Title: Time Crystals

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Abstract: I consider a class of simple classical systems which exhibit motion in their lowest-energy states and thus spontaneously break time-translation symmetry. Their Lagrangians have nonstandard kinetic terms and their Hamiltonians are multivalued functions of momentum, yet they are perfectly consistent and amenable to quantization. Field theoretical generalizations of these systems may have applications in condensed matter physics and cosmology.

Time Crystals

A.S. and Frank Wilczek
1202.2537
1207.2677
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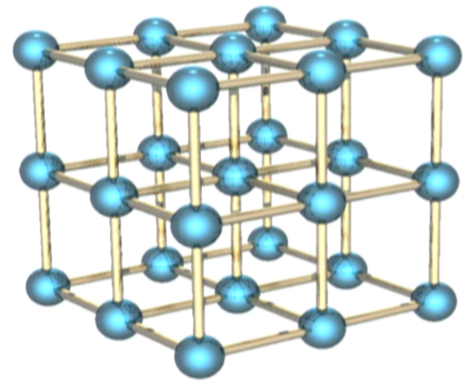
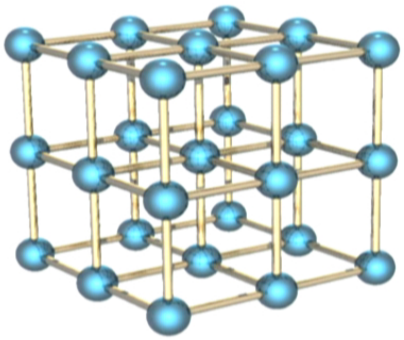
1210.3545

Time crystals?

- Crystals: periodic arrays in space
  - in ground state
  - spontaneously break space-translation symmetry
- Time crystals: periodic behavior in time
  - in ground state
  - spontaneously break time-translation symmetry
- Is this even possible?
  - It's not obvious…



Spontaneous space Crystals



- Field $\phi(t,x)$ angle-valued
- Potential

$$V(\phi) = -\frac{\kappa}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{\lambda}{4}\left(\frac{d\phi}{dx}\right)^4$$

- Minimized by $\left.\frac{d\phi}{dx}\right|_{\min} = \sqrt{\frac{\kappa}{\lambda}}$
- Min-energy solution $\phi = \sqrt{\frac{\kappa}{\lambda}}x + \phi_0$
- spontaneously breaks x-translation down to

$$x \to x + 2\pi\sqrt{\frac{\lambda}{\kappa}}\,n$$

... a crystal!

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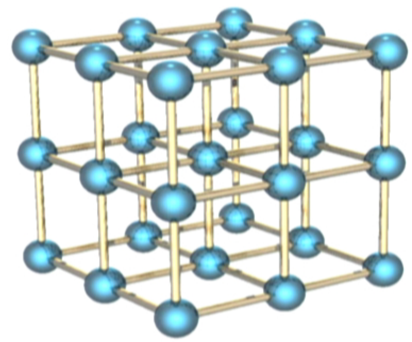

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... a crystal!

Try to break t-symmetry

- by doing a similar thing with kinetic term

$$H(p, \phi) = -\frac{\kappa}{2}p^2 + \frac{\lambda}{4}p^4$$

- Minimized by $p_{\min} = \sqrt{\frac{\kappa}{\lambda}}$

- But velocity $\dot{\phi} = \frac{\partial H}{\partial p} = -\kappa p + \lambda p^3\Big|_{p=p_{\min}} = 0$

- So minimum-energy solution is static, t-translation symmetry is **unbroken**.

- This is a **theorem**...

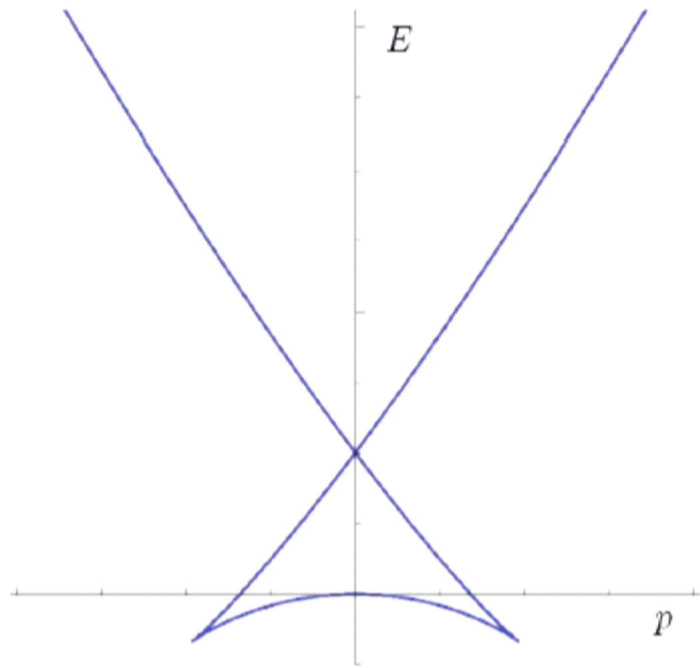
Lagrangian approach

- Try Lagrangian $L = -\frac{\kappa}{2}\dot{\phi}^2 + \frac{\lambda}{4}\dot{\phi}^4$
- Energy function $E = \frac{\partial L}{\partial \dot{\phi}} - L = -\frac{\kappa}{2}\dot{\phi}^2 + \frac{3\lambda}{4}\dot{\phi}^4$
  - Minimized by $\dot{\phi} = \sqrt{\frac{\kappa}{3\lambda}}$
- This is **not** static. How did we evade theorem?
- Momentum and velocity are nonlinearly related

$$p = -\kappa\dot{\phi} + \lambda\dot{\phi}^3$$

  - Hamiltonian a multivalued function of p

E vs. p



swallowtail catastrophe

- Hamiltonian $H(p,\phi)$ not differentiable at minima!

