

Title: Advanced General Relativity - Lecture 12

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Abstract:

Any linear map from one space of tensors to another is (comes from)  
a tensor

$$g_{ab} : V \rightarrow V^* \quad (g_{ab} v^b \leftarrow v^a)$$

$$(V_1 \otimes V_2 \otimes \dots)^* = V_1^* \otimes V_2^* \otimes \dots$$

$$\rightarrow T$$



Any linear map from one space of tensors to another is (comes from)  
a tensor.

eg  $g_{ab} : V \rightarrow V^*$  ( $g_{ab} v^b \leftarrow v^a$ )

$(\otimes \dots)^* = V_1^* \otimes V_2^* \otimes \dots$

$\xrightarrow{ab \dots c} S_{ab \dots c} T^{ab \dots c} \in \mathbb{R}$



$T^{ab\dots c} \longrightarrow S^{ab\dots c} T^{ab\dots c} \in \mathbb{R}$   
 (The map is "multilinear" on the factors that make up  $T$ )

$$V = T_p M$$

Example  $T^a_b \in V \otimes V^* \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



$T^{ab\dots c} \longrightarrow S^{ab\dots c} T^{ab\dots c} \in \mathbb{R}$   
 (The map is "multilinear" on the factors that make up  $T$ )

Example  $T^a_b \in V \otimes V^*$   $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$   $T^a_b = \sum_{k=1}^m v^a_{(k)} \omega_b^{(k)} \left( = \sum_k v_{(k)}^a \otimes \omega^{(k)}_b \right)$

$T: V \rightarrow V$   $(v^a \rightarrow T^a_b v^b)$

$T: V^* \rightarrow V^*$   $(\omega_a \rightarrow \omega_b T^b_a)$

$T: V^* \otimes V \rightarrow \mathbb{R}$   $(V \otimes V^* \rightarrow \mathbb{R})$

The tensor  $\delta^a_b$   $\text{id. } v^a \rightarrow \delta^a_b v^b$



$T^{ab\dots c} \longrightarrow S^{ab\dots c} T^{ab\dots c} \in \mathbb{R}$   
 (The map is "multilinear" on the factors that include  $T$ )

Example  $T^a_b \in V \otimes V^*$   $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$   $T^a_b = \sum_{k=1}^m v^a_{(k)} \frac{(k)}{b} \omega_b \left( = \sum_k v_{(k)}^a \otimes \omega_{(k)}^b \right)$

$T: V \rightarrow V \quad (v^a \rightarrow T^a_b v^b)$

$T: V^* \rightarrow V^* \quad (\omega_a \rightarrow \omega_b T^b_a)$

$T: V^* \otimes V \rightarrow \mathbb{R} \quad (V \otimes V^*)$

$\left[ \delta^a_b = \sum_{k=1}^m e^a_{(k)} e_{(k)}^b \right]$

The tensor  $\delta^a_b$  id.  $V \rightarrow V$

$\delta^a_b v^b$

$\delta^a_b \omega_a = \omega_b$

(proof  $\omega_a$

$\omega_a \delta^a_b v^b = \omega_b v^b$

$(\omega_a \delta^a_b - \omega_b) v^b = 0$



(proof  $w_a$  .

$$w_a \delta_b^a v^b = w_a v^a = w_b v^b$$

$$(w_a \delta_b^a - w_b) v^b = 0 \Rightarrow w_a \delta_b^a = w_b$$

$$\delta_b^a v^a = v^b \Leftrightarrow$$

$$\delta_b^a w_a = w_b$$



Tensor density  $\propto \sqrt{|g|}$  (scalar dens.  
axial tensor.

They are outerproducts of  $\otimes$  { "volume

Tensor density  $\sqrt{|g|}$  (density)  
axial tensor.

They are interpreted as  $n$ -way tensors with

$\left\{ \begin{array}{l} \text{"volume"} \\ \text{"orientation"} \end{array} \right\}$



$$w_a \delta_b^a v^b = w_a v^a = w_b v^b$$

$$(w_a \delta_b^a - w_b) v^b = 0 \Rightarrow w_a \delta_b^a = w_b$$

Tensor density  $\sqrt{|g|}$  (scalar density)  
axial tensor.

$j^a$  = electric current density

They are outerproducts of ordinary tensors with  $\otimes$

{ "volume"  
"orientation" }



$$w_a \delta_b^a v^b = w_a v^a = w_b v^b$$

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Tensor density  $\sqrt{|g|}$  (scalar density)  
axial tensor.

$j^a$  = electric current density (vector density)

They are outerproducts of ordinary tensors with  $\otimes$

$\left\{ \begin{array}{l} \text{"volume"} \\ \text{"orientation"} \end{array} \right\}$



$$(a^a b_a - a^b b_b) \delta^a_b = 0 \Rightarrow w_a \delta^a_b = w_b$$

$dV = \sqrt{-g} d^4x$   
 Tensor density  
 axial tensor  
 They are... with

$\int^a$  = electric current density (vector density)  
 power of  
 "volume"  
 "orientation"



$$(a^a b_a - a^b b_b) \delta^a_b = 0 \Rightarrow w_a \delta^a_b = w_b$$

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Bases & components

$$\dim V = n = 4$$



## Bases & components

$$\dim V = n = 4$$

$$\exists \text{ basis } \left\{ e_{(\alpha)}^a \mid \alpha = 0, 1, 2, 3 \right\}$$

