Title: 12/13 PSI - Gravitational Physics Review Lecture 12

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Abstract:

# Applications of Gauss-Codazzi

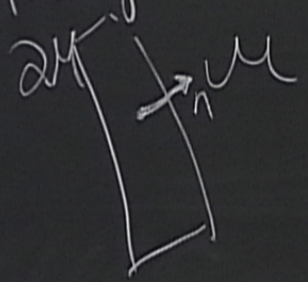
① Gravitational Action & the Gibbons-Hawking boundary term

Recall

$$\delta S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R_{ab}-\frac{1}{2}Rg_{ab}\right)\delta g^{ab}$$

$$-\frac{1}{16\pi G}\int_{\partial M} d^3x\sqrt{g}\left(D_n\delta g - n_a D_b \delta g^{ab}\right)$$

The boundary part of this variation contains normal derivatives of the metric - not normally set to zero in variational principle. Look for a boundary integral which will cancel this when varied.



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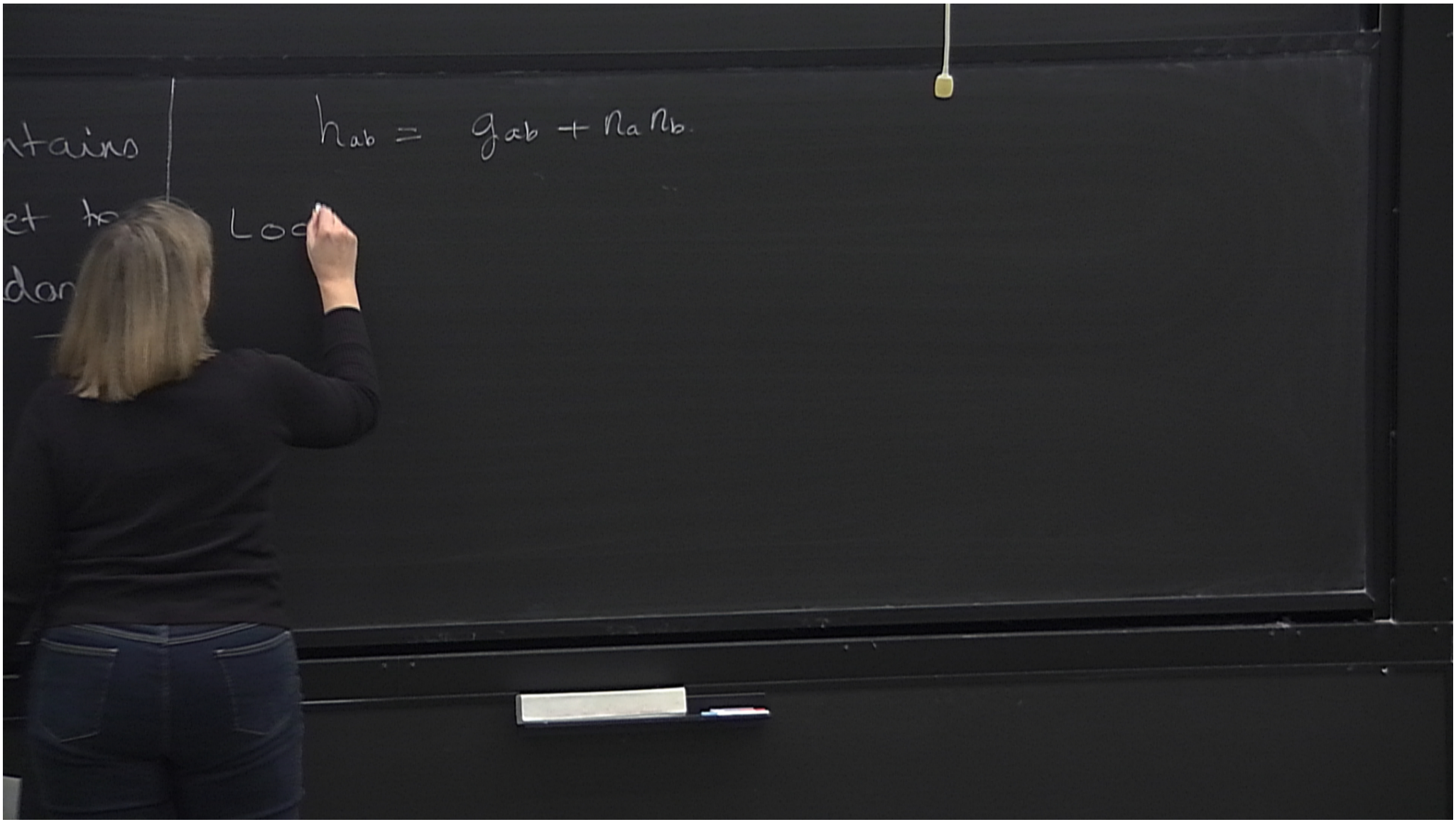
$$h_{ab} = g_{ab} + n_a n_b$$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$$h_{ab} = g_{ab} + n_a n_b$$ (Note: $\delta h_{ab} = 0$ for variational principle)

Look at $K$

$$\delta K = \delta(\nabla_a n^a) = \nabla_a \delta n^a + \delta \Gamma^a{}_{ab} n^b$$

