

Title: 12/13 PSI - Quantum Gravity Review Lecture 7

Date: Feb 05, 2013 10:15 AM

URL: <http://pirsa.org/13020048>

Abstract:

Quantization of the parametrized particle



$$C = p_t + \frac{p_q^2}{2m}$$

dim(reduced phase space) = 2

(generates time evolution)
 $\{t, C_t\} = 1$

particle

solution)

trafo's

q in

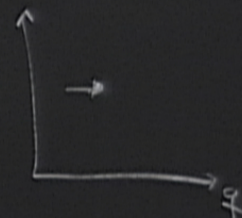
$$\tilde{C} = P_{\dot{q}}$$

generates gauge orbits
along the q -coordinate

$$\hat{C} = -i\hbar \frac{\partial}{\partial q}$$

$$\hat{C} \psi(q) = -i\hbar \frac{\partial}{\partial q} \psi(q)$$

$$e^{\frac{i}{\hbar} s \hat{C}} \psi(q) = e^{s \frac{\partial}{\partial q}} \psi(q) = \sum_{n=0}^{\infty} \frac{(s)^n}{n!} \frac{\partial^n}{\partial q^n} \psi(q)$$



particle

me evolution)

1

gauge trafo's

— $q \mapsto$

$$\hat{C} = P_\varphi$$

generates gauge orbits
along the q -coordinate

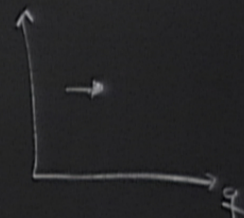
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$$= \psi(q+s)$$

Taylor
expansion



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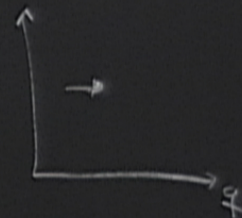
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Taylor
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$$\hat{C} \psi_{\text{phys}} = 0$$

$C(q, p) = 0$: only those phase space points are interesting

$\{f, C\}$: only gauge invariant fcts are interesting

Kinematic

Kinematical Phase space
→ Kinematical Hilbert space

$$f \rightarrow \hat{f}, \quad \widehat{\{f, g\}} = [\hat{f}, \hat{g}]$$

$$\hat{q} \psi(t, q) = q \psi(t, q)$$

$$\hat{p}_q \psi(t, q) = -i\hbar \frac{\partial}{\partial q} \psi(t, q)$$

$$\hat{t} \psi(t, q) = t \psi(t, q)$$

$$\hat{p}_t \psi(t, q) = -i\hbar \frac{\partial}{\partial t} \psi(t, q)$$

Kinematical Phase space
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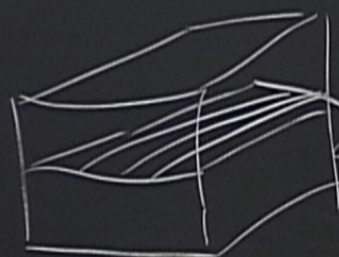
$$\hat{t} \psi(t, q) = t \psi(t, q)$$

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$$\mathcal{L}^2(\mathbb{R}^2, dq dt)$$

$$\langle \psi_1(q, t) | \psi_2(q, t) \rangle_{\text{complex conjugation}} = \iint \overline{\psi_1(q, t)} \psi_2(q, t) dq dt$$

Quantization of the parametrized particle



$$C = p_t + \frac{p_q^2}{2m} \quad (\text{free particle}) \quad (\text{generates time evolution})$$

$$\{t, C_t\} = 1$$

→ C generates gauge trafo's

→ $\hat{C}$ — " — $\hat{q}$

$$\Gamma C = p_t + \frac{p_q^2}{2m}$$

Hamiltonian
of the unperturbed system

$$C = P + h$$

$$\hat{C} = -i\hbar \frac{\partial}{\partial x} \psi + \hat{h} \psi$$

$$h_{\text{free}} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

$$\hat{C} \psi_{\text{phys}} = 0$$

$$C = p^2 + h$$

$$\hat{C} = -i\hbar \frac{\partial}{\partial x} \psi + \hat{h} \psi$$

$$h_{\text{free}} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

$$\hat{C} \psi_{\text{free}} = 0$$

$$C = p + \hbar$$

$$\hat{C} = -i\hbar \frac{\partial}{\partial t} \psi + \hat{h} \psi$$

$$h_{\text{free}} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

$$\hat{C} \psi_{\text{phys}} = 0 \Rightarrow \boxed{0 = -i\hbar \frac{\partial}{\partial t} \psi + \hat{h} \psi}$$