$$v^a \in V$$

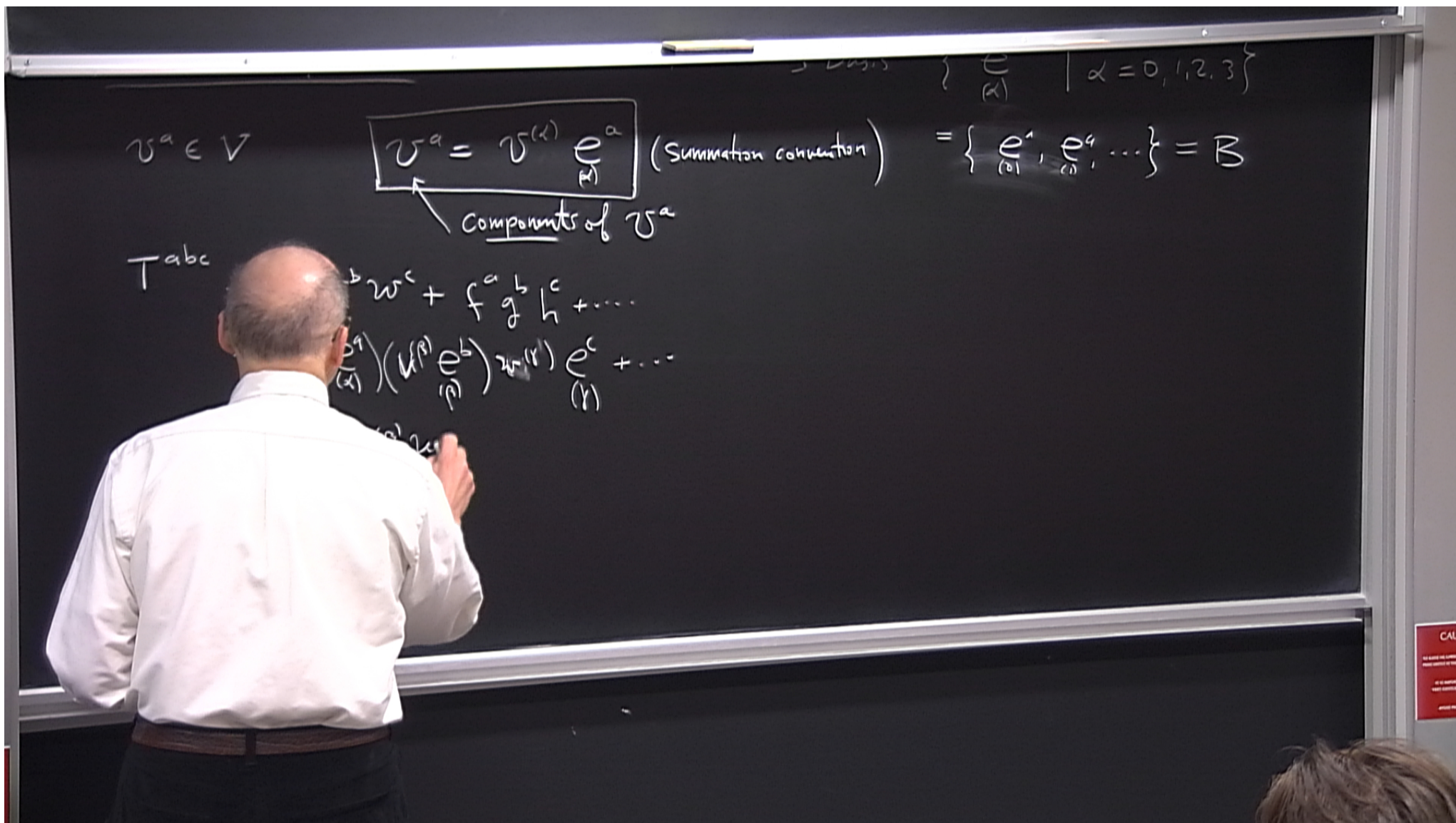
$$v^a = v^{(\alpha)} e_{(\alpha)}^a \quad (\text{summation convention})$$

$$= \{ e_{(0)}^a, e_{(1)}^a, \dots \} = B$$

↑  
components of  $v^a$

$$T^{abc} = u^a v^b w^c + f^a_g h^c + \dots$$







$$v^a \in V$$

$$v^a = v^{(i)} e^a_{(i)} \quad (\text{Summation convention})$$

components of  $v^a$

$$\{ e^a_{(i)} \mid a=0,1,2,3 \} = B$$

$T^{abc}$

$$v^a w^b + f^a g^b h^c + \dots$$

$$v^{(i)} e^a_{(i)} w^{(j)} e^b_{(j)} + \dots$$

$$v^{(i)} w^{(j)} + f^{(i)} g^{(j)} \dots \left] e^a_{(i)} e^b_{(j)} e^c_{(k)} \right.$$

