

Title: 12/13 PSI - Quantum Gravity Review Lecture 9

Date: Feb 07, 2013 10:15 AM

URL: <http://pirsa.org/13020050>

Abstract:

Quantum geometry

- Basic set of variables: Holonomy and Fluxes

smearing

1 dim

1 dim

$SU(2)$

Quantum geometry

- Basic set of variables: Holonomy and Fluxes $\rightarrow T^*SU(2)$
(for one edge)
- (smearing: 1 dim $\downarrow$ 1 dim)
- quantize one edge.

Quantum geometry

Basic set of variables: Holonomy and Fluxes $\rightarrow T^*SU(2)$
(for one edge)
smearing $\downarrow$ 1 dim $\downarrow$ 1 dim

* Quantize one edge. (Kinematical Hilbert space)

*

Quantum geometry

Basic set of variables: Holonomy and Fluxes $\rightarrow T^*SU(2)$
smearing $\downarrow$ 1 dim $\downarrow$ 1 dim (for one edge)

Quantize one edge. (Kinematical Hilbert space)
Gauß constraints $\rightarrow \delta(G)$

Quantum geometry

- Basic set of variables: Holonomy and Fluxes $\rightarrow T^*SU(2)$
(for one edge)
- $\xrightarrow{\text{smearing}} \downarrow 1 \text{ dim} \quad \downarrow 1 \text{ dim}$
- $\rightarrow$ * Quantize one edge. (Kinematical Hilbert space)
- * Gauss constraints $\rightarrow \mathcal{G}(\gamma) \rightarrow$ Quantization of triangle

Quantum geometry

the set of variables: Holonomy and Fluxes $\rightarrow T^*SU(2)$
smearing $\downarrow$ 1 dim $\downarrow$ 1 dim (for one edge)

* Quantize one edge. (Kinematical Hilbert space)

* Gauss constraints $\rightarrow \delta(\hat{G}) \rightarrow$ Quantization of triangle

* Flatness constraints $\rightarrow \delta(\hat{F})$

Quantum geometry

- Basic set of variables: Holonomy and Fluxes $\rightarrow T^*SU(2)$
(for one edge)
 $\xrightarrow{\text{smearing}} \downarrow 1 \text{ dim} \quad \downarrow 1 \text{ dim}$

$\rightarrow$ * Quantize edge. (Kinematical Hilbert space)

* Gauss constraints $\rightarrow \delta(\hat{G}_j) \rightarrow$ Quantization of triangle

* Hamiltonian constraints $\rightarrow \delta(\hat{H}) \rightarrow$ Quantization of tetrahedron

Quantum geometry

- Basic set of variables: Holonomy and Fluxes $\rightarrow T^*SU(2)$
(for one edge)
 $\downarrow$ $\downarrow$
smearing 1 dim 1 dim

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* Gauss constraints $\rightarrow \delta(\hat{G}) \rightarrow$ Quantization of triangle

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$\leftarrow \{6j\}$

Quantum geometry

Holonomies

• internal index $i = 1, 2, 3$

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→ transform wrt rotations $SO(3)$

$$(e^i)_a = R^i_j e^j_a$$

Holonomies

• internal index $i = 1, 2, 3$

→ transform wrt rotations $SO(3) \leftarrow$ "spin 1 representation"

$$(e^i)_a' = R^i_a e^i$$

$$R_{jk} = \delta_{jk} \cos \alpha + \underbrace{n_j n_k}_{\text{normal}} (1 - \cos \alpha) - \epsilon_{jkl} n^l \sin \alpha$$

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normal $\vec{n} \in S^2$ rotation angle

Holonomies

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rotation angle $\alpha \in [0, 2\pi]$

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normal $\vec{n} \in S^2$

rotation
angle $\in [0, \pi]$

$$0 \leq \alpha \leq \pi$$

$\cup \alpha = \pi$: identify "antipodal points"

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normal $\vec{n} \in S^2$

rotation
angle $\in [0, \pi]$

$0 \leq \alpha < \pi$

$\cup \alpha = \pi$: identify "antipodal points on S^2 : $\vec{n} \sim -\vec{n}$ "

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$SU(2)$: special unitary 2×2 matrices
 $\hookrightarrow \det = 1$

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

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$SU(2)$: special unitary 2×2 matrices
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$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

$$g = \exp(d \vec{n} \cdot \vec{T}_n)$$

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$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

$$g = \exp(d \vec{n} \cdot \vec{T})$$

" $\mathfrak{su}(2)$ Lie algebra basis"

$SU(2)$: special, unitary 2×2 matrices
 $\hookrightarrow \det(\cdot) = 1$

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

$$g = \exp(d \vec{n} \cdot \underset{\substack{\uparrow \\ \text{"su(2) Lie algebra basis" }}}}{T_j}) = \mathbb{1} \cos \frac{d}{2} - i \vec{n} \cdot \vec{\sigma} \sin \frac{d}{2}$$

"su(2) Lie algebra basis"

$$T_j = -\frac{i}{2} \sigma_j \leftarrow \text{Pauli matrices}$$

special unitary 2×2 matrices
 $\hookrightarrow \det(\cdot) = 1$

$\begin{pmatrix} b \\ \bar{a} \end{pmatrix}$

$$U(\vec{n}, \alpha) = \mathbb{1} \cos \frac{\alpha}{2} - i \vec{n} \cdot \vec{G} \sin \frac{\alpha}{2}$$

" $\mathfrak{su}(2)$ Lie algebra basis"

$$T_j = -\frac{i}{2} G_j \leftarrow \text{Pauli matrices}$$

$$\begin{aligned} \alpha &\in [0, 2\pi] \\ \vec{n} &\in S^2 \end{aligned}$$

normal $\vec{p} \in S^2$

angle $\in [0, \pi]$

$0 \leq \alpha < \pi$
or $\alpha = \pi$: identify "antipodal points" on

$SU(2)$: special unitary 2×2 matrices
 $\hookrightarrow \det(\cdot) = 1$

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

$$g = \exp\left(\alpha \vec{n} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \mathbb{1} \cos \frac{\alpha}{2} - i \vec{n} \cdot \vec{G} \sin \frac{\alpha}{2}$$

" $su(2)$ Lie algebra basis"

$$T_j = -\frac{i}{2} G_j \leftarrow \text{Pauli matrices}$$

$$\begin{array}{l} \alpha \in [0, 2\pi] \\ \vec{n} \in S^2 \\ \hline \Rightarrow \text{3-sphere } S^3 \end{array}$$

