Title: Quantum Gravity as Random Geometry

Date: Feb 27, 2013 02:00 PM

URL: http://pirsa.org/13020132

Abstract: Matrix
models, random maps and Liouville field
theory are prime tools which connect

random
geometry and quantum gravity in two dimensions. The
tensor track is a new program to extend this
connection to higher dimensions through

the
corresponding notions of tensor models, colored
triangulations and tensor group field theories.

Quantum Gravity as Random Geometry

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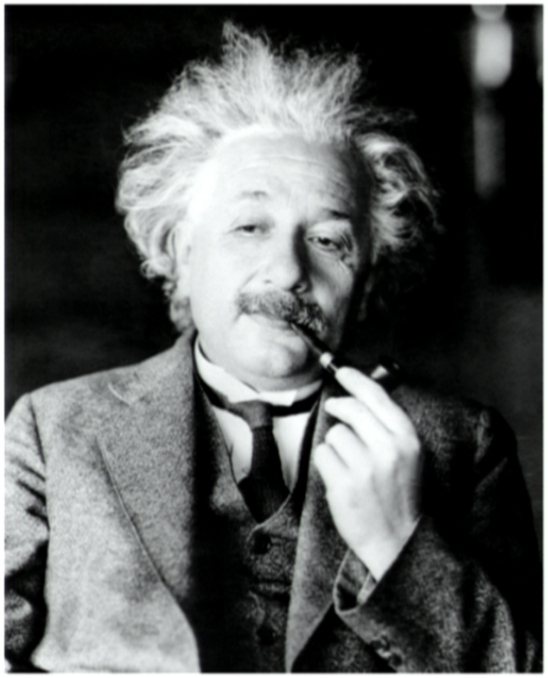
Laboratoire de Physique théorique
Université Paris-Sud
and Perimeter Institute

Perimeter Institute Colloquium,
February 27, 2013

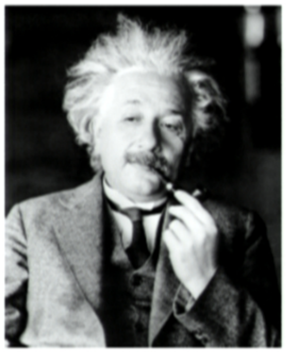
General Relativity

Einstein linked gravity to space-time curvature



In this way he gave new physical meaning to geometry.

Quantum Gravity as (Large) Random Geometry



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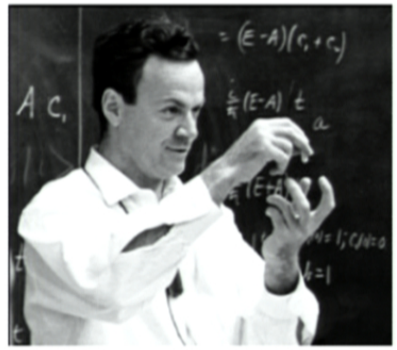

= ?

Quantizing Gravity $\simeq$ Randomizing Geometry ?

$$Z \simeq \int Dg \quad e^{i S_{EH}(g)}$$

But what is the measure Dg? On which underlying space-time? Should one also sum on topologies? How to relate quantum gravity to classical space and time as we know them?

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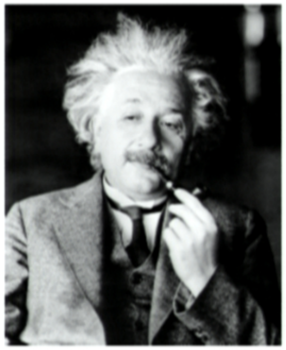
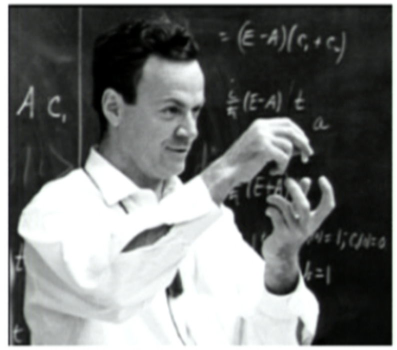
 +  = ?

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Probability and Enumerative Combinatorics

In probability theory careful counting is critical.

We feel quantizing gravity requires mathematical techniques coming from at least four main areas: quantum field theory, geometry, probability theory and enumerative combinatorics.

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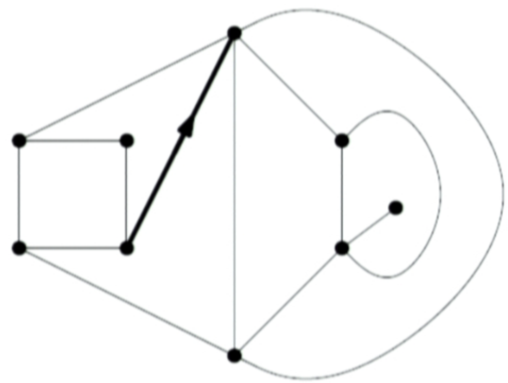
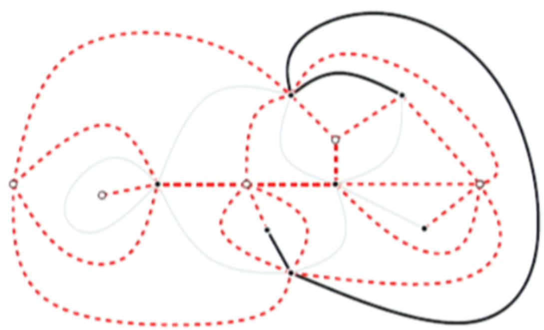
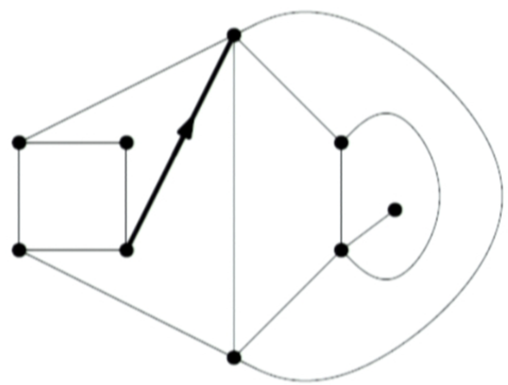


Rooted planar quadrangulations are simple objects



They are quite universal...



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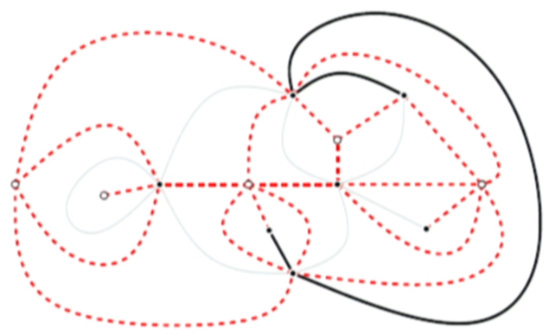

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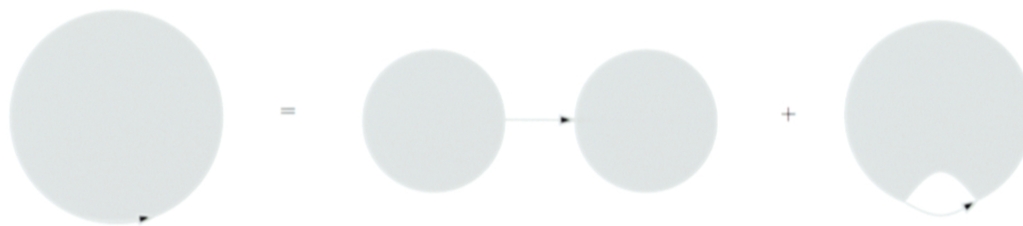
Counting Planar Graphs à la Tutte (1963)

Q_n= number of rooted planar quadrangulations with n faces

Adding boundaries Tutte found in 1963 a quadratic recursive equation (à la Polchinski).



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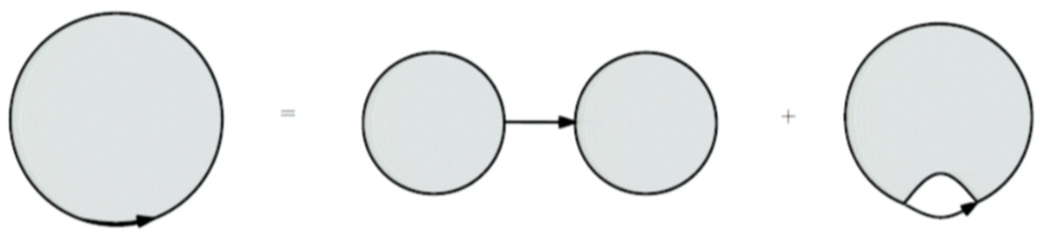

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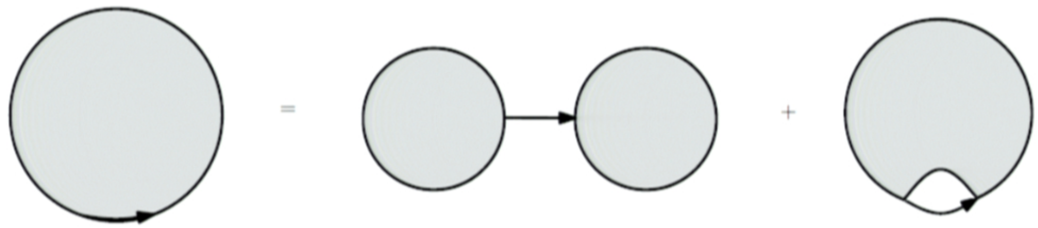

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Counting Planar Graphs à la 'tHooft and Brezin-Itzykson-Parisi-Zuber (1978)

Why planar quadrangulations?