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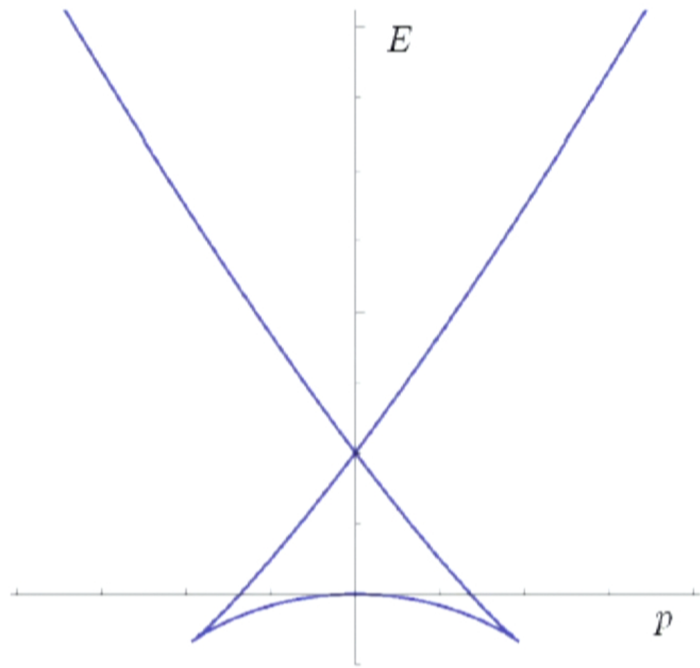

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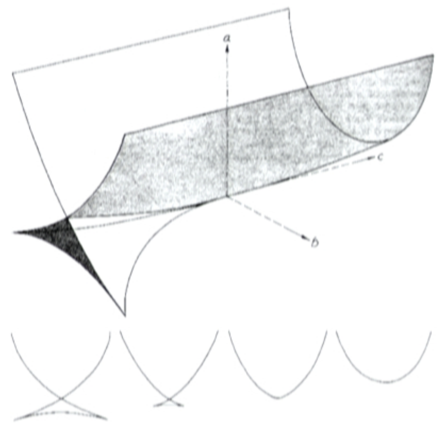


swallowtail
catastrophe

- Plot E vs. p vs. κ : one of the fundamental "catastrophes" of Thom's Catastrophe Theory.
- In our case the catastrophe is associated with the transition from positive to negative values of κ − which results in the formation of a time crystal.
- Get higher catastrophes from higher-order polynomials in $\dot{\phi}$.

Add potential energy

- We have constructed a Lagrangian with minimum energy solutions that are periodic in time (if ϕ is angular) – time crystals.
- What happens if we add a potential energy term?

$$L = -\frac{\kappa}{2}\dot{\phi}^2 + \frac{\lambda}{4}\dot{\phi}^4 - V(\phi)$$

- Then ϕ wants to minimize KE and PE at the same time – incompatible: can't satisfy $\phi = \phi_{\rm min}$ and $\dot{\phi} = \sqrt{\frac{\kappa}{3\lambda}}$ simultaneously.
- Even for non-minimum energy solution there is a problem:
  – Equation of motion $(3\lambda\dot{\phi}^2 - \kappa)\ddot{\phi} = -V'(\phi)$

    is problematic when $\dot{\phi} = \pm\sqrt{\frac{\kappa}{3\lambda}}$

  – Requires infinite acceleration.
    - try to solve anyway…

Turning points

$$t(\phi) = \int^{\phi} \frac{d\phi}{\pm\sqrt{\frac{\kappa}{3} \pm \sqrt{\left(\frac{\kappa}{3}\right)^2 + \frac{4}{3}(E - V(\phi))}}}$$

- Argument of inner square root is non-negative iff

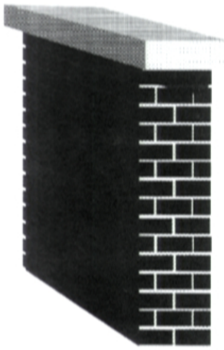
$$V(\phi) \ \leq \ \frac{\kappa^2}{12} + E = \Delta$$

- At turning point $\dot{\phi} = \pm\sqrt{\frac{\kappa}{3\lambda}}$

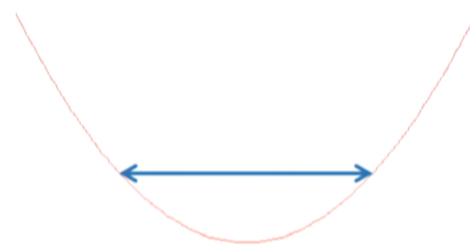
ϕ runs right into hard potential wall, flips over

from $\dot{\phi} = +\sqrt{\frac{\kappa}{3}}$ to $\dot{\phi} = -\sqrt{\frac{\kappa}{3}}$

– For these values of velocity, infinite acceleration is consistent with eqn of motion.



Nearly minimum-energy solutions



- Close to the bottom of potential velocity is $\dot{\phi} \sim \pm\sqrt{\frac{\kappa}{3}}$

  with equality at turning points.
- ϕ oscillates back and forth with nearly constant speed.
- At bottom, ϕ oscillates with infinite frequency.
- Reconciles apparently contradictory conditions

  $\phi = \phi_{\min}$ and $\dot{\phi} = \pm\sqrt{\frac{\kappa}{3\lambda}}$
- Quantum effects will lift minimum energy state away from minimum.

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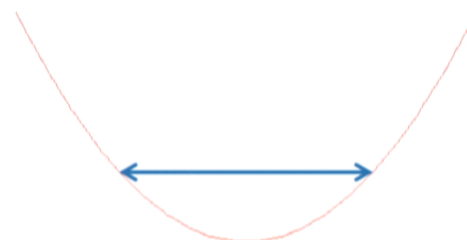