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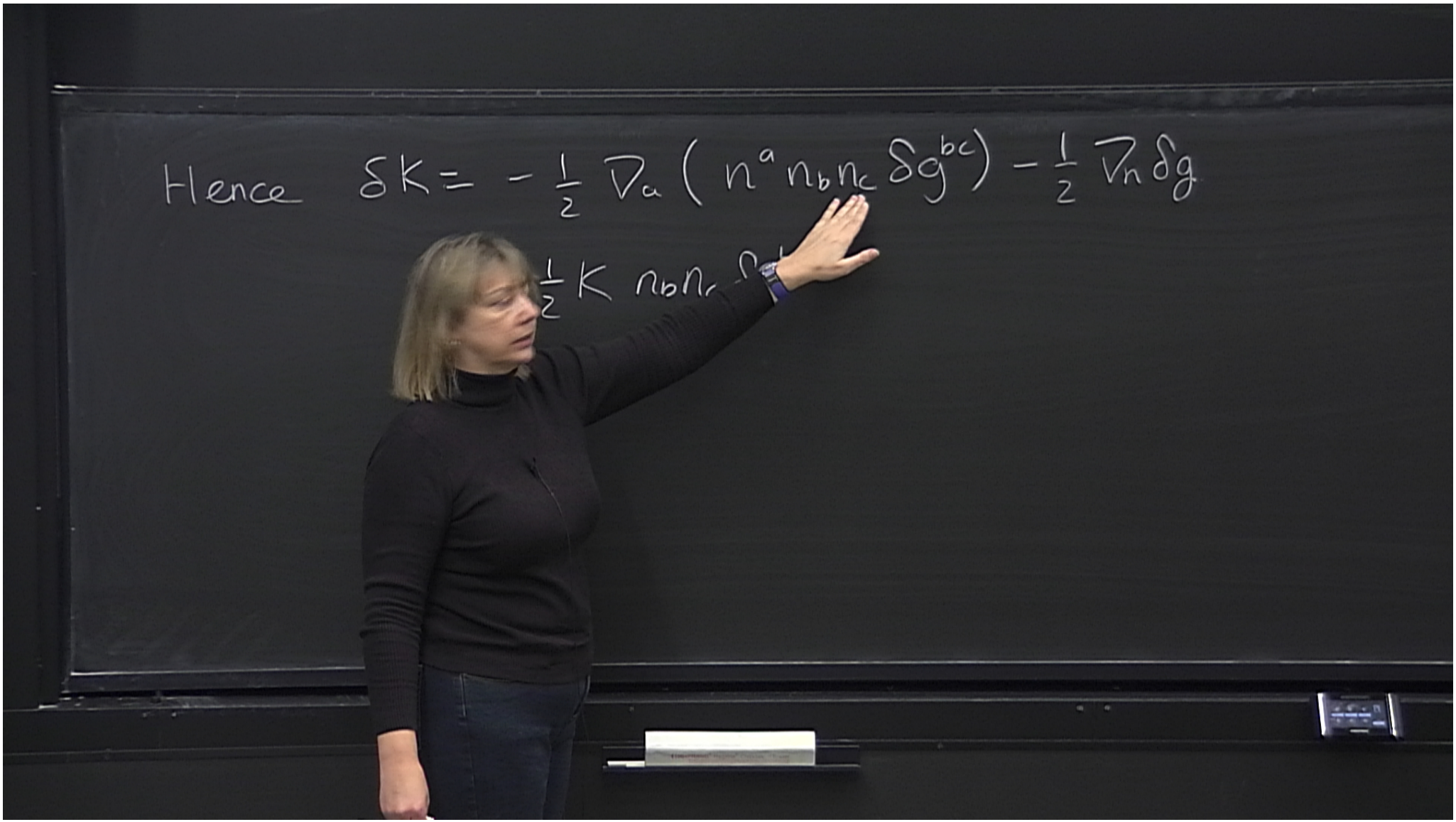
$$h_{ab} = g_{ab} + n_a n_b .$$ (Note: $\delta h_{ab} = 0$ for variational principle)

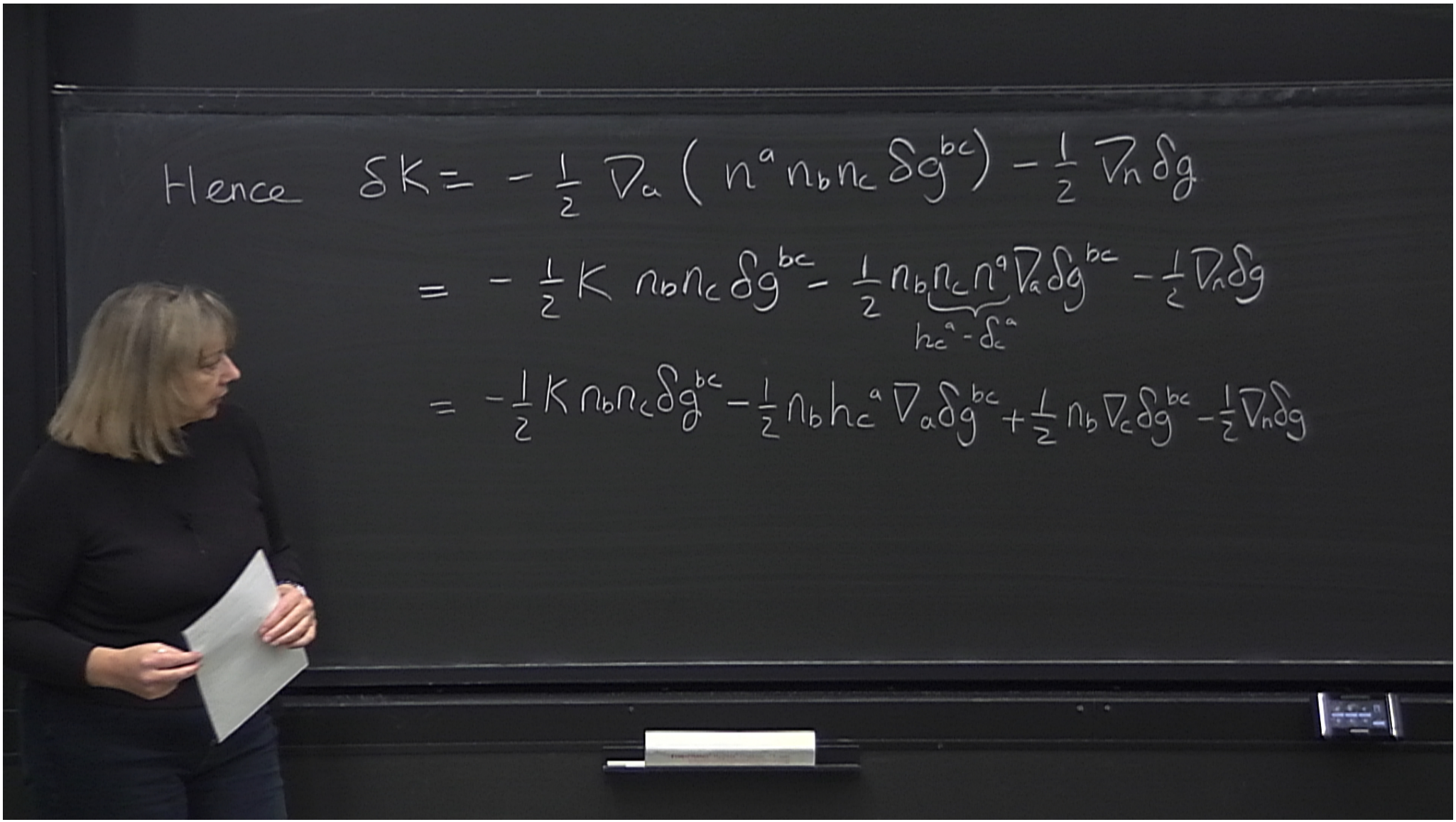
Look at $K$

$$\delta K = \delta(\nabla_a n^a) = \nabla_a \delta n^a + \delta \Gamma^a{}_{ab} n^b .$$

