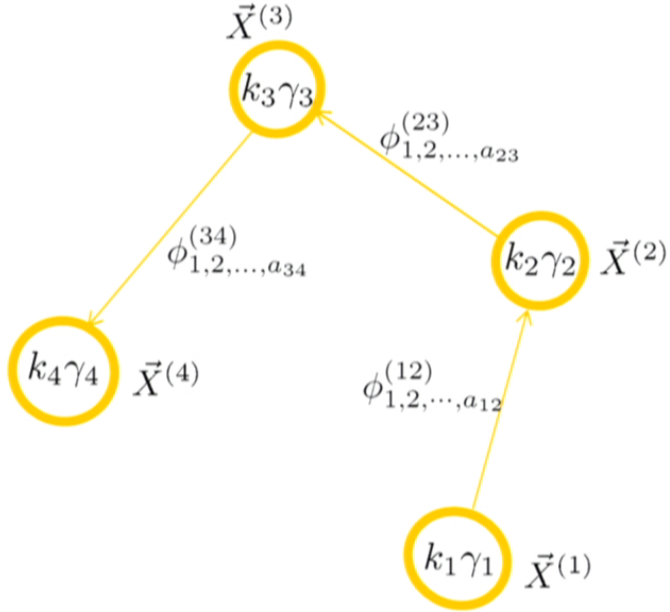

$\vec{X}^{(1)}$ $\vec{X}^{(2)}$ $\vec{X}^{(3)}$ $\vec{X}^{(4)}$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \quad \phi^{(23)}_{1,2,\ldots,a_{23}} \quad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

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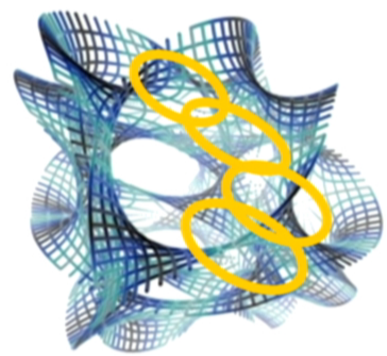
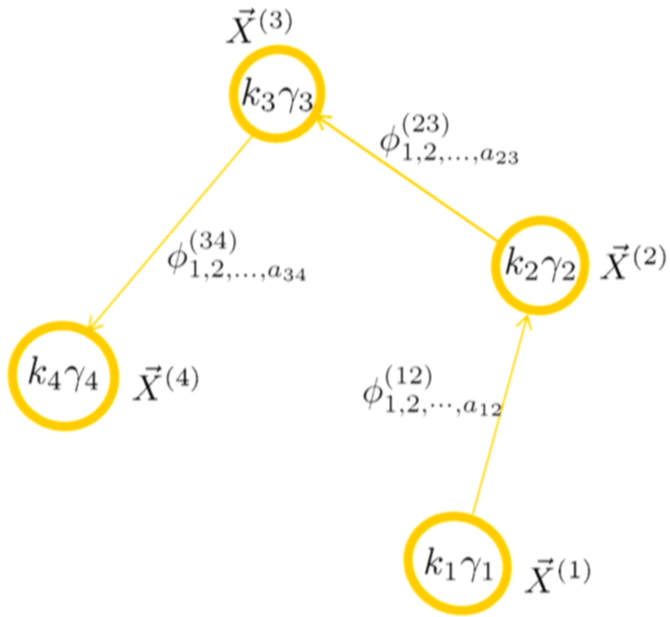


Higgs phase : $\mathcal{M}_{\mathrm{H}} = \{\phi^{(12)}, \ldots | D^{(i)} = \xi^{(i)} 1_{k_i \times k_i}\} / \prod_i U(k_i)$



$$\Omega_{\mathrm{Higgs}}\left(\sum_i k_i\gamma_i; \xi^{(i)}\right)$$

$$\sim \chi\left(\mathcal{M}_{\mathrm{H}}\right)$$

$$= \sum_l (-1)^l \dim\left[H^l\left(\mathcal{M}_{\mathrm{H}}\right)\right]$$

Higgs phase : $\mathcal{M}_{\rm H} = \{\phi^{(12)}, \dots | D^{(i)} = \xi^{(i)} 1_{k_i \times k_i}\} / \prod_i U(k_i)$