$T^{(i)(j)(k)} \leftarrow$  components of  $T^{abc}$  w.r.t.  $B$

$$T^{000} \dots T^{333}$$



$$T = u \otimes v \otimes w + \dots$$

$$= u^{(1)} e_{(1)} \otimes v^{(1)} e_{(1)} \otimes w^{(1)} e_{(1)} + \dots$$

$u^{(1)}$

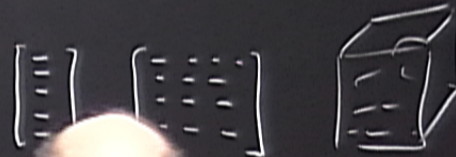


$$\begin{aligned}
 T &= u \otimes v \otimes w + \dots \\
 &= u^{(1)} e_{(1)} \otimes v^{(1)} e_{(1)} \otimes w^{(1)} e_{(1)} + \dots \\
 &\quad (u^{(1)} v^{(1)} w^{(1)} + \dots) e_{(1)} \otimes e_{(1)} \otimes e_{(1)}
 \end{aligned}$$

$$e_1 = u \otimes$$



$$\begin{aligned}
 T &= u \otimes v \otimes w + \dots \\
 &= u^{(\alpha)} e_{(\alpha)} \otimes v^{(\beta)} e_{(\beta)} \otimes w^{(\gamma)} e_{(\gamma)} + \dots \\
 &= (u^{(\alpha)} v^{(\beta)} w^{(\gamma)} + \dots) e_{(\alpha)} \otimes e_{(\beta)} \otimes e_{(\gamma)}
 \end{aligned}$$



eg  $T = u \otimes v \otimes w \Rightarrow T^{(\alpha, \beta, \gamma)} = u^{(\alpha)} v^{(\beta)} w^{(\gamma)} \quad (\alpha, \beta, \gamma = 0, 1, 2, 3)$

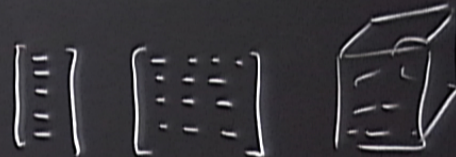
$e_{(\alpha)} \otimes e_{(\beta)}$  is a basis for  $V \otimes V$



$$T = u \otimes v \otimes w + \dots$$

$$= u^{(\alpha)} e_{(\alpha)} \otimes v^{(\beta)} e_{(\beta)} \otimes w^{(\gamma)} e_{(\gamma)} + \dots$$

$$= (u^{(\alpha)} v^{(\beta)} w^{(\gamma)} + \dots) e_{(\alpha)} \otimes e_{(\beta)} \otimes e_{(\gamma)}$$



eg  $T = u \otimes v \otimes w \Rightarrow T^{(\alpha, \beta, \gamma)} = u^{(\alpha)} v^{(\beta)} w^{(\gamma)} \quad (\alpha, \beta, \gamma = 0, 1, 2, 3)$

$e_{(\alpha)} \otimes e_{(\beta)}$  is a basis for  $V \otimes V$   $(e_{\binom{0}{0}} \otimes e_{\binom{0}{0}}, e_{\binom{0}{0}} \otimes e_{\binom{1}{0}}, \dots, e_{\binom{3}{3}} \otimes e_{\binom{3}{3}})$



$$= (u^{(\alpha)} v^{(\beta)} w^{(\gamma)} + \dots) e_{(\alpha)} \otimes e_{(\beta)} \otimes e_{(\gamma)}$$

eg  $T = u \otimes v \otimes w \Rightarrow T^{(\alpha, \beta, \gamma)} = u^{(\alpha)} v^{(\beta)} w^{(\gamma)} \quad (\alpha, \beta, \gamma = 0, 1, 2, 3)$

ie  $\boxed{e_{(\alpha)} \otimes e_{(\beta)} \text{ is a basis for } V \otimes V} \quad (e_{(0)} \otimes e_{(0)}, e_{(0)} \otimes e_{(1)}, \dots, e_{(3)} \otimes e_{(3)})$

$$\dim(V \otimes W) = \dim V \times \dim W$$



Dual bases



Dual bases  $B \Rightarrow B^*$  basis for  $V^*$ ,  $B^* = \{ \epsilon_a^{(2)} \}$



Dual bases  $B \Rightarrow B^*$  basis for  $V^*$ ,  $B^* = \{ e_a^{(\alpha)} \mid \alpha=0 \dots 3 \}$   $B \leftrightarrow B^*$

$$e_a^{(\alpha)} e_{(p)}^a = \delta_{(p)}^{(\alpha)}$$



Dual bases  $B \Rightarrow B^*$  basis for  $V^*$ ,  $B^* = \{ e_a^{(\alpha)} \mid \alpha=0..3 \}$   $B \leftrightarrow B^*$

def  $\boxed{e_a^{(\alpha)} e_{(\beta)}^a = \delta_{(\beta)}^{(\alpha)}}$   $\stackrel{(\alpha)}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} 1 & \text{if } \alpha=\beta \\ 0 & \text{else} \end{cases}$



Dual bases  $B \Rightarrow B^*$  basis for  $V^*$ ,  $B^* = \{ e_a^{(\alpha)} \mid \alpha = 0 \dots 3 \}$   $B \leftrightarrow B^*$

def

$$e_a^{(\alpha)} e_{(\beta)}^a = \delta_{(\beta)}^{(\alpha)}$$

$$= \delta_{(\alpha)}^a e_{(\beta)}^a$$

$$\delta_{(\beta)}^{(\alpha)} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{else} \end{cases}$$



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$$\Rightarrow e_{(\alpha)}^a e_b^{(\alpha)} = \delta_b^a$$

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def

$$e_a^{(\alpha)} e_{(\beta)}^a = \delta_{(\beta)}^{(\alpha)}$$

$$\delta_{(\beta)}^{(\alpha)} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow e_{(\alpha)}^a e_b^{(\alpha)} = \delta_b^a \quad (\text{completeness rel'n})$$

$$\begin{aligned} \delta_b^a &= e_a^{\dot{a}} e_{\dot{a}b} + e_a^{\dot{a}} e_{\dot{a}b} + \dots \\ &= e_{\dot{0}}^{\dot{a}} e_{\dot{a}b} + e_{\dot{1}}^{\dot{a}} e_{\dot{a}b} + \dots \end{aligned}$$



