Title: A classification of symmetry enriched topological phases with exactly solvable models

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Abstract: Two types of topological phases have attracted a lot of attention in condensed matter physics:

symmetry protected

topological(SPT) phases and topologically ordered phases.
On the one hand, SPT phases are protected by given global symmetries while there is no topological order in the bulk. On the other hand, topologically ordered phases do not require symmetry and feature topological ground state degeneracy. In this talk, I present a
classification of phases with both topological orders and global symmetries, equipped with local bosonic exactly solvable models. This classification, in some sense, organizes previous pieces of understandings on SPT phases, topological orders, symmetry fractionalizations, into a single framework. Solution of the exactly solvable models and measurable consequences will be discussed.

A classification of symmetry enriched topological phases with exactly solvable models

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Feb 2013, Perimeter Institute

Outline

- Motivation
- The classification and connections with previous works
- The gauge-only exactly solvable models
  Dijkgraaf-Witten TQFT, Solving excitations
- Adding symmetry: symmetry fractionalization and beyond
- Discussion and future directions

The "Standard Model" of condensed matter

- Emergence: "more is different"

  Phases and Phase transitions

Landau's Fermi Liquid (metals)

Landau Theory of broken symmetry.

- Successfully describes a large set of phenomena in solids

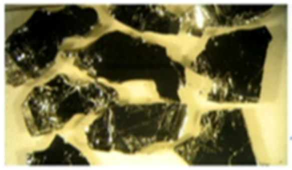
- Motivation

Beyond the "Standard Model" in solids?

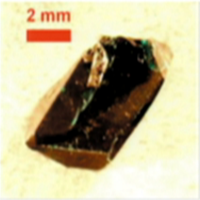
- Previously, violations only in extreme conditions:

  1D systems, 2DEG in strong magnetic field

- Recent experimental progresses: non-Landau phases in solids



e.g. topological insulators, quantum spin liquids



Bi_2Se_3

Herbertsmithite $ZnCu_3(OH)_6Cl_2$

➡ Force us to look beyond the "Standard Model":

New patterns of emergence or new types of orders

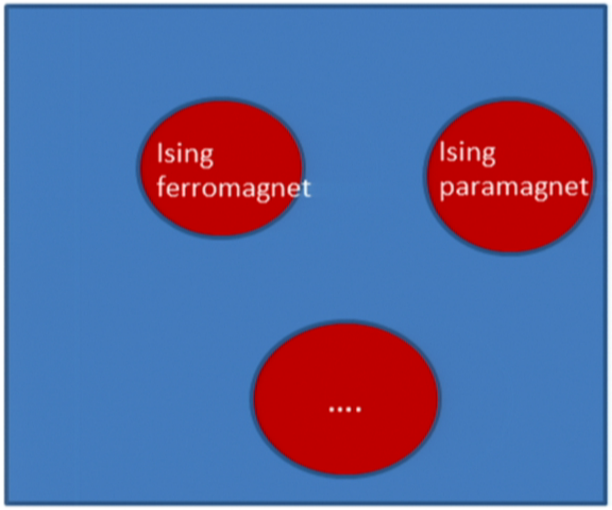
Issues: How to systematically describe these new orders?

What is the new theoretical framework ?

- Motivation

A modern view of gapped quantum phases

Gapped quantum phases

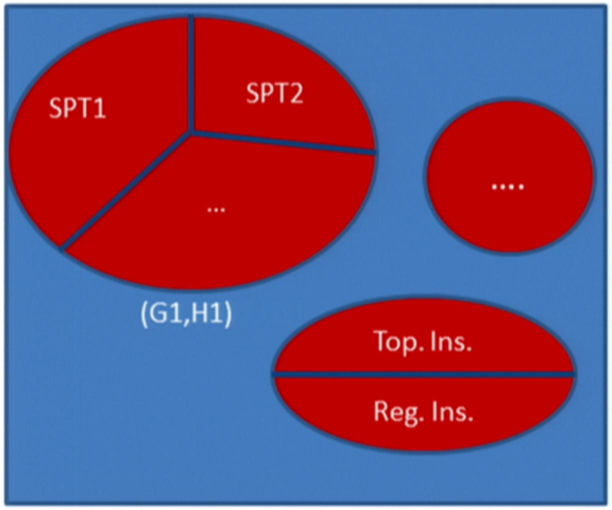


The "standard model"

- Motivation

A modern view of gapped quantum phases

Gapped quantum phases
without topological order



No anyons in the bulk but has symm.

Symmetry protected topological (SPT) phases:

(G,H) is not enough

↑

More than one phases with the same (G,H)

How to characterize these new orders?

Measurement: gapless edge states

Theoretical framework:

- Free fermion:
  K-theory (Kitaev)

- Boson systems:
  group cohomology
  (Chen, Gu, Wen, Turner, Pollmann,....)

- Motivation

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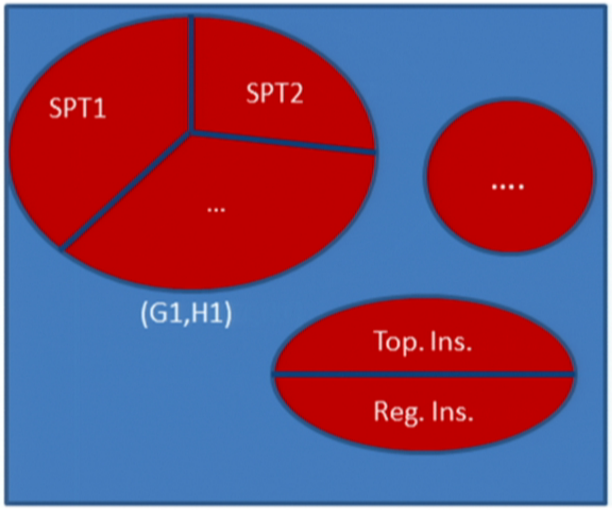

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Gapped quantum phases
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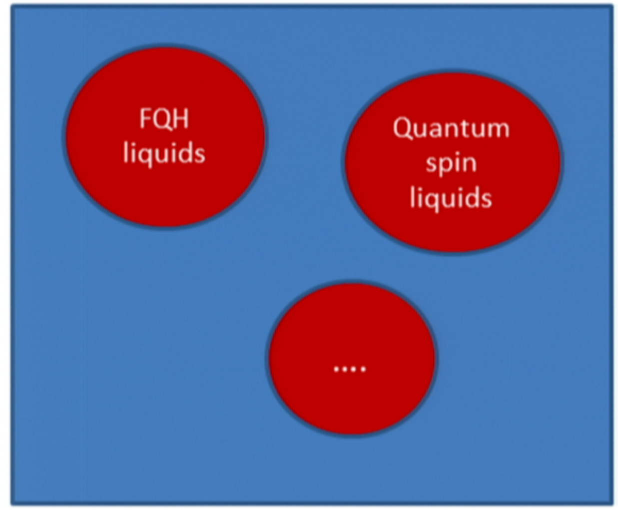

Anyons in the bulk do not require symm.

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Even without any symmetry, there are many phases:

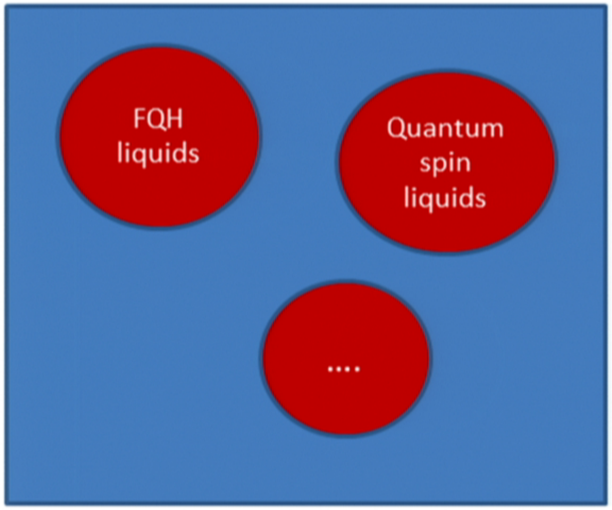
How to characterize these new orders?

- Measurement:

anyon statistics, topological ground state degeneracy…

- Theoretical framework:
- Gauge theories
- Tensor category
  (Levin, Wen, …)
- Pattern of zeros…
  (Wen, Wang, …)

Gapped quantum phases
With topological order



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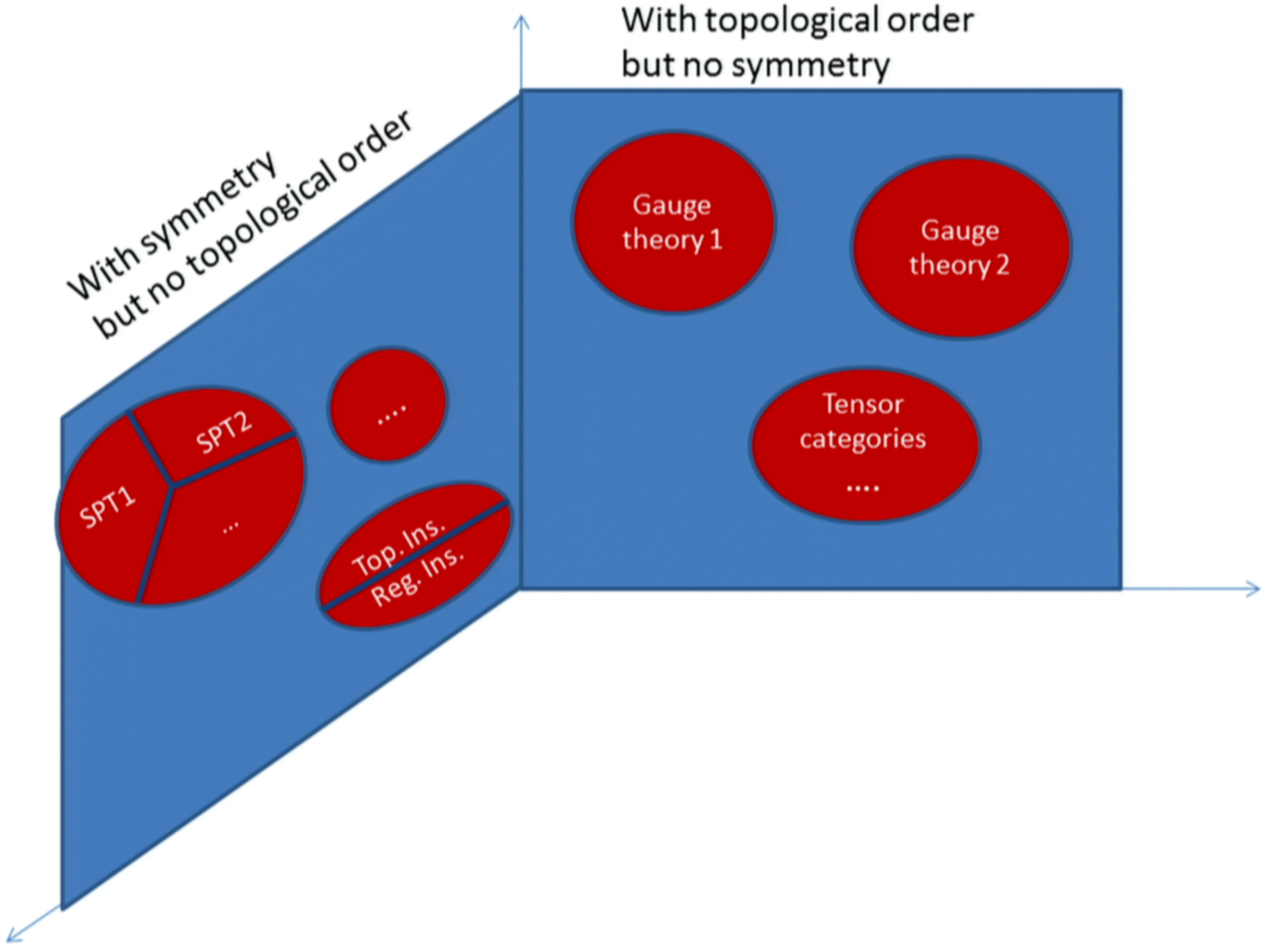