marginal stability wall

$\xi^{(1)} > \xi^{(2)} > \xi^{(3)} > \xi^{(4)}$

$\chi(\mathcal{M}_{\rm H}) = 0$

$\chi(\mathcal{M}_{\rm H}) = a_{12} \times a_{23} \times a_{34}$

$a_{ik} = \langle \gamma_i, \gamma_k \rangle$

$$\mathcal{M}_{\rm H}\left(\sum_i \gamma_i ; \xi^{(i)}\right) = 0$$

$$\mathcal{M}_{\rm H}\left(\sum_i \gamma_i ; \xi^{(i)}\right) = CP^{a_{12}-1} \times CP^{a_{32}-1} \times CP^{a_{34}-1}$$

large & positive
FI constants

small & positive
FI constants

$$\Omega_{\text{Higgs}} = \Omega_{\text{Coulomb}}$$

F. Denef 2002 + A. Sen 2011

large & positive
FI constants

small & positive
FI constants

$$\Omega_{\rm Higgs} = \Omega_{\rm Coulomb}$$

F. Denef 2002 + A. Sen 2011

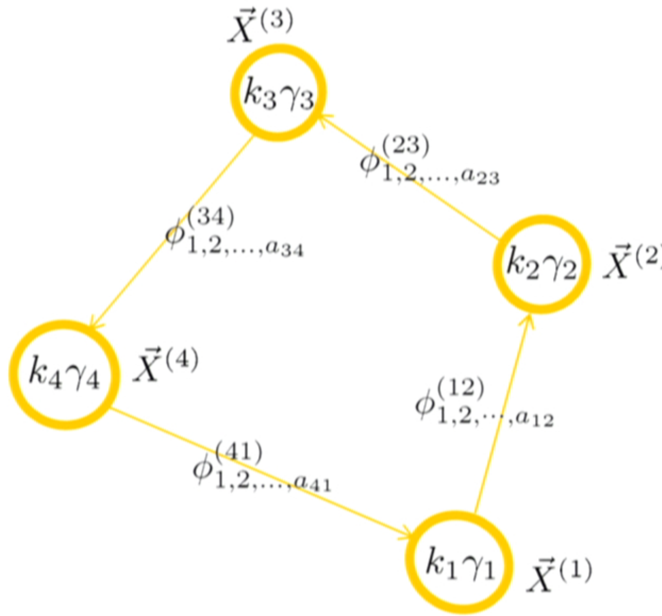
?

which apparently fails for some quivers with loops

Denef + Moore 2007

$$|\Omega_{\text{Higgs}}| \;\geq\; |\Omega_{\text{Coulomb}}|$$

$$(\,|\Omega_{\text{Higgs}}| \;>>\; |\Omega_{\text{Coulomb}}|\,)$$



$$W(\phi) = \mathrm{tr}\left[\phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}\right]$$

also, all known wall-crossing formulae need input data

$$\Omega^+(\gamma) = \Omega^-(\gamma)$$

how to count & figure out these wall-crossing safe states ?

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$$\Omega^+(\gamma) = \Omega^-(\gamma)$$

how to count & figure out these wall-crossing safe states ?

example 1 : elementary objects such as certain 2r+f hypermultiplet dyons in Seiberg-Witten theory of rank r and f flavors