Dual bases  $B \Rightarrow B^*$  basis for  $V^*$ ,  $B^* = \{ e_a^{(\alpha)} \mid \alpha=0 \dots 3 \}$   $B \leftrightarrow B^*$

def  $\boxed{e_a^{(\alpha)} e_{(\beta)}^a = \delta_{(\beta)}^{(\alpha)}}$   $\delta_{(\beta)}^{(\alpha)} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{else} \end{cases}$

$\Rightarrow \boxed{e_{(\alpha)}^a e_b^{(\alpha)} = \delta_{(\alpha)}^a \text{ (pleteness rel'n)}}$   $\delta_b^a = e_a^{\dot{a}} e_{\dot{a}b} + e_a^{\dot{b}} e_{\dot{b}b} + \dots$   
 $= e_a^{\dot{0}} e_{\dot{0}b} + e_a^{\dot{1}} e_{\dot{1}b} + \dots$

$v^a = \delta_b^a v^b = e_{\dot{a}b}^a v^b$



Dual bases  $B \Rightarrow B^*$  basis for  $V^*$ ,  $B^* = \{ e_a^{(\alpha)} \mid \alpha = 0 \dots 3 \}$   $B \leftrightarrow B^*$

def  $\boxed{e_{(\beta)}^a = \delta_{(\beta)}^{(\alpha)}}$

$$\delta_{(\beta)}^{(\alpha)} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{else} \end{cases}$$

$\boxed{e_{(\alpha)}^a e_b^{(\alpha)} = \delta_b^a}$  (completeness rel'n)

$$\delta_b^a = e_0^a e_b^{(0)} + e_1^a e_b^{(1)} + \dots = e_0^a e_b^{(0)} + e_1^a e_b^{(1)} + \dots$$

$$v^b = e_{(\alpha)}^a \underbrace{(e_b^{(\alpha)} v^b)}_{v^{(\alpha)}}$$

$\boxed{v^{(\alpha)} = e_a^{(\alpha)} v^b}$  etc



Dual bases  $B \Rightarrow B^*$  basis for  $V^*$ ,  $B^* = \{ e^{(\alpha)}_a \mid \alpha=0 \dots 3 \}$   $B \leftrightarrow B^*$

def  $\boxed{e^{(\alpha)}_{(\beta)} e^a = \delta^{(\alpha)}_{(\beta)} a}$

$$\delta^{(\alpha)}_{(\beta)} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{else} \end{cases}$$

$\boxed{e^{(\alpha)}_{(\beta)} e_b = \delta^{(\alpha)}_{(\beta)} b}$  (completeness rel'n)

$$\delta^a_b = e^a_{\dot{e}_b} + e^a_{\dot{e}'_b} + \dots$$

$$= e^a_{\dot{0}} \dot{e}_b + e^a_{\dot{1}} \dot{e}'_b + \dots$$

$$v^{(\alpha)}_b = e^{(\alpha)}_{(\gamma)} \underbrace{(e^{(\gamma)}_b v^b)}_{v^{(\gamma)}} = e^{(\alpha)}_{(\gamma)} v^{(\gamma)}$$

$\boxed{v^{(\alpha)} = e^{(\alpha)}_a v^a}$  etc



$$\psi_n = \langle n | \psi \rangle$$

$$1 = \sum_n |n\rangle \langle n|$$



Dual bases  $B \Rightarrow B^*$  basis for  $V^*$ ,  $B^* = \{ e_a^{(\alpha)} \mid \alpha=0 \dots 3 \}$   $B \leftrightarrow B^*$

def  $\boxed{e_a^{(\alpha)} e_{(\beta)}^a = \delta_{(\beta)}^{(\alpha)}}$   $\stackrel{(\alpha)}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} 1 & \text{if } \alpha=\beta \\ 0 & \text{else} \end{cases}$

$\Rightarrow \boxed{e_{(\alpha)}^a e_b^{(\alpha)} = \delta_b^a}$   $(\text{rel'n})$   $\delta_b^a = e_a^a e_b^a + e_a^b e_b^a + \dots$   
 $= e_a^a e_b^a + e_a^b e_b^a + \dots$

$v^a = \delta_b^a v^b = e_{(\alpha)}^a \underbrace{e_b^{(\alpha)}}_{v^b} \dots$  etc



Dual bases  $B \Rightarrow B^*$  basis for  $V^*$ ,  $B^* = \{ e_a^{(\alpha)} \mid \alpha=0 \dots 3 \}$   $B \leftrightarrow B^*$

def  $\boxed{e_a^{(\alpha)} e_{(\beta)}^a = \delta_{(\beta)}^{(\alpha)}}$

$$\delta_{(\beta)}^{(\alpha)} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{else} \end{cases}$$

$\Rightarrow \boxed{e_{(\alpha)}^a e_b^{(\alpha)} = \delta_b^a}$  (completeness rel'n)

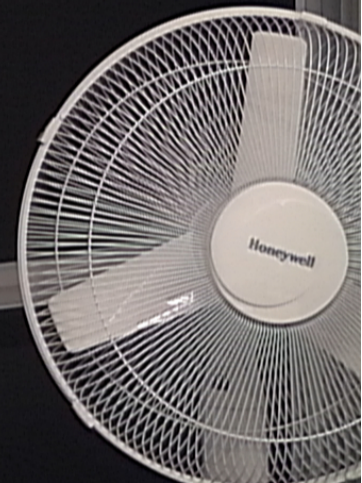
$$\delta_b^a = e_a^a e_b^{(a)} + e_a^a e_b^{(a)} + \dots = e_a^a e_b^{(a)} + e_a^a e_b^{(a)} + \dots$$