QFT answer: because they are dual to the Feynman graphs which dominate the $1/N$ expansion of a matrix model...

$$Z = \int dM \exp(-\frac{1}{2}\mathrm{Tr}M^2 + \frac{\lambda}{N}\mathrm{Tr}M^4)$$
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The Cori-Vauquelin-Schaeffer Map

The connection with random metrics and their (random) geodesics remained obscure. Recent progress came from better combinatoric counting.

Plane trees are well counted by Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

$$(n+2)Q_n = 2 \cdot 3^n C_n . \quad .$$

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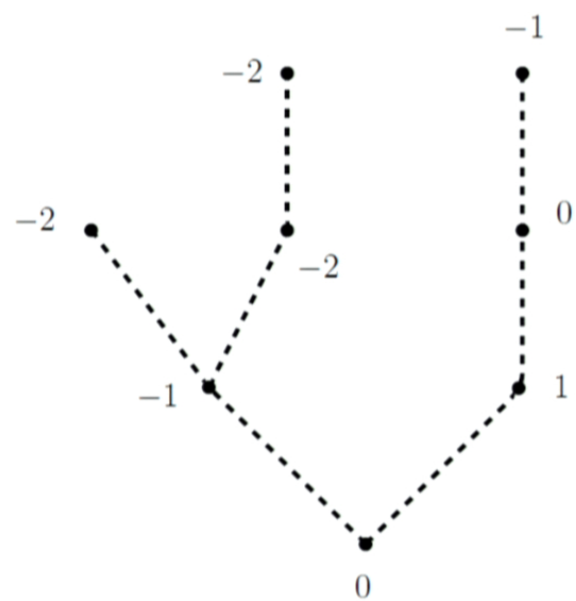

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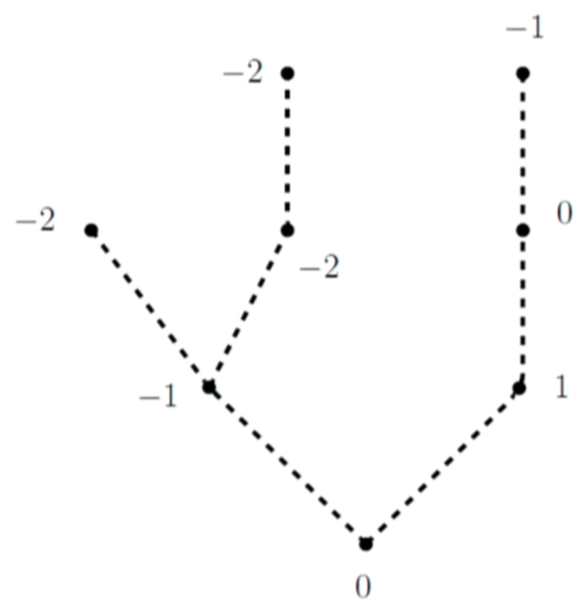

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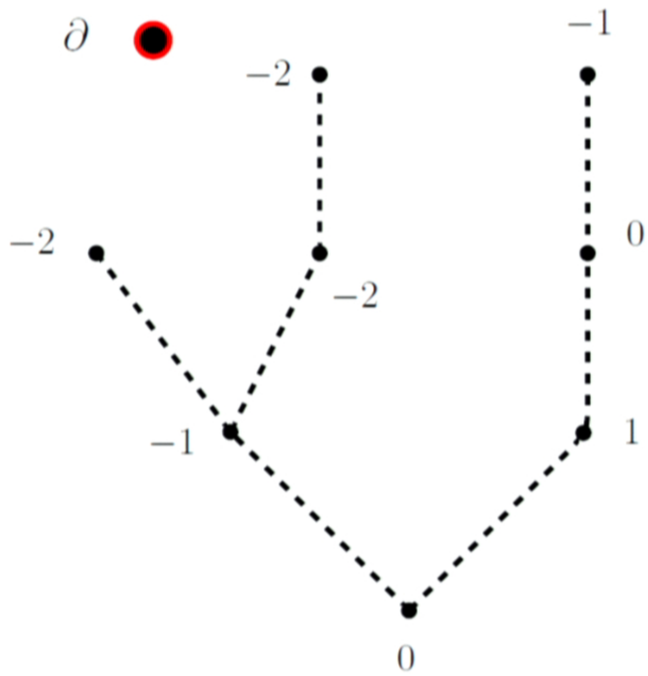

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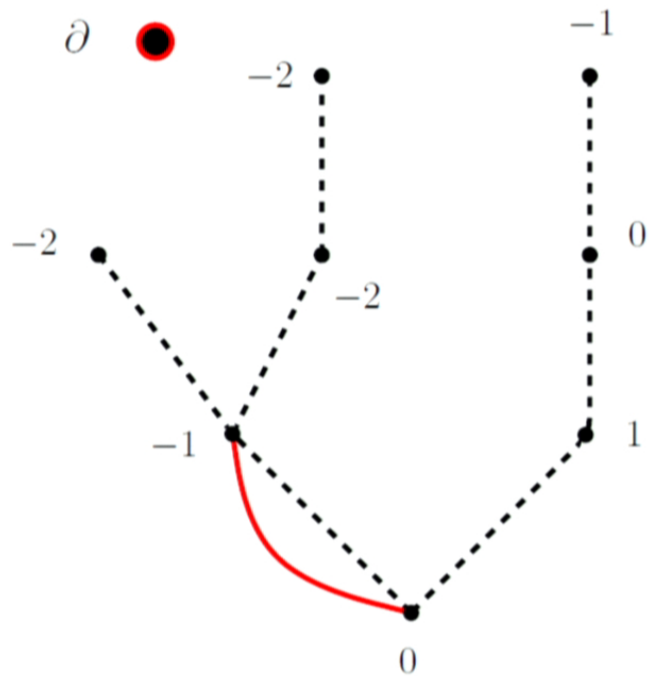
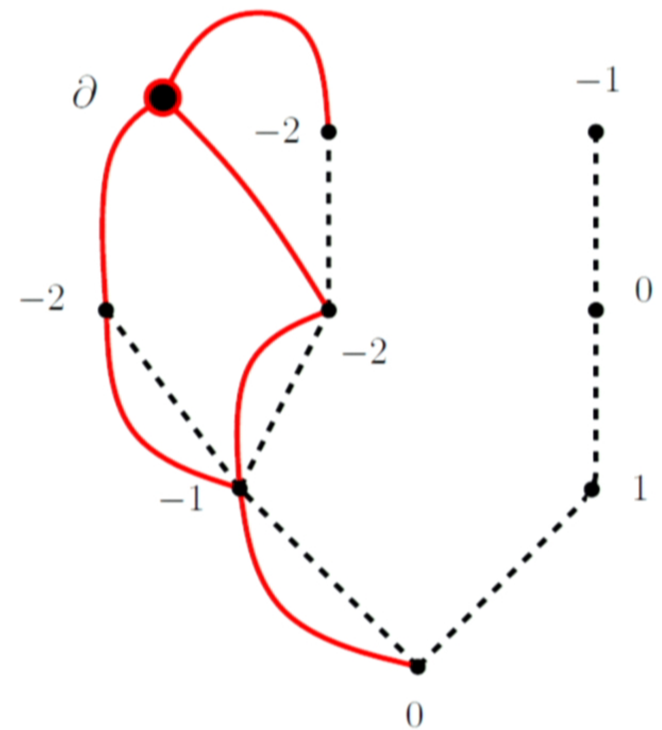


Large Quadrangulations



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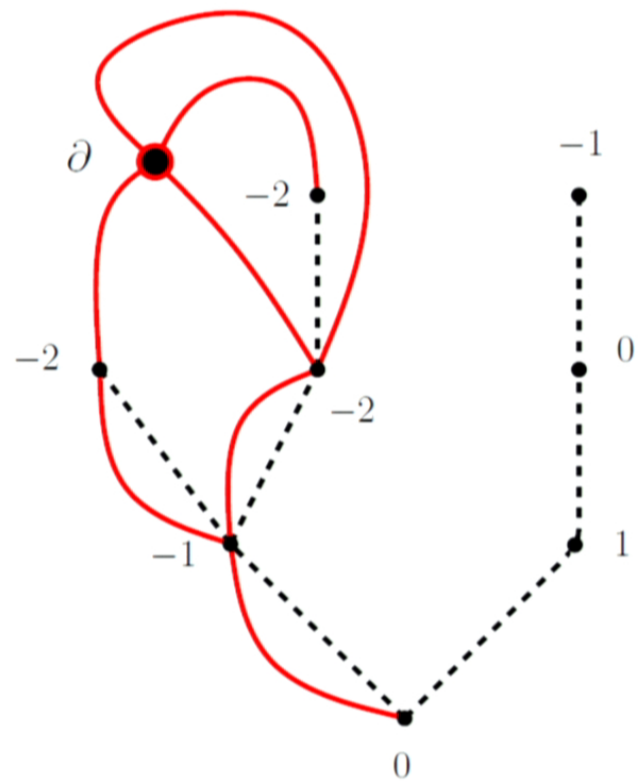