$$C = p + h$$

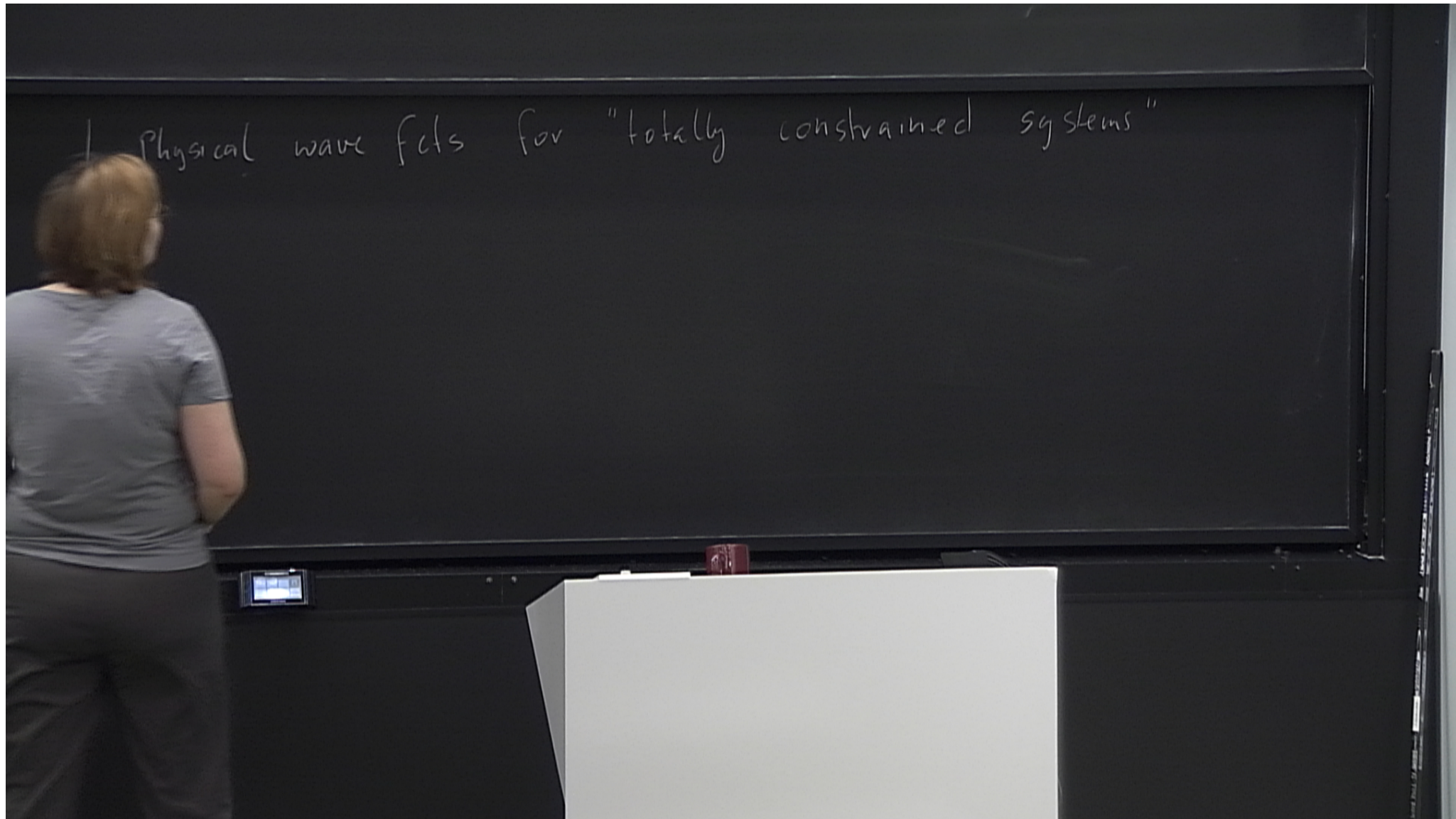
$$\hat{C} = -i\hbar \frac{\partial}{\partial t} \psi + \hat{h} \psi$$

$$h_{\text{free}} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2}$$

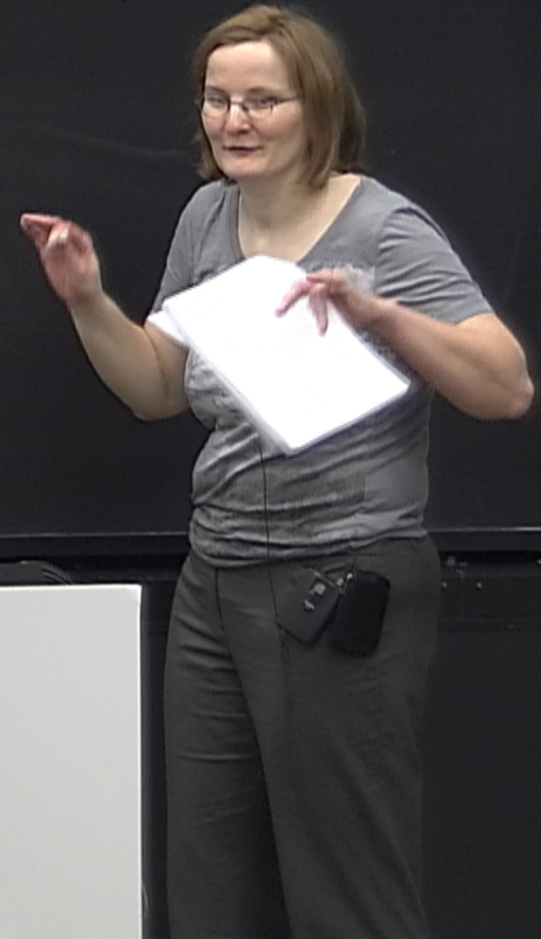
$$\hat{C} \psi_{\text{phys}} = 0 \Rightarrow$$