$v^a = \delta_b^a v^b = e_{(\alpha)}^a \underbrace{(e_b^{(\alpha)} v^b)}_{v^{(\alpha)}} \quad \boxed{v^{(\alpha)} = e_a^{(\alpha)} v^a}$  etc

eg. also  $T^a_b = T_{(\beta)}^{(\alpha)} e_{(\alpha)}^a e_b^{(\beta)}$  where  $T_{(\beta)}^{(\alpha)} = e_a^{(\alpha)} e_b^{(\beta)} T^a_b$



Contraction





Contraction  $T^a_b \rightarrow T^a_a = T^a_b \delta^b_a = T^a_b e^a_a e^b_b$



Contraction  $T^a_b \rightarrow T^a_a = T^a_b \delta^b_a = T^a_b e^b_a = T^{(Y)}_{(Y)}$



Contraction  $T^a_b \rightarrow T^a_a = T^a_b \delta^b_a = T^a_b e^b_a = T^{(y)}_{(y)}$

Contra. is summation ("partial trace")

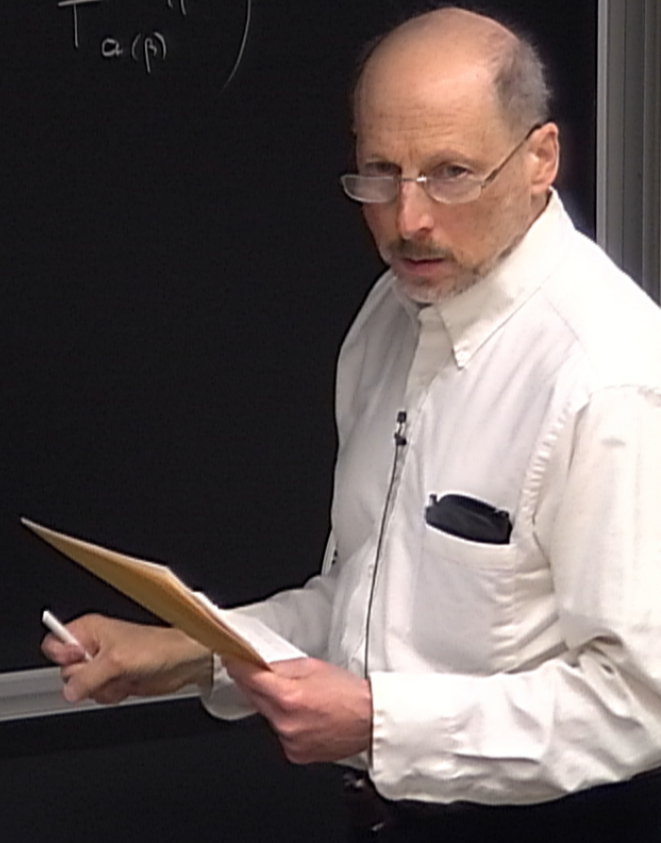
$$R_a = T^b_{ab}, \quad R_a = T^c_{ab} \delta^b_c = T^c_{ac}$$



Contraction  $T^a_b \rightarrow T^a_a = T^a_b \delta^b_a = T^a_b e^b_a = T^{(Y)}_{(Y)}$

Contra. is summation ("partial trace")

$$R_a = T_{ab}^b, \quad R_a = T_{ab}^c \delta_c^b = T_{ab}^c e^b_{(P)} e^a_{(P)} (= "T_{a(P)}^{(P)}")$$





Contraction  $T^a_b \rightarrow T^a_a = T^a_b \delta^b_a = T^a_b e^b_a = T^{(Y)}_{(Y)}$

Contro. is summation ("partial trace")

$$R_a = T_{ab}^b, \quad R_a = T_{ab}^c \delta_c^b = T_{ab}^c e_{(p)}^b e_c^{(p)} (= \text{"} = \text{"})$$

$$\Rightarrow R_{(x)} = e_{(x)}^a R_a = T_{ab}^c e_{(x)}^a e_{(p)}^b e_c^{(p)}$$



$$\Gamma_a = \Gamma_{ab}^b, \quad R_a = T_{ab}^c \delta_c^b = T_{ab}^c e_{(b)}^b e_c^{(b)} (= T_{a(b)}^{(b)})$$

$$\Rightarrow R_{(x)} = e_{(x)}^a R_a = T_{ab}^c e_{(x)}^a e_{(b)}^b e_c^{(b)} = T_{(x)(b)}^{(b)}$$