But $\delta \Gamma^a{}_{ab} = \frac{1}{2} g^{ab} \delta g_{ab,c} = -\frac{1}{2} \nabla_c (\delta g^{ab} g_{ab})$

Hence $\delta K = -\frac{1}{2} \nabla_a \left( n^a n_b n_c \delta g^{bc} \right) - \frac{1}{2} \nabla_n \delta g$


Hence $\delta K = -\frac{1}{2}\nabla_a\left(n^a n_b n_c \delta g^{bc}\right) - \frac{1}{2}\nabla_n \delta g$
$\frac{1}{2} K n_b n_c$


Hence $\delta K = -\frac{1}{2}\nabla_a\left(n^a n_b n_c \delta g^{bc}\right) - \frac{1}{2}\nabla_n \delta g$
$= -\frac{1}{2}K\, n_b n_c \delta g^{bc} - \frac{1}{2} n_b \underbrace{n_c n^a}_{h_c{}^a - \delta_c{}^a} \nabla_a \delta g^{bc} - \frac{1}{2}\nabla_n \delta g$
$= -\frac{1}{2}K\, n_b n_c \delta g^{bc} - \frac{1}{2} n_b h_c{}^a \nabla_a \delta g^{bc} + \frac{1}{2} n_b \nabla_c \delta g^{bc} - \frac{1}{2}\nabla_n \delta g$

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$$= -\frac{1}{2}K\, n_b n_c \delta g^{bc} - \frac{1}{2} n_b h_c{}^a \nabla_a \delta g^{bc} + \frac{1}{2} n_b \nabla_c \delta g^{bc} - \frac{1}{2}\nabla_n \delta g$$

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$$= -\frac{1}{2}K\, n_b n_c \delta g^{bc} \underbrace{- \frac{1}{2} n_b h_c{}^a \nabla_a \delta g^{bc}}_{-\frac{1}{2}\overset{(3)}{\nabla}_a\left(n_b \delta g^{bc}\right)} + \boxed{\frac{1}{2} n_b \nabla_c \delta g^{bc} - \frac{1}{2}\nabla_n \delta g}$$

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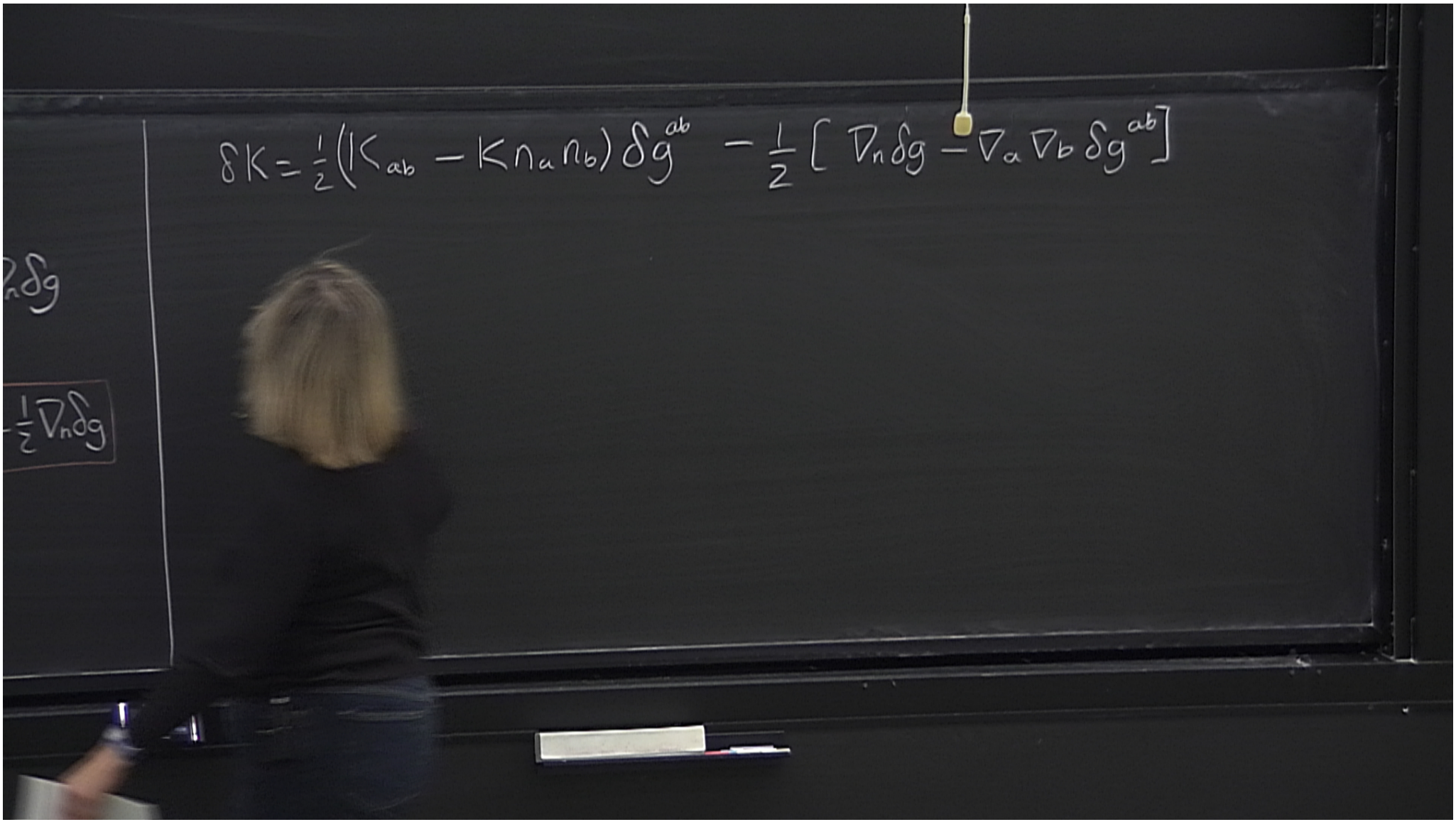
$$= -\frac{1}{2}K\, n_b n_c \delta g^{bc} \underbrace{- \frac{1}{2} n_b h_c{}^a \nabla_a \delta g^{bc}}_{-\frac{1}{2}\overset{(3)}{\nabla}_a\left(n_b \delta g^{bc}\right) + \frac{1}{2} K_{bc}\delta g^{bc}} + \boxed{\frac{1}{2} n_b \nabla_c \delta g^{bc} - \frac{1}{2}\nabla_n \delta g}$$