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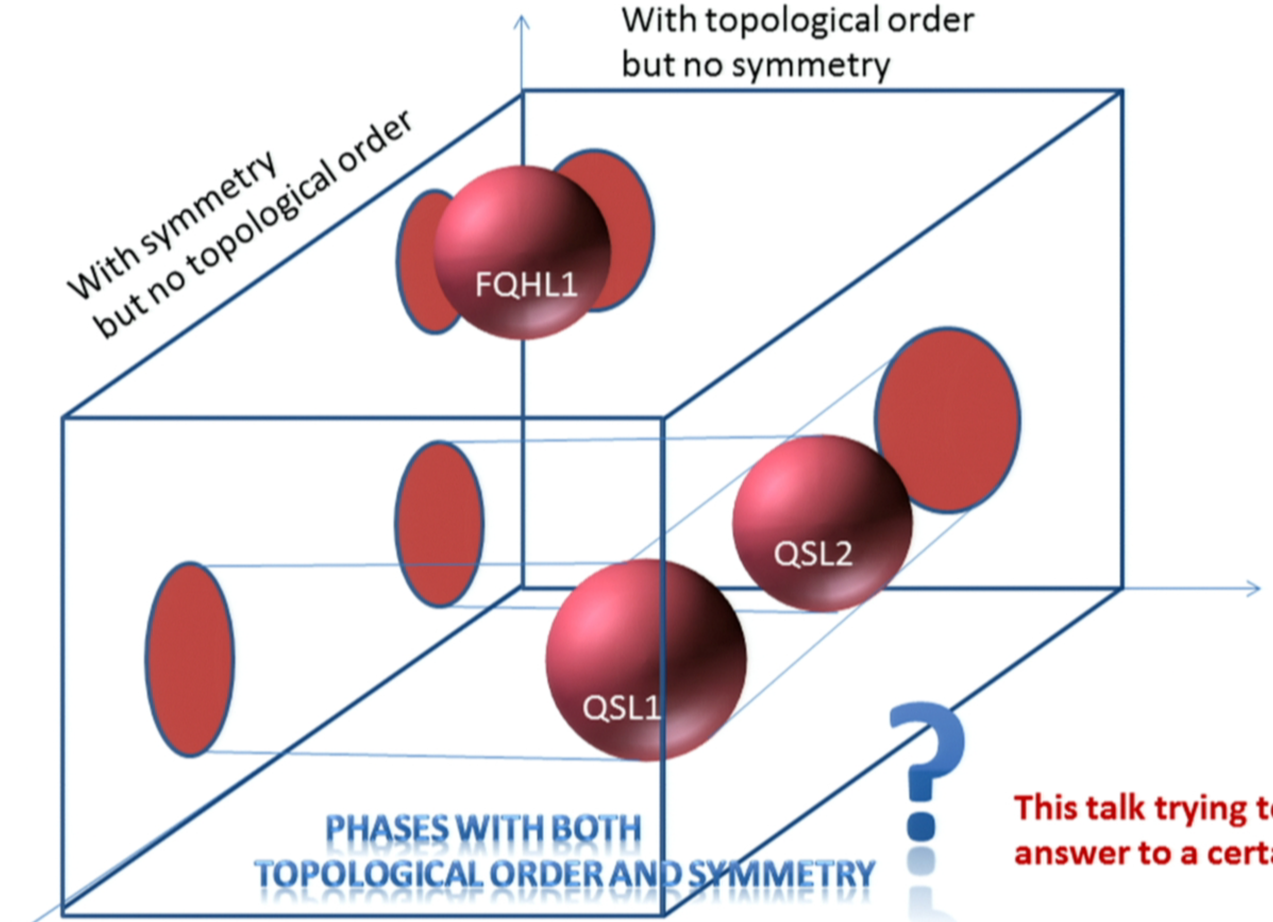

i.e., symmetry enriched topological (SET) phases

This talk trying to provide answer to a certain level

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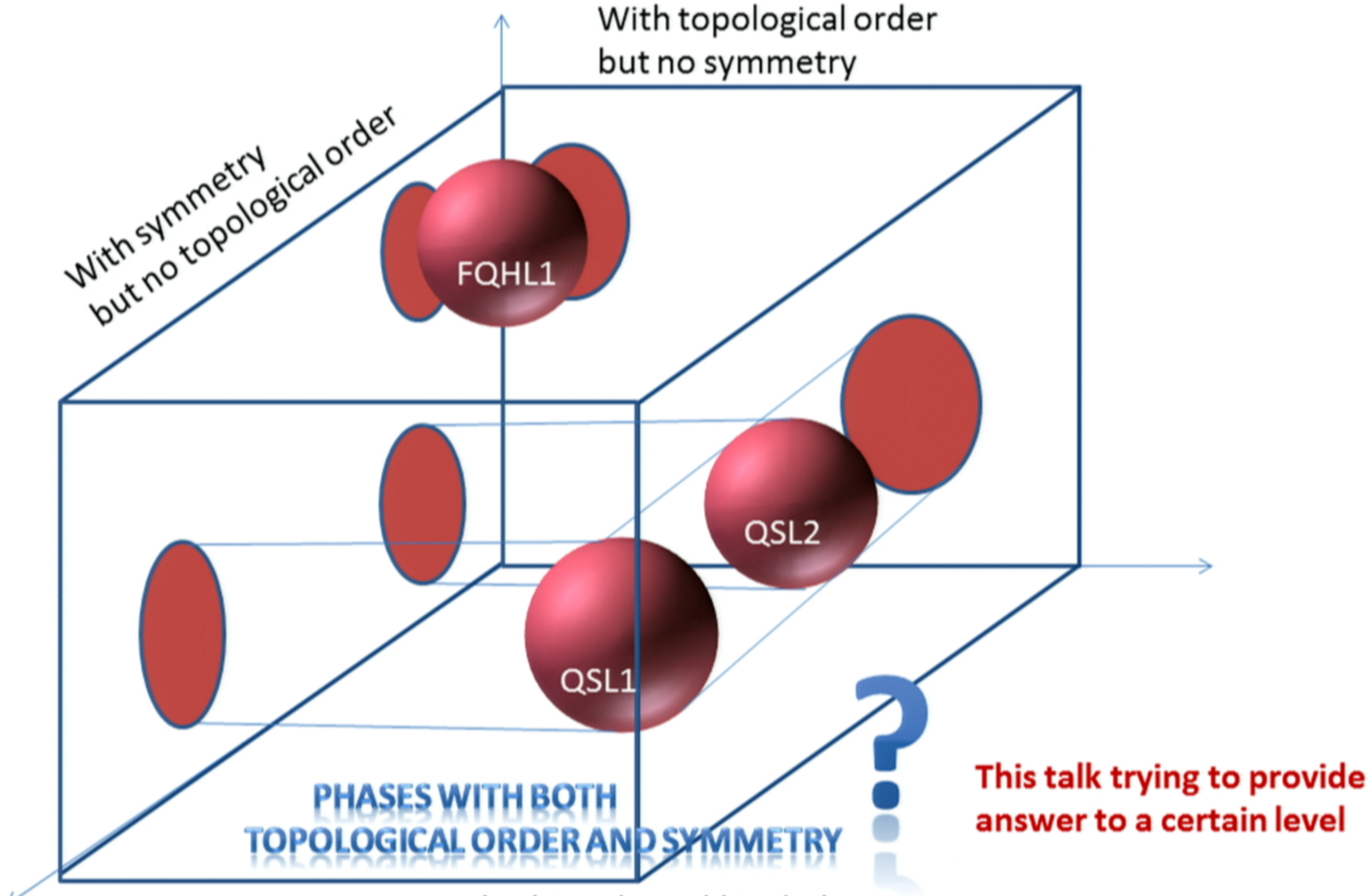

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A classification of SET phases

- We consider bosonic gapped quantum phases in 2+1 dimension, with a topological order described by a gauge group GG, and a on-site global symmetry group SG.

(SG is not a space-group symmetry; it's like spin rotation symmetry on a site.)

These phases are classified by the 3rd cohomology group:

$$H^3(SG \times GG, U(1))$$

For example, when $SG=Z_2$ and $GG=Z_2$,

$$H^3(Z_2 \times Z_2, U(1))=Z_2{}^3$$

--- 8 Ising disordered phases with topological orders described by Z_2 gauge group.

And we provide an exact solvable local bosonic model for each phase in our classification.

- Classification

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Connections with previous works

$H^3(SG \times GG, U(1))$

- When GG=0. (no topological order)

How many different gapped quantum phases in the presence of a global symmetry SG?

Chen et al. (2011) showed that these phases (so-called Symmetry protected topological phases, or SPT phases) are classified by:

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How many different topological orders that are described by a gauge group GG?

Dijkgraaf-Witten (1990) showed these are classified by:

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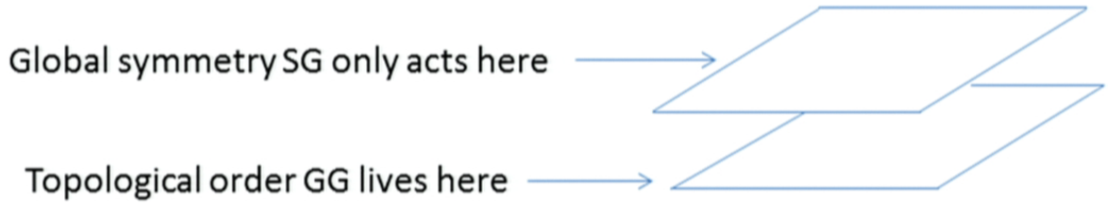
- Classification

When both SG and GG are present, a physical intuition:

- What if the topological order and the global symmetry decouple from each other?
- This is always possible: a bilayer system



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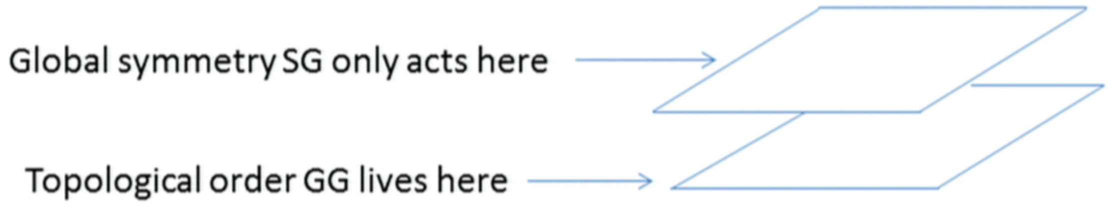

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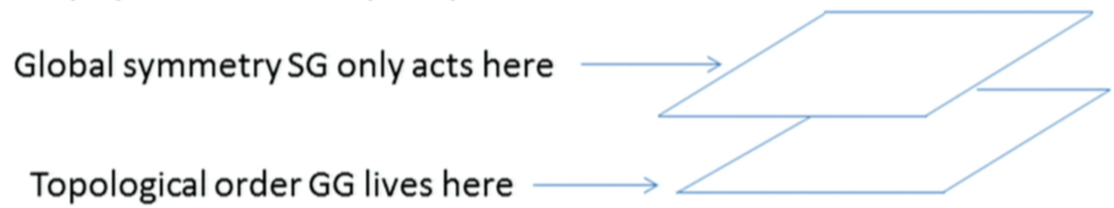

- The classification should at least include:

(1) SPT phases labeled by $H^3(SG,U(1))$.