$\swarrow$  free  
 $\nwarrow$  dummy



$$\Rightarrow K_{(x)} = e^a_{(x)} R_a = T^c_{ab} e^a_{(x)} e^b_{(y)} e^c_{(z)} = T^c_{(x)(y)(z)} \leftarrow \text{dummy}$$

tensor addition : "pointwise" add components

⊗

contraction



$$\delta_a = \delta_{ab}, \quad K_a = T_{ab} \delta_c = \delta_{ab} \quad e^a_{(p)} e_c^{(p)} = T_{a(p)}^{(p)}$$

$$\Rightarrow R_{(x)} = e^a_{(x)} R_a = T_{ab}^c e^a_{(x)} e^b_{(p)} e_c^{(p)} = T_{(x)(p)}^{(p)}$$

free  $\leftarrow$  dummy

tensor addition : "pointwise" add

⊗ (outer) mult. co (in all ways)

contraction sum

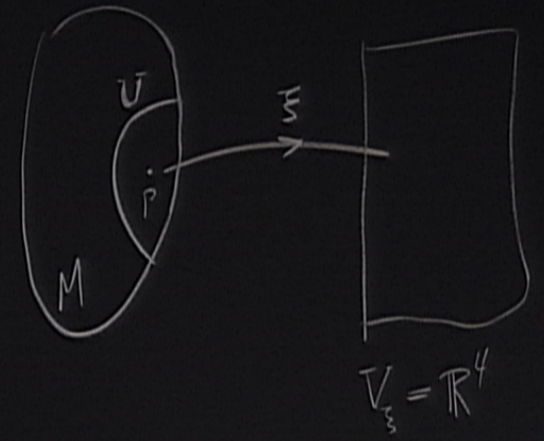




Coord bases for TM

$$V_p = \langle \partial/\partial x^i \rangle$$

$$1 = \sum_{i=1}^n \langle \partial/\partial x^i \rangle \langle dx^i \rangle$$

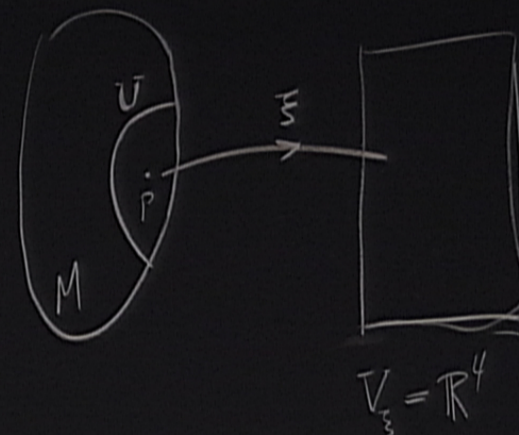




Coord bases for TM

$\mathbb{R}^4$  has a natural basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$



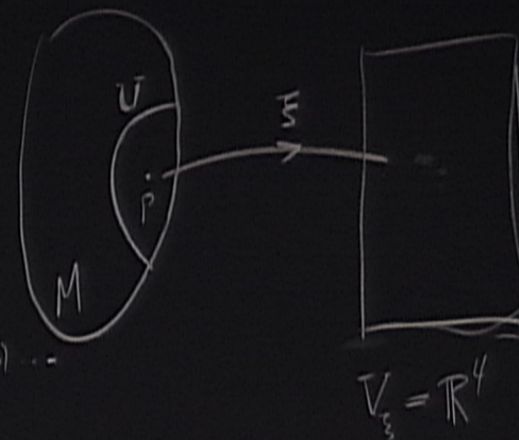


Coord bases for TM

$\mathbb{R}^4$  has a natural basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \dots \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$= (\mathbb{R}^4)^*$  also has ...  $(1\ 0\ 0\ 0), (0\ 1\ 0\ 0) \dots$   
 $\equiv \mathbb{R}^4$





$\mathbb{R}^4$  has a natural basis

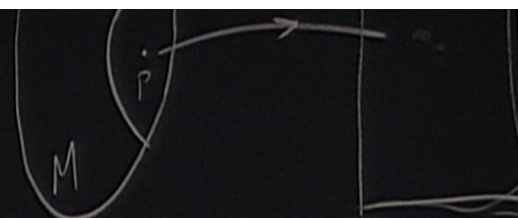
$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \dots \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$(\mathbb{R}^4)^*$  also has ...  $(1\ 0\ 0\ 0), (0\ 1\ 0\ 0) \dots$

$$\cong \mathbb{R}^4$$

$$\xi: T_p M \rightarrow \mathbb{R}^4$$

components of  $U^a \in T_p M$  are  $\xi_*(U^a) \in \mathbb{R}^4$



$$V_p = \mathbb{R}^4$$

$$Q = P + dP$$

A diagram showing a point  $P$  and a vector arrow pointing from  $P$  to a point labeled  $Q = P + dP$ .





Coord Bases for  $TM$

$\mathbb{R}^4$  has a natural basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \dots \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

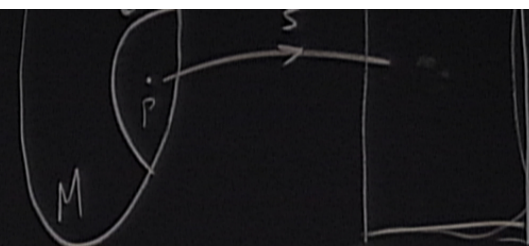
$(\mathbb{R}^4)^*$  also has ...  $(1\ 0\ 0\ 0), (0\ 1\ 0\ 0) \dots$

$\cong \mathbb{R}^4$

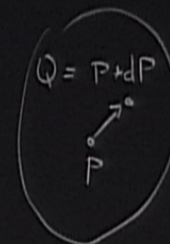
$$\xi_* : T_p M \rightarrow \mathbb{R}^4$$