$\delta K = \frac{1}{2}(K_{ab} - K n_a n_b)\delta g^{ab} - \frac{1}{2}[\nabla_n \delta g - \nabla_a \nabla_b \delta g^{ab}]$
$\nabla_n \delta g$
$-\frac{1}{2}\nabla_n \delta g$

$$\delta K = \frac{1}{2}\left(K_{ab} - K n_a n_b\right)\delta g^{ab} - \frac{1}{2}\left[\nabla_n \delta g - n_a \nabla_b \delta g^{ab}\right]$$

$$\Rightarrow \delta\left(\sqrt{-g}\,K\right) = \frac{1}{2}\left(K_{ab} - K h_{ab}\right)\delta g^{ab} - \frac{1}{2}\left[\nabla_n \delta g - n_a \nabla_b \delta g^{ab}\right]$$

$\nabla_n \delta g$

$\boxed{-\frac{1}{2}\nabla_n \delta g}$

$$\delta K = \frac{1}{2}\left(K_{ab} - K n_a n_b\right)\delta g^{ab} - \frac{1}{2}\left[\nabla_n \delta g - n_a \nabla_b \delta g^{ab}\right] + \text{total deriv}$$

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Hence $\delta K = -\frac{1}{2}\nabla_a\left(n^a n_b n_c \delta g^{bc}\right) - \frac{1}{2}\nabla_n \delta g$

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$$= -\frac{1}{2}K\, n_b n_c \delta g^{bc} \underbrace{-\frac{1}{2} n_b h_c{}^a \nabla_a \delta g^{bc}}_{-\frac{1}{2}\overset{(3)}{\nabla}_c\left(n_b \delta g^{bc}\right) + \frac{1}{2}K_{bc}\delta g^{bc}} + \boxed{\frac{1}{2} n_b \nabla_c \delta g^{bc} - \frac{1}{2}\nabla_n \delta g}$$

$\delta$

$\Rightarrow \delta($

$$\delta K = \frac{1}{2}\left(K_{ab} - K n_a n_b\right)\delta g^{ab} - \frac{1}{2}\left[\nabla_n \delta g - n_a \nabla_b \delta g^{ab}\right] + \text{total deriv}$$

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If define the full gravitational action as

$$S_{EH} = \frac{1}{8\pi G}\int K\sqrt{g}$$

$$+ \frac{1}{2} K_{bc} \delta g^{bc}$$

$$\delta S_{GRAV} = - \frac{1}{16\pi G} \int_{\mathcal{M}} G_{ab} \delta g^{ab} \sqrt{g}\, d^4x - \frac{1}{16\pi G} \int_{\partial\mathcal{M}} (K_{ab} - K h_{ab}) \delta g^{ab} \sqrt{g}\, d^3x$$

$$+\frac{1}{2}K_{bc}\,\delta g^{bc}$$

$$\delta S_{GRAV} = -\frac{1}{16\pi G}\int_{\mathcal{M}} G_{ab}\,\delta g^{ab}\sqrt{g}\,d^4x - \frac{1}{16\pi G}\int_{\partial\mathcal{M}} (K_{ab} - K h_{ab})\,\delta g^{ab}\sqrt{g}\,d^3x$$

$\downarrow$

$=0$ for variational principle

GIBBONS–HAWKING

$h_{ab})\delta g^{ab}\sqrt{g}\,d^3x$

or variational
principle
ild.


$\Delta\tau = 8\pi G M$

Since $R = 0$ $\quad S_{EH} = 0$

Action is determined by $S_{GH}$

$$I_E = \frac{1}{8\pi G}\int d^3x\,\sqrt{h}\,K$$

Introduce a cut-off boundary at large r:

$$ds_3^2 = \left(1 - \frac{2GM}{R}\right) dt^2 + R^2 d\Omega_{II}^2$$

large r:

$$-\frac{2}{R}\sqrt{1-\frac{2GM}{R}} - \frac{GM}{R^2}\Big/\sqrt{1-\frac{2GM}{R}}$$

$$I_{G-H} = \frac{1}{8\pi G}\int dt.\sin\theta d\theta d\varphi . R^2\left(1-\frac{2GM}{R}\right)^{1/2}\left[-\frac{2}{R}\sqrt{1-\frac{2GM}{R}} - \frac{GM}{R^2\sqrt{1-\frac{2GM}{R}}}\right]$$

$$= -\frac{\beta}{2G}\left[2R\left(1-\frac{2GM}{R}\right) + GM\right] = -\frac{\beta R}{G} + \frac{3}{2}\beta M$$

large r:

$$-\frac{2}{R}\sqrt{1-\frac{2GM}{R}} - \frac{GM}{R^2}\Big/\sqrt{1-\frac{2GM}{R}}$$