$$0 = -i\hbar \frac{\partial}{\partial t} \psi + \hat{h} \psi$$

Schrödinger
equation!



Physical wave fcts for "totally constrained systems"
require to solve the (quantum)



Physical wave fcts for "totally constrained systems"
require to solve the (quantum) dynamics.

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Schrödinger
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$\leadsto$ (Wheeler deWitt eqn.)

Physical wave fcts
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Physical wave fcts for "totally constrained systems"
require to solve the (quantum) dynamics.

$$\langle \psi_{\text{phys}} | \psi_{\text{phys}} \rangle_{\text{kin}} = \iint \overline{\psi_{\text{phys}}(q,t)} \psi_{\text{phys}}(q,t) \, dq \, dt$$

nger
h
!

rtt eqn.)

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Physical wave fcts for "totally constrained systems"
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$$\langle \psi_{\text{phys}} | \psi_{\text{phys}} \rangle_{\text{kin}} = \iint \overline{\psi_{\text{phys}}(q_2, t)} \psi_{\text{phys}}(q_1, t) dq_1 dt$$

$$\psi_{\text{phys}}(q, t) = e^{-i\hat{H}t/\hbar} \psi_{\text{phys}}(q, 0)$$

Physical wave fcts for "totally constrained systems"
require to solve the (quantum) dynamics.

$$\langle \psi_{\text{phys}} | \psi_{\text{phys}} \rangle_{\text{kin}} = \iint \overline{\psi_{\text{phys}}(q,t)} \psi_{\text{phys}}(q,t) dq dt$$

$$\psi_{\text{phys}}(q,t) = \underbrace{e^{-i\hat{H}t/\hbar}}_{\text{unitary operator}} \psi_{\text{phys}}(q,0)$$

$$\langle U\psi, \psi \rangle = \langle \psi, U^{-1}\psi \rangle$$

$$\langle e^{-i\hat{H}t/\hbar} \psi_2(q,0), e^{-i\hat{H}t/\hbar} \psi_1 \rangle_{q-\text{integr.}} = \langle \psi_2, \psi_1 \rangle_{q-\text{integr.}}$$

Physical wave fct for "totally constrained systems"
 require to solve the (quantum) dynamics.

$$\langle \psi_2^{\text{phys}}(q,t), \psi_1^{\text{phys}}(q,t) \rangle_{q-\text{integr.}} \quad d q \, d t$$

$$\langle \psi_2^{\text{phys}}(q,t), \psi_1^{\text{phys}}(q,t) \rangle_{q-\text{integr.}}$$

$$\psi_2^{\text{phys}}(q,t)$$

$$\psi_1^{\text{phys}}(q,t)$$

$$= \langle \psi_1, \hat{U}^{-1} \psi_2 \rangle$$

$$\langle e^{-i\hat{H}t/\hbar} \psi_2(q,0), e^{-i\hat{H}t/\hbar} \psi_1(q,0) \rangle_{q-\text{integr.}} = \langle \psi_2(q,0), \psi_1(q,0) \rangle_{q-\text{integr.}}$$