example 2 : single-center black holes

$$\Omega^{+} = \boxed{\Omega_{\text{Invariant}}} + \Omega^{+}_{\text{Coulomb}}$$

$$\Omega^{-} = \boxed{\Omega_{\text{Invariant}}} + \Omega^{-}_{\text{Coulomb}}$$

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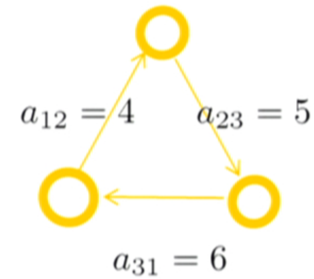
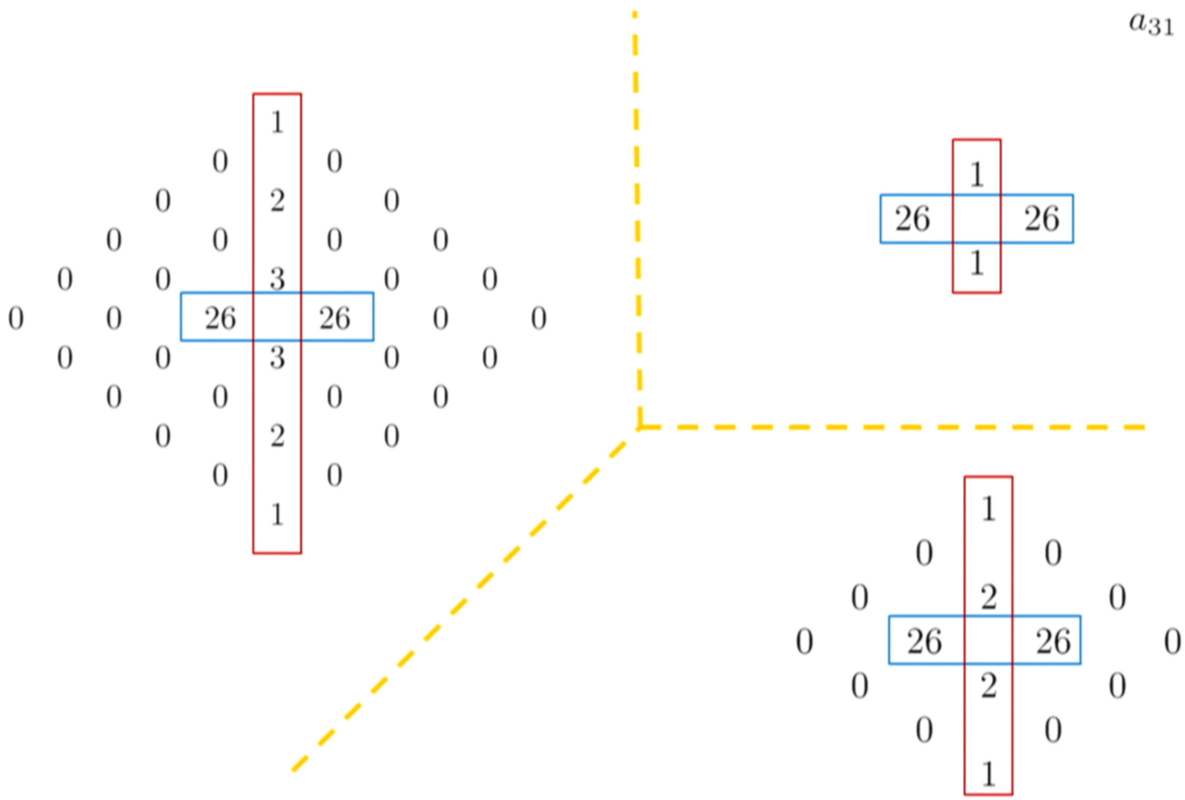
$$\Omega^+ = \boxed{\Omega_{\text{Invariant}}} + \Omega^+_{\text{Coulomb}}$$