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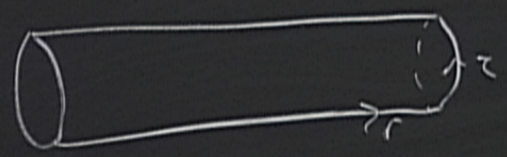
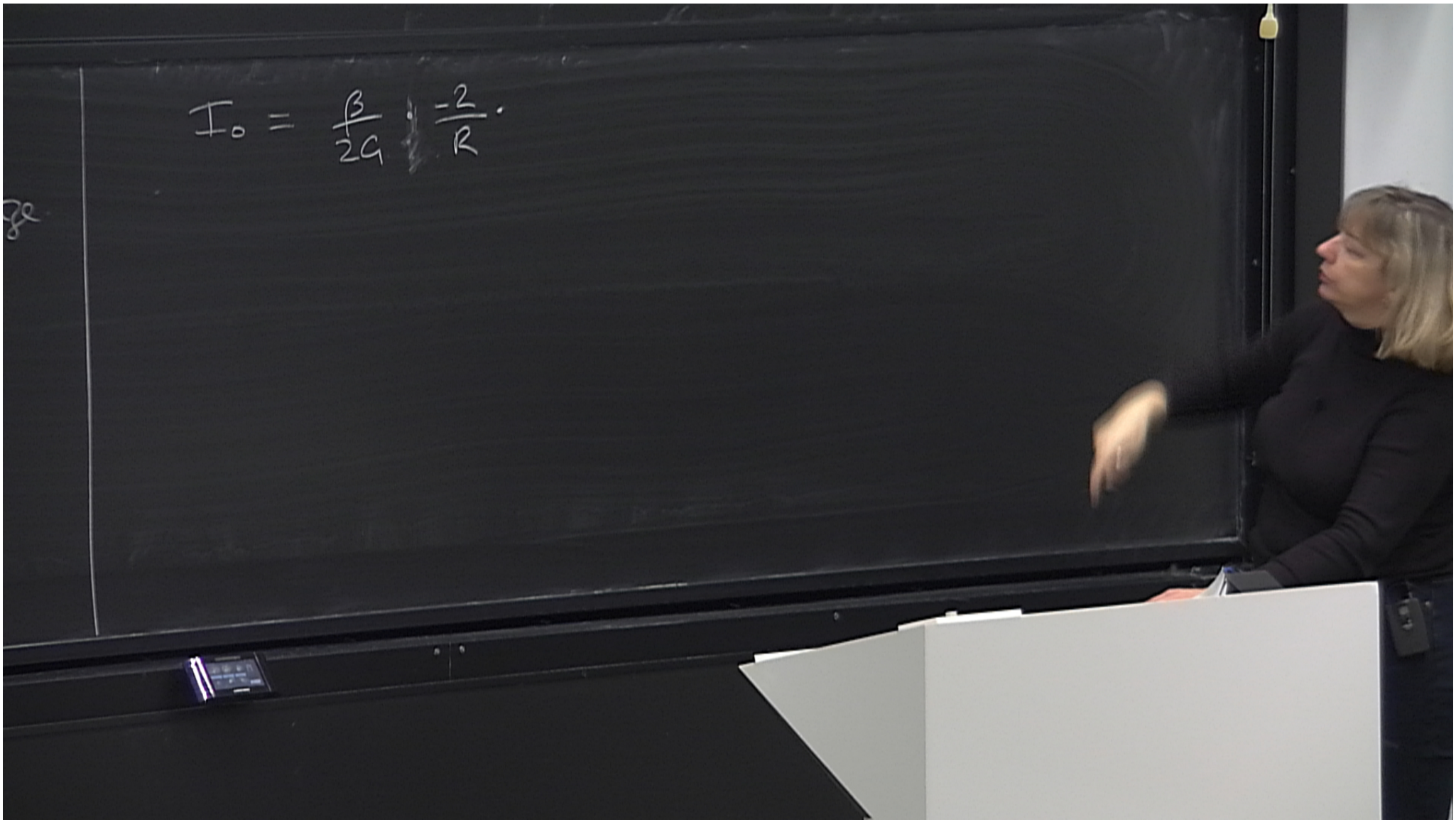
linearly divergent

But flat space at finite temp will also have divergent $S_{GH}$, so must renormalize




$I_0 = \frac{\beta}{2G} \frac{-2}{R}$.

But flat space at finite temp will also have divergent $S_{GH}$, so must renormalize


$$ds^2_{\substack{3D\\ FLAT\\ \text{at } r=R}} = \left(1-\frac{2GM}{R}\right)d\tau^2 + R^2 d\Omega_{II}^2$$

For flat space, $K = -\frac{2}{R}$


$$dT^2 + R^2 d\Omega_{II}^2 \qquad \Delta T = \sqrt{1-\frac{2GM}{R}}\,\Delta\tau$$

$$\boxed{d\tau^2\left(1-\frac{2GM}{R}\right) + R^2 d\Omega_{II}^2}$$

$I_0 =$

$=$

Regulange

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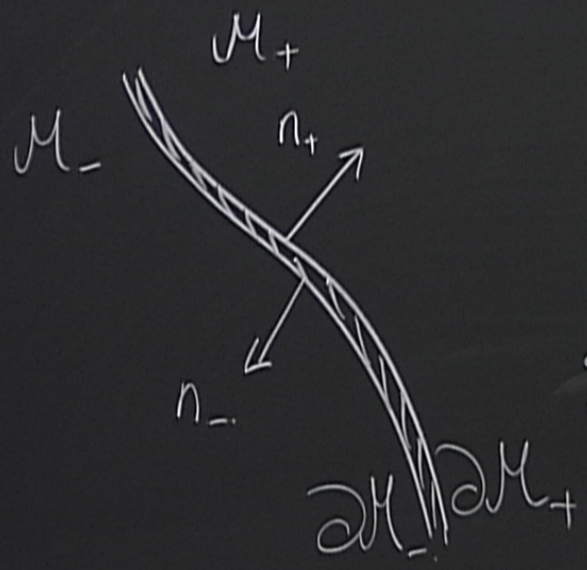
Regulange