$\vec{n} \in S^2$

$\alpha \in [0, 2\pi]$

$\cup \alpha = \pi$: identify "antipodal points on S^2 " : $n \sim -n$

$SU(2)$: special unitary 2×2 matrices
 $\hookrightarrow \det(\cdot) = 1$

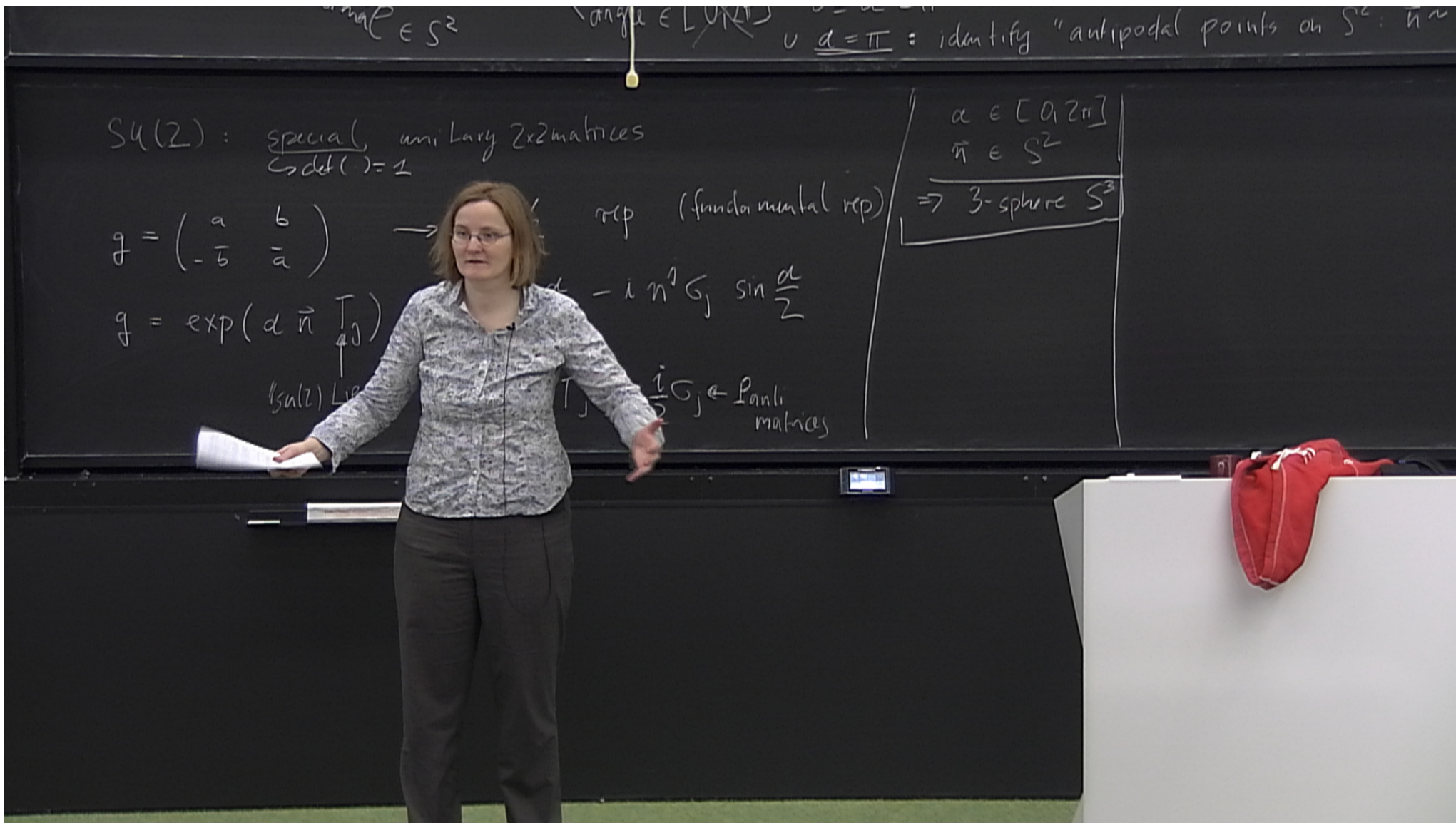
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" $SU(2)$ Lie algebra basis"

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$$\vec{n} \in S^2$$

$$\alpha \in [0, 2\pi]$$

$\cup \alpha = \pi$: identify "antipodal points on S^1 " : $U(1) \rightarrow SU(2)$

$SU(2)$: special unitary 2×2 matrices
 $\hookrightarrow \det(\cdot) = 1$

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \rightarrow \text{spin } \frac{1}{2} \text{ rep (fundamental rep)}$$

$$g = \exp\left(\alpha \vec{n} \cdot \vec{T}\right) = \mathbb{1} \cos \frac{\alpha}{2} - i \vec{n} \cdot \vec{\sigma} \sin \frac{\alpha}{2}$$

$\vec{T}$: " $SU(2)$ Lie algebra basis"

$T_j = -\frac{i}{2} \sigma_j \leftarrow \text{Pauli matrices}$

$$\begin{aligned} \alpha &\in [0, 2\pi] \\ \vec{n} &\in S^2 \\ \hline &\Rightarrow \text{3-sphere } S^3 \end{aligned}$$

spin 1 = spin $\frac{1}{2} \otimes$ spin $\frac{1}{2}$



$$\vec{n} \in S^2$$

$$\alpha \in [0, 2\pi]$$

$\cup \alpha = \pi$: identify "antipodal points on S^2 " : $n \sim -n$

$SU(2)$: special unitary 2×2 matrices
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$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \rightarrow \text{spin } \frac{1}{2} \text{ rep (fundamental rep)}$$

$$g = \exp\left(\alpha \vec{n} \cdot \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}\right) = 1 \cos \frac{\alpha}{2} - i \vec{n} \cdot \vec{\sigma} \sin \frac{\alpha}{2}$$

$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$ "SU(2) Lie algebra basis"
 $T_j = -\frac{i}{2} \sigma_j \leftarrow \text{Pauli matrices}$

$$\begin{array}{l} \alpha \in [0, 2\pi] \\ \vec{n} \in S^2 \\ \hline \Rightarrow \text{3-sphere } S^3 \end{array}$$

$$\text{spin } 1 = \text{spin } \frac{1}{2} \otimes \text{spin } \frac{1}{2} \\ (\oplus \text{spin } 0)$$

$$\det(\cdot) = 1$$

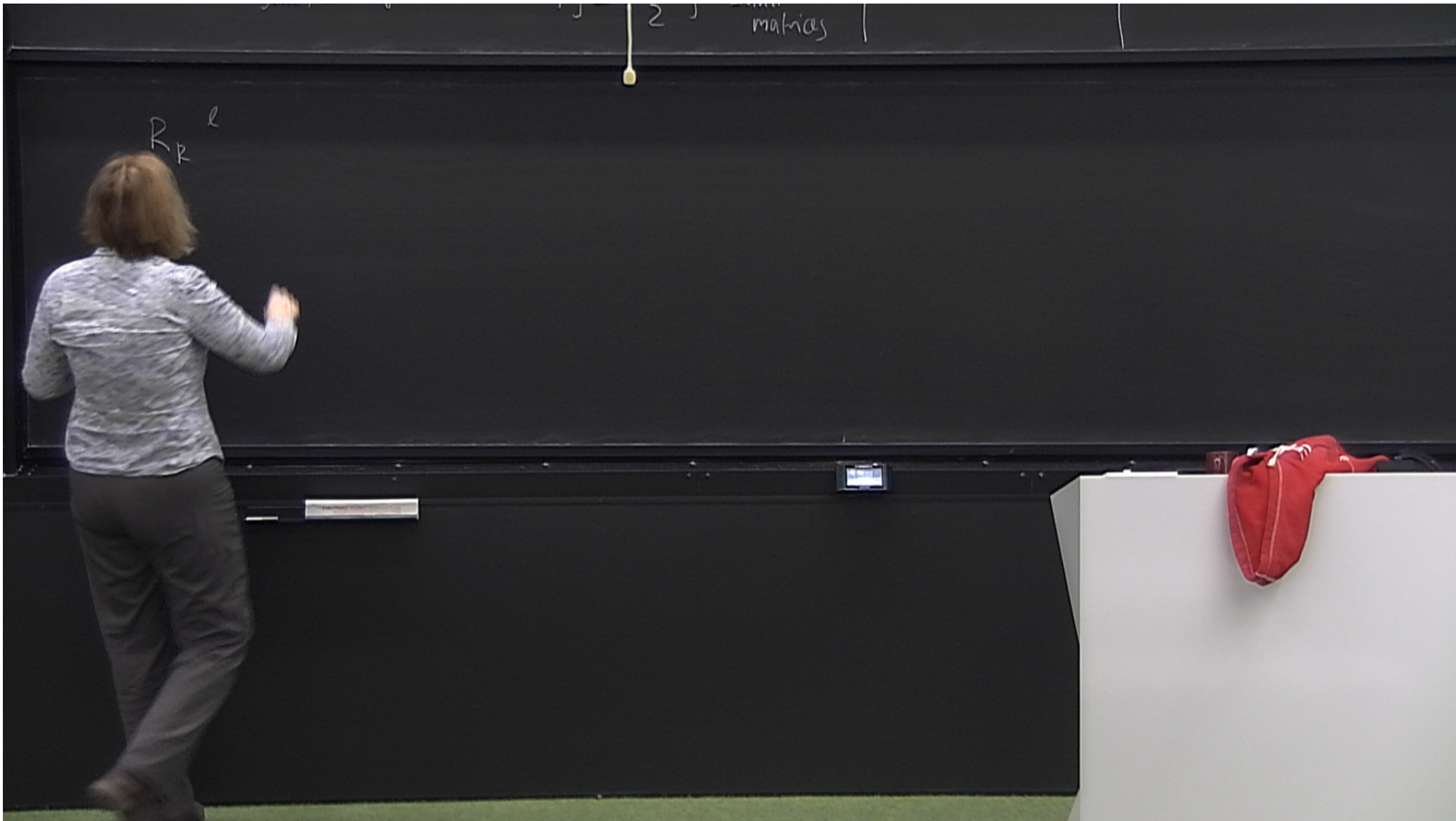
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$$T_j = -\frac{i}{2} \sigma_j \leftarrow \text{Pauli matrices}$$