(2) Dijkgraaf-Witten phases labeled by $H^3(GG,U(1))$.

In fact, the classification should at least contain:

$$H^3(SG,U(1)) \times H^3(GG,U(1))$$

Any EXTRA phases have non-trivial interplay between symmetry and topological order.

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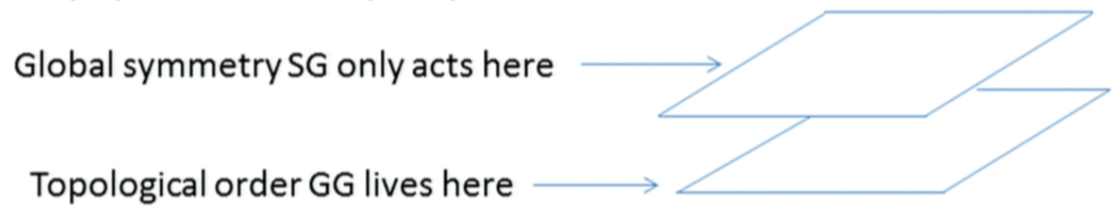

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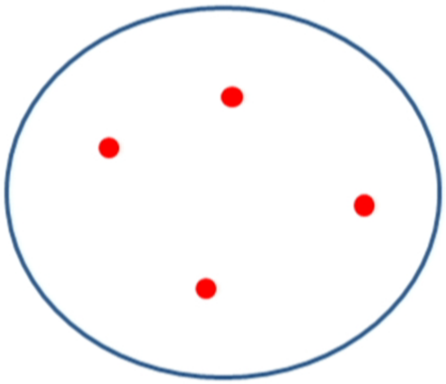
What did we know about symmetry enriched topological phases before?

- Projective Symmetry Group (PSG) classification (Wen 2001)

A general framework to describe "symmetry fractionalization".

Quasiparticles do not need to form representations of the symmetry group, they only need to form projective representations.

The basic assumption behind the PSG:
Can decouple a global symmetry transformation
into a product of local transformations
---- each local transformation acts on one quasiparticle.



Consider a global symmetry Transformation ***U*** on an excited state with 4 quasiparticles.

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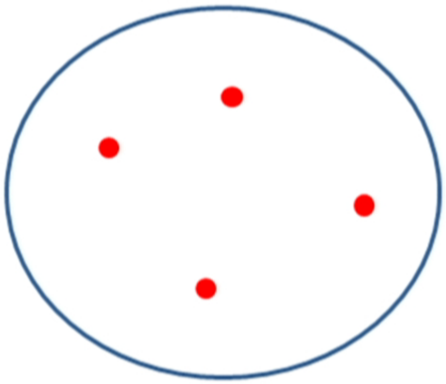

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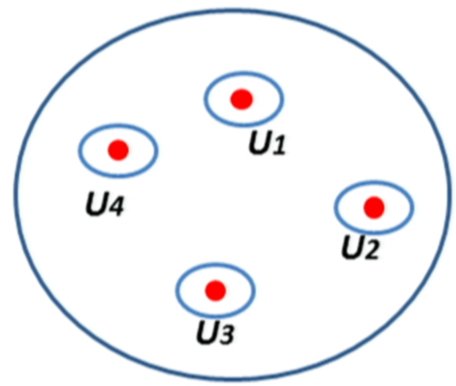
I think this assumption is the essence of “symmetry fractionalization.”



Assumption: exist local U_i’s
$U=U_1 U_2 U_3 U_4$

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Example:

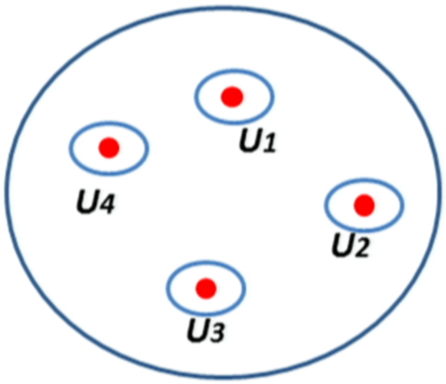
Let's try to fractionalize Ising symmetry (σ^2=1) for Z_2 gauge charges:

Note: Total gauge charge = 0 (mod 2)

$\Rightarrow$ Two possibilities:

$$U(\sigma)^2=\pm 1$$

Corresponding to: $H^2(Z_2, Z_2)=Z_2$.



The algebraic structure underlying PSG classification is:

$$H^2(SG, GG)$$ (when GG is abelian.)

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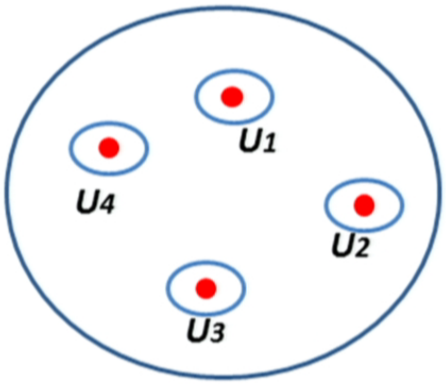

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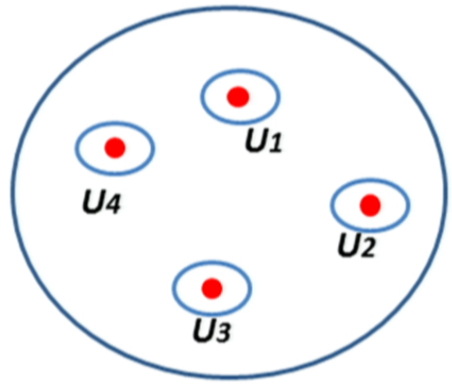

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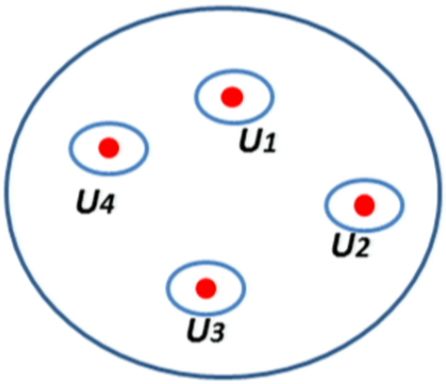

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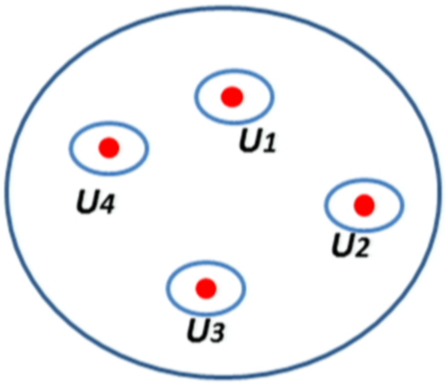

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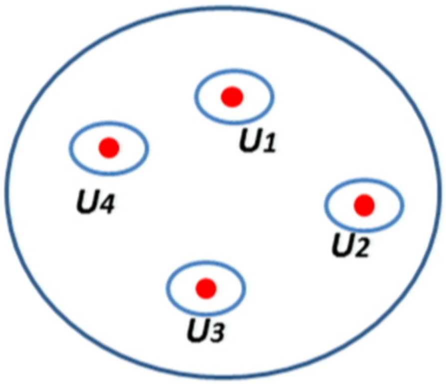

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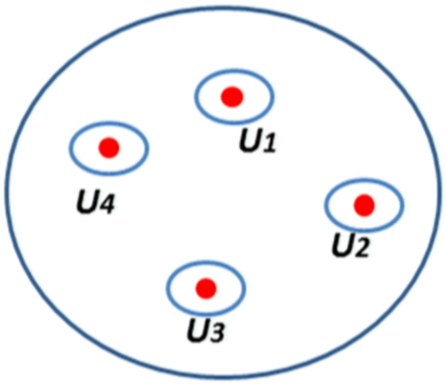

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$$H^3(SG \times GG, U(1))$$

$$= H^3(SG, U(1)) \times H^3(GG, U(1)) \times H^2(SG, GG) \times \text{(Extra term)}$$

$H^3(SG, U(1))$: SPT phases (symmetry protected topological phases)

$H^3(GG, U(1))$: Dijkgraaf-Witten classification of Topological orders

$H^2(SG, GG)$: Symmetry fractionalization classes (PSG) (for gauge fluxes)

(Extra term): New phases.