② Israel Eqns - give Einstein's eqns for a th




$$\mathcal{M} = \mathcal{M}_+ \cup \mathcal{M}_-$$

$$\Sigma = \partial\mathcal{M}_- = \partial\mathcal{M}_+$$

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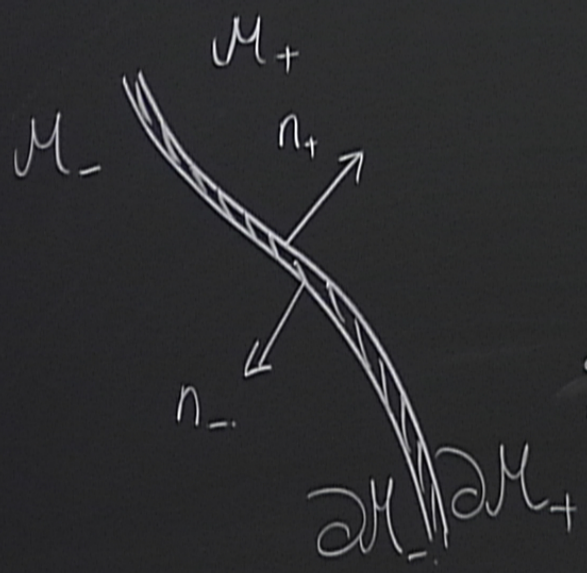


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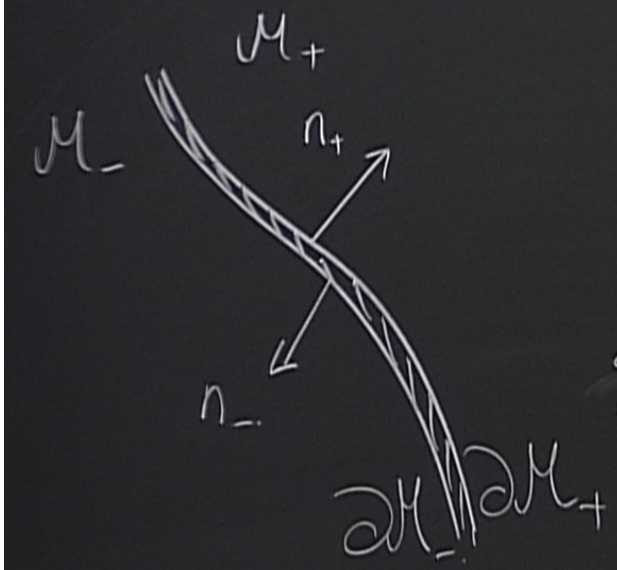
② <u>Israel Eqns</u> - give Einstein's eqns for a thin sh