$$\Omega^- = \boxed{\Omega_{\text{Invariant}}} + \Omega^-_{\text{Coulomb}}$$

an Abelian example

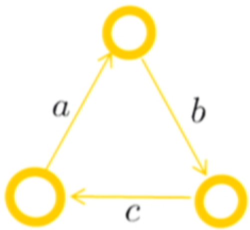
$$\dim H^{(p,q)}(\mathcal{M}_H)$$





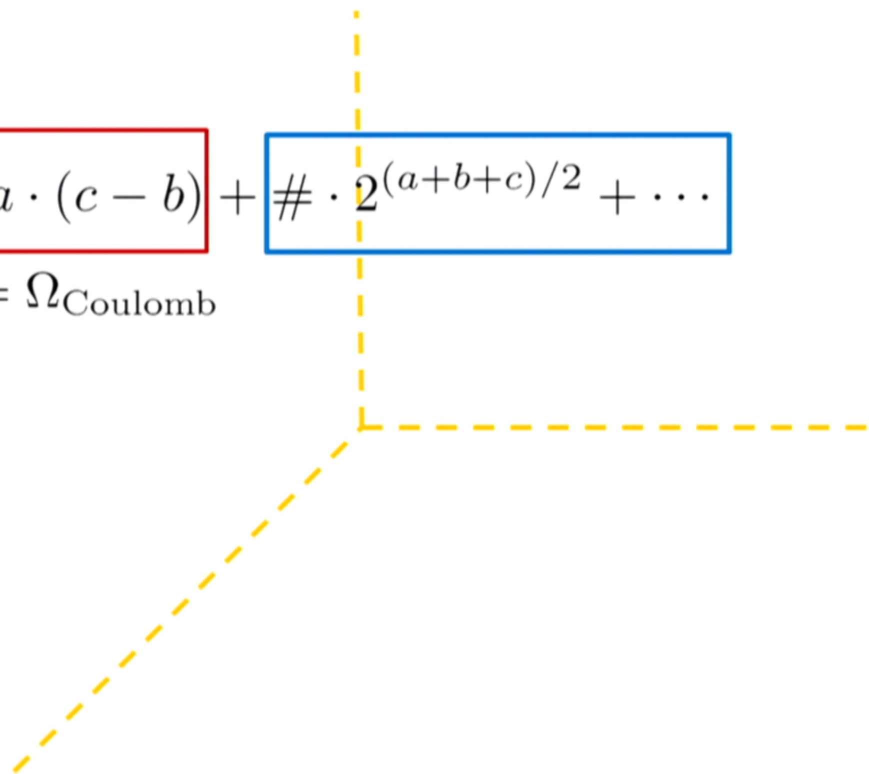
an Abelian example

Denef + Moore 2007



$$\chi(\mathcal{M}_H) \sim a \cdot (c - b) + \# \cdot 2^{(a+b+c)/2} + \cdots$$

$$a \cdot (c - b) = \Omega_{\text{Coulomb}}$$



$$|\Omega_{\text{Higgs}}| \;\geq\; |\Omega_{\text{Coulomb}}|$$

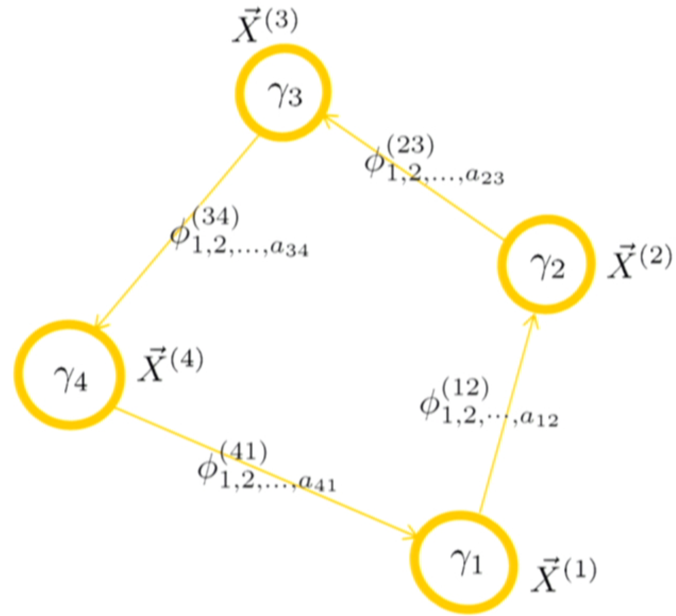
$$\{\phi^{(12)}, \ldots\}$$

$$\Big\downarrow \; D \;\sim\; // \prod_i U(1)_i$$

$$X_{\text{H}}$$

$$\Big\downarrow \; F \;\sim\; \partial_\phi W = 0$$

$$\mathcal{M}_{\text{H}}$$



$$W(\phi) \sim \phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}$$

two conjectures

S.J. Lee + Z.L. Wang + P.Y., 2012
cf) Bena + Berkooz + de Boer + El-Showk + d. Bleeken, 2012