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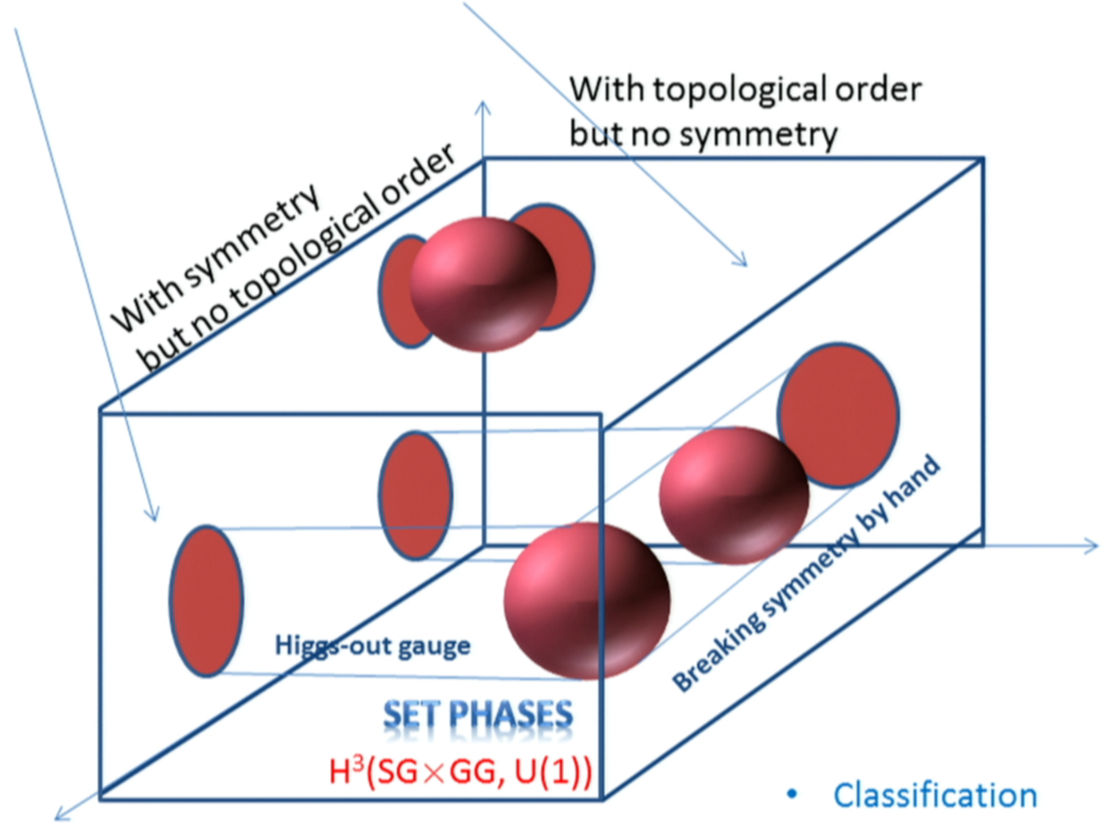

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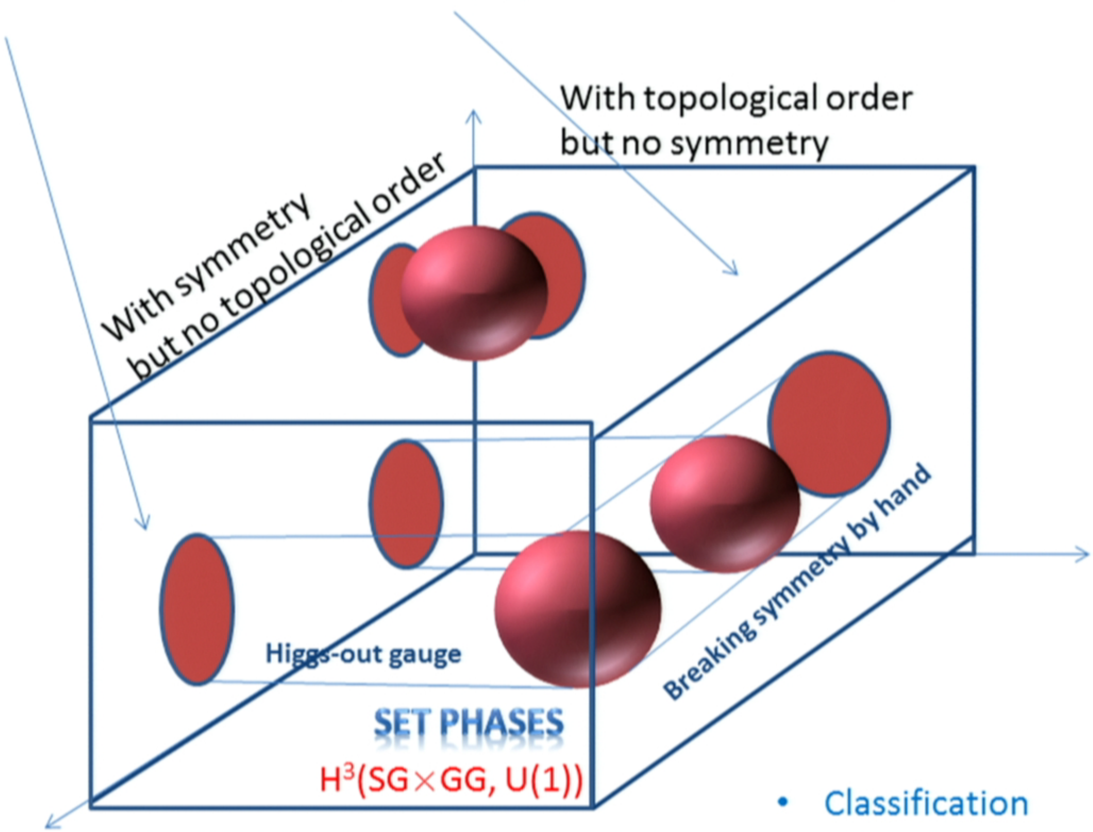

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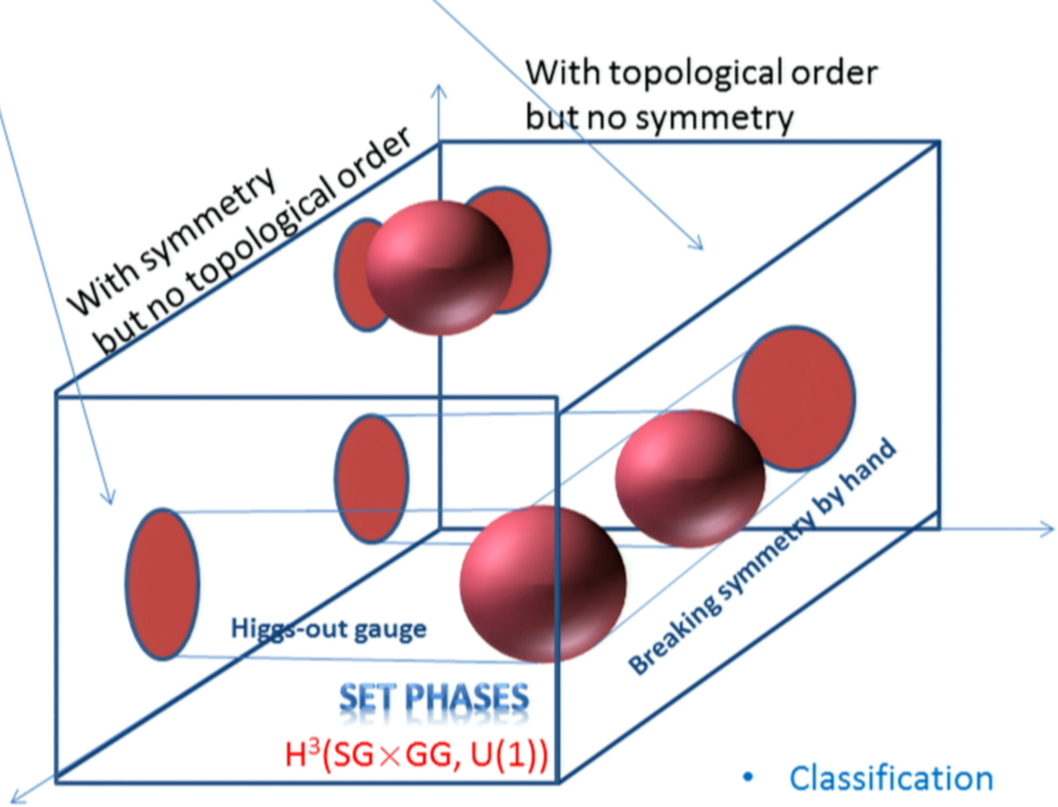

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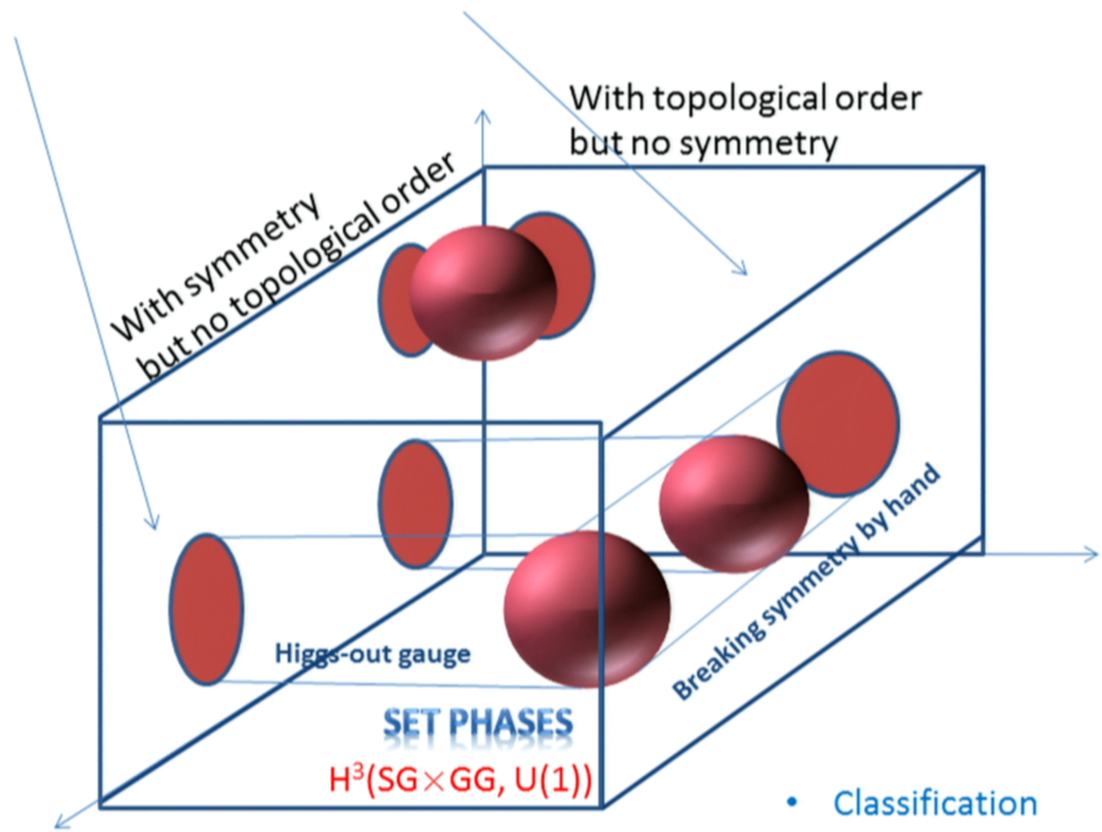

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What is contained in our classification?