The energy momentum of the sc

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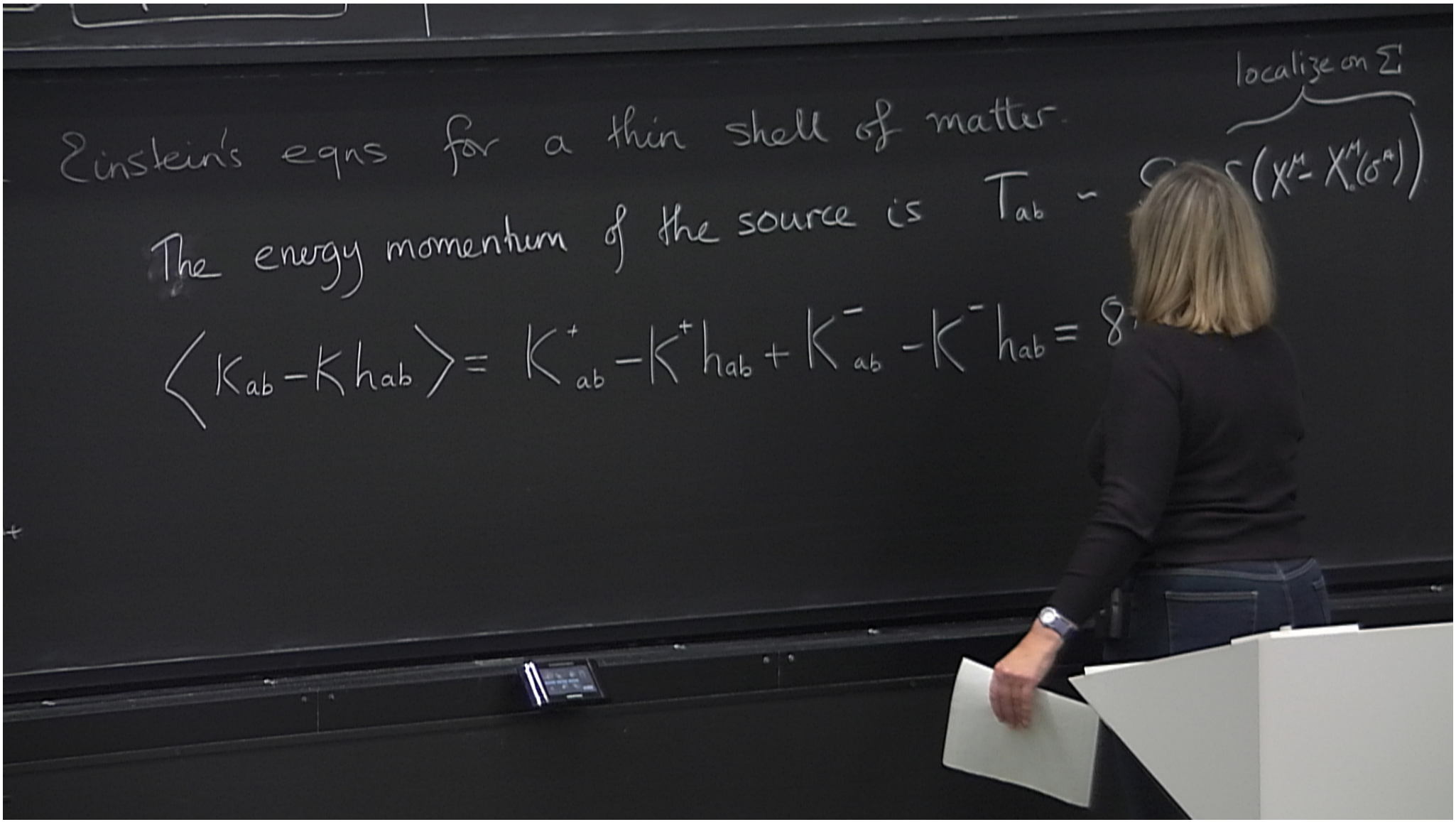
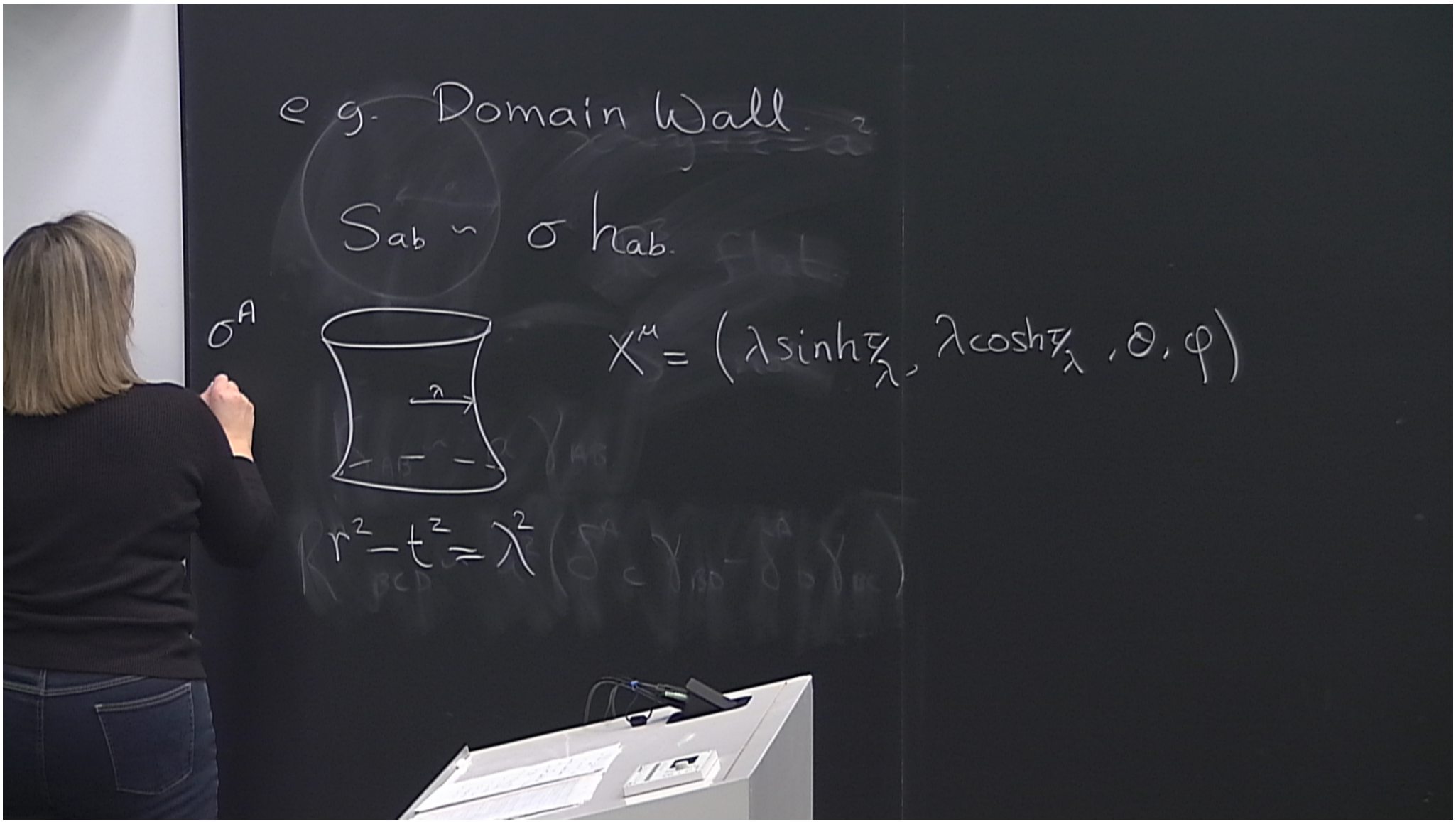
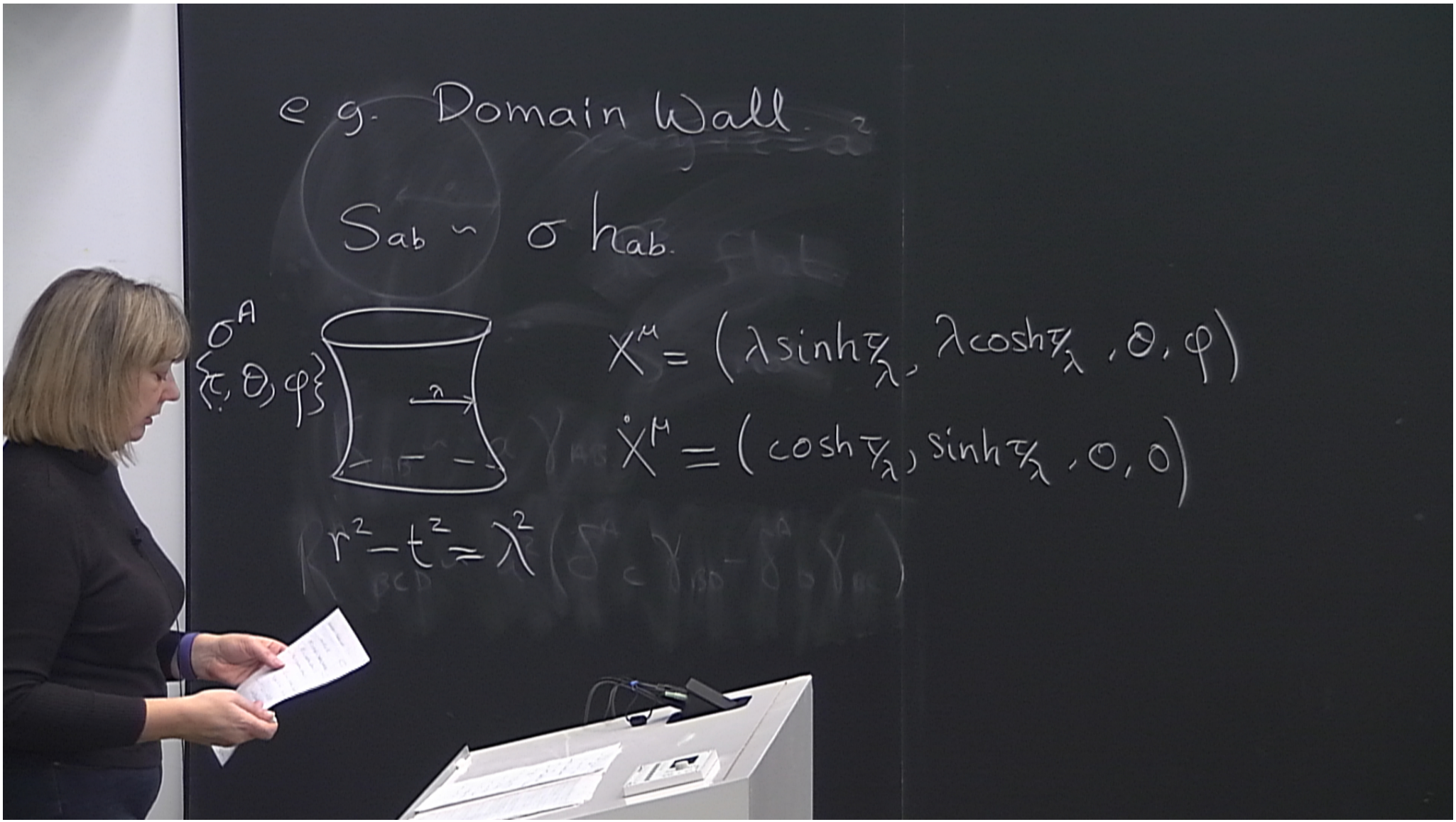