$$\Rightarrow \text{3-sphere } S^3$$

$$\text{spin } 1 = \text{spin } \frac{1}{2} \otimes \text{spin } \frac{1}{2} \quad (\oplus \text{spin } 0)$$



$$R_R^L(g) T_L =$$

$\nearrow$
 2×2 matrices

matrices

$$R_k^L(g) T_k = g^{-1} \cdot T_k \cdot g \quad (\text{adjoint rep of } \text{SU}(2))$$

$\nearrow$
2x2 matrices

$$\boxed{R_R^{\ell}(g) T_{\ell} = g^{-1} \cdot T_R \cdot g}$$

$\swarrow$
 2×2 matrices

(adjoint rep of $SU(2)$)

$$\boxed{R_k^{\ell}(g) T_{\ell} = g^{-1} \cdot T_k \cdot g}$$

↖
2x2 matrices

(adjoint rep of $SU(2)$)

$$U^k R_k^{\ell}(g) T_{\ell} = g^{-1} (U^k T_k) g$$

$$\boxed{R_k^l(g) T_l = g^{-1} \cdot T_k \cdot g}$$

↙
2x2 matrices

(adjoint rep of $SU(2)$)

$$\Rightarrow v^k R_k^l(g) T_l = g^{-1} (v^k T_k) \cdot g$$

$$(R(g) \cdot v)^j T_j = g (v^k T_k) \cdot g^{-1}$$

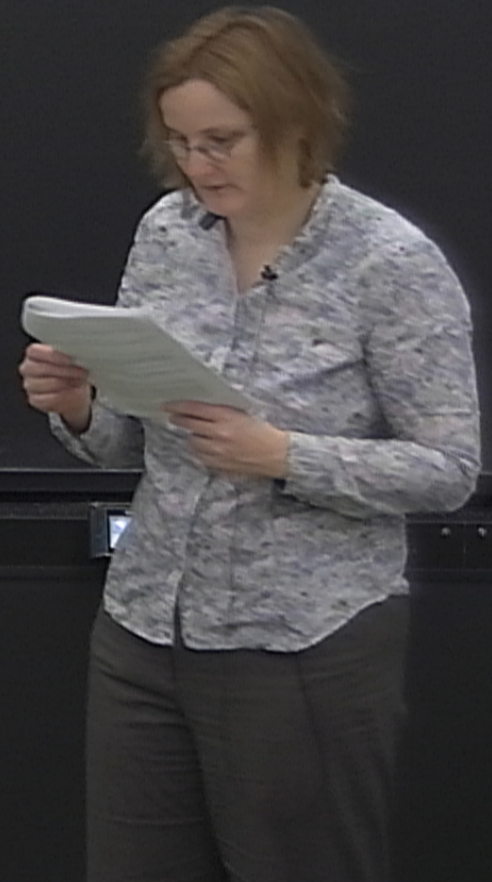
$$\boxed{R_k^{\ell}(g) T_{\ell} = g^{-1} \cdot T_k \cdot g}$$

↙
2x2 matrices

(adjoint rep of $SU(2)$)

$$\Rightarrow v^k R_k^{\ell}(g) T_{\ell} = g^{-1} (v^k T_k) \cdot g$$

$$(R(g) \cdot v)^j T_j = g (v^k T_k) \cdot g^{-1}$$



$$\boxed{R^{\ell}(g) T_{\ell} = g^{-1} \cdot T_k \cdot g}$$

↖
2x2 matrices

(adjoint rep of
SU(2))

$$\Rightarrow \sigma^k R^{\ell}(g) T_{\ell} = g^{-1} (\sigma^k T_k) \cdot g$$

$$(R(g) \cdot \sigma)^j T_j = g (\sigma^k T_k) \cdot g^{-1}$$

$$\boxed{R_R^{\ell}(g) T_{\ell} = g^{-1} \cdot T_k \cdot g}$$

↙
2x2 matrices

(adjoint rep of $SU(2)$)

$$\Rightarrow v^k R_{(k)}^{\ell}(g) T_{\ell} = g^{-1} (v^k T_k) \cdot g$$

$$(R(g) \cdot v)^j T_j = g (v^k T_k) \cdot g^{-1}$$

$$(R(g))^{-1} = R(g^{-1})$$

$$\boxed{R_R^{\ell}(g) T_{\ell} = g^{-1} \cdot T_k \cdot g}$$

2x2 matrices

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rep of

$$\mathbb{R}^3 \simeq \mathfrak{su}(2)$$

spanned by $\{T_j\}_{j=1,2,3}$

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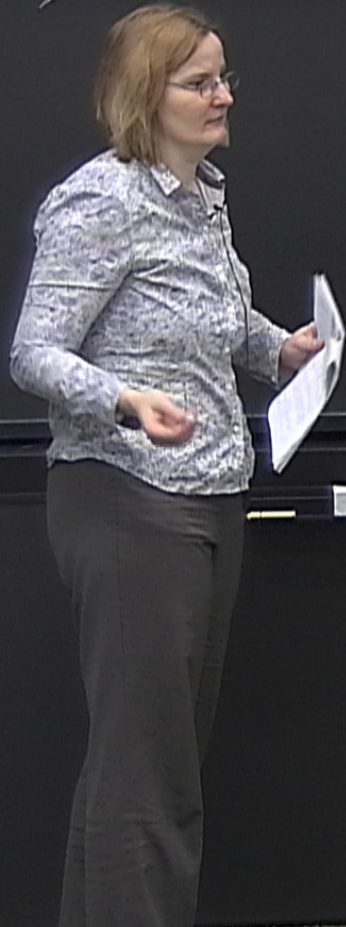
spanned by $\{T_j\}_{j=1,2,3}$

$$U \longrightarrow U T_j$$

mass constraints $\rightarrow d(\vec{F}) \rightarrow$ Quantization of tetrahedron

Now introduce "Lie algebra valued" objects

$\Rightarrow$



mass constraints $\rightarrow d(\vec{F}) \rightarrow$ Quantization of tetrahedron