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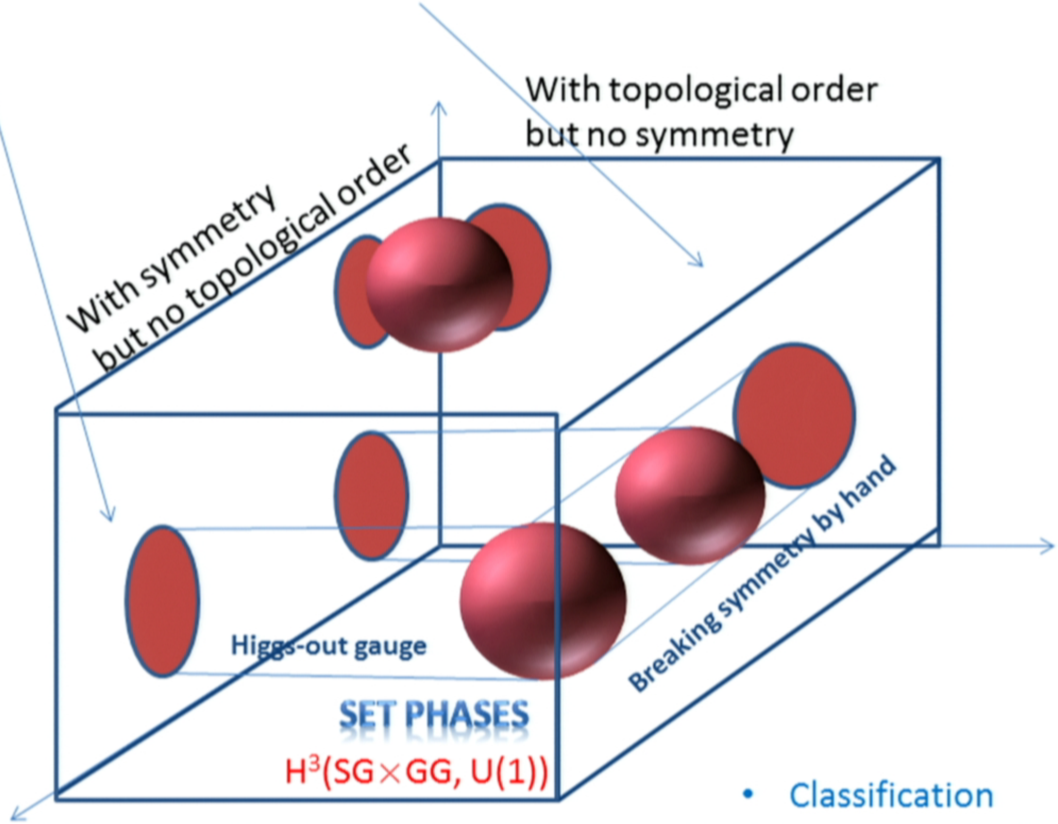

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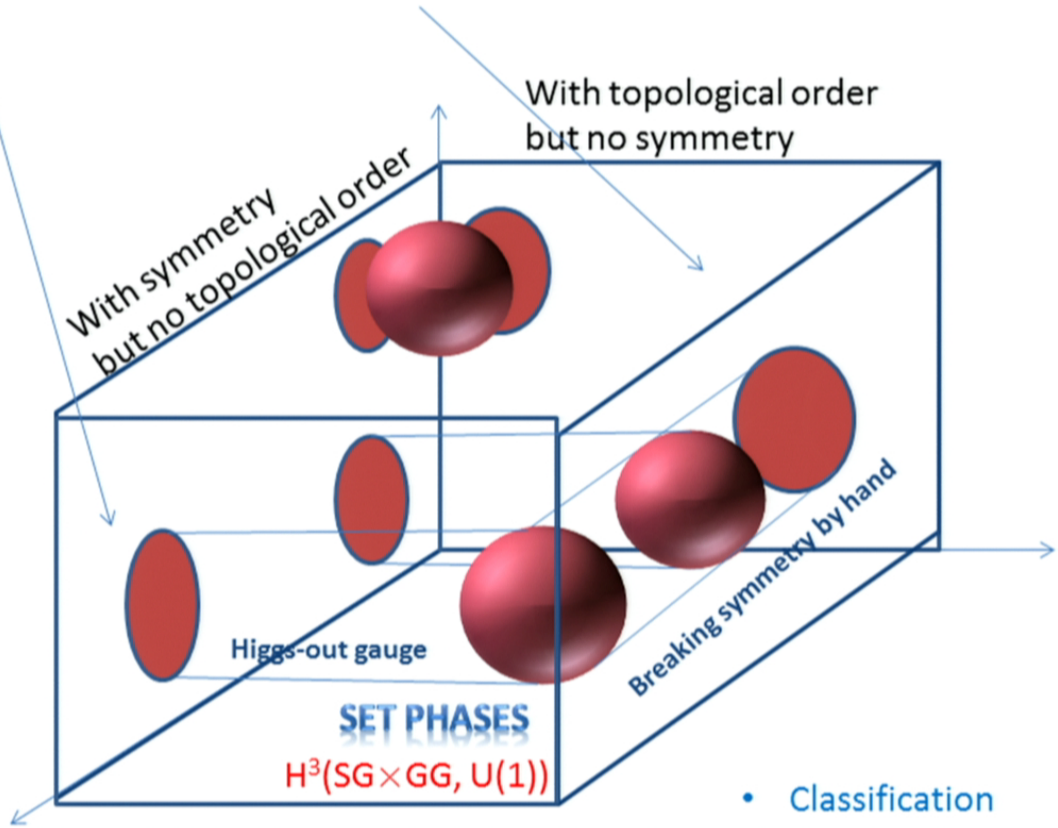

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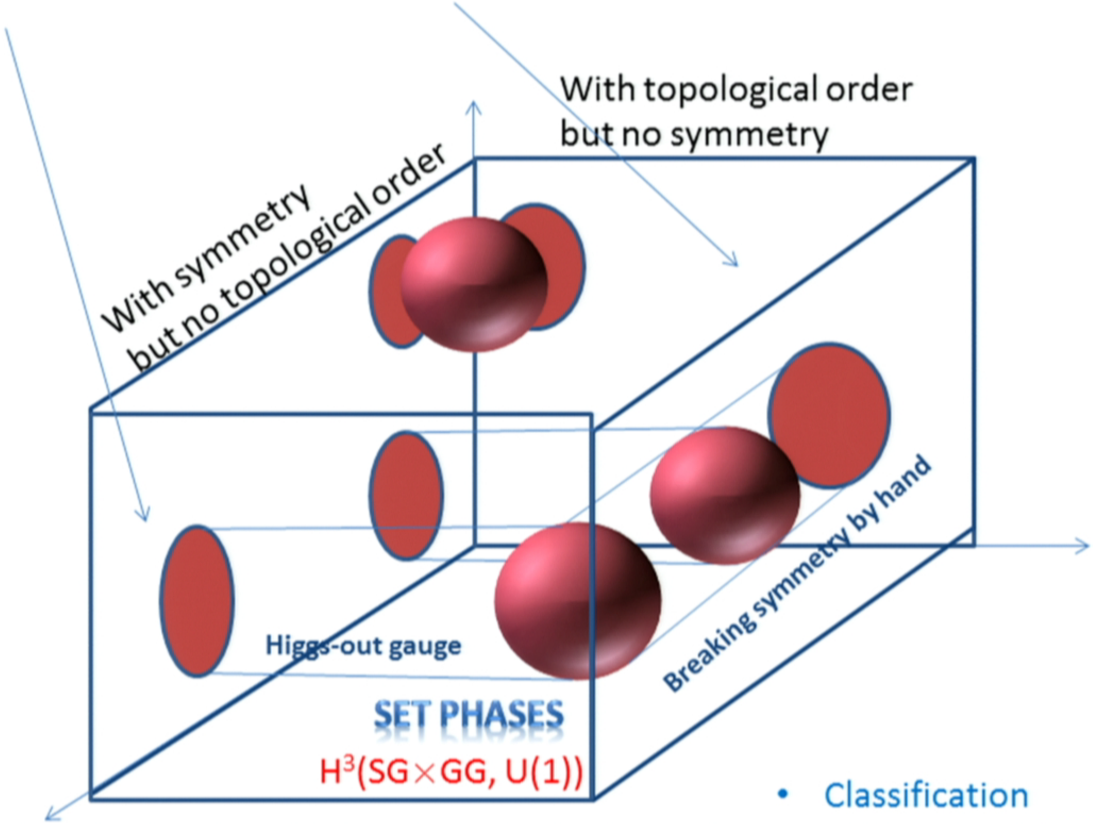


- Classification

Why our classification is reasonable

- A nice work by Levin and Gu (2012) points out the underlying relation between the SPT phases and the Dijkgraaf-Witten phases:

SPT phases with on-site global symmetry group G

Both classified by $H^3(G,U(1))$

"Gauge" the global symmetry

Dijkgraaf-Witten phases with gauge group G

- Our work is a generalization of Levin-Gu work :

"partially dualize" the gauge group of the DW topological orders.

- Classification

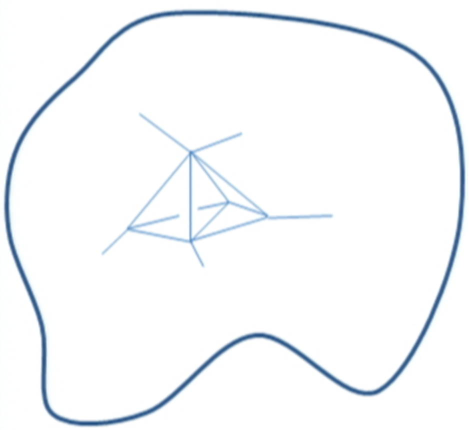
Dijkgraaf-Witten topological invariants

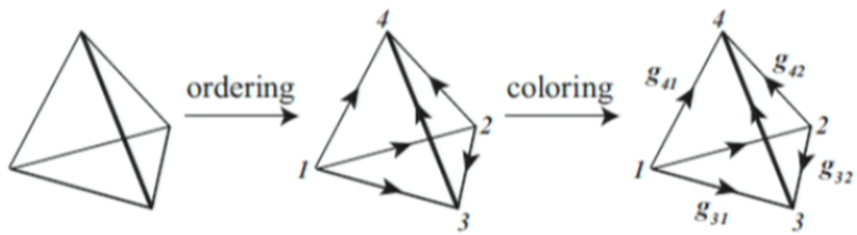
- An element(3-cocyle) $\omega \in H^3(G,U(1))$, defines a mapping:
  ω (x,y,z)→U(1), $\forall$ x,y,z $\in$ G, satisfying "certain conditions".