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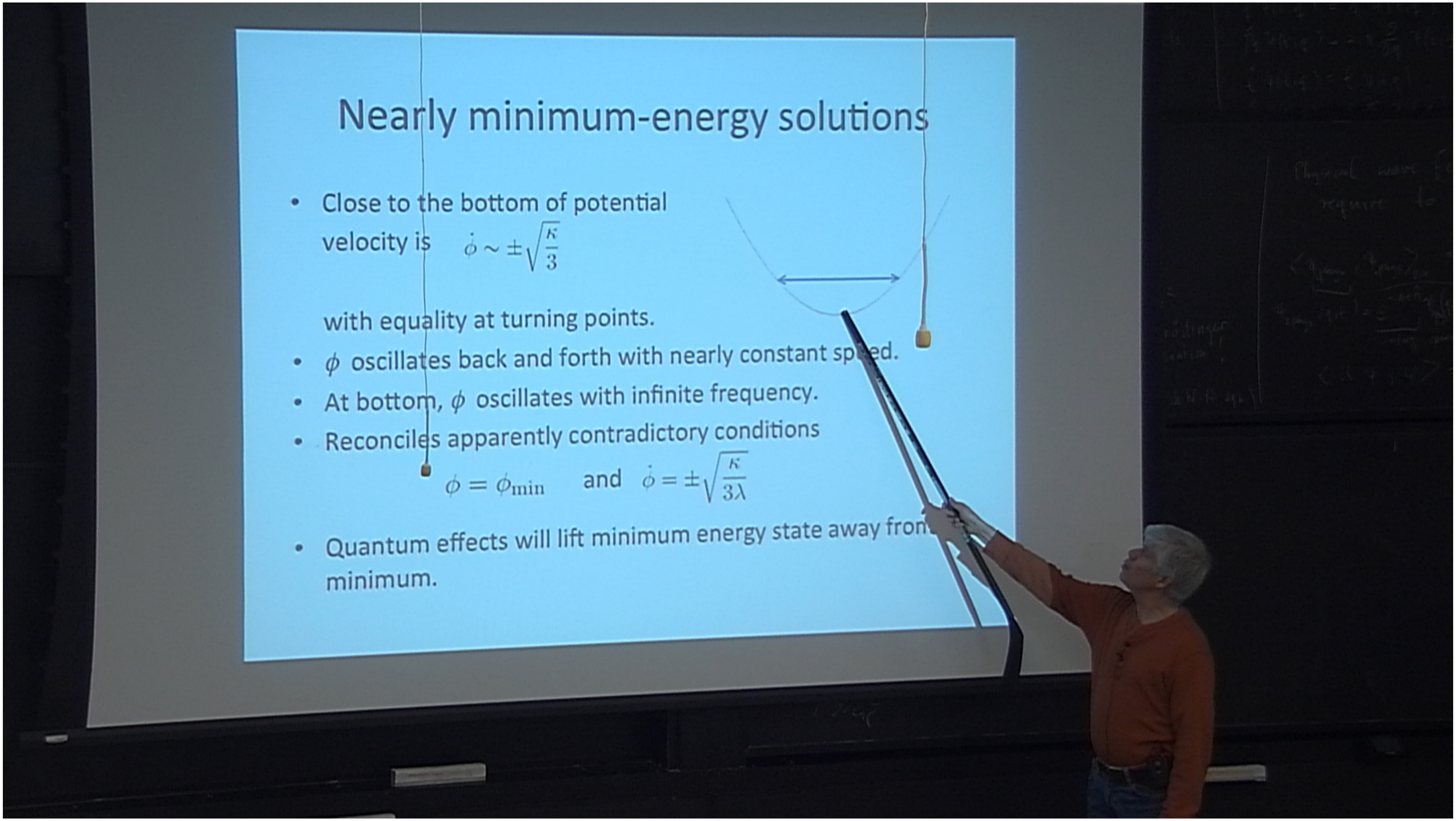
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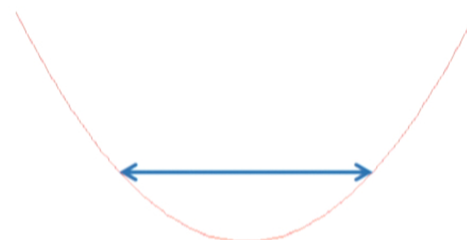

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Semiclassical quantization

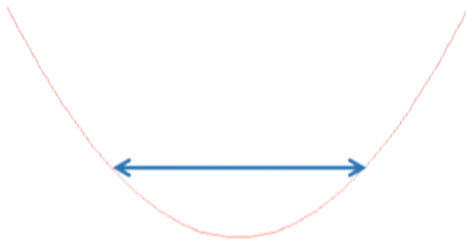
- BS formula

$$S = \oint p\,d\phi = \int (\dot\phi^3 - \kappa\dot\phi)d\phi = 2\pi\hbar(n+\delta)$$

  - use $\delta = 1$ for hard-wall potential
  - approximate $\dot\phi = \pm\sqrt{\frac{\kappa}{3\lambda}}$
  - turning points for $V(\phi) \approx \frac{1}{2}\mu(\phi - \phi_0)^2$

- Ground state energy

$$\boxed{E_{\min} = \frac{27\pi^2}{128}\frac{\hbar^2\mu}{\kappa^3}}$$



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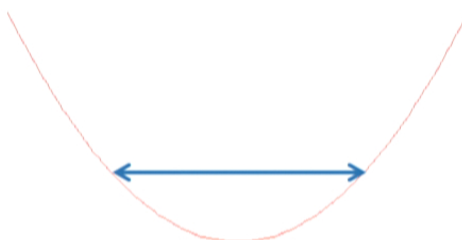

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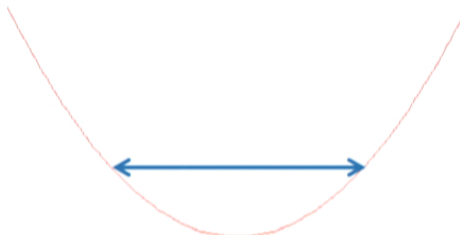

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Generalizations

- Two degrees of freedom

$$L = \frac{1}{4}(\dot{\psi}_1^2 + \dot{\psi}_2^2 - \kappa)^2 - V(\psi_1, \psi_2)$$

- e.g. "Double Mexican Hat"

$$V = -\frac{\mu}{2}(\psi_1^2 + \psi_2^2) + \frac{\lambda}{4}(\psi_1^2 + \psi_2^2)^2$$

$$L = \frac{1}{4}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 - \kappa)^2 + \frac{\mu}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

- Has minimum-energy solutions without turning points:
  - just go around bottom of hat at constant velocity

$$\dot{\phi} = \pm\sqrt{\kappa/3\rho_0}$$

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Fields

- Fields $\phi(t, x)$
- Can consider both higher-order time derivatives and higher-order gradient terms
  - can get **space-time crystals** by mixing these, *i.e.* waves:

$$E = \frac{\kappa'}{2}\left(\dot{\phi}^2 - v^2(\frac{d\phi}{dx})^2\right)^2 + \cdots$$

  - Get propagating waves as minimum-energy solutions.
  - Can engineer charge-density waves etc.
- Set $v = c$ to get relativistic fields…

Relativistic fields

- Quartic in derivatives $\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 + \lambda(\partial_\mu \phi)^4 + \cdots$
  - higher derivative terms arise naturally in effective field theories
- Energy density

$$\mathcal{E} = \frac{1}{2}\left((\partial_0\phi)^2 + (\nabla\phi)^2\right) + 3\lambda\left((\partial_0\phi)^2 + (\nabla\phi)^2\right)\left((\partial_0\phi)^2 - (\nabla\phi)^2\right)$$

  - Not bounded below: wrong sign of $(\nabla\phi)^4$

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  - Not bounded below: wrong sign of $(\nabla\phi)^4$
- Problem is cured at next order! $2n^{\text{th}}$-order term gives

$$\mathcal{E}_{2n} = ((2n-1)(\partial_0\phi)^2 + (\nabla\phi)^2)((\partial_0\phi)^2 - (\nabla\phi)^2)^{n-1}$$

  - For stability: highest order should be $4k$+2.
  - For some parameter ranges minimum energy solutions are homogeneous $\nabla\phi = 0$ - solution are pure time crystals.

Cosmology

- Such $\mathcal{L}$'s have been proposed as a source of inflationary vacuum energy:
  - k-inflation [Armendariz-Picon, Damour, Mukhanov 99]
  - ghost condensation [Arkani-Hamed, Cheng, Luty, Mukohyama 04]
  - interesting but different:
    - non-equilibrium, external t-dependent background
    - typically a potential for ϕ is forbidden by PQ symmetry
      - but PQ symmetries get broken..