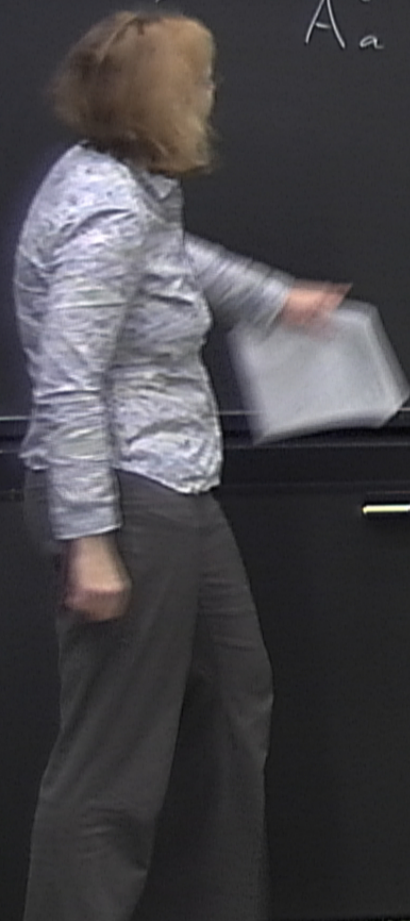
Now introduce "Lie algebra valued" objects

$$\Rightarrow A^i \rightarrow A_a^i T_i = A_a$$

mass constraints $\rightarrow d(\vec{F}) \rightarrow$ Quantization of tetrahedron

Now introduce "Lie algebra valued" objects

$$\Rightarrow A_a^i \rightarrow A_a^i T_i =: A_a$$



mass constraints $\rightarrow d(\hat{F}) \rightarrow$ Quantization of tetrahedron

Now introduce "Lie algebra valued" objects

$$\Rightarrow \begin{aligned} A_a^i &\rightarrow A_a^i T_i =: A_a \\ F_{ab}^i &\rightarrow F_{ab}^i T_i =: F_{ab} \end{aligned}$$

mass constraints $\rightarrow d(\vec{F}) \rightarrow$ Quantization of tetrahedron

Now introduce "Lie algebra valued" objects

$\Rightarrow$

$$A^i \rightarrow A_a^i T_i =: A_a$$

$$\rightarrow F_{ab}^i T_i =: F_{ab}$$

$$\rightarrow E^a{}^i T_i =: E^a$$

mass constraints $\rightarrow d(\vec{F}) \rightarrow$ Quantization of tetrahedron

Now introduce "Lie algebra valued" objects

$$\Rightarrow \begin{aligned} A^i &\rightarrow A_a^i T_i =: A_a \\ &\rightarrow F_{ab}^i T_i =: F_{ab} \\ E^a &\rightarrow E^a_j T_j =: E^a \end{aligned}$$

mass constraints $\rightarrow d(\vec{F}) \rightarrow$ Quantization of tetrahedron

Now introduce "Lie algebra valued" objects

Under $SL(2)$ "rotation" g :

$\Rightarrow$

$$A_a^i \rightarrow A_a^i T_i =: A_a$$

$$F_{ab}^i \rightarrow F_{ab}^i T_i =: F_{ab}$$

$$E^a_i \rightarrow E^a_i T_i =: E^a$$

mass constraints $\rightarrow d(\vec{F}) \rightarrow$ Quantization of tetrahedron

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$$E^a_i \rightarrow E^a_i T_i =: E^a$$

Under $SL(2)$ "rotation" g :

$$E^a = g$$

rep. of

$$\mathbb{R}^3 \simeq \mathfrak{su}(2)$$

spanned by $\{T_j\}_{j=1,2,3}$

$$U \longrightarrow U T_j$$

(2 x 2 matrix)

$$R(g_1) R(g_2)$$

Quantization of Yang-Mills

(g-valued) objects

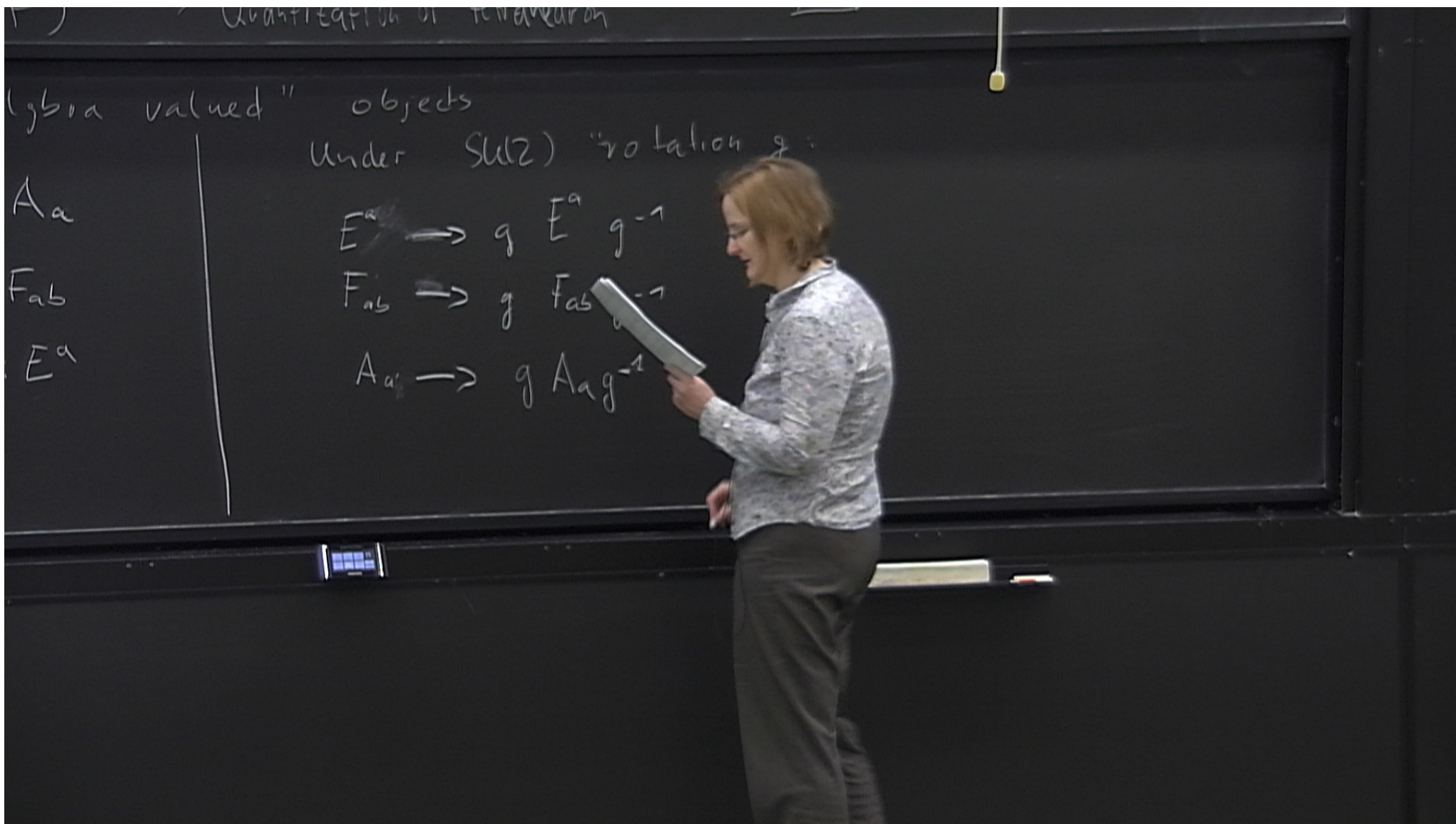
Under $SU(2)$ rotation g :