- Math foundation of the models: Dijkgraaf-Witten (DW) topological invariants for n-manifolds:

Theorem-1 (e.g, closed 3-manifolds): Choose a 3-cocyle ω, then for any

Closed-manifold → Triangulation→Ordering→Coloring





Constraints for coloring: zero-flux rule
$g_{31} = g_{32} \cdot g_{21}$

Assign a phase to the colored tetrahetron: $\omega(g_{43}, g_{32}, g_{21})$

- Models

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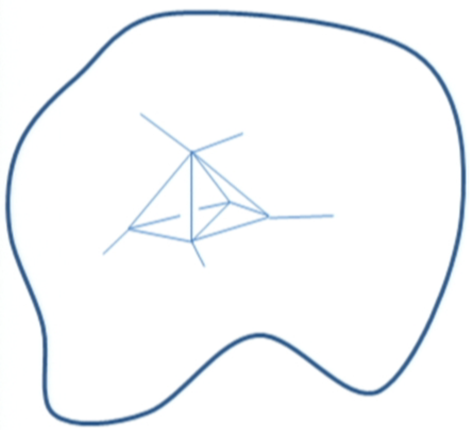

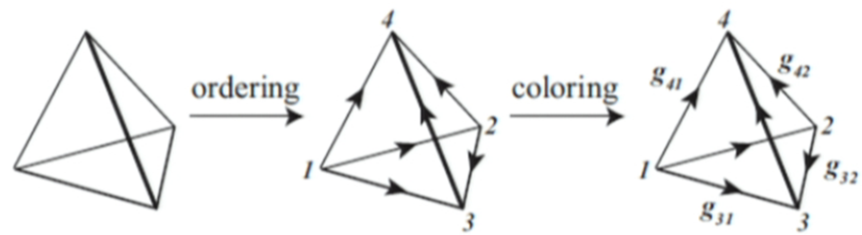

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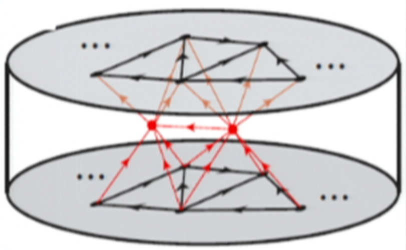
Theorem-2 (e.g, open 3-manifolds): Choose a 3-cocyle, for any open-manifold:
First, Triangulation→Ordering→Coloring for the surface, Fixing them,
Then extend them into the bulk: Triangulation→bulk Ordering→bulk Coloring



$$\text{DW inv.} \equiv (\text{norm. factor}) \sum_{[\text{all possible colorings}]} (\text{prod. of phases for every tetrahedron})$$

Independent of triangulation & ordering in the bulk! --- consequence of the "certain conditions"
Only dependent on (1) manifold topology
(2) the 3-cocycle
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- Models

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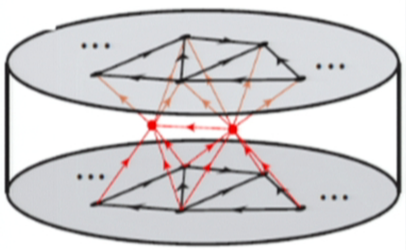

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- Models

Dijkgraaf-Witten TQFT

- DW topological invariant is a topological quantum field theory (TQFT).

e.g., Consider 3-manifold = closed 2-manifold $\times$ [0,1]

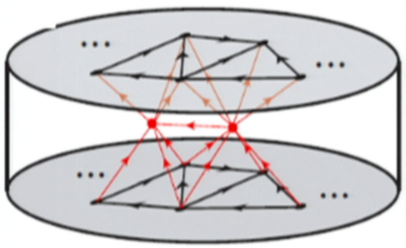
DW inv = quantum amplitude from bottom surface to top surface

namely, a mapping on the Hilbert space of the closed 2-manifold

Img (mapping) = Ground State Sector of DW TQFT

e.g., choose 2-manifold = 2-torus, $G=Z_2$ and the trivial cocycle

➔ Img (mapping) = 4-dim G.S.Sec. (same as Toric code)



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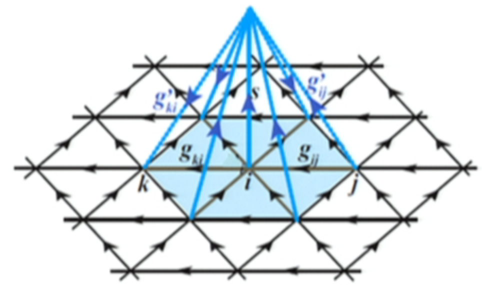
- Models

The exactly solvable models

- Consider model with GG only first, choose triangular lattice for simplicity:

Hilbert space spanned by (spin)states on bonds $\in$ GG

$$H = -\sum_t Q_t - \sum_p \hat{B}_p \prod_{t\in p} Q_t,$$



Q_t :(on plaquette) projector enforcing zero-flux rule

$\hat{B}_p$:(on site) projector enforcing "no gauge charge".

$$B_p = \frac{1}{|G|}\sum_{s\in G} B_p^s.$$

B_p^s operator=Local spin-flipping: amplitute = prod of cocycle phases

Obvious: Q_t mutually commute, Q_t and $\hat{B}_p$ commute.

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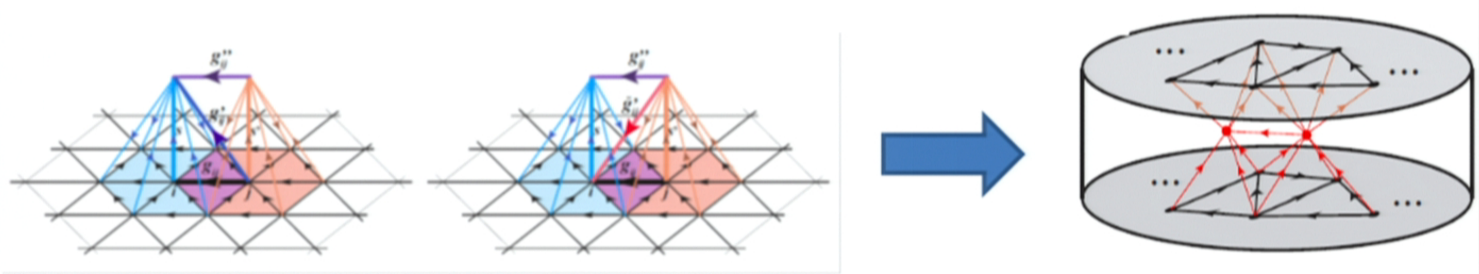

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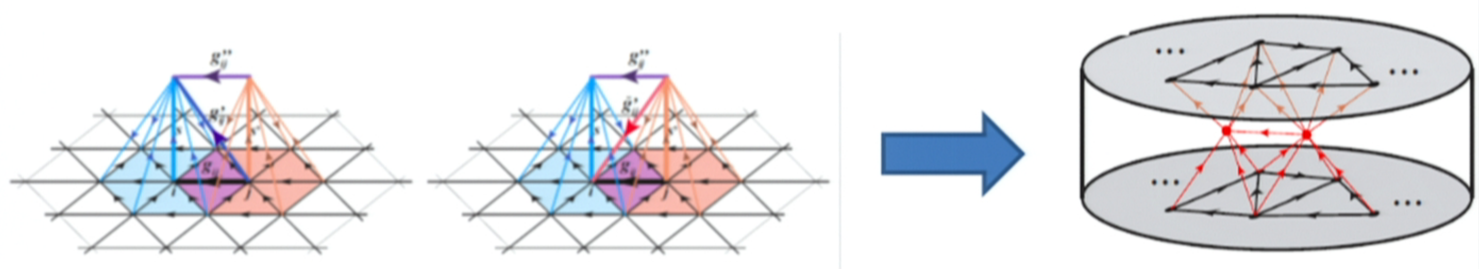

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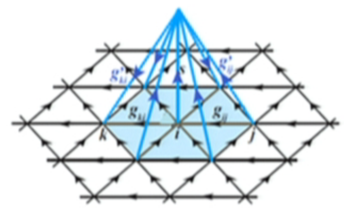
Solve excitations:

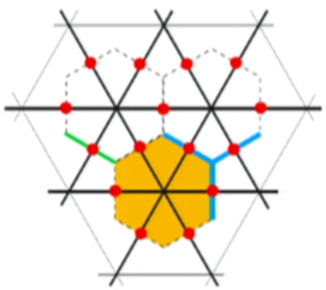
- Warm-up: $$H = -\sum_t Q_t - \sum_p \hat{B}_p \prod_{t\in p} Q_t,$$

With G=Z_2 & trivial cocycle, model goes back to Toric code on honeycomb lattice.

$$Q_v = \frac{1}{2}(1 + \prod_{i\in v} \sigma_z^{(i)}) \qquad \hat{B}_p^{TC} = \frac{1}{2}(1 + \prod_{l\in ph} \sigma_x^{(l)}),$$

s=+1 s=-1



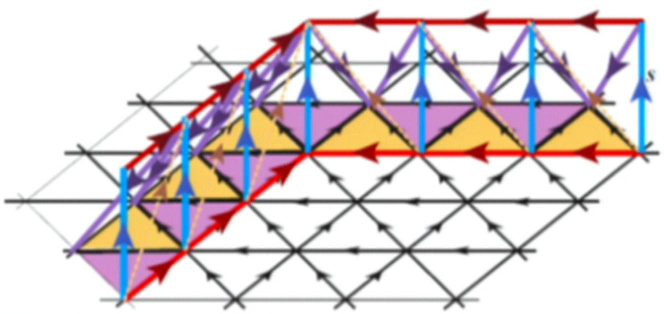


Recall: How to construct excitations of toric code?

Loop operators (e.g. product of $\prod_{l\in ph} \sigma_x^{(l)}$ over a region)

cutting loop operators → string operators

Excitations created at ends of string



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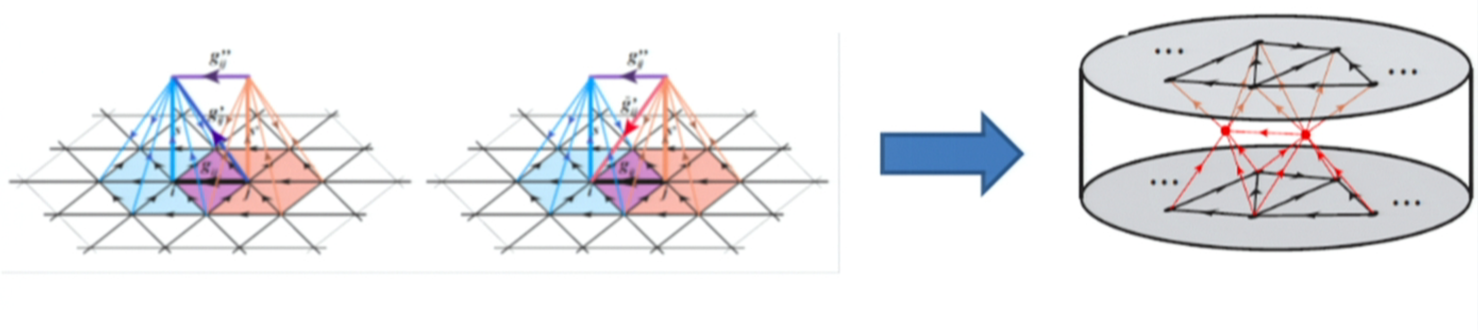

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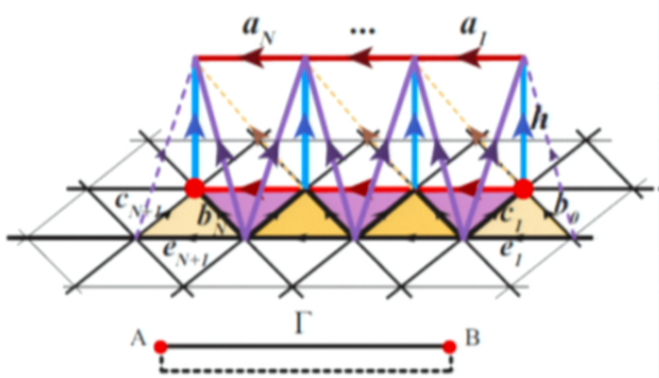
- Similar thing can be done here. (choose abelian G for simplicity)