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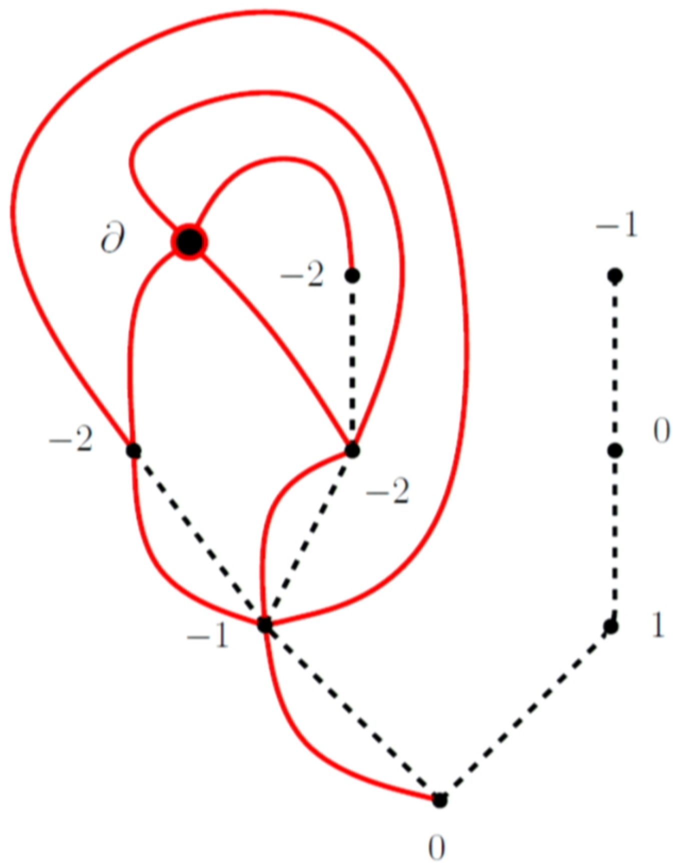

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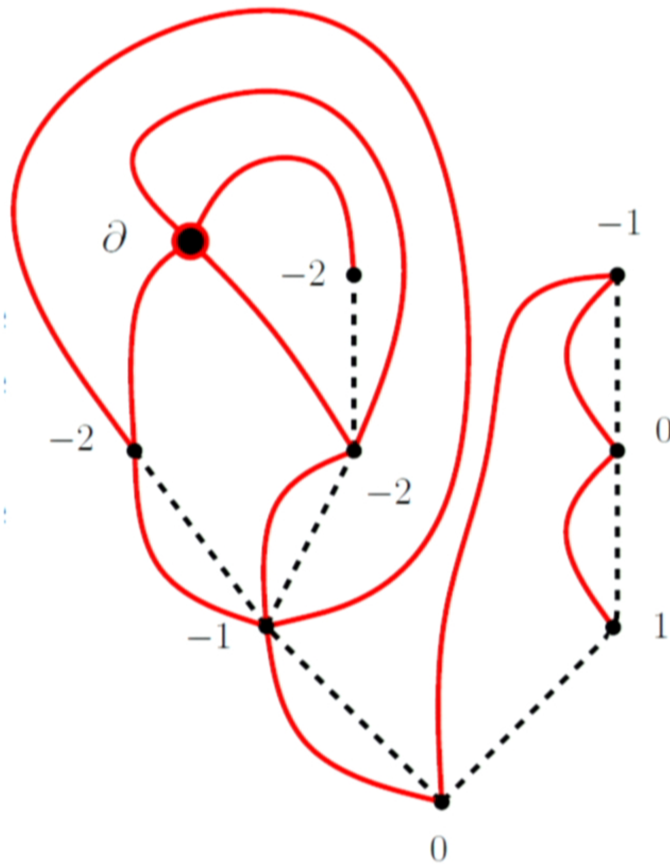

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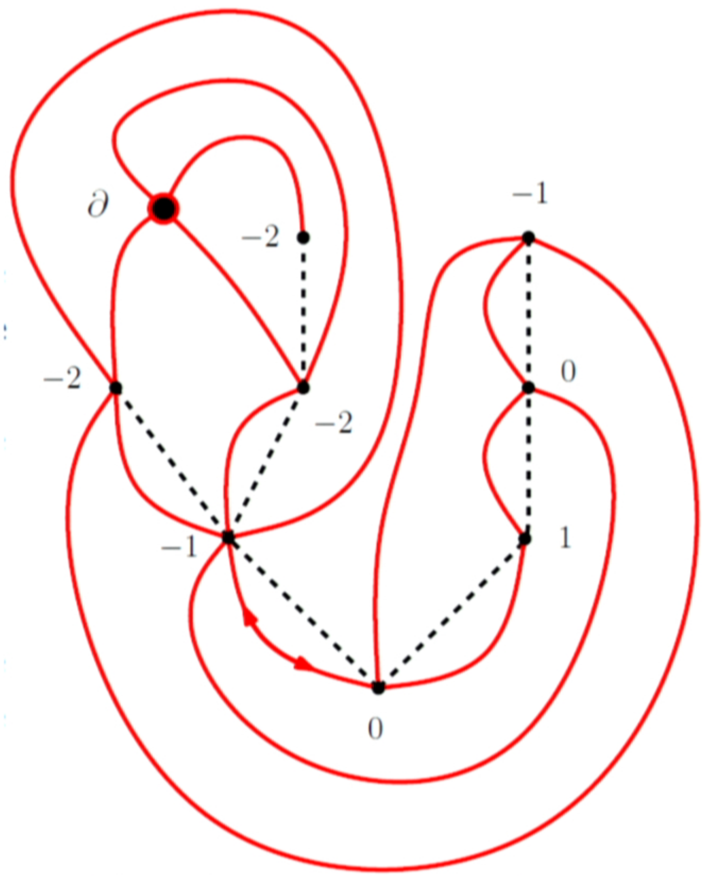


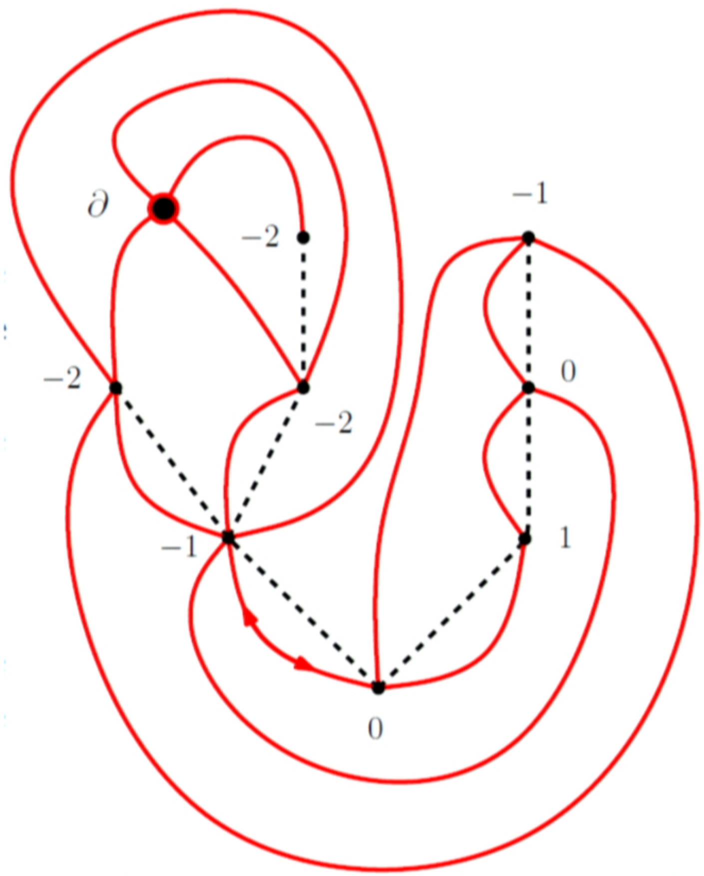
2D Random Geometry à la Le-Gall-Miermont

Theorem (Le Gall, Miermont (2007-2011)

Equidistributed planar quadrangulations of order n converge after rescaling the graph distance by $n^{-1/4}$ (in the Gromov-Hausdorff sense), towards a universal random compact space, called the brownian 2-sphere.

This space has Hausdorff dimension 4 and is almost surely homeomorphic to the two-dimensional sphere.