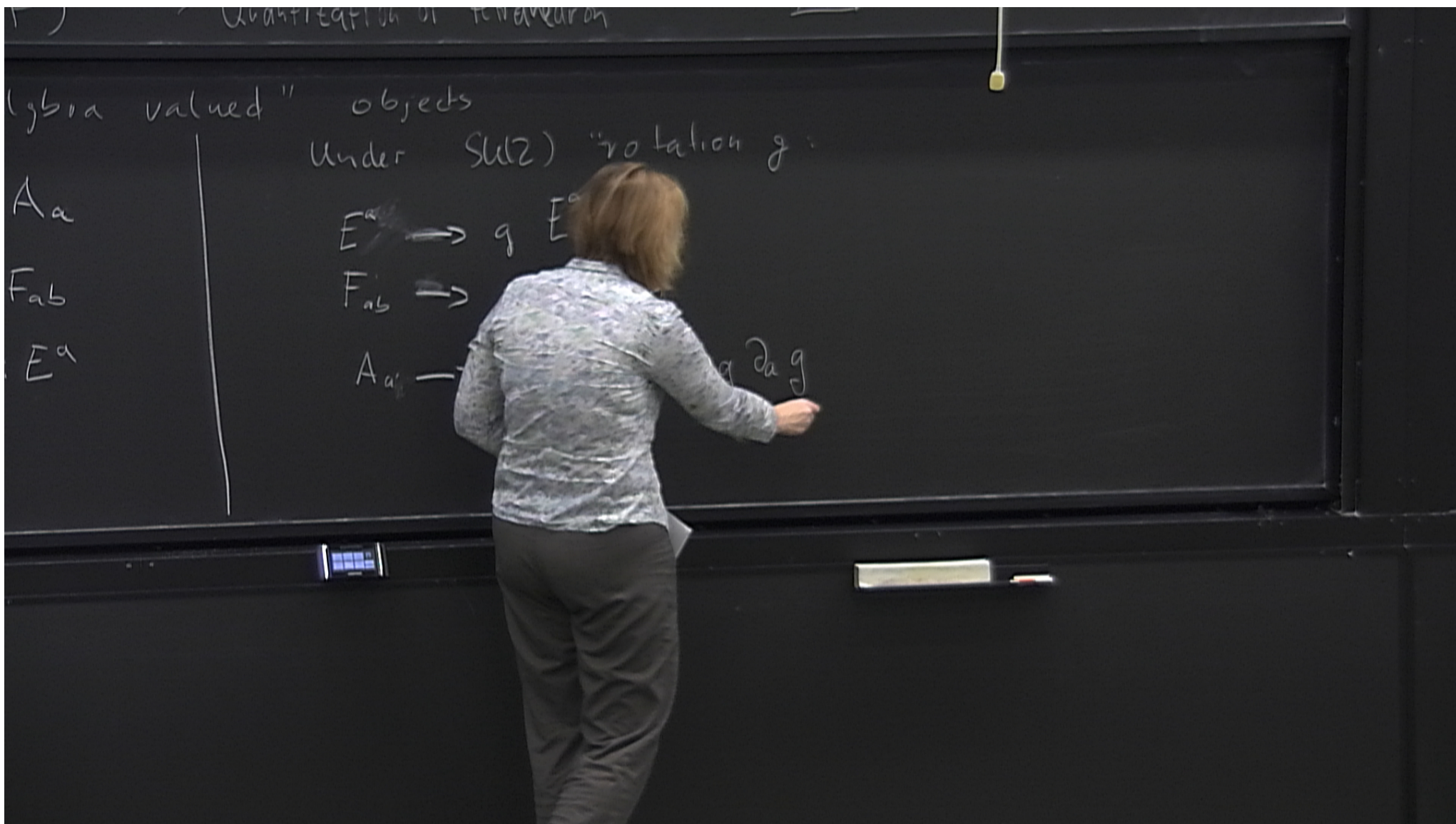
A_a

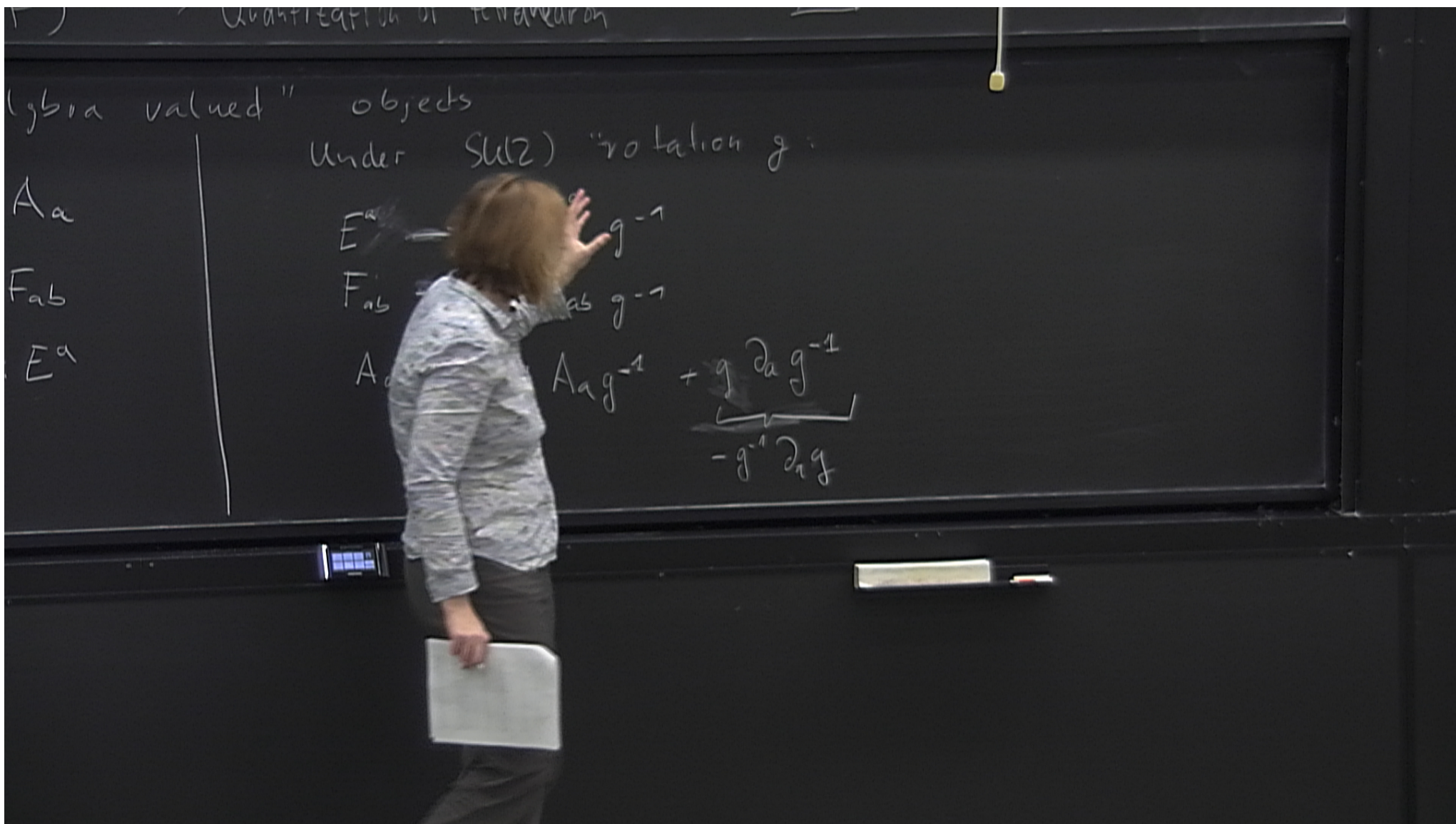
$$E^a \rightarrow g E^a g^{-1}$$

$$F_{ab} \rightarrow g F_{ab} g^{-1}$$

$$A_{a'} \rightarrow$$







Quantization of Yang-Mills

(g-valued" objects

Under $SL(2)$ rotation g :

$A_a \rightarrow g A_a g^{-1}$

$F_{ab} \rightarrow g F_{ab} g^{-1}$

$E^a \rightarrow g A_a g^{-1} + \underbrace{g \partial_a g^{-1}}_{-g^{-1} \partial_a g}$

(g-valued) objects

Under $SL(2)$ rotation g :