The components of  $u^a \in T_p M$   $\xi_*(u^a) \in \mathbb{R}^4$

$$(dP)^a \rightarrow (dP)^a$$

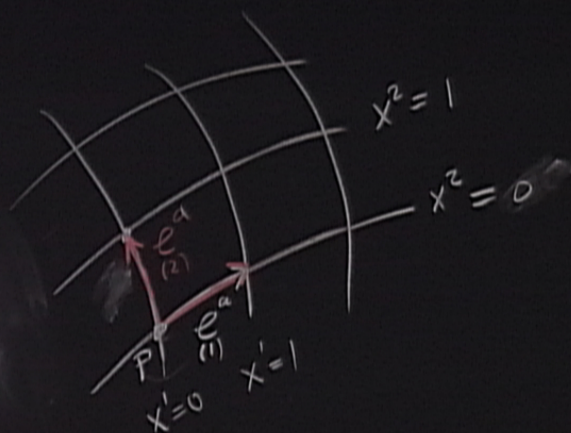


$$V_\xi = \mathbb{R}^4$$





Dual basis



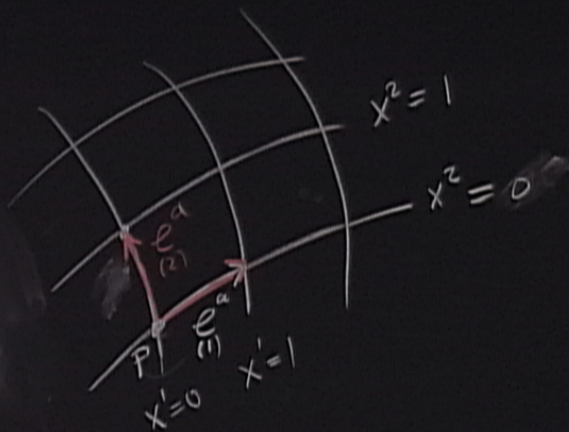
$\alpha = 1, 2$

$$e^a = \left[ \frac{dY(\lambda)}{d\lambda} \right]^a \quad \text{if} \quad \xi(Y(\lambda)) = (\lambda, 0, 0, 0)$$





$d=1,2$



$$e_a = \left[ \frac{dY(\lambda)}{d\lambda} \right]^a \quad \text{if} \quad \xi(Y(\lambda)) = (\lambda, 0, 0, 0)$$

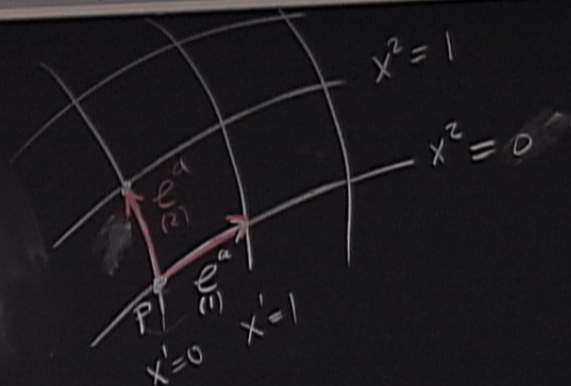
Dual basis

Gradient operator  $\partial$

$$\partial f = \text{grad } f$$







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Dual basis

Gradient operator  $\partial$

$$\partial f = \text{grad } f$$

$$dP^a(\partial_a f) = df$$

$$\left( \frac{dY(\lambda)^a}{d\lambda} \partial_a f = \frac{d}{d\lambda} (f(Y(\lambda))) \right)$$





Dual bases  $B \Rightarrow B^*$  basis for  $V^*$ ,  $B^* = \{ \epsilon_a^{(1)} \mid a=0 \dots 3 \}$   $B \leftrightarrow B^*$

Claim  $dP^a \rightarrow df$  is linear in  $dP$  Then  $\partial_a f$



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Claim  $dP^a \rightarrow df$  is linear in  $dP$ . Then  $\partial_a f(P) \in (T_P^* M)^* = T_P^* M$



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$df \rightarrow df$  is linear in  $dP$  Then  $\partial_a f(P) \in (T_P^* M)^* = T_P^* M$

$$\partial_a f = \frac{\partial f}{\partial x^{(a)}}$$



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Claim  $d\pi^* \rightarrow df$  is linear in  $dP$  Then  $\partial_a f(P) \in (T_P^* M)^* = T_P^* M$

$$\partial_{(a)} \frac{f}{x^{(\lambda)}}$$

eg  $\lambda=2$

$\omega$



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Claim  $dP^a \rightarrow df$  is linear in  $dP$  Then  $\partial_a f(P) \in (T_P^* M)^* = T_P^* M$

$$\partial_{(\lambda)} f = \frac{\partial f}{\partial x^{(\lambda)}} \quad \text{eg } \lambda=2$$

$$\gamma(\lambda) = \overset{0 \ 1 \ 2 \ 3}{\xi^{-1}}(0, 0, \lambda, 0)$$

$$\omega_a = \partial_a f \quad \omega_2 = \partial_2 f = e_2^a (\partial_a f) = \frac{d\gamma(\lambda)}{d\lambda} \partial_a f$$



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Claim  $dP^a \rightarrow df$  is linear in  $dP$  Then  $\partial_a f(P) \in (T_P^* M)^* = T_P^* M$

$$\partial_{(i)} f = \frac{\partial f}{\partial x^{(i)}} \quad \text{eg } i=2$$

$$\gamma(\lambda) = \xi^{-1}(0, 0, \lambda, 0) \quad \tilde{f} = f \circ \xi^{-1}$$

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$$\partial_{(a)} f = \frac{\partial f}{\partial x^{(a)}} \quad \text{eg } a=2$$