- Could some of the characteristic features of higher-derivative Lagrangians (such as hard-wall turning points) lead to observable cosmological signatures, like bounces?

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Quantization

- Problem: momentum is not a good operator
  - $\dot{\phi}$ is better
  - So use noncanonical phase space coordinates $(\phi, \dot{\phi})$

    with symplectic form $\omega = d\phi \wedge dp = (3\dot{\phi}^2 - \kappa) d\phi \wedge d\dot{\phi}$

    and Poisson bracket $\{\phi, \dot{\phi}\} = \dfrac{1}{3\dot{\phi}^2 - \kappa}$
- Quantize: $[\phi, \dot{\phi}] = \dfrac{i\hbar}{3\dot{\phi}^2 - \kappa}$ ⇨ $\hat{\phi} = \dfrac{i\hbar}{3\dot{\phi}^2 - \kappa} \dfrac{\partial}{\partial \dot{\phi}}$
- Hamiltonian:

$$H = \tfrac{3}{4}\dot{\phi}^4 - \tfrac{\kappa}{2}\dot{\phi}^2 + V\left(\frac{i\hbar}{3\dot{\phi}^2 - \kappa} \frac{\partial}{\partial \dot{\phi}}\right)$$

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Branched Quantization

- Another approach: work directly with “bad” momentum

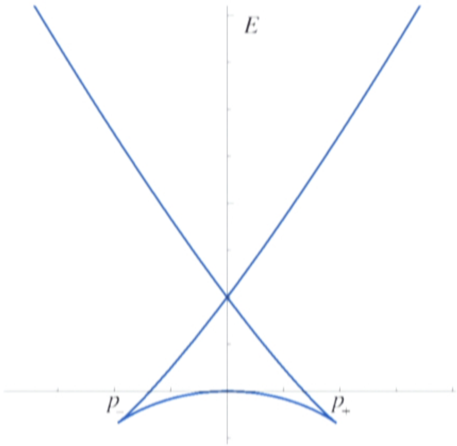
- Solve Hamiltonian

$$H(\phi, p) = \mathrm{Sw}(p) + V(\phi)$$

where $\mathrm{Sw}(p)$ is the multivalued “swallowtail” function

- Requires “unfolding” the catastrophe:
  - Break wavefunction $\psi(p)$ into 3 pieces

$$\begin{array}{ll} \psi_1(p) & -\infty < p \leqslant p_+ \\ \psi_2(p) & p_- \leqslant p \leqslant p_+ \\ \psi_3(p) & p_- \leqslant p < \infty \end{array}$$



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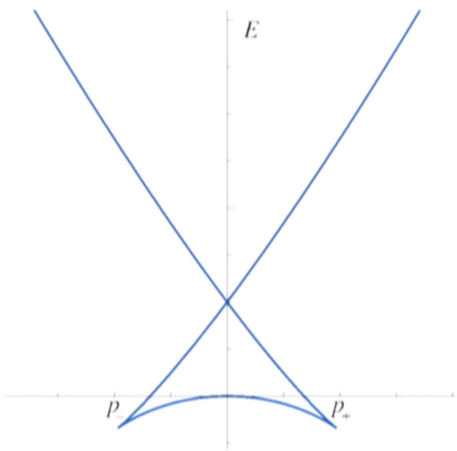

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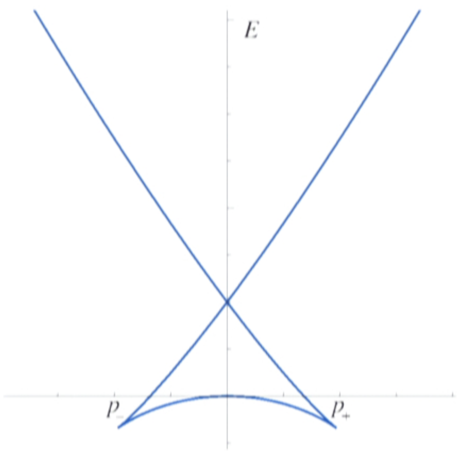

Branched Quantization

- Impose consistent boundary conditions at endpoints of these 3 intervals: conservation of probability current, i.e. **unitarity**.
- Probability density on each branch $\mu = 1, 2, 3$

$$\frac{\partial \rho_\mu}{\partial t} = i\left(\psi_\mu^* H \psi_\mu - (H^* \psi_\mu^*)\psi_\mu\right) = -\frac{i\alpha}{2}\left(\psi_\mu^* \frac{\partial^2 \psi_\mu}{\partial p^2} - \frac{\partial^2 \psi_\mu^*}{\partial p^2}\psi_\mu\right)$$

- Continuity equation for total probability $\rho \equiv \sum \rho_\mu$

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial p} = 0\,; \quad j \equiv \sum_\mu \frac{i\alpha}{2}\left(\psi_\mu^* \frac{\partial \psi_\mu}{\partial p} - \frac{\partial \psi_\mu^*}{\partial p}\psi_\mu\right)$$

- A consistent choice identifies enpoints of adjacent branches:

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Branched Quantization

- Impose consistent boundary conditions at endpoints of these 3 intervals: conservation of probability current, i.e. **unitarity**.
- Probability density on each branch $\mu = 1, 2, 3$

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Branched Quantization

- Another approach: work directly with "bad" momentum

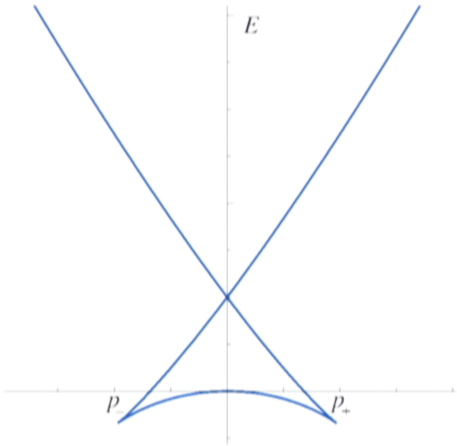
- Solve Hamiltonian

$$H(\phi, p) = \mathrm{Sw}(p) + V(\phi)$$

where $\mathrm{Sw}(p)$ is the multivalued "swallowtail" function

- Requires "unfolding" the catastrophe:
  - Break wavefunction $\psi(p)$ into 3 pieces

$$\begin{array}{ll} \psi_1(p) & -\infty < p \leqslant p_+ \\ \psi_2(p) & p_- \leqslant p \leqslant p_+ \\ \psi_3(p) & p_- \leqslant p < \infty \end{array}$$



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- These boundary conditions insure unitarity, self-adjointness of the Hamiltonian.
  - They define a **self-adjoint extension** of H
- But they are not the only possible choice.
  - e.g. reflecting boundary conditions

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Quantum Wires

- Consider two semi-infinite wires with endpoints a and b:

- What are all the possible unitary boundary conditions?
  - Identification
    $$\begin{aligned} \psi(a) &= \psi(b) \\ \frac{\partial \psi}{\partial x}(a) &= -\frac{\partial \psi}{\partial x}(b) \end{aligned}$$
  - Dirichlet/Neumann
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General boundary conditions

- Define $u \equiv \begin{pmatrix} \psi(a) & \psi(b) & \frac{\partial\psi}{\partial x}(a) & \frac{\partial\psi}{\partial x}(b) \end{pmatrix}^T$
- Use projection operator Π to enforce boundary conditions

$$u = \Pi\xi$$

- For example:

Dirichlet

$$\Pi_{\mathrm{D}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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  conservation of probability ⇨

$$u^\dagger J u = 0 \quad \text{where} \quad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

- A *hermitian symplectic* condition.
  - Projection onto a Lagrangian subspace
- Projected $u = \Pi\xi$ will satisfy condition if $\Pi^\dagger J \Pi = 0$
  - Π_D and $\Pi_=$ work.
- In fact, a whole $\mathrm{U}(2)$ of unitary boundary conditions
  - $\mathrm{U}(n)$ for n endpoints.

[Balachandran et. al. 1995]

Interpolating boundary conditions

- Can smoothly interpolate between Dirichlet and Identification conditions:

$$\Pi(\theta) = \frac{1}{2}\begin{pmatrix} c^2 & c^2 & cs & cs \\ c^2 & c^2 & cs & cs \\ cs & cs & 1+s^2 & -c^2 \\ cs & cs & -c^2 & 1+s^2 \end{pmatrix}$$

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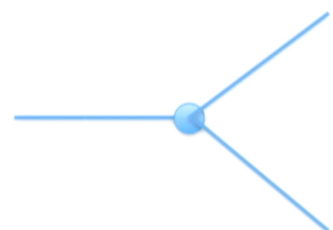

Operator deformations

- Topology change is smooth in Hilbert space; interpolating b.c.'s do not have a purely geometric interpretation in real space.
- Equivalently to above, can deform Hamiltonian by boundary operators.
- Cut a wire by adding a potential term

$$V(x) \;=\; v\delta(x-a)$$

  - interpolate between identification and Dirichlet as $v \;\rightarrow\; \infty$
- Implement Neumann with $V = \delta'(x-a)$.
- Identification requires a nonlocal operator