A_a

$$E^a \rightarrow g E^a g^{-1}$$

F_{ab}

$$F_{ab} \rightarrow g F_{ab} g^{-1}$$

E^a

$$A_a \rightarrow g A_a g^{-1} + \underbrace{g \partial_a g^{-1}}_{-g^{-1} \partial_a g}$$

$$-g^{-1} \partial_a g \in \partial_a (g g^{-1}) = 0$$

$\gamma \rightarrow$

$$\gamma \rightarrow \gamma^k = R_{(\gamma)}^k \gamma$$

$z \rightarrow$

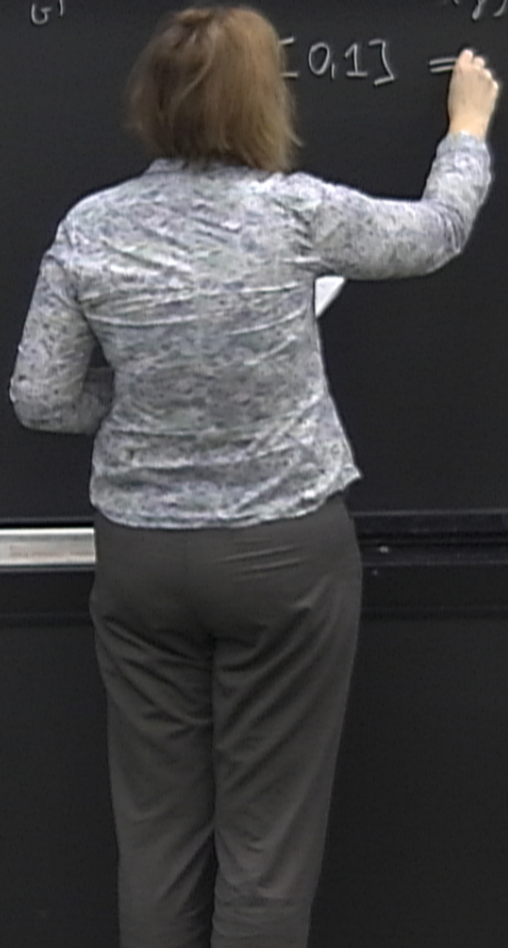
$$z \xrightarrow{\gamma} z' = R_{(\gamma)}^{\mu} z_{\mu}$$

$z \rightarrow$

$U(s)$

$$z \xrightarrow{\gamma} z_k = R_{(\gamma)}^k z$$

$$[0,1] =$$



$z \rightarrow$

$v(s)$

$$z \xrightarrow{\gamma} z = R_{(\gamma)}^k z$$

$$\gamma: [0,1] \Rightarrow \mathcal{M}$$

$\sigma_j \rightarrow$

$$\sigma_j \xrightarrow{\gamma} \sigma_j^k = R_{(\gamma)}^k \sigma_j$$

$$U(s) = h(s) \sigma$$

$$\gamma: [0,1] \rightarrow \mathcal{M}$$

$$s \in [0,1]$$

$$h(s) \in SU(2)$$

$\sigma^i \rightarrow$

$$\sigma^i \xrightarrow{\gamma} \sigma^k = R^k_{(Y)^i} \sigma^i$$

$$U(s) = h(s) U(0) h^{-1}(s)$$

$$\gamma: [0,1] \rightarrow \mathcal{M}$$

$$s \in [0,1]$$

$h(s) \in SU(2)$: holonomy

$$D_a U^i = \partial_a U^i + \epsilon^i_{jk} A^k_a U^j$$

$$D_a U = \partial_a U + [$$

$$\sigma^i \rightarrow$$

$$\sigma^i \xrightarrow{\gamma} \sigma^k = R_{(\gamma)}^k{}_i \sigma^i$$

$$\boxed{U(s) \quad (s) \quad U(0) \quad h(s)}$$

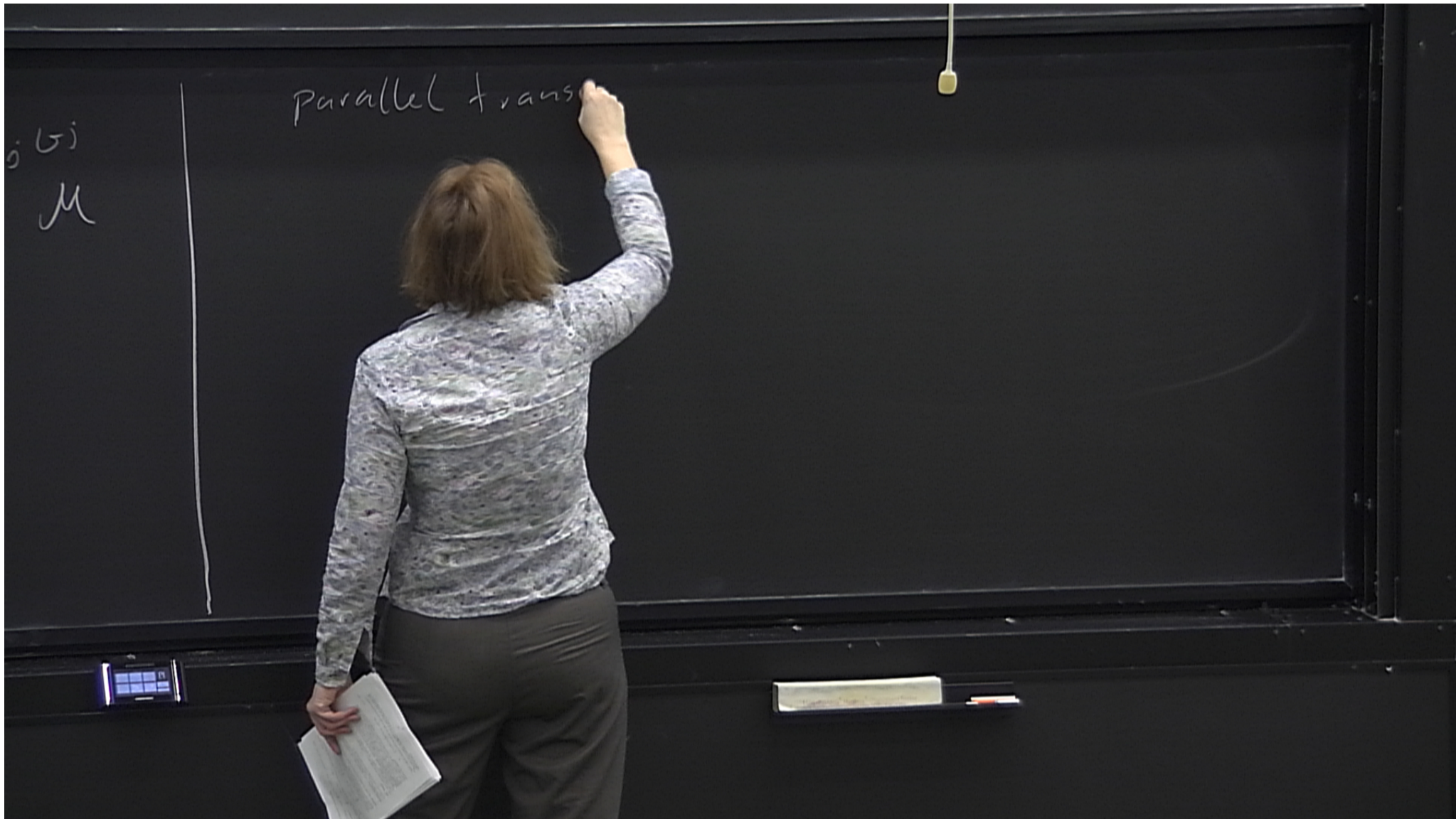
$$\gamma: [0,1] \rightarrow \mathcal{M}$$