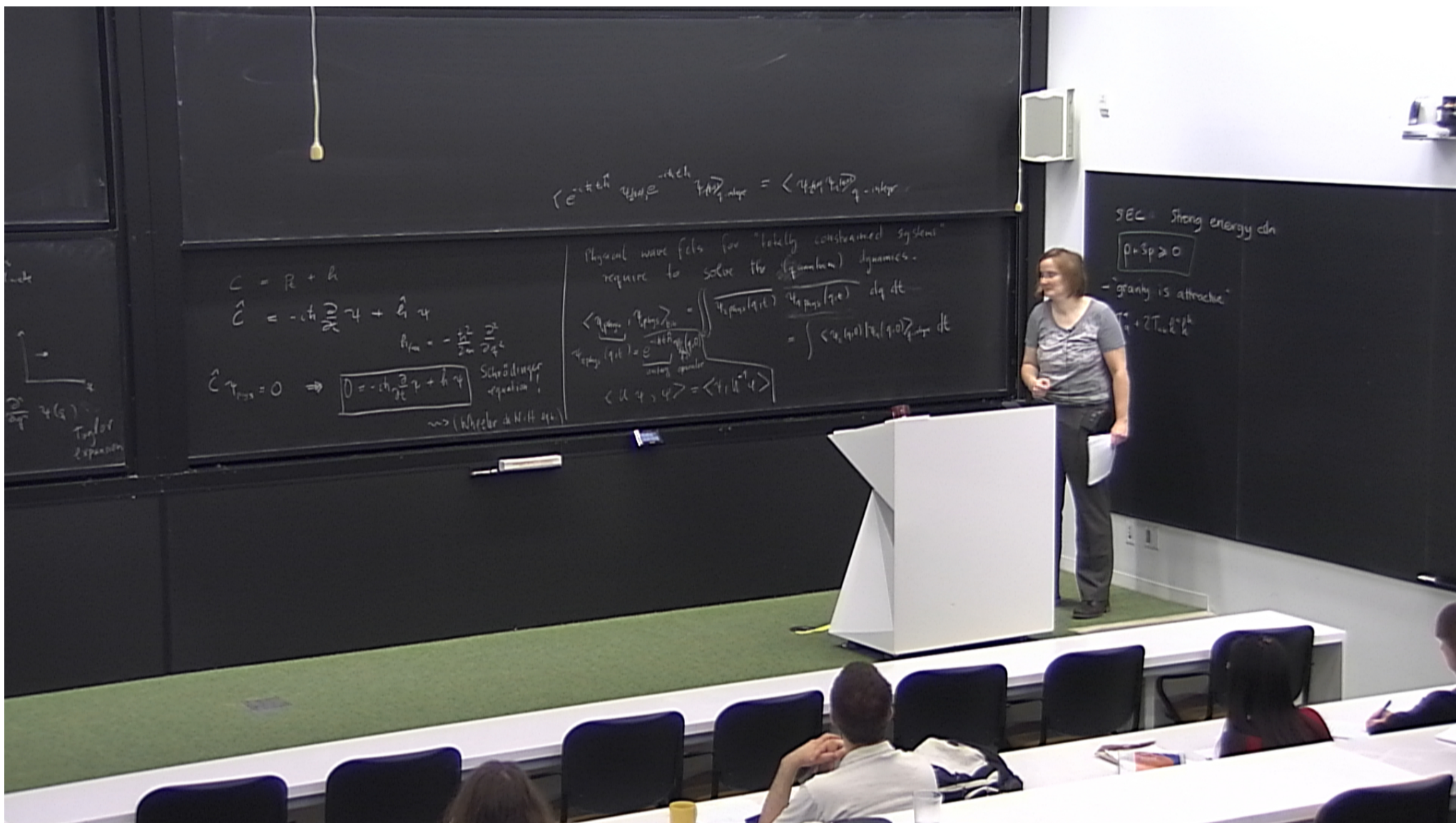
Physical wave fcts for "totally constrained systems" require to solve the (quantum) dynamics

$$\langle \psi_{2\text{phys}}, \psi_{1\text{phys}} \rangle_{\text{kin}} = \iint \overline{\psi_{2\text{phys}}(q,t)} \psi_{1\text{phys}}(q,t) dq dt$$

$$\psi_{2\text{phys}}(q,t) = \underbrace{e^{-i\hat{H}t/\hbar}}_{\text{unitary operator}} \psi_{2\text{phys}}(q,0)$$

$$\langle U\psi, \psi \rangle = \langle \psi, U^{-1}\psi \rangle$$

eq. 1)



$$e^{-i\hat{H}t/\hbar} \psi_2(q,0) e^{-i\hat{H}t/\hbar} \psi_1(q,0) \Big|_{q\text{-integr.}} = \langle \psi_2(q,0) | \psi_1(q,0) \rangle_{q\text{-integr.}}$$

Physical wave fcts for "totally constrained systems" require to solve the (quantum) dynamics.