Large Quadrangulations



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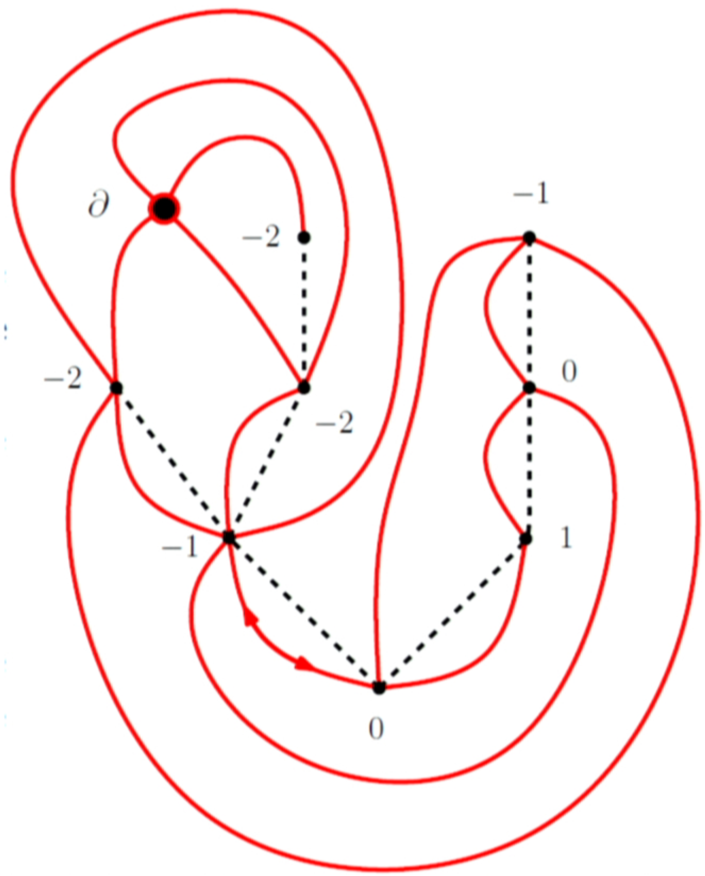
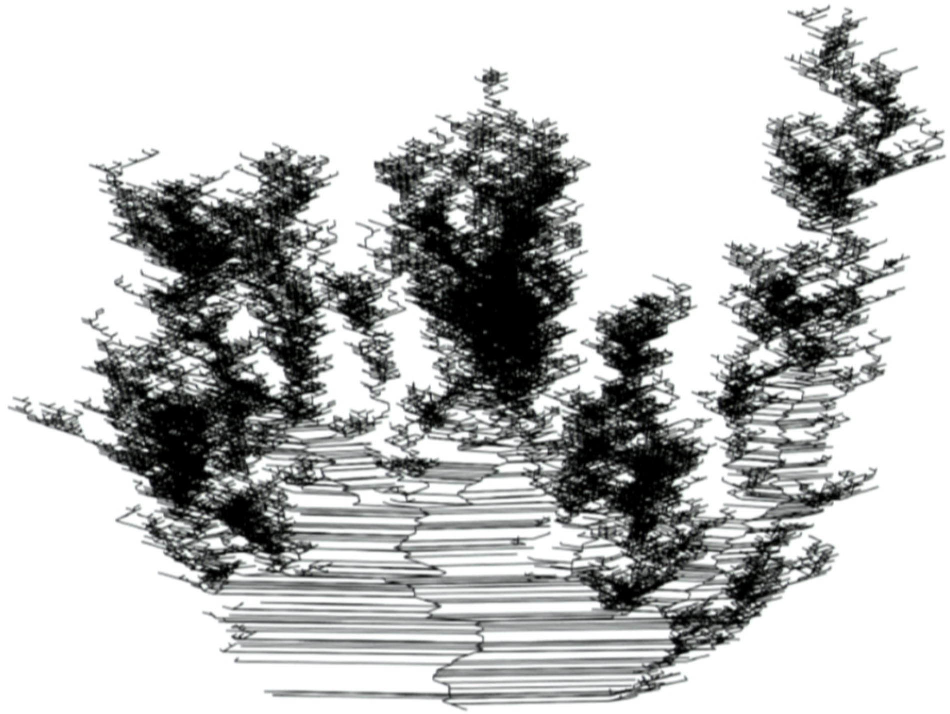


A Look at Large Random Quadrangulations



The Probabilist's View: The Brownian Snake, Part Profile

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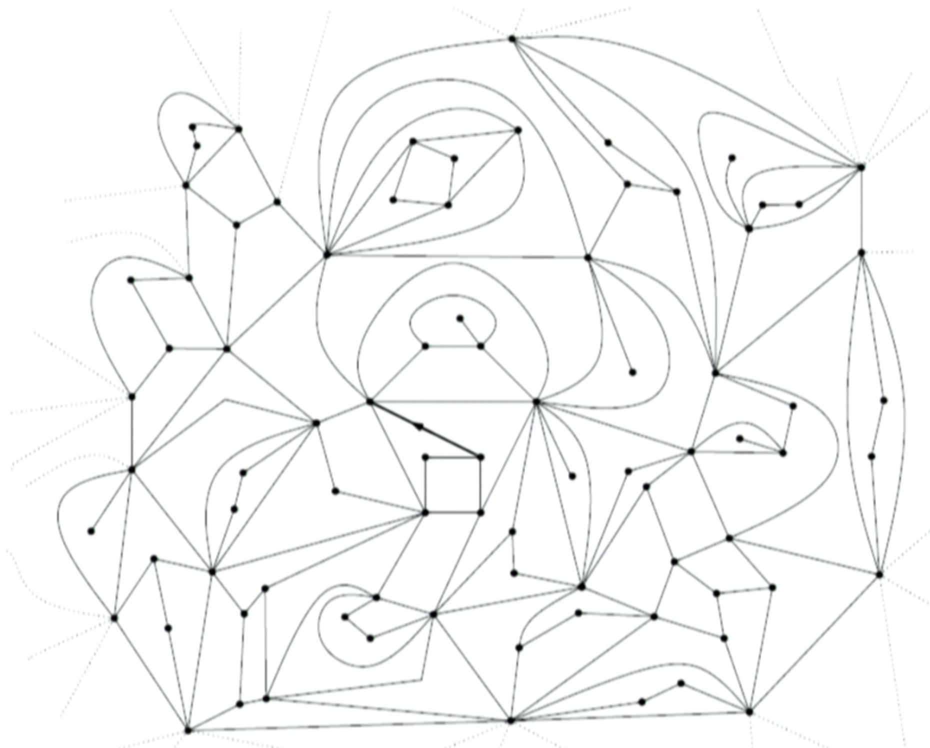
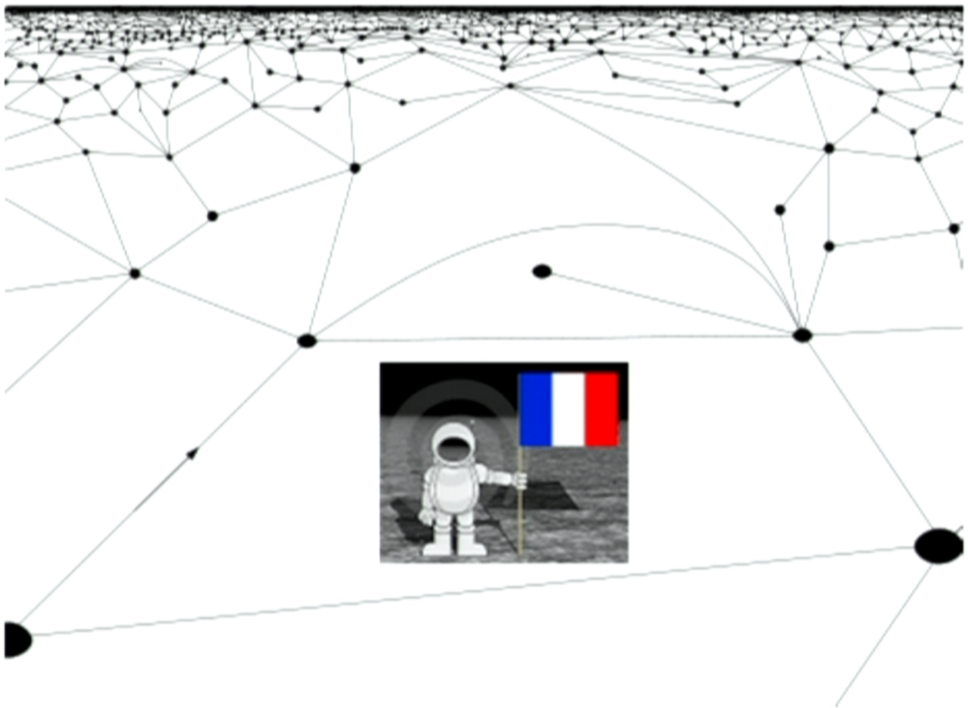