localize on Σ
Einstein's eqns for a thin shell of matter
The energy momentum of the source is $T_{ab} \sim$
$\langle K_{ab} - K h_{ab} \rangle = K^+_{ab} - K^+ h_{ab} + K^-_{ab} - K^- h_{ab} = 8$


e.g. Domain Wall
$S_{ab} \sim \sigma h_{ab}$
$\sigma^A$
$\lambda$
$X^\mu = (\lambda \sinh \tau/\lambda, \lambda \cosh \tau/\lambda, \theta, \varphi)$
$r^2 - t^2 = \lambda^2$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$\sigma^A$
$\{\tau, \theta, \varphi\}$
$\lambda$
$r^2 - t^2 = \lambda^2$
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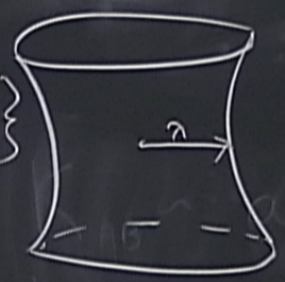
$\dot{X}^\mu = (\cosh \tau/\lambda, \sinh \tau/\lambda, 0, 0)$

e.g. Domain Wall

$$S_{ab} \sim \sigma h_{ab}$$

$\sigma^A$
$\{\tau, \theta, \varphi\}$




$$X^\mu = (\lambda \sinh \tau/\lambda, \lambda \cosh \tau/\lambda, \theta, \varphi)$$

$$\dot{X}^\mu = (\cosh \tau/\lambda, \sinh \tau/\lambda, 0, 0)$$

$$r^2 - t^2 = \lambda^2 \qquad n_\mu = (-\sinh \tau/\lambda, \cosh \tau/\lambda, 0,$$

$\tau/\lambda,\ \lambda\cosh\tau/\lambda,\ \theta,\ \varphi)$

$\tau/\lambda),\ \sinh\tau/\lambda,\ 0,\ 0)$

$\tau/\lambda),\ \cosh\tau/\lambda,\ 0,\ 0)$

$$K_{\theta\theta} = -\Gamma^{r}_{\theta\theta}\, n_r = r\cosh\tau/\lambda = \lambda\cosh^2\tau/\lambda.$$

$\lambda, \lambda\cosh\tau/\lambda, \theta, \varphi)$

$\tau/\lambda), \sinh\tau/\lambda, 0, 0)$

$\tau/\lambda), \cosh\tau/\lambda, 0, 0)$

$$K_{\theta\theta} = -\Gamma^r_{\theta\theta}\, n_r = r\cosh\tau/\lambda = \lambda\cosh^2\tau/\lambda$$

$$= -\frac{1}{\lambda}\, g_{\theta\theta} = -\frac{1}{\lambda}\, h_{\theta\theta}$$

$$K_{\tau\tau} = -n_\nu\, D_\tau \dot{X}^\nu$$

$$= \sinh\tau/\lambda\left[\frac{1}{\lambda}\sinh\frac{\tau}{\lambda}\right] - \cosh\frac{\tau}{\lambda}\left[\frac{1}{\lambda}\cosh\frac{\tau}{\lambda}\right]$$

$\tau/\lambda,\ \lambda\cosh\tau/\lambda,\ \theta,\ \varphi)$

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$$K_{\tau\tau} = -n_\nu\, D_\tau \dot{X}^\nu$$

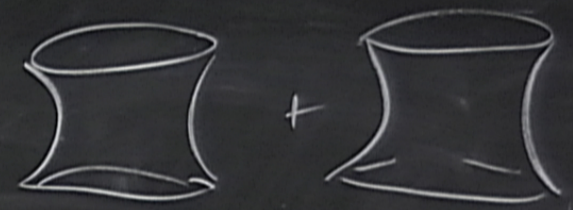
$$= \sinh\tau/\lambda\left[\frac{1}{\lambda}\sinh\frac{\tau}{\lambda}\right] - \cosh\frac{\tau}{\lambda}\left[\frac{1}{\lambda}\cosh\frac{\tau}{\lambda}\right]$$

$$= -1/\lambda = -\frac{1}{\lambda}\, g_{\tau\tau}$$

$$+ \frac{1}{2} K_{bc} \partial g^{bc}$$

So $K_{AB} = -\frac{1}{\lambda} \gamma_{AB}$.

If take 2 mirror images



$$-\frac{1}{\lambda} h_{ab} - \left(-\frac{1}{\lambda} h_{ab}\right) h_{ab} = 4\pi G \sigma h_{ab}$$

$$\frac{2}{\lambda} = 4\pi G \sigma$$