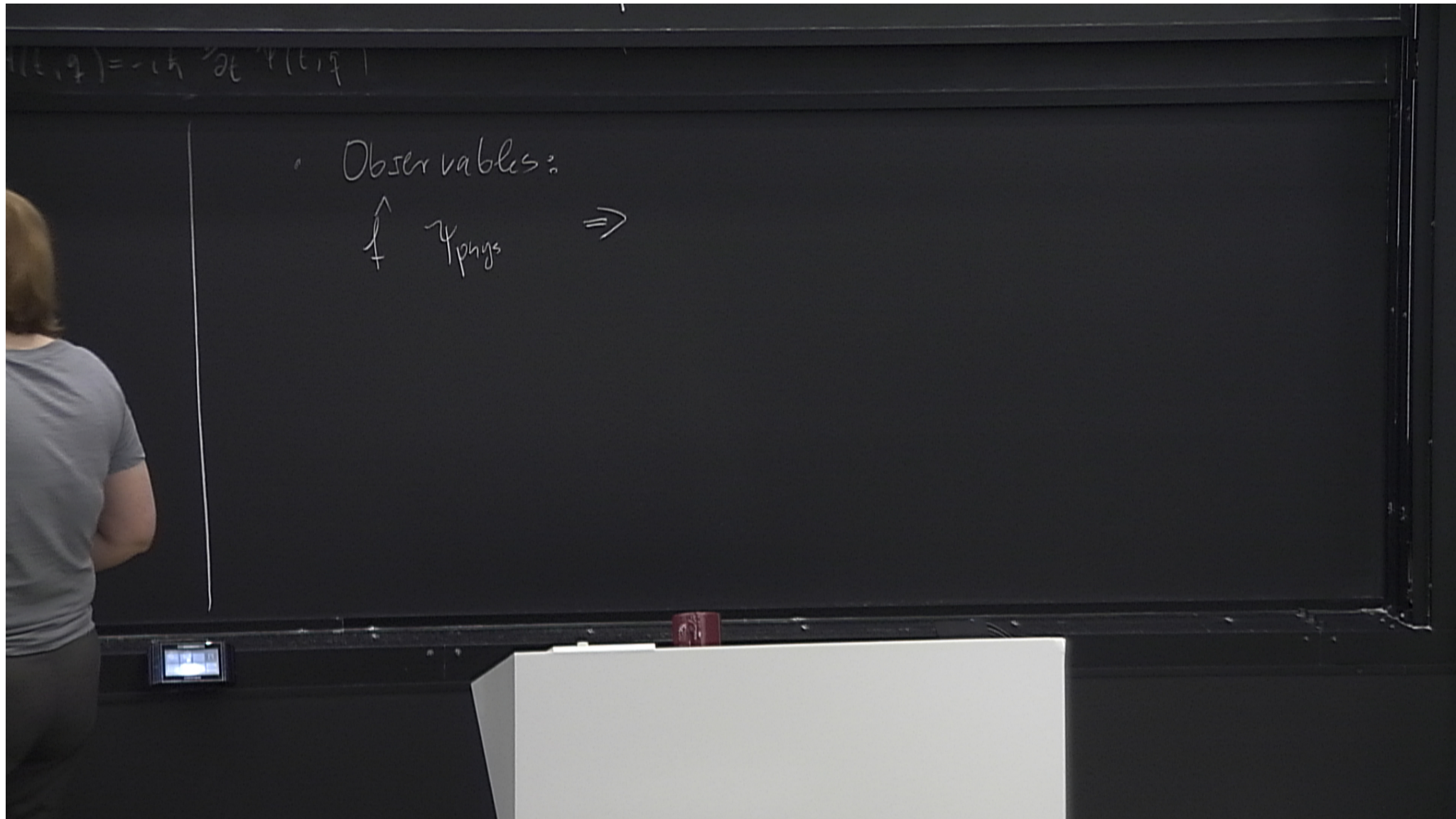
$$\langle \psi_{2\text{phys}} | \psi_{1\text{phys}} \rangle_{\text{kin}} = \int \overline{\psi_{2\text{phys}}(q,t)} \psi_{1\text{phys}}(q,t) dq dt$$

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$$= \int \langle \psi_2(q,0) | \psi_1(q,0) \rangle_{q\text{-integr.}} dt$$

$$\langle U \psi, \psi \rangle = \langle \psi, U^{-1} \psi \rangle$$

- $\hat{C} \psi_{\text{phys}} = 0$
- need new inner product
→ physical Hilbert space



$$i\hbar \frac{\partial}{\partial t} \psi(t, q) = \hat{H} \psi(t, q)$$

• Observables:

$$\hat{f} \psi_{\text{phys}}$$

$\Rightarrow$

$$\hat{C} \hat{f} \psi_{\text{phys}} = 0 \quad ?$$

$$\int \frac{d^3q}{(2\pi)^3}$$

$$i\hbar \frac{\partial}{\partial t} \psi(t, q) = \hat{H} \psi(t, q)$$

Observables:

$$\hat{f} \psi_{\text{phys}}$$

$\Rightarrow$

$$\hat{C} \hat{f} = \hat{f} \hat{C}$$

$$= 0 \quad ?$$

Need this

$$[\hat{C}, \hat{f}_{\text{phys}}] = 0$$

$$\frac{d}{dt} \int dq \psi^* \psi$$

$$i\hbar \frac{\partial}{\partial t} \psi(t, q) = \hat{H} \psi(t, q)$$

Observables:

$$\hat{F} \psi_{\text{phys}}$$

$\Rightarrow$

$$\hat{C} \hat{F} \psi_{\text{phys}} = 0$$

Need this

$$\hat{F} \hat{C} \psi_{\text{phys}}$$

$$[\hat{C}, \hat{F}_{\text{phys}}] = 0$$

$$\Leftrightarrow \{C, F_{\text{phys}}\}$$

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$\hat{F}_{\text{phys}}$

$\Rightarrow$

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$$\hat{F}_{\text{phys}} \hat{C} \psi_{\text{phys}}$$

$$[\hat{C}, \hat{F}_{\text{phys}}] = 0 \quad \Leftrightarrow \quad \{C, F_{\text{phys}}\} = 0$$