The Topological View

A Look at Large Random Quadrangulations



Landing on the Brownian sphere

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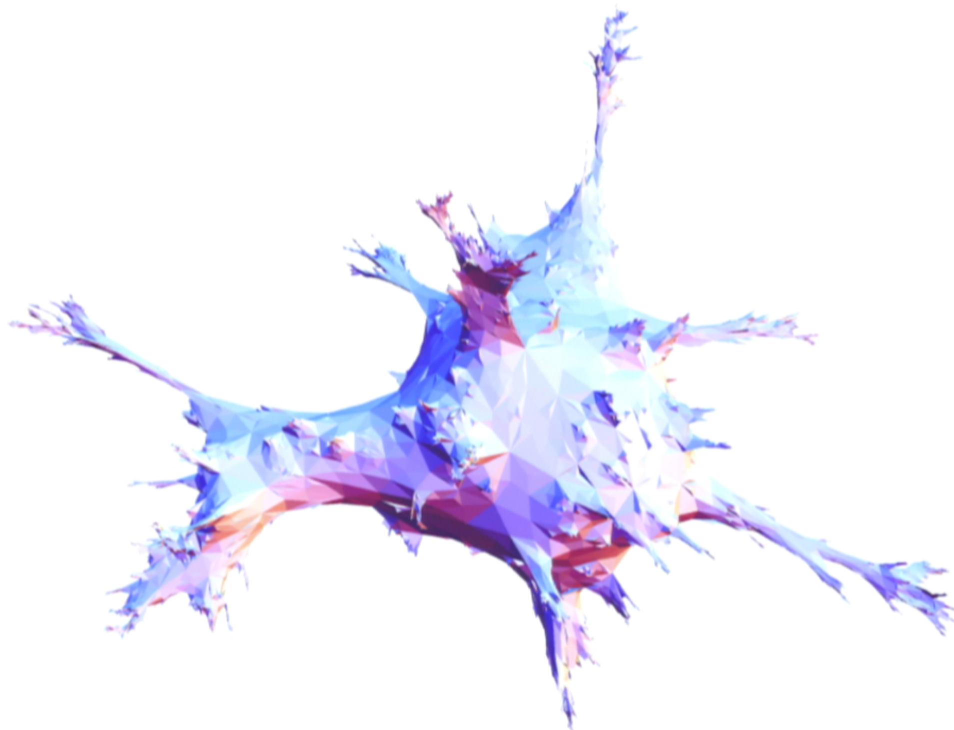

Artist's view in 3D (Courtesy: Marckert)

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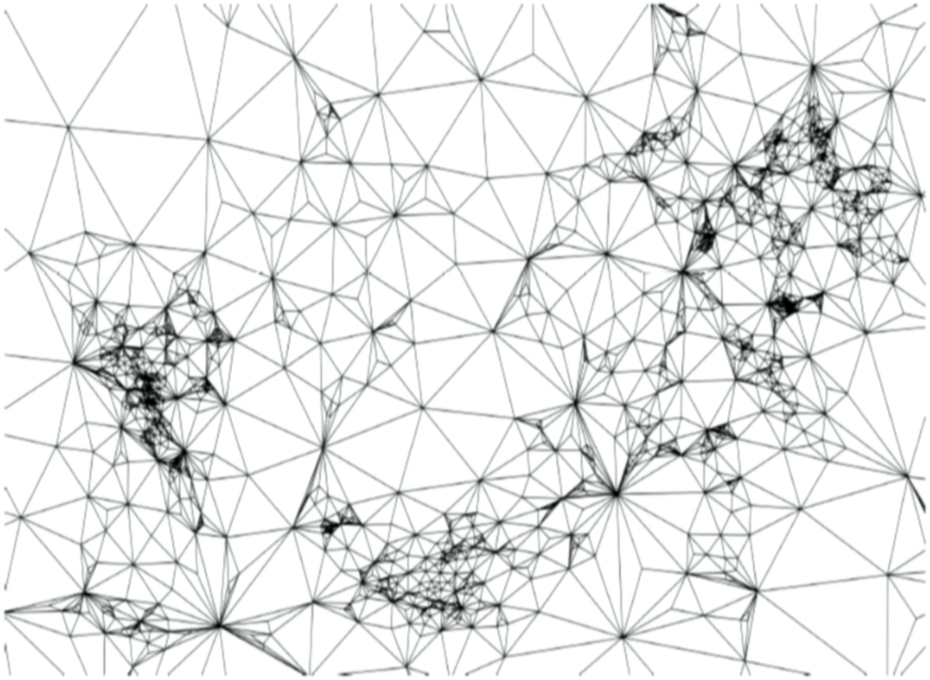
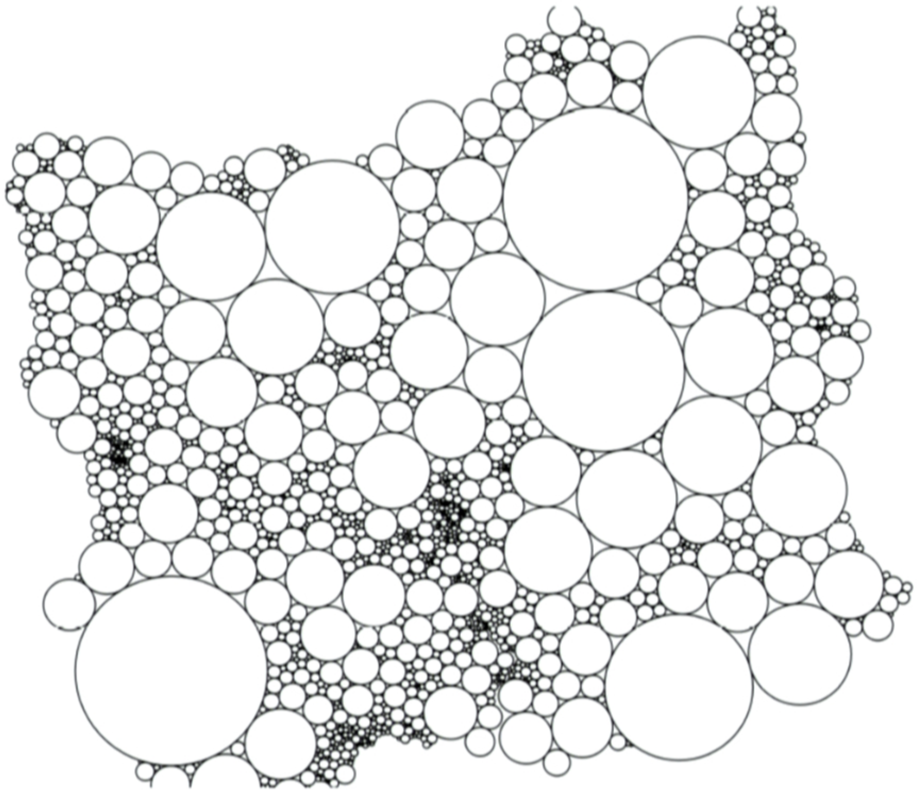


Uniformized Through Riemann Mapping Theorem

A Look at Large Random Quadrangulations



Using the Circle Packing Theorem (Courtesy: Krikun)

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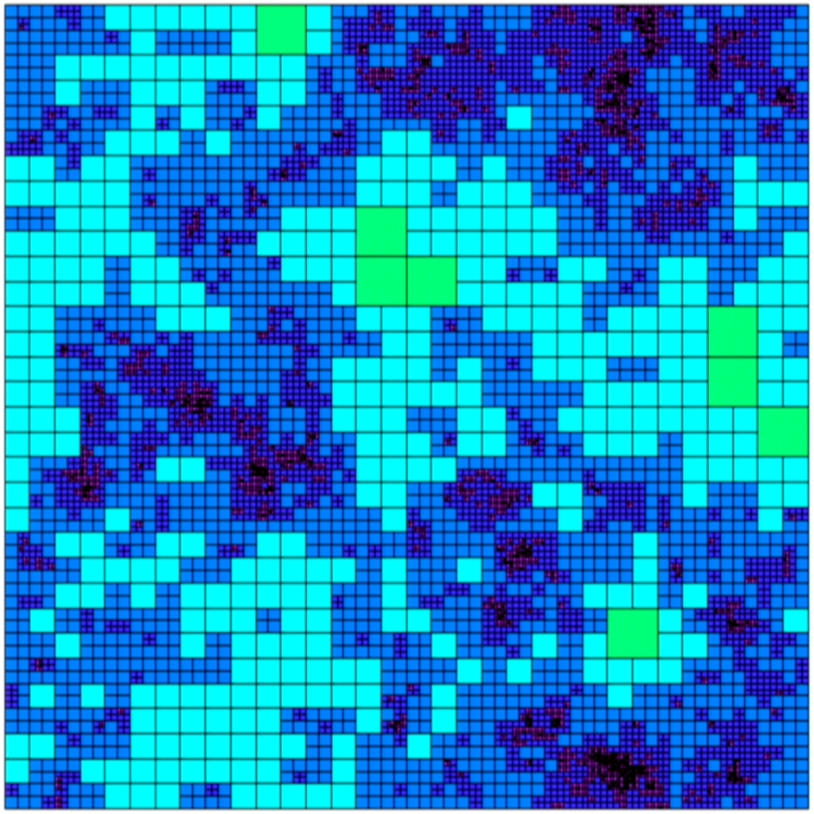
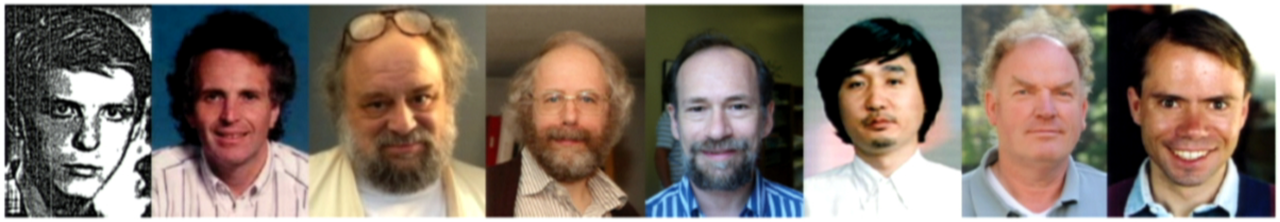


The Liouville Theory (Courtesy: Duplantier)

2D Random Geometry à la KPZ-DDK-DS (1984-2011)



There exists a relationship between critical exponents x and Δ of matter on a fixed (x) and on a random (Δ) geometry.

$$x = \frac{\gamma^2}{4}\Delta^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta$$

The matter type is characterized by a number $\gamma \in [0, 2[$, related to the Schramm-Loewner evolution parameter κ through $\gamma = \sqrt{\min(\kappa, 16/\kappa)}$, and to the central charge $c = \frac{(8-3\kappa)(\kappa-6)}{2\kappa}$ (for Ising, $c = 1/2, \kappa = 3, \gamma = \sqrt{3}$).