  Generalized string operator == Ribbon operator



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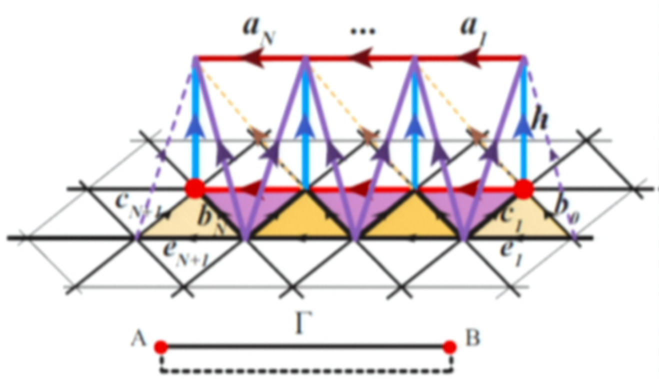

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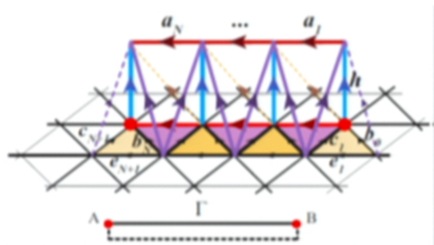
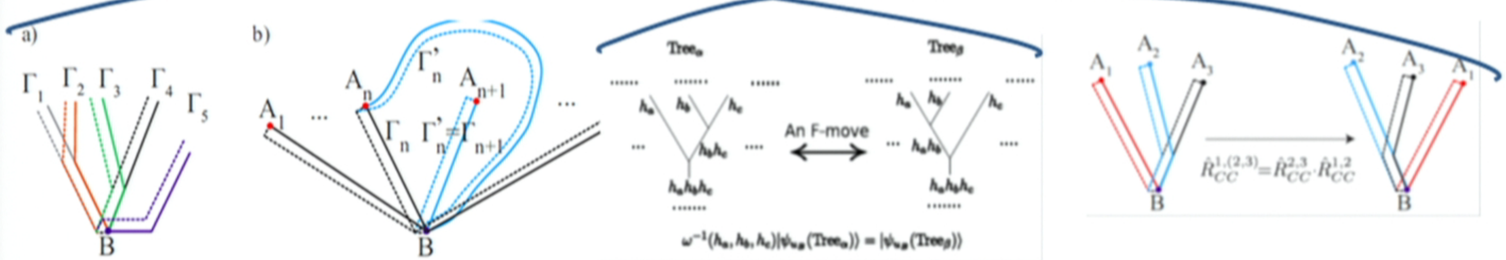


Topological order:
Braiding and fusion of particles can be fully understood based on the extended twisted ribbon algebra

Braiding algebra

Fusion algebra



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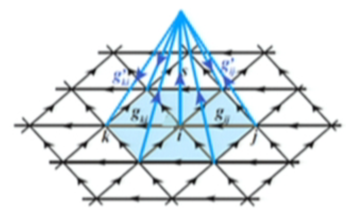

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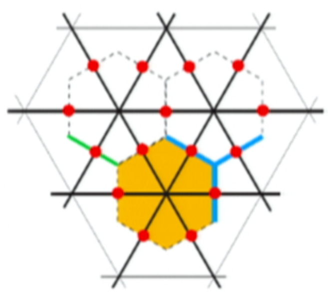

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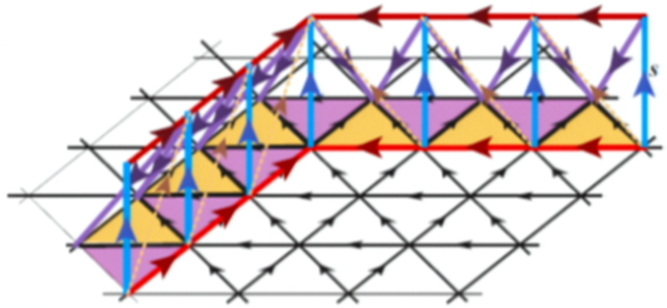

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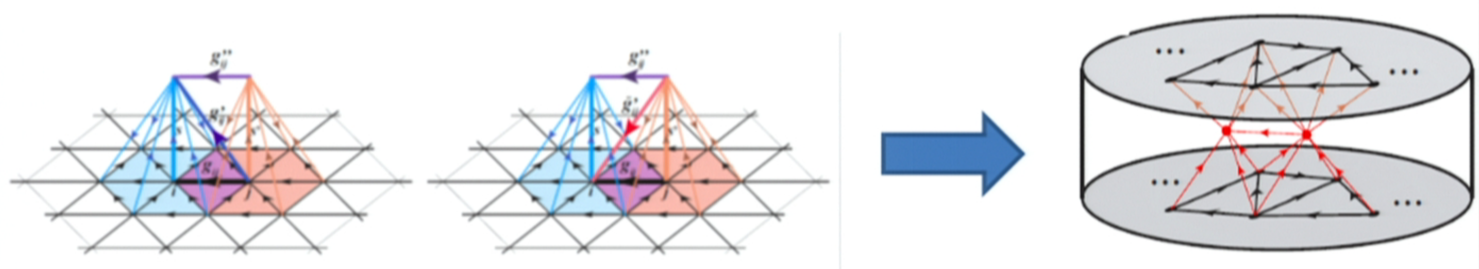

- Models

Solve excitations

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- By the end of the day ---

topological order (Fusion/braiding) are described by quasi-quantum double $D^\omega(G)$.

An interesting example:

for $G=Z_2^3$, can choose a cocycle such that $D^\omega(G) = D_4$ non-abelian gauge theory. (Bais et. al.)

- The twisted extended ribbon algebra is crucial (will be used soon again):

F: ribbon operator, D: local operators at ends of ribbon

$$\begin{aligned}
D_m(C)\, D_n(C) &= \Omega^k_{mn}\, D_k(C) \\
F^m(\Gamma)\, F^n(\Gamma) &= \Lambda^{mn}_k\, F^k(\Gamma) \\
F^m(\Gamma)\, D_i(A) &= \Lambda^{jk}_i\, \Omega^m_{kl}\, D_j(A)\, F^l(\Gamma) \\
D_i(B)\, F^m(\Gamma) &= \Lambda^{kj}_i\, \Omega^m_{lk}\, F^l(\Gamma)\, D_j(B)
\end{aligned}$$

where we defined the tensors

$$\begin{aligned}
\Omega^k_{ij} &\equiv \delta_{h_i,h_j}\, \delta_{h_k,h_i}\, \delta_{g_k,g_i g_j}\, c_{h_k}(g_i, g_j) \\
\Lambda^{ij}_k &\equiv \delta_{g_i,g_j}\, \delta_{g_k,g_i}\, \delta_{h_k,h_i h_j}\, c_{g_k}(h_i, h_j),
\end{aligned}$$

- Excitations

Adding symmetry

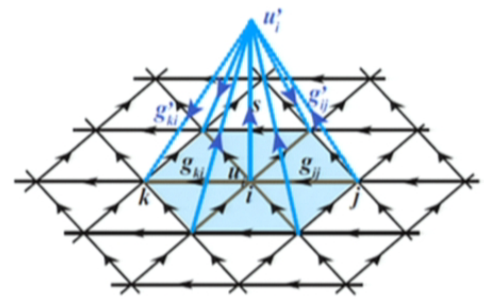
(1) With G=SG$\times$GG, firstly treat G as gauge group

(2) obtain the gauge-only model by choosing $\omega \in H^3(G, U(1))$

(3) Dualize SG degrees of freedom to sites:

$$g_{ij} \equiv h_{ij} \cdot u_i \cdot u_j^{-1} \in G$$

GG SG



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Hamiltonian looks the similar as before, now has global symmetry SG.

(1) Still use ribbons to create topological excitations
(2) Certainly we know how an excited state transform under SG.
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We found gauge charge always has trivial symmetry fractionalization here.

- SET

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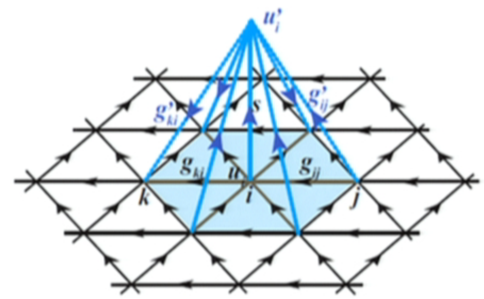

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- SET

Example 2: $GG=Z_2 \times Z_2$, $SG=Z_2$

$H^3(SG \times GG, U(1))$ Z_2^7

$= H^3(SG, U(1)) \times H^3(GG, U(1)) \times H^2(SG, GG) \times$ (Extra term)

Z_2 ↑ SPT

Z_2^3 ↑ DW topological order

Z_2^2 ↑ Symmetry fractionalization for two visons in two gauge sectors:

$$\sigma^2 = 1$$
$$\tilde{\sigma}^2(\text{sector-1}) = \pm 1$$
$$\tilde{\sigma}^2(\text{sector-2}) = \pm 1$$

Z_2 ↑ New phase! beyond symmetry fractionalization

- SET

Example 2: $GG=Z_2\times Z_2$, $SG=Z_2$

- (Extra term) = $\mathrm{Tor}[H^1(SG,U(1)),H^2(GG,U(1))]$

New phases beyond symmetry fractionalization

Choosing the non-trivial cocycle, solving the model, we found:

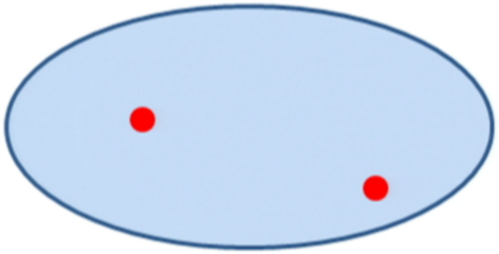
Ising global symmetry is interchanging quasiparticle types:

(e1,m1,e2,m2) → (e1+m2,m1,e2+m1,m2)

This cannot be described by symmetry fractionalization. (violate basic assumption -- cannot be implemented by local operators.)

One measurable consequence:

A single pair of visions has symmetry protected 2-fold degeneracy.



- SET

Discussion

- This classification and exact solvable models can be easily generalized to 3+1d:

$$H^4(SG \times GG, U(1))$$

New interplays between SG and extended excitations (flux loops)!