$\{\phi^{(12)}, \ldots\}$ $\quad H^*(\mathcal{M}_H)$

$\downarrow D$ $\quad = \; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$

X_{H}

complete intersection $\quad \uparrow i \;\sim\; \partial_\phi W = 0$

$\mathcal{M}_{\mathrm{H}}$

two conjectures

$\{\phi^{(12)}, \ldots\}$ $\quad H^*(\mathcal{M}_H) = \sum H^{(p,q)}(\mathcal{M}_H)$

$\downarrow D$ $\quad = i^*[H^*(X_H)] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$

X_{H} $\quad \text{tr}_{i^*(H(X))}(-1)^{p+q-d}y^{2p-d} \quad \text{tr}_{\text{Intrinsic}}(-1)^{p+q-d}y^{2p-d}$

complete intersection $\uparrow i$ $\quad \Updownarrow \quad \Updownarrow$

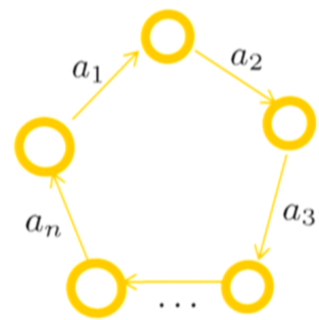
$\mathcal{M}_{\text{H}}$ $\quad \Omega_{\text{Coulomb}} \quad \Omega_{\text{Invariant}}$

S.J. Lee + Z.L.Wang + P.Y., 2012
cf) Bena + Berkooz + de Boer + El-Showk + d. Bleeken, 2012

the total equivariant index of a cyclic Abelian quiver is

$$\Omega^{(k)}_{\text{Higgs}}(y) = \text{tr}_{H^*(\mathcal{M}^{(k)}_H)}(-1)^{2J_3}y^{2J_3+2I} = \sum(-1)^{p+q-d}y^{2p-d}h^{(p,q)}(\mathcal{M}^{(k)}_H)$$

$$= (-y)^{-d_k}\chi_{t=-y^2}(\mathcal{M}^{(k)}_H)$$



and can be decomposed into two parts

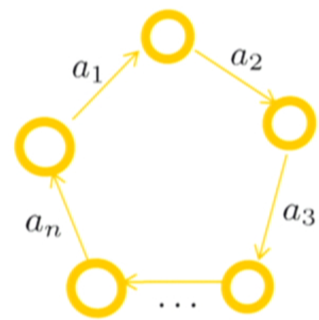
$$\Omega^{(k)}_{\text{Higgs}}(y) = \boxed{(-1)^{d_k} y^{a_k} \prod_{i \neq k} \frac{y^{a_i} - y^{-a_i}}{y - y^{-1}} + \Delta\Omega_{\{a_i\}}(y)}$$

$$+ \boxed{\frac{(-y)^{n+2-\sum_i a_i}}{(y^2-1)^n} \prod_i \oint_{\omega_i = 1} \frac{d\omega_i}{2\pi i} \left[\prod_i \left(\frac{1 - y^2 \omega_i}{1 - \omega_i} \right)^{a_i} \right] \cdot \frac{1}{1 - y^2 \prod_i \omega_i} - \Delta\Omega_{\{a_i\}}(y)}$$

$$X^{(k)}_{\text{H}} = \prod_{i \neq k} CP^{a_i - 1}$$

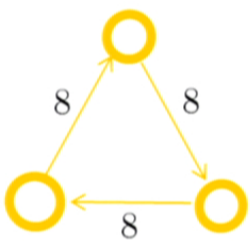
embedding map $\uparrow$ i

$$\mathcal{M}^{(k)}_{\text{H}}$$



wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$



$\dim H^{(p,q)}(\mathcal{M}_H)$

						1						
					0		0					
				0		2		0				
			0		0		0		0			
		0		0		3		0		0		
	0		0		0		0		0		0	
1		322		6803		4 + 18212		6803		322		1
	0		0		0		0		0		0	
		0		0		3		0		0		
			0		0		0		0			
				0		2		0				
					0		0					
						1						