$$s \in [0,1]$$

$h(z)$: holonomy

$$\sigma^i + \epsilon^j{}_k A_a^k \sigma^a$$

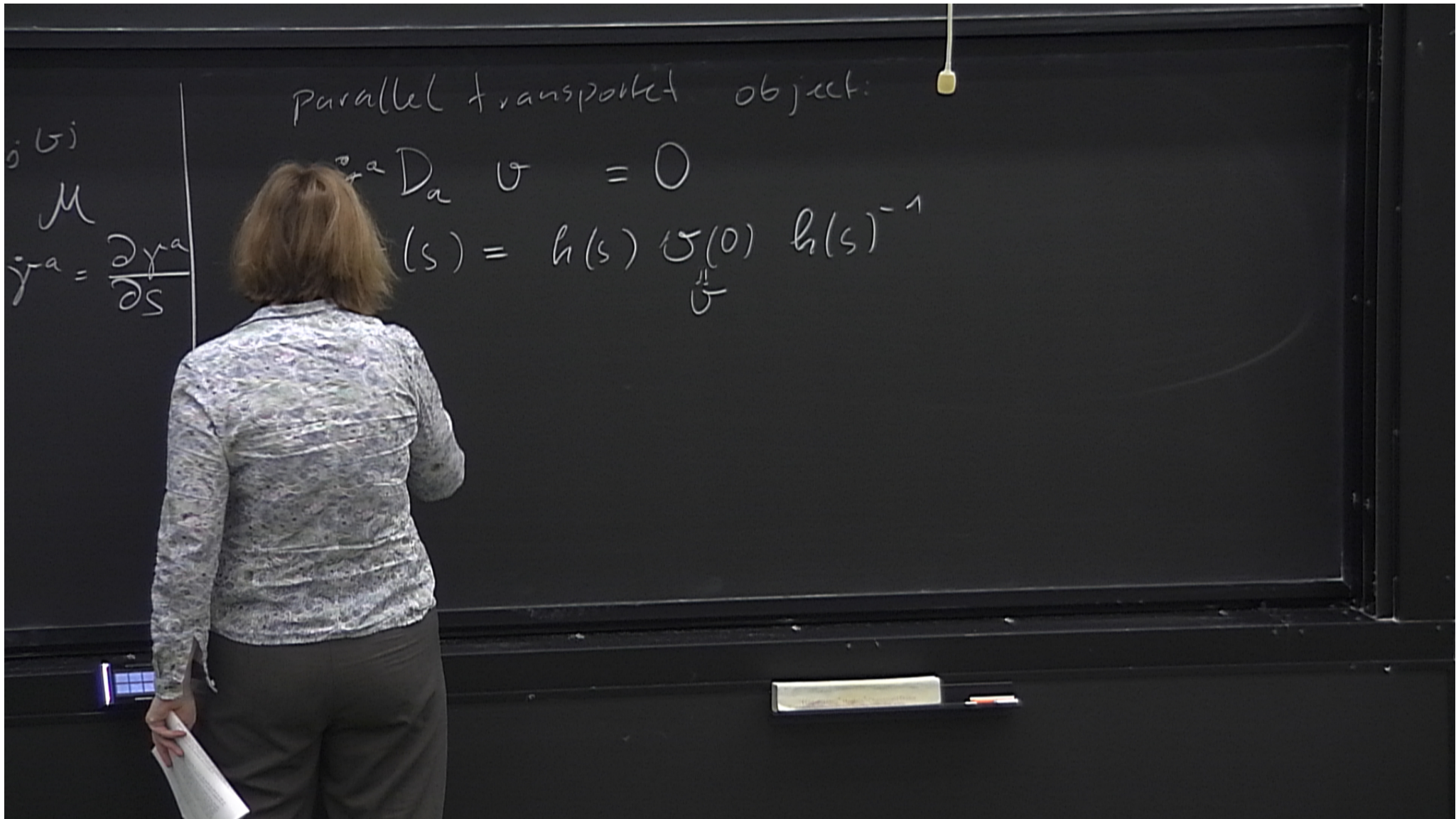
$$+ [A_a, U]$$



parallel transport object:

$$\dot{y}^a D_\gamma = 0$$

γ is
 M
 $\dot{y}^a = \frac{\partial y^a}{\partial s}$



$$g_{ij}$$

$$M$$

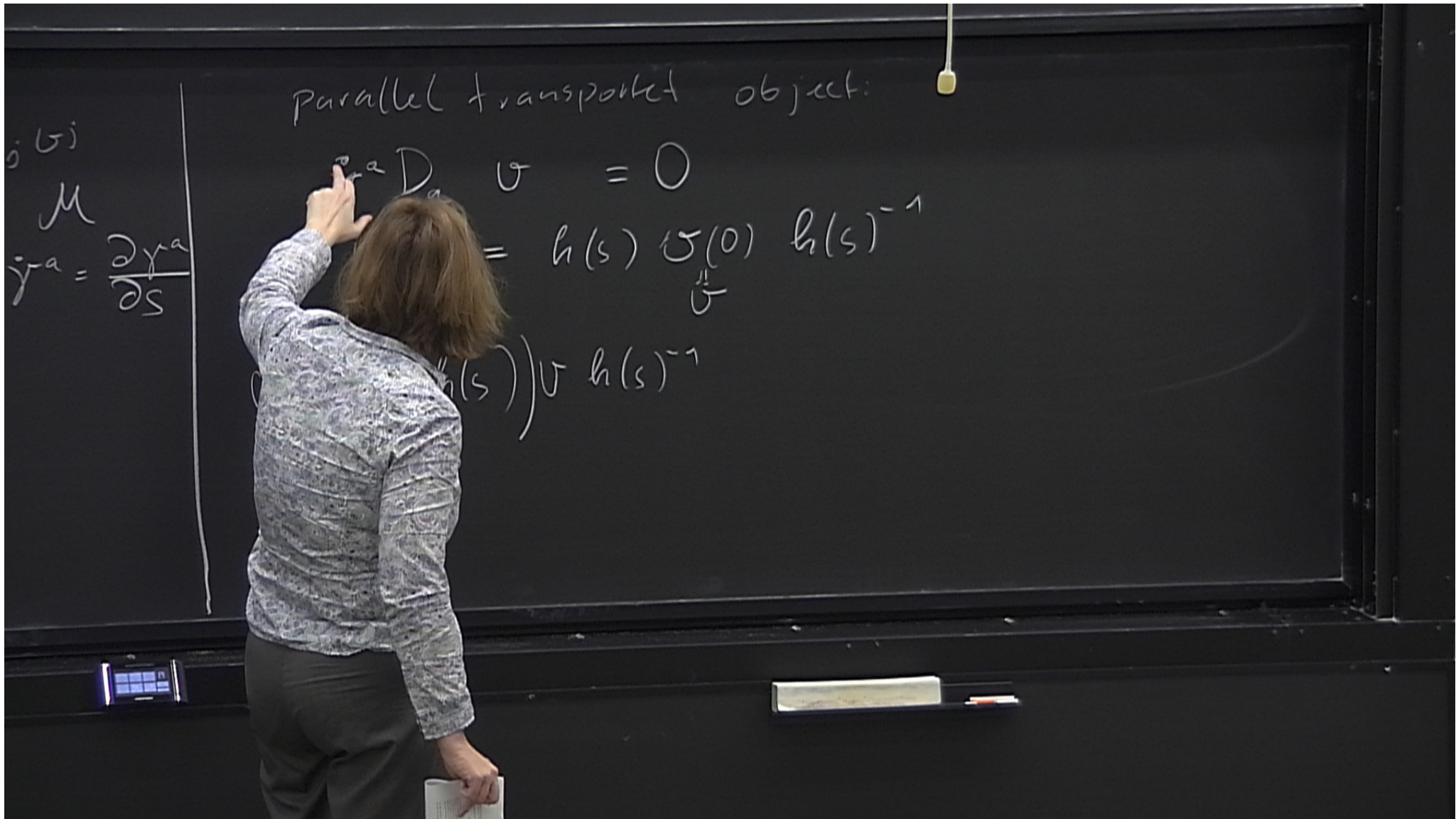
$$g^a = \frac{\partial y^a}{\partial s}$$