Dirac quantization

- $\hat{C} \psi_{\text{phys}} = 0$

- need new inner product
→ physical Hilbertspace

$$\langle \psi_{1\text{phys}}(q,t) | \psi_{2\text{phys}}(q,t) \rangle = \int \overbrace{\psi_{1\text{phys}}(q,t) \psi_{2\text{phys}}(q,t)}^{\text{does not depend } t} \underline{dq}$$

Dirac quantization

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$$\langle \psi_1(q,t) | \psi_2(q,t) \rangle = \int \overbrace{\psi_1(q,t) \psi_2(q,t)}^{\text{does not depend } t} \underline{dq}$$

Dirac quantization

$$\hat{C} \psi_{\text{phys}} = 0 \Rightarrow \langle \hat{C} \rangle_{\psi_{\text{phys}}} = 0, \quad \langle \Delta \hat{C}^2 \rangle_{\psi_{\text{phys}}} = 0$$

- need new inner product
 $\rightarrow$ physical Hilbertspace

$$\langle \psi_1(q,t) | \psi_2(q,t) \rangle = \int \overbrace{\psi_1(q,t) \psi_2(q,t)}^{\text{does not depend } t} dq$$

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What do we do?

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$$\gamma_{k_1} (q, t)$$

What do we do?

$$\delta(\hat{C}) \sim (q, t)$$

What do we do?