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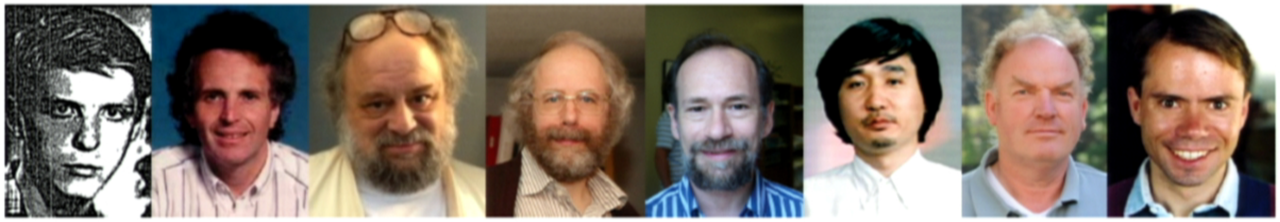

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Lessons to draw

2d Random geometry can be based on the careful counting of large triangulations or on the continuum (Liouville) picture. The two pictures should be equivalent, but the first one is particularly convincing from a conceptual point of view.

Random 2d planar geometry can be interpreted as trees or branched polymers equipped with fluctuation fields (the labels). These fields generate space-time shortcuts which change the Hausdorff dimension from 2 to 4.

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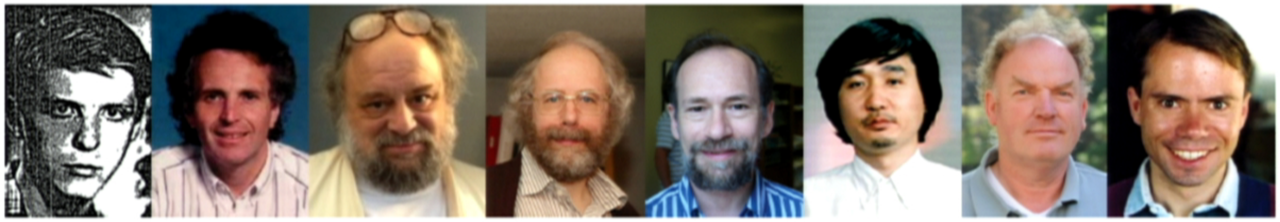

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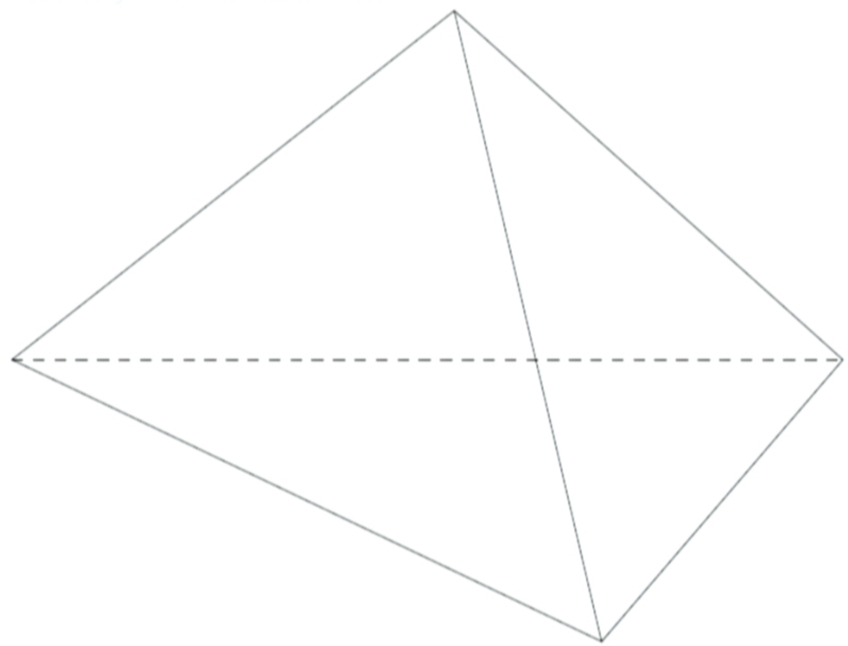
Any d-dimensional triangulation uniquely defines a $(d-1)$ vertex-colored triangulation, its barycentric subdivision.



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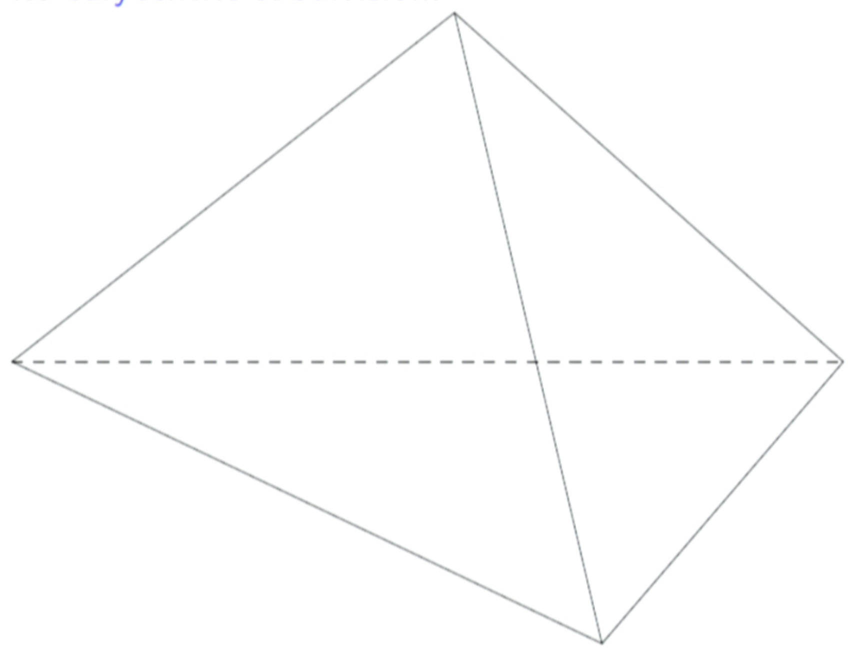

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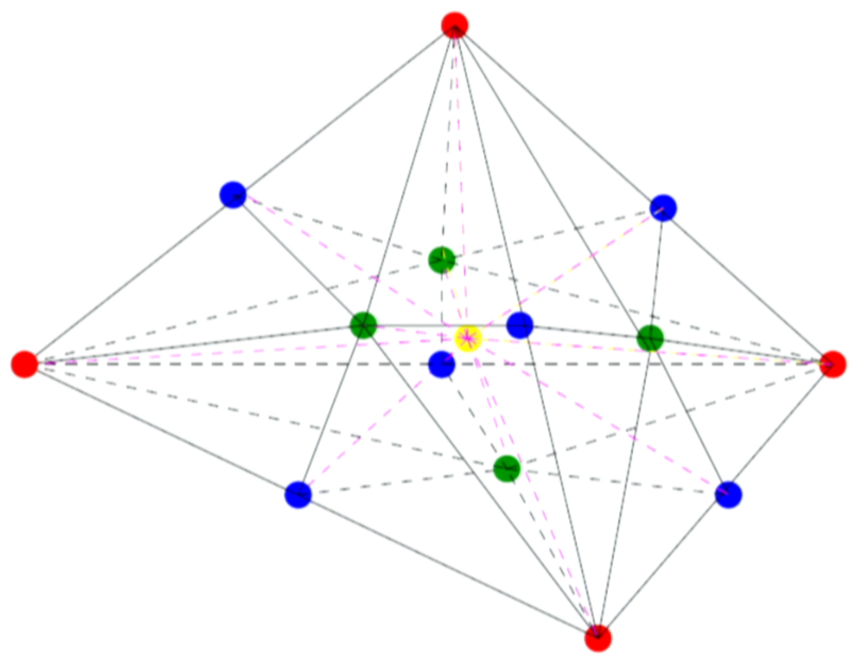