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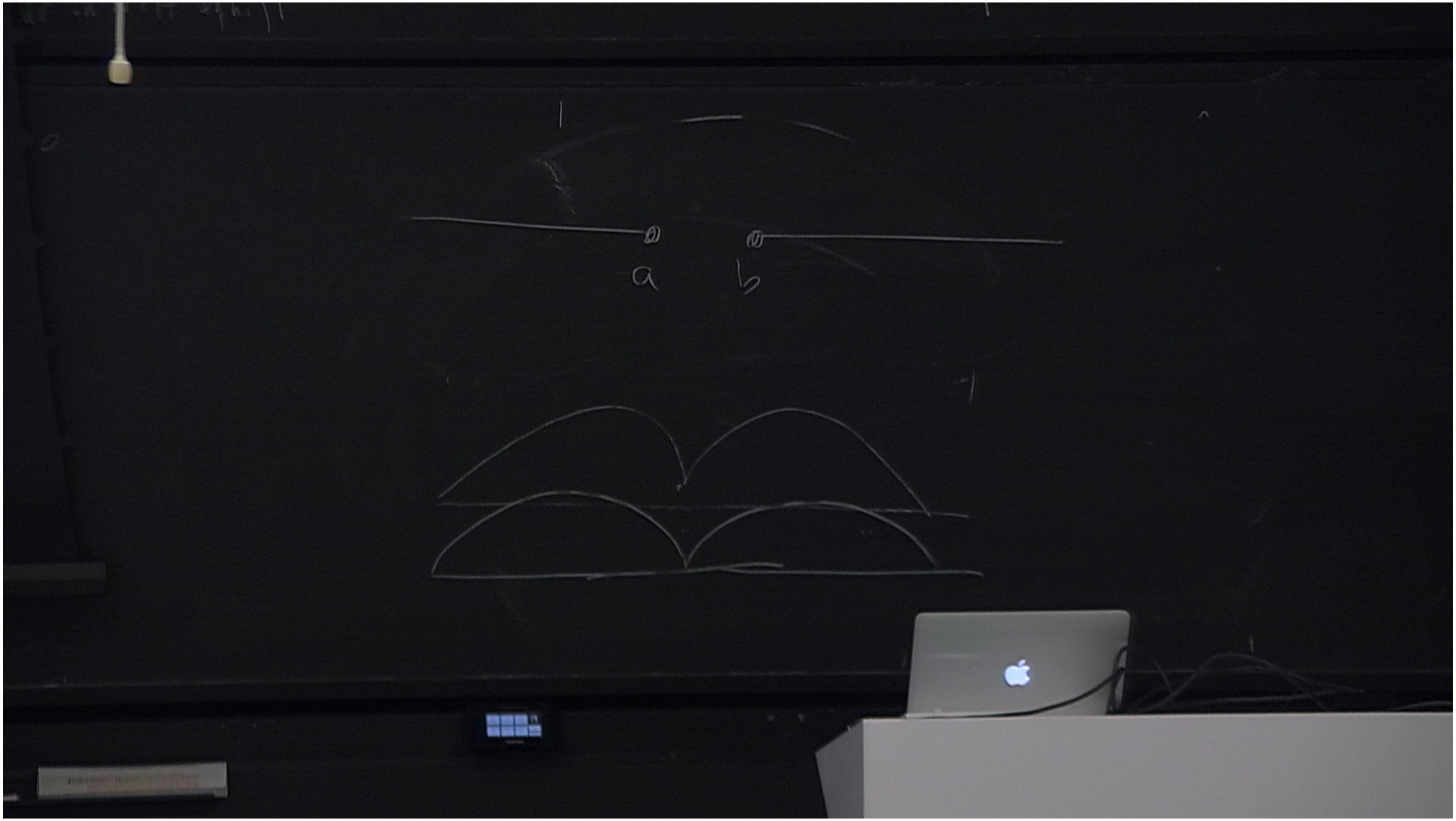
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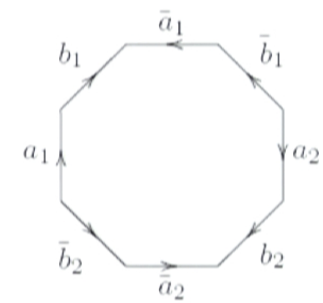
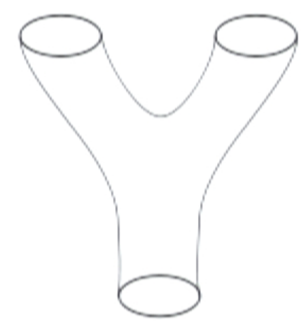
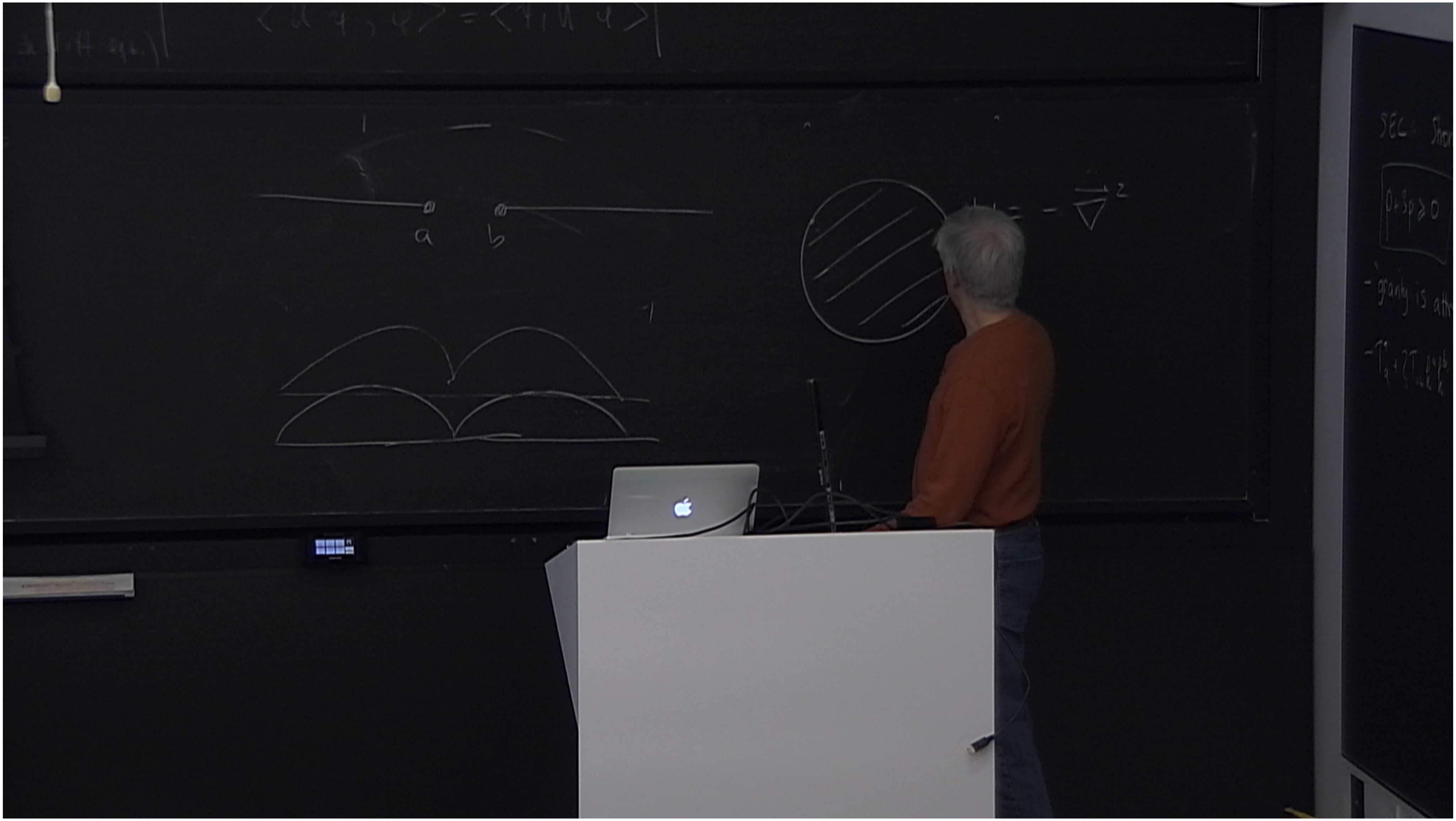
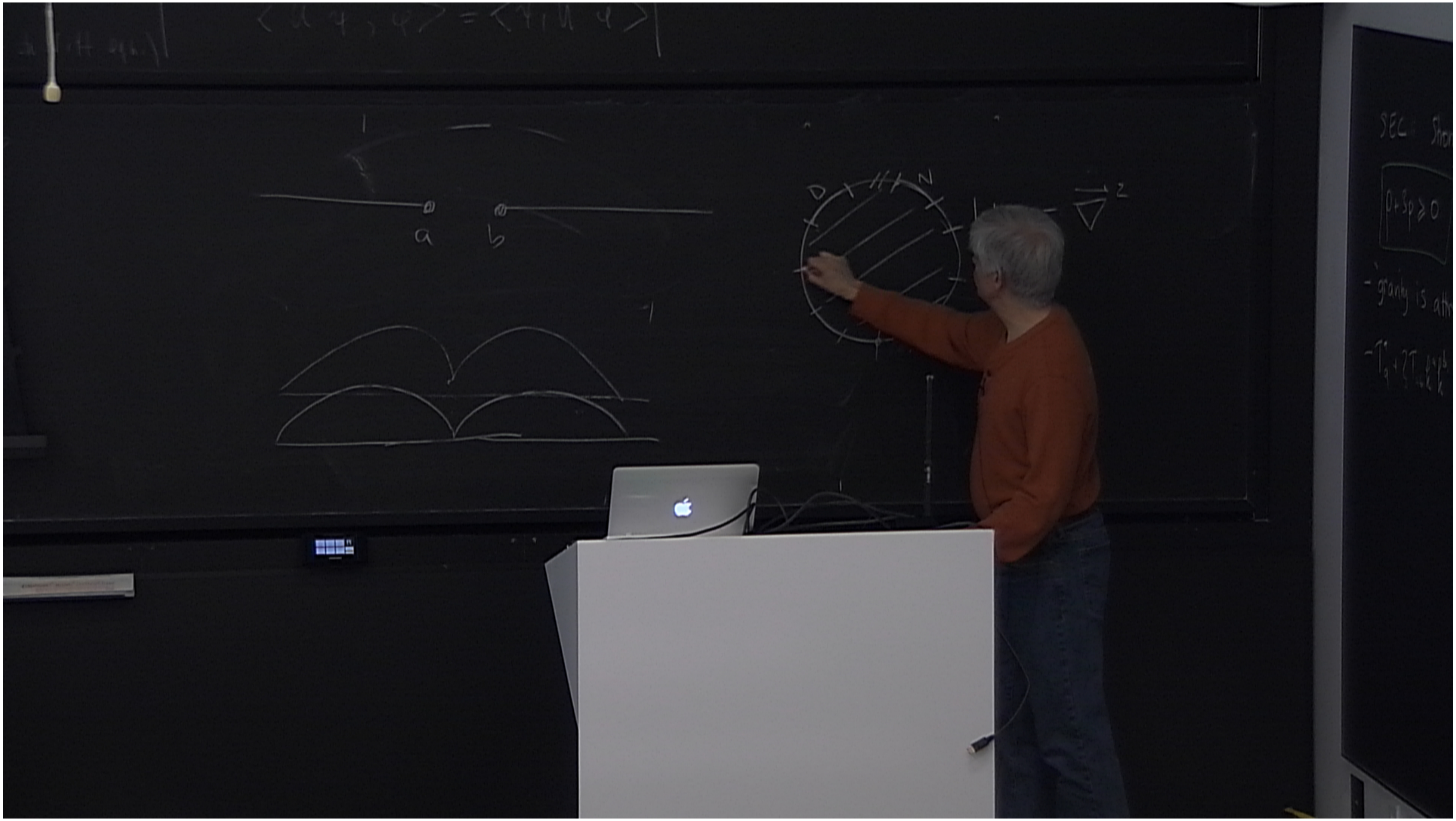
Extensions