$$\delta(\hat{C}) \psi(q, t)$$

$$\delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} dk$$

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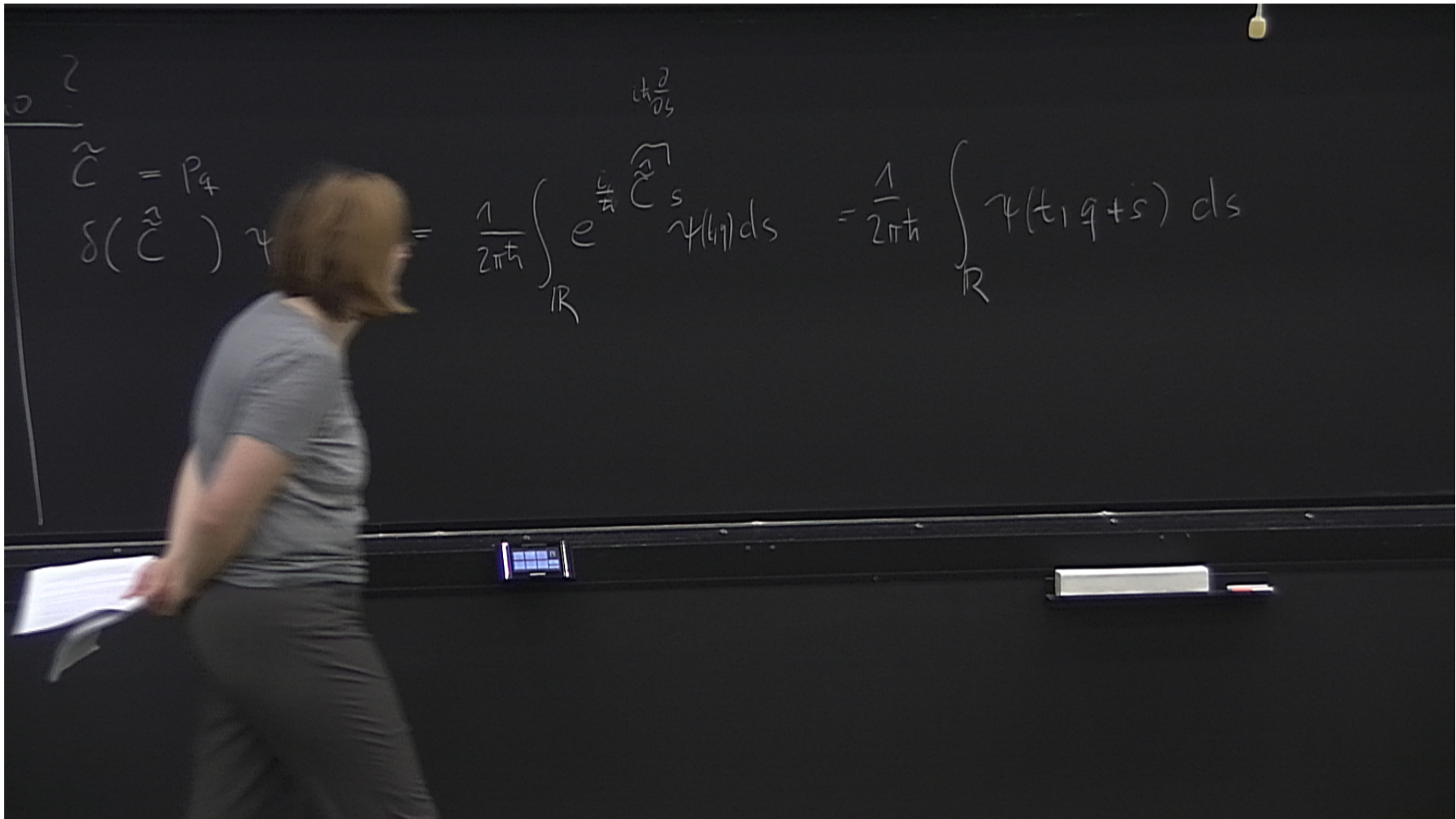
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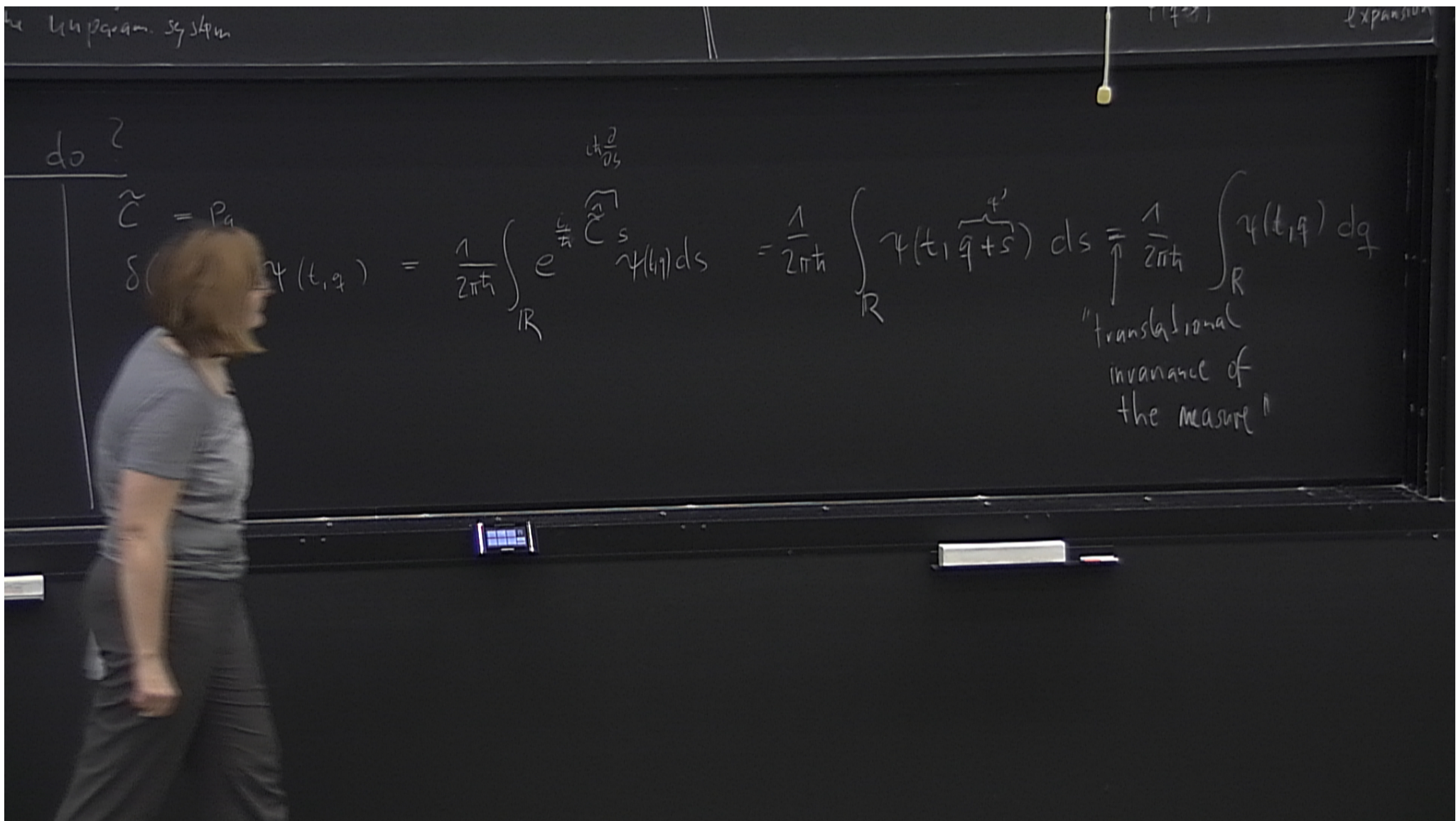
$$\delta(\hat{C}) \psi$$

$$\tilde{C} = P_4$$

$$\delta(\tilde{C}) \psi(t, q) = \frac{1}{2\pi\hbar} \int e^{\frac{i}{\hbar} \tilde{C} s} ds$$

$$\delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itx} dt$$





the Lippmann-Schwinger

expansion

do ?