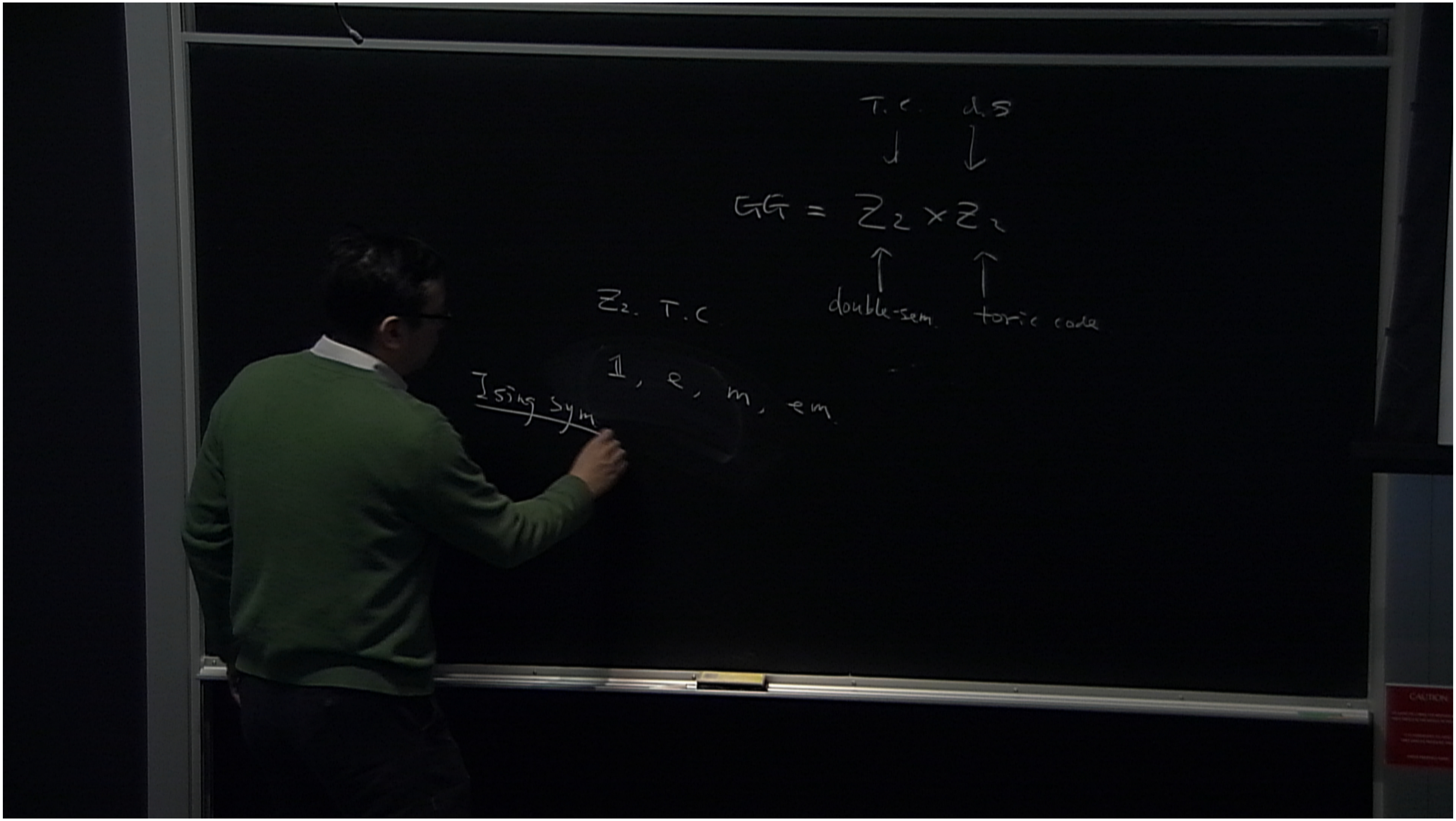
- Hung&Wen's generalization:

$H^d(G,U(1))$, with G being any extension of SG by GG: $G/GG=SG$

The missing symmetry fractionalization of gauge charges are captured here by non-trivial group extensions

- Discussion

Thank you.


T.C.
d.S
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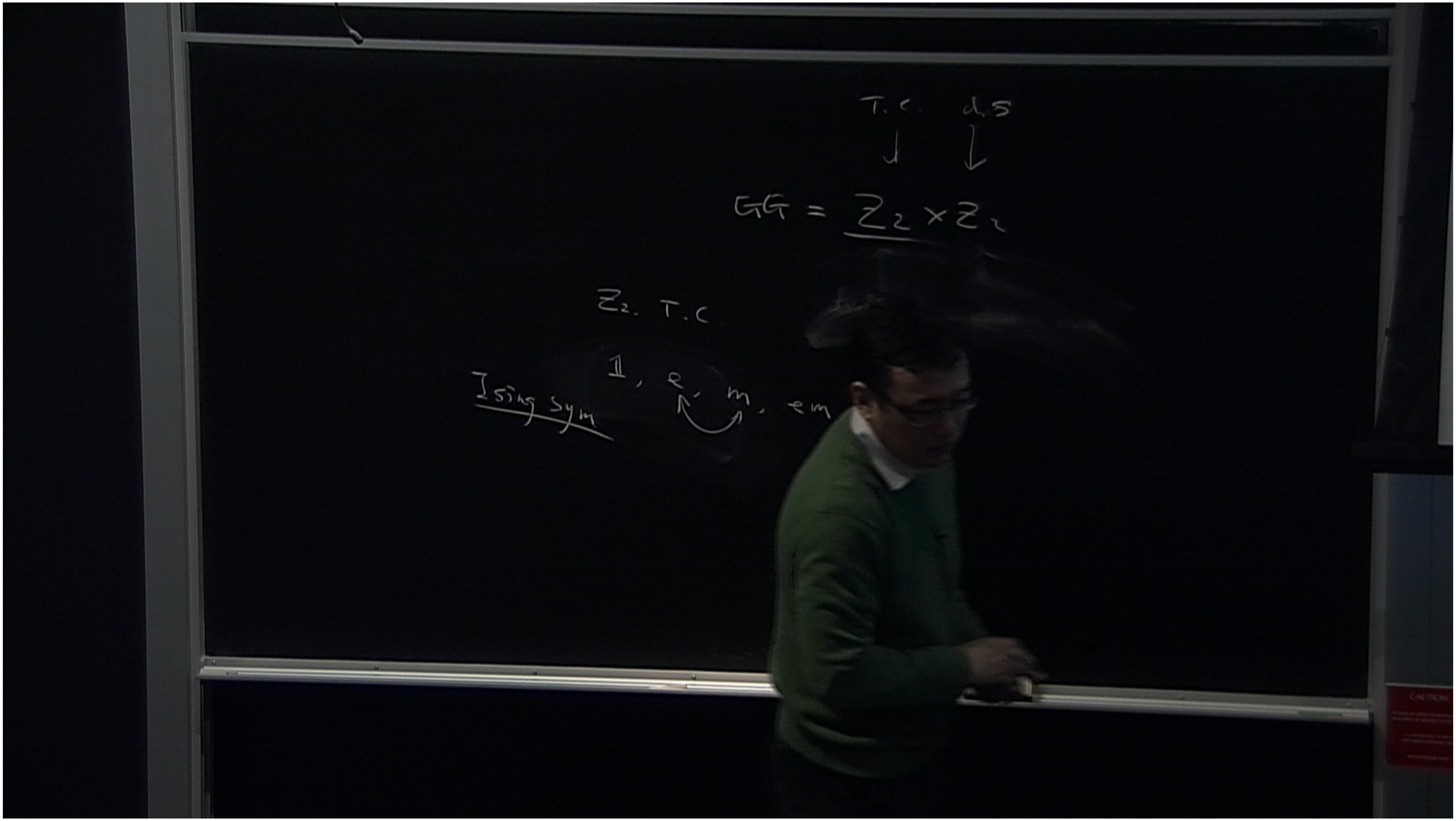
double-sem.
toric code
$\mathbb{Z}_2$. T.C
$\mathbb{1}$, e, m, em
Ising sym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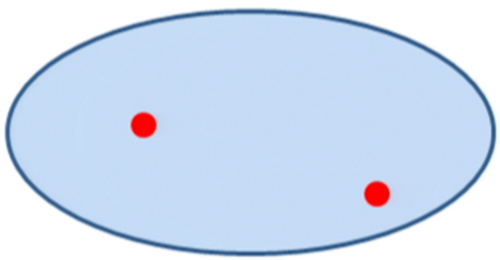

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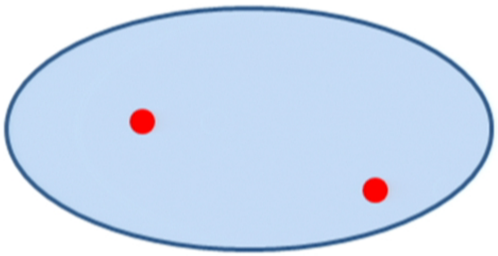
This cannot be described by symmetry fractionalization. (violate basic assumption -- cannot be implemented by local operators.)

One measurable consequence:

A single pair of visions has symmetry protected 2-fold degeneracy.



- SET

Example 2: $GG=Z_2 \times Z_2$, $SG=Z_2$

- (Extra term) = Tor[$H^1(SG,U(1)),H^2(GG,U(1))$]

New phases beyond symmetry fractionalization

Choosing the non-trivial cocycle, solving the model, we found:

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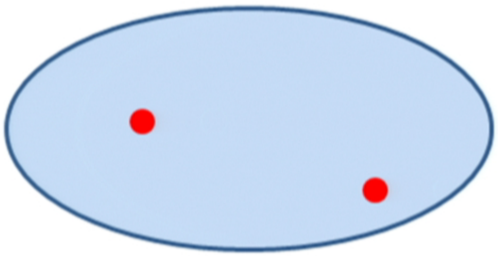


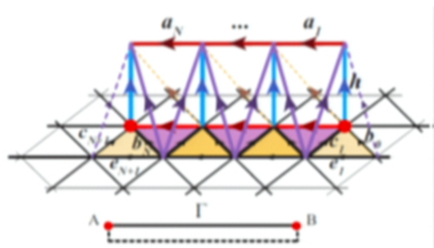
- SET

Solve excitations:

$$H = -\sum_t Q_t - \sum_p \hat{B}_p \prod_{t\in p} Q_t,$$

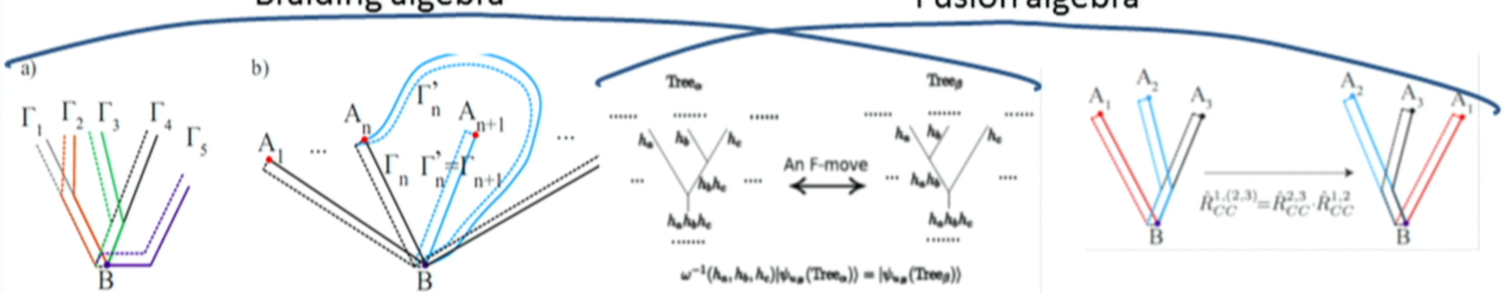
- Similar thing can be done here. (choose abelian G for simplicity)

  Generalized string operator == Ribbon operator



Topological order:
Braiding and fusion of particles can be fully understood based on the extended twisted ribbon algebra

Braiding algebra Fusion algebra



- Excitations