parallel transport object:

$$g^a D_a \psi = 0$$

$$\psi(s) = h(s) \psi(0) h(s)^{-1}$$

$$0 = \left(\frac{d}{ds} h(s) \right) \psi h(s)^{-1}$$



$$\begin{aligned}
 & \text{on } M \\
 & \gamma^a = \frac{\partial x^a}{\partial s}
 \end{aligned}$$

parallel transported object:

$$\gamma^a D_a \psi = 0$$

$$\psi(s) = h(s) \underbrace{\psi(0)}_{\psi} h(s)^{-1}$$

$$0 = \left(\frac{d}{ds} h(s) \right) \psi h(s)^{-1} + h(s) \psi \left(\frac{d}{ds} h(s)^{-1} \right)$$

$$g_{ij}$$

$$M$$

$$g^a = \frac{\partial x^a}{\partial s}$$

parallel transport object:

$$g^a D_a \psi = 0$$

$$\psi(s) = h(s) \psi(0) h(s)^{-1}$$

$$0 \stackrel{!}{=} \underbrace{\left(\frac{d}{ds} h(s) \right)}_{\text{partial derivative}} \psi h(s)^{-1} + h(s) \psi \left(\frac{d}{ds} h(s)^{-1} \right)$$

$$+$$

$$g_{ij}$$

$$M$$

$$\dot{y}^a = \frac{\partial y^a}{\partial s}$$

parallel transport object:

$$\dot{y}^a D_a v = 0$$

$$v(s) = h(0) h(s)^{-1}$$

$\dot{y}^a$ · partial derivative

$$0 = \left(\frac{d}{ds} h(s) \right)$$

$$+ \dot{y}^a A_a$$

$$h(s) v \left(\frac{d}{ds} h(s)^{-1} \right)$$

$$\omega_i$$

$$M$$

$$= \frac{\partial \gamma^a}{\partial s}$$

parallel transported object:

$$\dot{\gamma}^a D_a \psi = 0$$

$$\psi(s) = h(s) \psi(0) h(s)^{-1}$$

$$0 = \underbrace{\dot{\gamma}^a \cdot \text{partial derivative}}_{\dot{\gamma}^a} \left(\frac{d}{ds} h(s) \right) \psi h(s)^{-1} + h(s) \psi \left(\frac{d}{ds} h(s)^{-1} \right)$$

$$+ \underbrace{\dot{\gamma}^a A_a h(s) \psi h(s)^{-1} - h(s) \psi h(s)^{-1} \dot{\gamma}^a A_a}_{\dot{\gamma}^a [A_a, \psi(s)]}$$

$$g_{ij}$$

$$M$$

$$\dot{y}^a = \frac{\partial y^a}{\partial s}$$

parallel transport object:

$$\dot{y}^a D_a \psi = 0$$

$$\psi(s) = U(s) \psi(0) h(s)^{-1}$$

$\dot{y}^a \cdot \text{partial}$

$$0 = \left(\frac{d}{ds} h(s) \right)^{-1} + h(s) \psi \left(\frac{d}{ds} h(s)^{-1} \right)$$

$$+ \left(\psi h(s)^{-1} - h(s) \psi h(s)^{-1} \right) \dot{y}^a A_a$$

$$\dot{y}^a [A_a \psi(s)]$$

$$j(s)$$

$$M$$

$$\dot{\gamma}^a = \frac{\partial \gamma^a}{\partial s}$$

parallel transported object:

$$\dot{\gamma}^a D_a \psi = 0$$

$$\psi(s) = h(s) \psi(0) h(s)^{-1}$$

partial derivative

$$\left(\frac{d}{ds} h(s) \right) \psi h(s)^{-1} + h(s) \psi \left(\frac{d}{ds} h(s)^{-1} \right)$$

$$+ \underbrace{\dot{\gamma}^a A_a h(s) \psi h(s)^{-1} - h(s) \psi h(s)^{-1} \dot{\gamma}^a A_a}_{\dot{\gamma}^a [A_a, \psi(s)]}$$

Defining eqn for holonomy
(Extremely convention dependent)

$$\left[\frac{d}{ds} h(s) = -\dot{\gamma}^a A_a(\gamma(s)) h(s) \right]$$

$$h(0) = \underline{1}$$

$$h_\gamma(s) = P \exp \left[- \int_\gamma A \right]$$

$\{T_a\}$ Lie algebra basis
 $T_j = -\frac{i}{2} \sigma_j \leftarrow$ Pauli matrices

$$P \exp \left[- \int_0^s A_a \dot{\gamma}^a ds \right]$$

"path ordered exponential"

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^s ds_n \int_0^{s_n} ds_{n-1} \dots \int_0^{s_2} ds_1 A(s_n) A(s_{n-1}) \dots A(s_1)$$

$$s_n \geq s_{n-1} \geq \dots \geq s_1$$

$\{S_{\alpha}\}$ Lie algebra basis
 $T_j = -\frac{i}{2} \sigma_j \leftarrow$ Pauli matrices

$$P \exp \left[- \int_0^s A_a \dot{y}^a ds \right]$$

"path ordered exponential"

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^s ds_n \int_0^{s_n} ds_{n-1} \dots \int_0^{s_2} ds_1 \underbrace{A(s_n) A(s_{n-1}) \dots A(s_2) A(s_1)}_{A_a \dot{y}^a(s_i)}$$

$$s_n \geq s_{n-1} \geq \dots \geq s_1$$

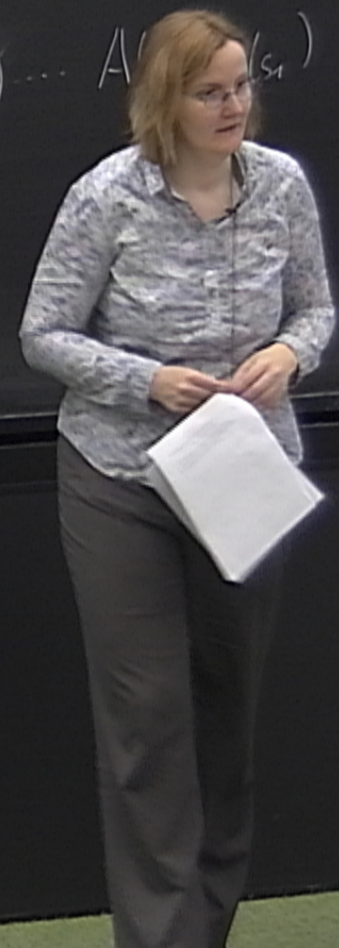
(sult) Lie algebra basis $T_j = -\frac{i}{2} \sigma_j \leftarrow$ Pauli matrices
 $\tau, T, \tau, T, \dots, T_n$

$$P \exp \left[- \int_0^s A_a \dot{\gamma}^a ds \right]$$

"path ordered exponential"

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^s ds_n \int_0^{s_n} ds_{n-1} \dots \int_0^{s_2} ds_1 \underbrace{A(s_n) A(s_{n-1}) \dots A(s_1)}_{A_a(\dot{\gamma}^a(s_i))}$$

$$s_n \geq s_{n-1} \geq \dots \geq s_1$$



$\{S_{\alpha i}\}$ Lie algebra basis
 $T_j = -\frac{i}{2} \sigma_j \leftarrow$ Pauli matrices

$$P \exp \left[- \int_0^s A_a \dot{y}^a ds \right]$$

"path ordered exponential"

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^s ds_n \int_0^{s_n} ds_{n-1} \dots \int_0^{s_2} ds_1$$

$$A(s_n) A(s_{n-1}) \dots A(s_2) A(s_1)$$

$A_a(\dot{y}^a(s_i))$

$$s_n \geq s_{n-1} \geq \dots \geq s_1$$

holonomy: group element associated to a