$$\tilde{C} = P_0$$

$i\hbar \frac{\partial}{\partial s}$

$$S(t, q) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} e^{i \frac{L}{\hbar} \tilde{C} s} \psi(q) ds = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} \psi(t, \overbrace{q+s}^{q'}) ds = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} \psi(t, q) dq$$

"translational invariance of the measure"

the Lippmann-Schwinger

(14.2)

expansion

do ?

$$\tilde{C} = P_q$$

$$\delta(\tilde{C}) \psi(t, q) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} e^{i \frac{C}{\hbar} s} \overbrace{\psi(t, q)}^{t, \frac{\partial}{\partial s}} ds = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} \psi(t, \overbrace{q+s}^{q'}) ds = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} \psi(t, q') dq'$$

"translational invariance of the measure"

the Lagrangian system

(14-2)

expansion

do?

$$\tilde{C} = P_q$$

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$\Rightarrow$ constant is q

$\Rightarrow \delta(\tilde{C})$ is a "group" averaging over the gauge orbit

"translational invariance of the measure"

do ?

$$\tilde{C} = P_q$$

$$\delta(\tilde{C}) \psi(t, q) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} e^{i \frac{L}{\hbar} \tilde{C} s} \psi(t, q) ds = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} \psi(t, \overbrace{q+s}^{q'}) ds = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} \psi(t, q') dq'$$

$\Rightarrow$ constant is q

$\Rightarrow \delta(\tilde{C})$ is a "group" averaging over the gauge orbit

"translational invariance of the measure"

$\leadsto$ (Wheeler deWitt eqn.)

$$\langle \psi, \psi \rangle = \langle \psi, \psi \rangle$$

$$\langle \delta(\hat{C}) \psi_2, \delta(\hat{C}) \psi_1 \rangle_{\text{kin}} = \infty$$

$\leadsto$ (Wheeler deWitt eqn.)

$$\langle \psi, \psi \rangle = \langle \psi, \psi \rangle$$

$$\langle \delta(\hat{C}) \psi_2, \delta(\hat{C}) \psi_1 \rangle_{\text{kin}} = \infty$$

$$\langle \delta(\hat{C}) \psi_2,$$

$\leadsto$ (Wheeler de Witt eqn.)

$$\langle U \psi, \psi \rangle = \langle \psi, U \psi \rangle$$

$$\langle \delta(\hat{C}) \psi_2, \delta(\hat{C}) \psi_1 \rangle_{\text{kin}} = \infty$$

$$\langle \delta(\hat{C}) \psi_2, \psi_1 \rangle_{\text{Hilb}}, \delta(\hat{C}) \psi_1 \rangle$$

$$\langle \delta(\hat{C}) \psi_2, \delta(\hat{C}) \psi_1 \rangle_P$$

physical (circled)
kinematical

$\leadsto$ (Wheeler deWitt eqn.)

$$\langle U\psi, \psi \rangle = \langle \psi, U\psi \rangle$$

$$\langle \delta(\hat{C})\psi_2, \delta(\hat{C})\psi_1 \rangle_{\text{kin}} = \infty$$

$$\langle \delta(\hat{C})\psi_2, \psi_1 \rangle_{\text{Hilbert}} = \langle \psi_2, \delta(\hat{C})\psi_1 \rangle$$

$$\langle \underbrace{\delta(\hat{C})\psi_2}_{\text{phys. state}}, \underbrace{\delta(\hat{C})\psi_1}_{\text{kinemat.}} \rangle_{\text{phys}} = \langle \psi_2, \delta(\hat{C})\psi_1 \rangle_{\text{kin}}$$

$\leadsto$ (Wheeler de Witt eqn.)

$$\langle U \psi, \psi \rangle = \langle \psi, U \psi \rangle$$

$$\langle \delta(\hat{C}) \psi_2, \delta(\hat{C}) \psi_1 \rangle_{\text{kin}} = 0$$

$$\langle \delta(\hat{C}) \psi_2, \psi_1 \rangle \stackrel{\text{Hofe}}{=} \langle \psi_2, \delta(\hat{C}) \psi_1 \rangle$$

$$\langle \underbrace{\delta(\hat{C}) \psi_2}_{\text{phys. state}}, \underbrace{\delta(\hat{C}) \psi_1}_{\text{kinemat.}} \rangle_{\text{phys}} = \langle \psi_2, \delta(\hat{C}) \psi_1 \rangle_{\text{kin}}$$