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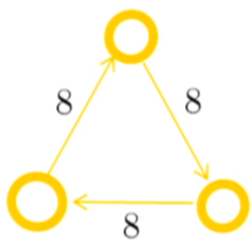

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						1						
					0		0					
				0		2		0				
			0		0		0		0			
		0		0		3		0		0		
	0		0		0		0		0		0	
1		322		6803		4 + 18212		6803		322		1
	0		0		0		0		0		0	
		0		0		3		0		0		
			0		0		0		0			
				0		2		0				
					0		0					
						1						

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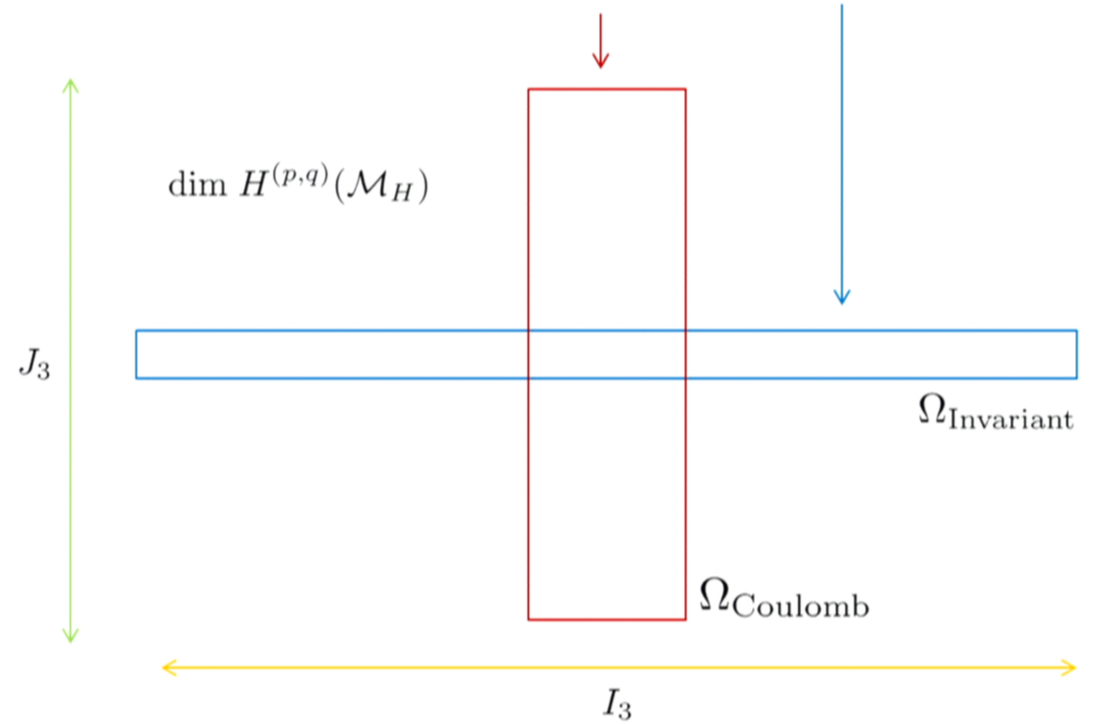

wall-crossing states vs. wall-crossing-safe states

Seung-Joo Lee + Zhao-Long Wang + P.Y., 2012

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Updownarrow \qquad \mathrm{tr}(-1)^{p+q-d}y^{2p-d} \qquad \Updownarrow$$

Ω_{Coulomb} | $\Omega_{\text{Invariant}} = \Omega_{\text{Higgs}} - \Omega_{\text{Coulomb}}$

many-body bound states
wall-crossing

single-center states
wall-crossing-safe

angular momentum | R-charge

summary

1. wall-crossing formulae from direct & subtle real-space index for SW BPS dyons with ab initio low energy dynamics or more generally, for the Coulomb phase of BPS quivers

2. equivalence to Kontsevich-Soibelman (when $\Omega_{\mathrm{Invariant}} = 0$) KS rational invariants from statistics orbifolding

3. quiver invariants, or wall-crossing safe BPS states; interpretation via single-center black hole entropy

4. quiver invariants for non-Abelian quivers with scaling regime → difficulties (again) on the Coulomb side