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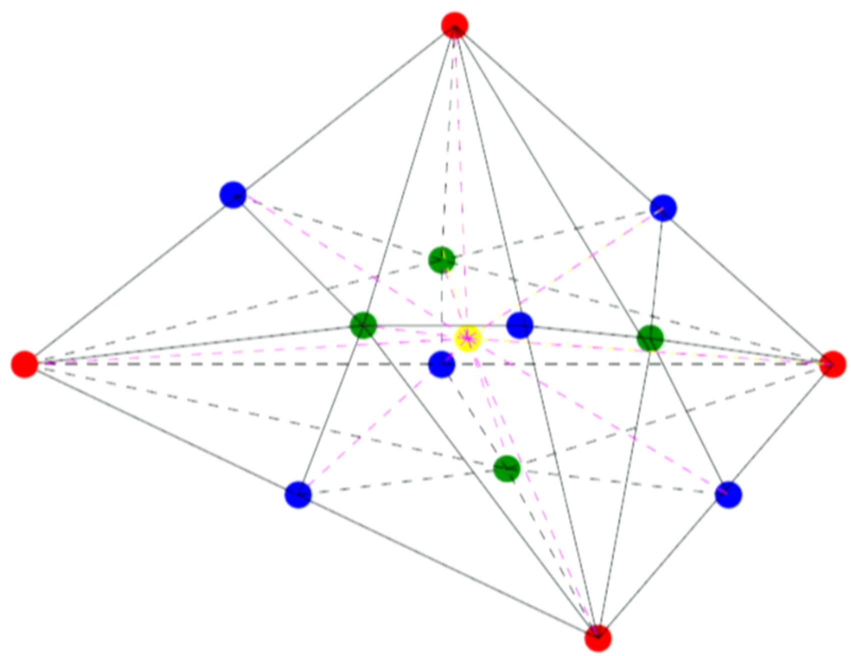

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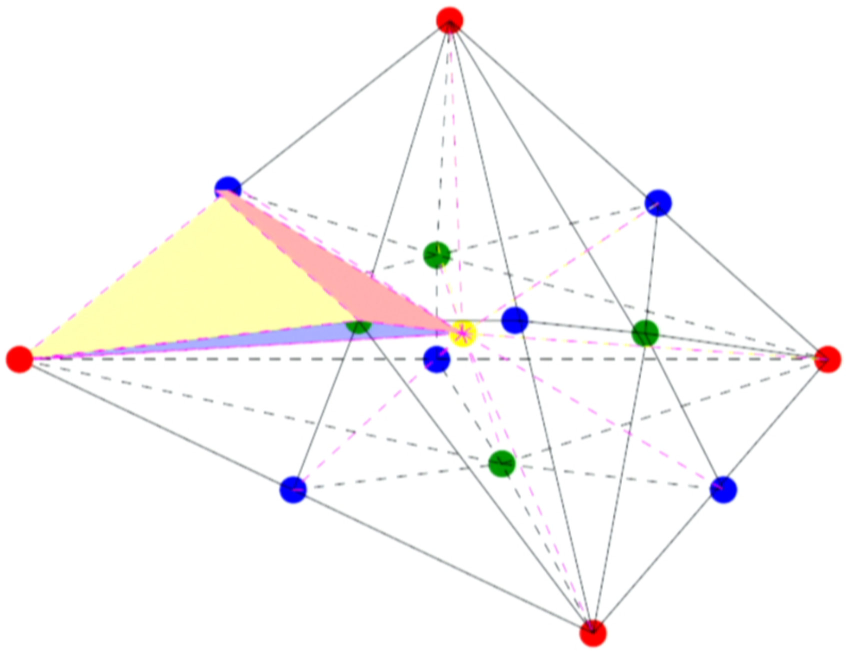

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Random Tensors

Vector Models are probability measures for random vectors of size N.

Matrix models are probability measures for N by N random matrices M.

Tensor models are probability measures for tensors of higher rank $D > 2$, with eg N^D coefficients.

Universal properties when N gets large stem from the existence of a $1/N$ expansion.

There is an algebraic link between random (unsymmetrized) tensors of rank D and $D + 1$ colored triangulations, namely classical invariant theory.

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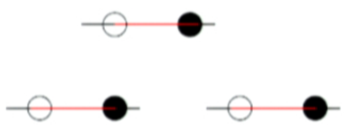

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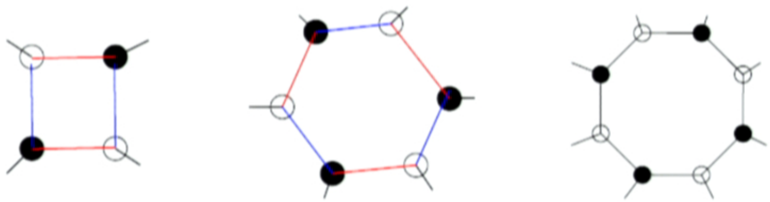

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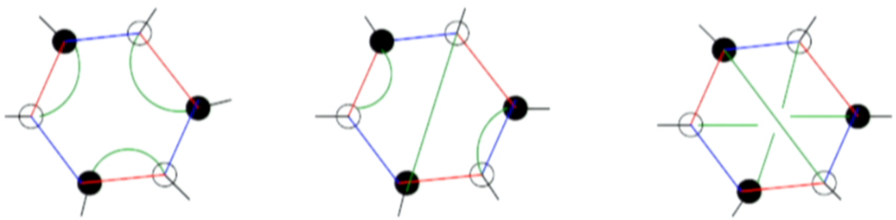

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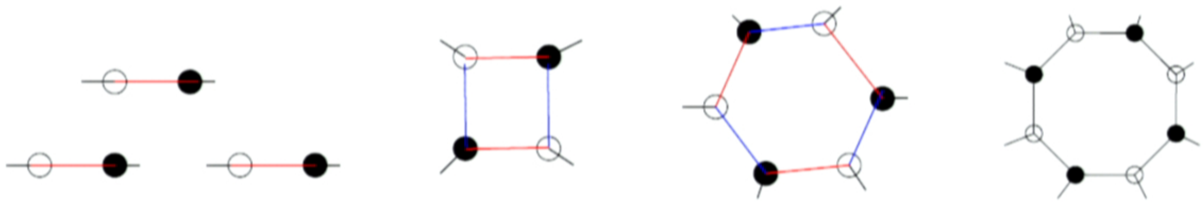

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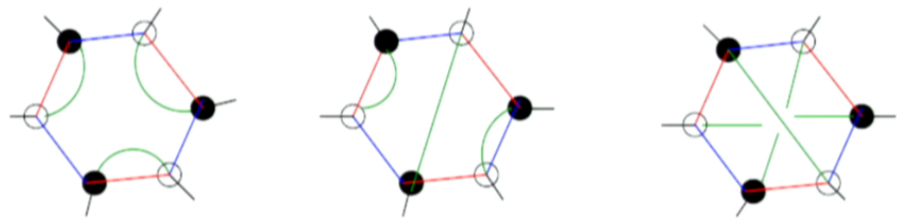

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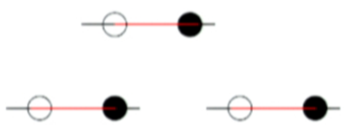

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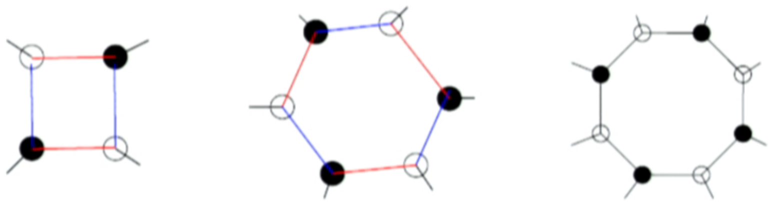

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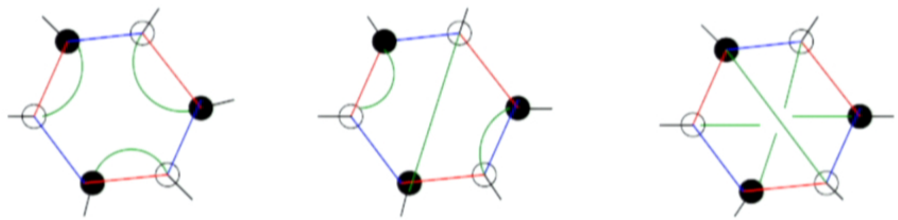
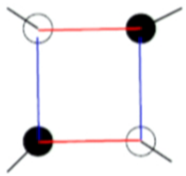


Tensor Invariants

Invariants, II

The algebraic invariants associated to the vector and matrix drawings are

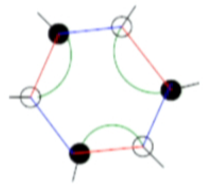
$$= \sum_i \bar{\phi}_i \phi^i$$



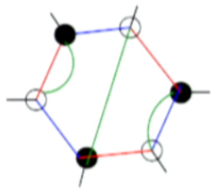
$$= \sum_{i,j,k,l} \bar{M}_{ij} M^{ik} \bar{M}_{lk} M^{lj} = Tr\,[\ M^\dagger M M^\dagger M\ ]$$

Invariants, III

The algebraic invariants associated to the tensorial drawings are



$$= \sum_{i,j,k,l,m,n,p,q,r} \bar{T}_{ijp} T^{ikq} \bar{T}_{lkq} T^{lmr} \bar{T}_{nmr} T^{njp}$$



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and so on...