- Higher dimensions, surgery
- Applications:
  - change topology of compactified spaces
  - Make parameters dynamical – a $\mathrm{U}(n)$ sigma-model in uncompactified space.
    - in 4D, "instantons" $\pi_3(\mathrm{U}(n))$ mediate topology change.
  - String vertex?
    - Change of topology from one circle to two.




a
b
SEC
ρ+3p ≥ 0


a
b
N
z
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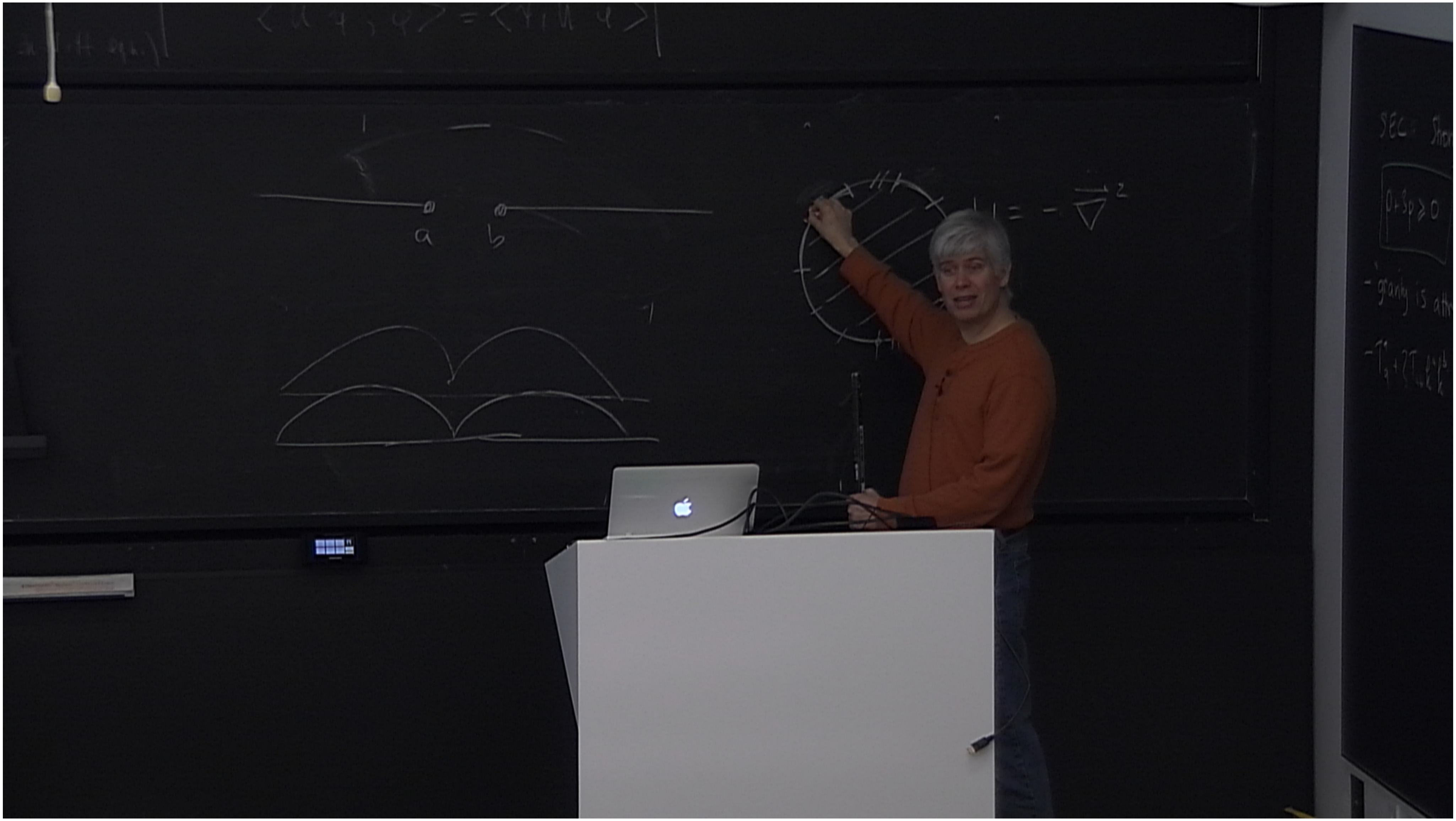

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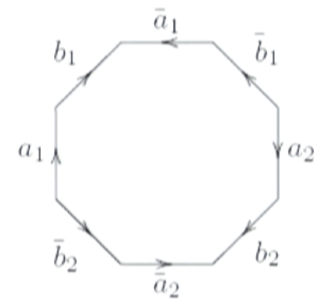