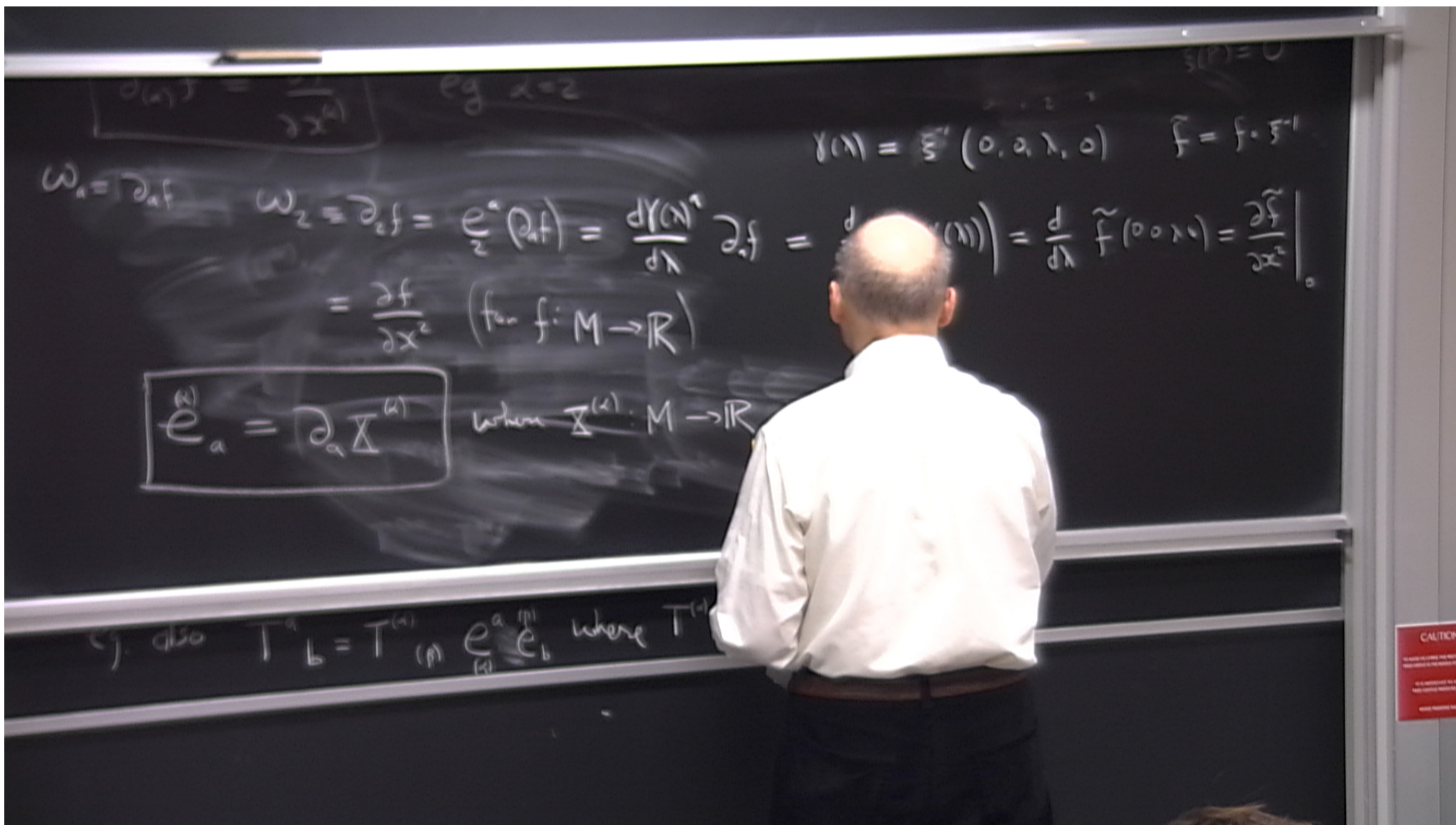
$$\xi(P) = 0$$

$$\gamma(\lambda) = \xi^{-1}(0,0,\lambda,0) \quad \tilde{f} = f \circ \xi^{-1}$$

$$\omega_a = \partial_a f \quad \omega_2 = \partial_2 f = e_2^a (\partial_a f) = \frac{d\gamma(\lambda)}{d\lambda} \partial_a f = \frac{d}{d\lambda} (f(\gamma(\lambda))) = \frac{d}{d\lambda} \tilde{f}(0,0,\lambda,0) = \left. \frac{\partial \tilde{f}}{\partial x^2} \right|_0$$

$$= \frac{\partial f}{\partial x^2}$$







$$\partial_{(1)} = \frac{\partial}{\partial x^{(1)}} \quad \text{eg } x=2$$

$$\omega_a = \partial_a f \quad \omega_2 = \partial_2 f = e_2^* (\partial_a f) = \frac{dY(N)}{d\lambda} \partial_a f = \frac{d}{d\lambda} (f(Y(N))) = \frac{d}{d\lambda} \tilde{f}(0, \lambda, 0) = \left. \frac{\partial \tilde{f}}{\partial x^2} \right|_0$$

$$= \frac{\partial f}{\partial x^2} \quad (f: M \rightarrow \mathbb{R})$$

$$\boxed{e_a^{(n)} = \partial_a X^{(n)}} \quad \text{where } X^{(n)}: M \rightarrow \mathbb{R} \quad \text{given } \xi(P) = (X^{(1)}(P), X^{(2)}(P), \dots)$$

↑  
X is const regarded as scalar field on M.

$$\text{eg. also } T^a_b = T^{(a)}_{(b)} e^a_{(1)} e^b_{(1)} \quad \text{where } T^{(a)}_{(b)} = e^a_{(1)} e^b_{(1)} T^a_b$$