Random Vectors and Matrices

- iid random vectors have a Gaussian limit as $N \to \infty$ (eg in the sense of the central limit theorem);
- invariant random vector models (eg Gaussian plus invariant interactions) have a $1/N$ expansion, dominated by bubble chains;
- iid (centered) random matrices such as GUE have a Gaussian limit as $N \to \infty$, and the invariant observables such as eigenvalues converge to the Wigner-Dyson distribution;
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Random Tensors

Random tensors are best analyzed using unsymmetrized colored models. These models

- triangulate pseudo-manifolds (Gurau, 2010)
- admit a $1/N$ expansion (Gurau 2010), whose leading graphs, called melons triangulate only spheres in any dimension (Gurau, R., 2011)
- have computable phase transitions (Bonzom, Gurau, Riello, R. 2011) to a leading melonic phase of branched polymers (Gurau, Ryan, 2013)
- Matter fields can be included (Bonzom et al)
- In short: the theory of $U(N)^{\otimes D}$-invariant random tensors is universal (Gurau 2011; Bonzom, Gurau, R. 2012), and different from the theory of random matrices.

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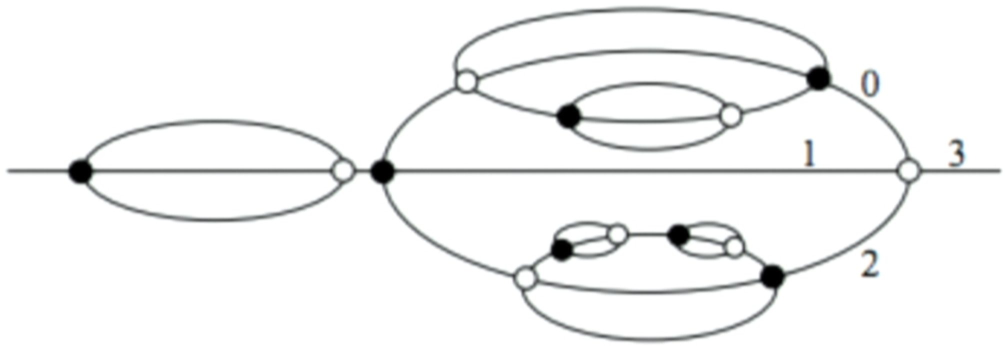
Melonic Graphs



Melonic graphs are trees (branched polymers) without random labels. But we know that a rich structure of labels hides in the sub-melonic contributions, because random tensors in particular include all random matrix models.

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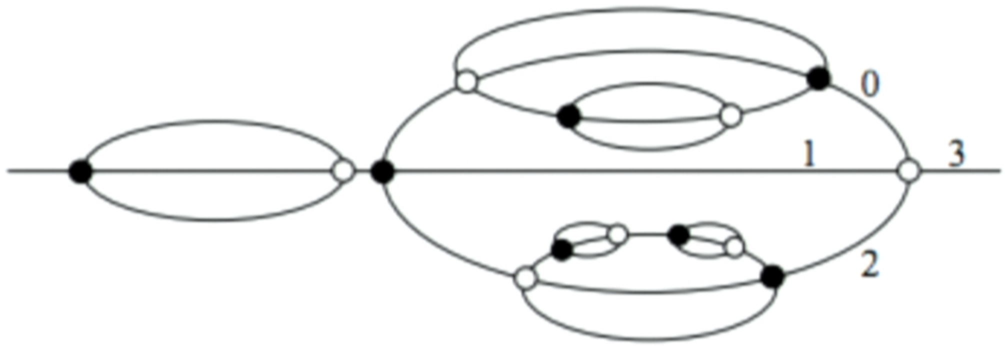

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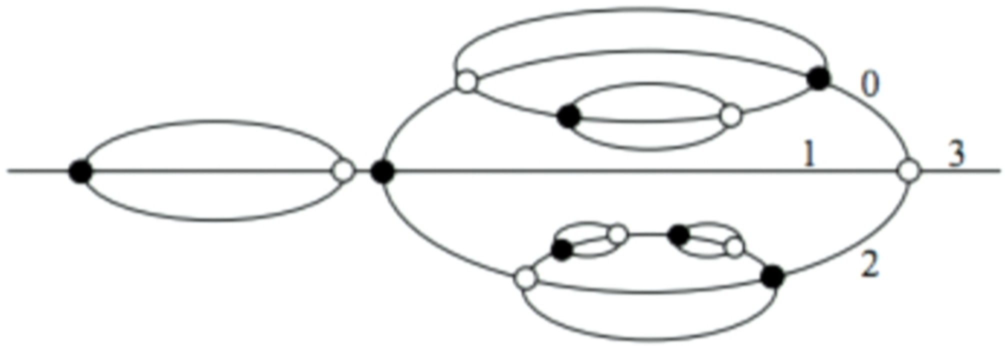

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Models/Field Theories

One could distinguish

- Invariant models, with an action fully invariant under $U(N)^{\otimes D}$. These are the analogs of ultralocal quantum field theories.
- Field theories, which have invariant interactions but a propagator which softly break this invariance. This breaking launches their renormalization group flow, just like the soft non-locality of the propagator launches ordinary renormalization group flow in ordinary quantum field theory.

Building Rules for TFT's

What could be the natural building rules for TFT's?

- Replace rotation and translation invariance by color permutation symmetry
- Replace locality by tensor invariance (use invariant interactions, but not invariant propagator)
- Natural propagator in group space: Laplacian
- Replace clustering by decay of correlation functions in the number and type of the boundary components (external legs)

Renormalization

- All standard model interactions (except gravity, until now....) are renormalizable
- Renormalizability is approximate scale invariance over many scales
- Renormalizable (marginal) interactions are the natural ones for physics because they survive long RG flows
- There exist renormalizable tensorial field theories (Ben Geloun, R, 2011).
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- Asymptotic freedom (AF) seems generic in the tensor world because the wave-function renormalization is stronger and dominates the coupling renormalization [Ben Geloun, Ben Geloun and Dine, 2012]
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Enumerative Combinatorics in 3D and 4D

Until now it seemed difficult to find analogues of CVS map in 3D. Even the Gromov conjecture is unproven today.

Gromov Conjecture: The number ST_n of triangulations of the sphere with n tetrahedra is exponentially bounded in n:

$$ST_n \leq K^n \tag{1}$$

Random tensors could bring new ideas there.

Uniformization in 3D

The transfer to a non-diffeo-invariant reference frame (eg a fixed sphere) is an essential step in understanding quantum gravity. It requires some form of uniformization.



Uniformization in 3D is possible thanks to the works of Thurston, Hamilton and Perelman.

Results

- Random Geometry in higher diemnsions can be better understood analytically through the use of random tensors and colored triangulations.

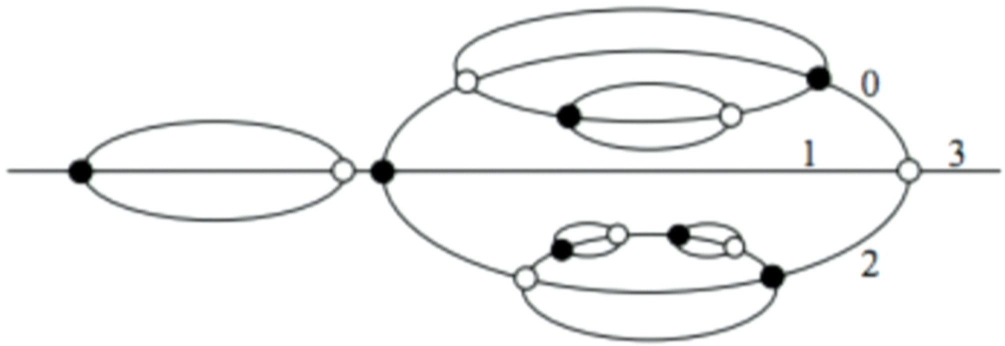
- Associated renormalizable tensor field theories exist, are typically asymptotically free hence point towards the generic presence of phase transitions.

Suggestions

To quantize gravity this suggests

- no need for supersymmetry
- no need for Kaluza-Klein dimensions
- no need for any ultraviolet cutoff at Planck scale

Melonic Graphs



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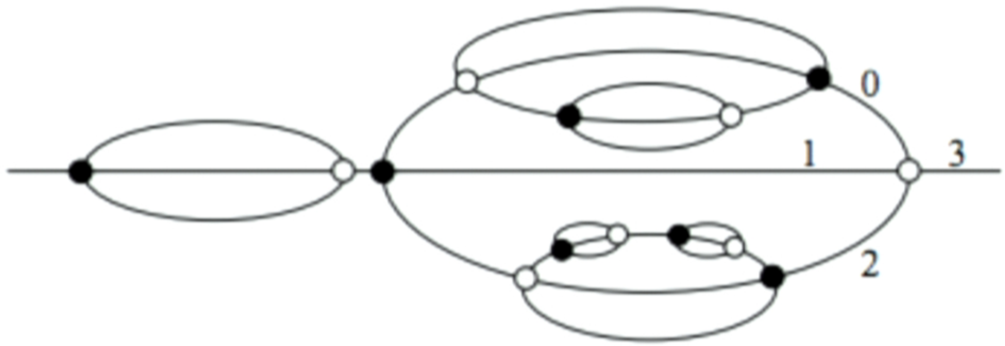

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