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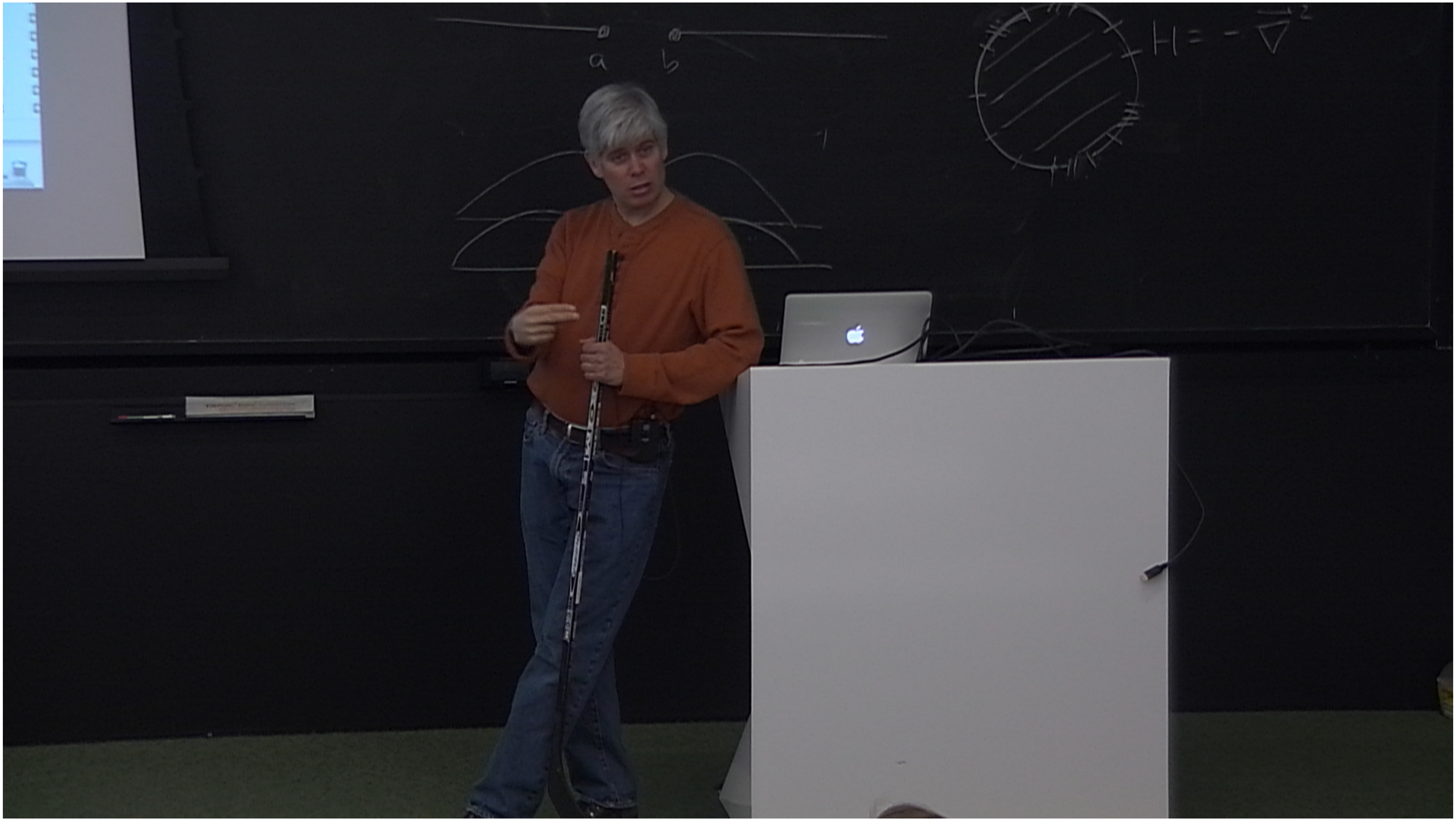
Back to time crystals

- The answer, after a long diversion, is that we can quantize time crystals in many consistent ways.
- It will be interesting to find out which (if any) of these quantizations are realized in nature.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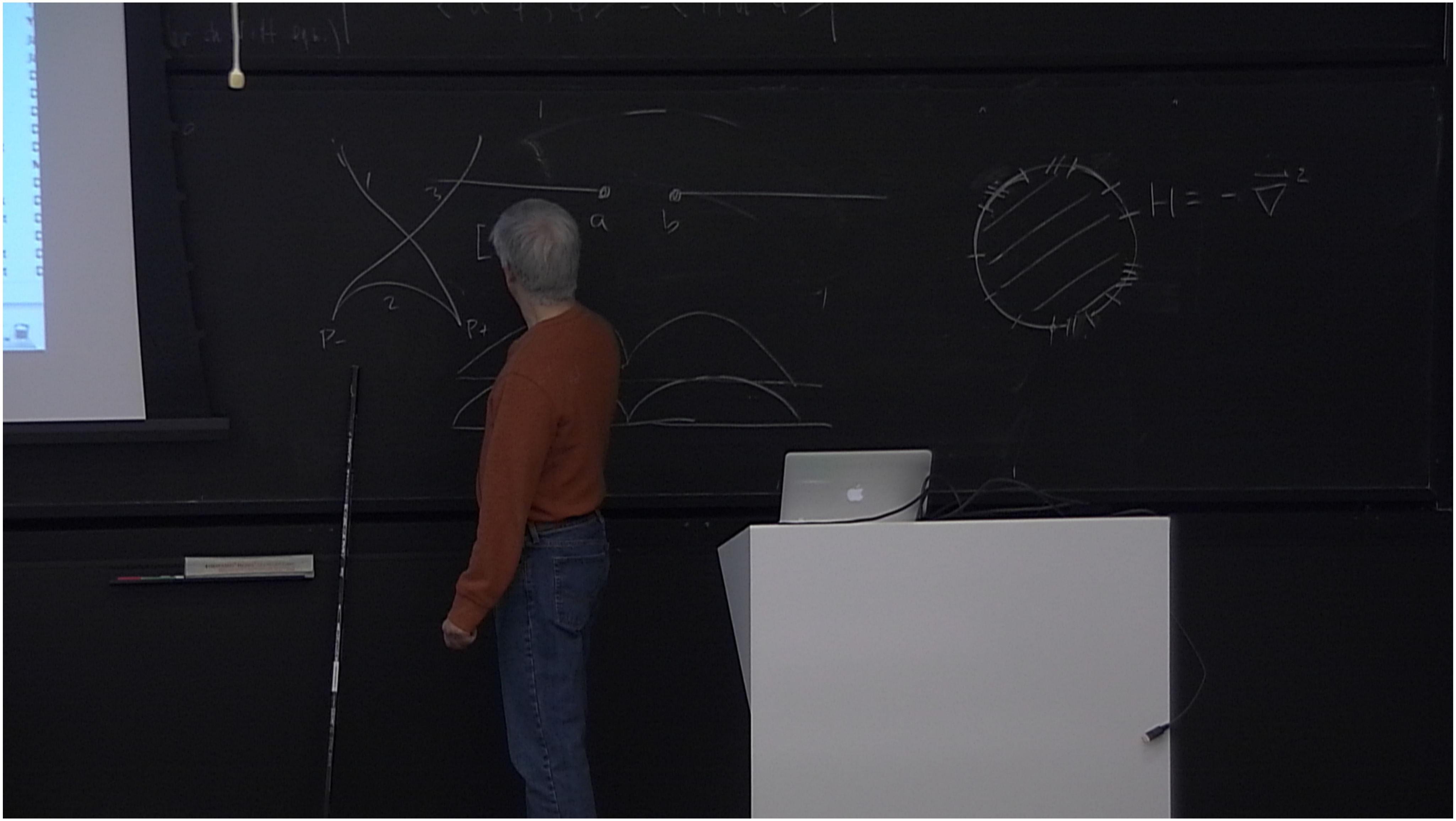
H = - ∇²


P-
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