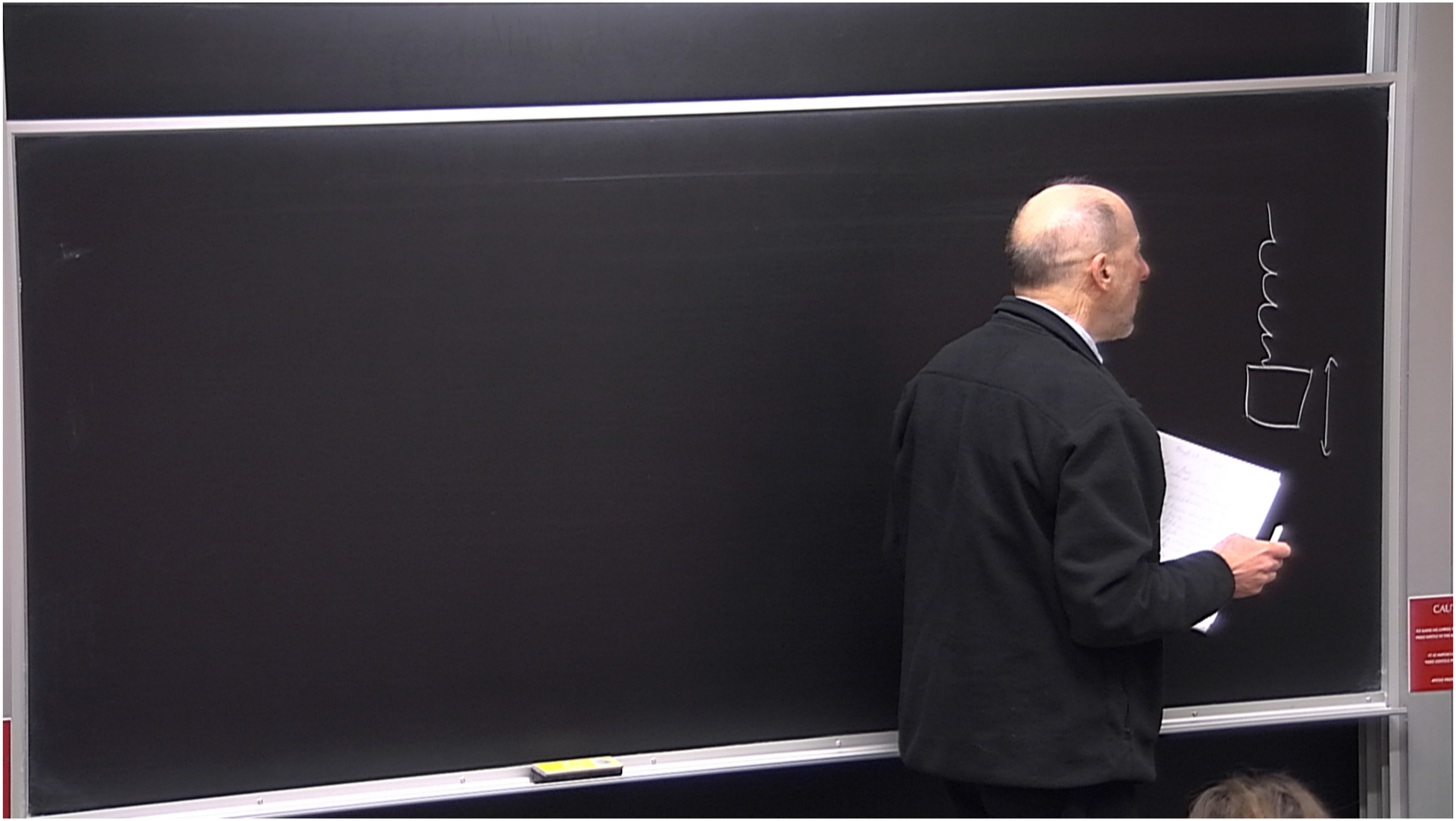
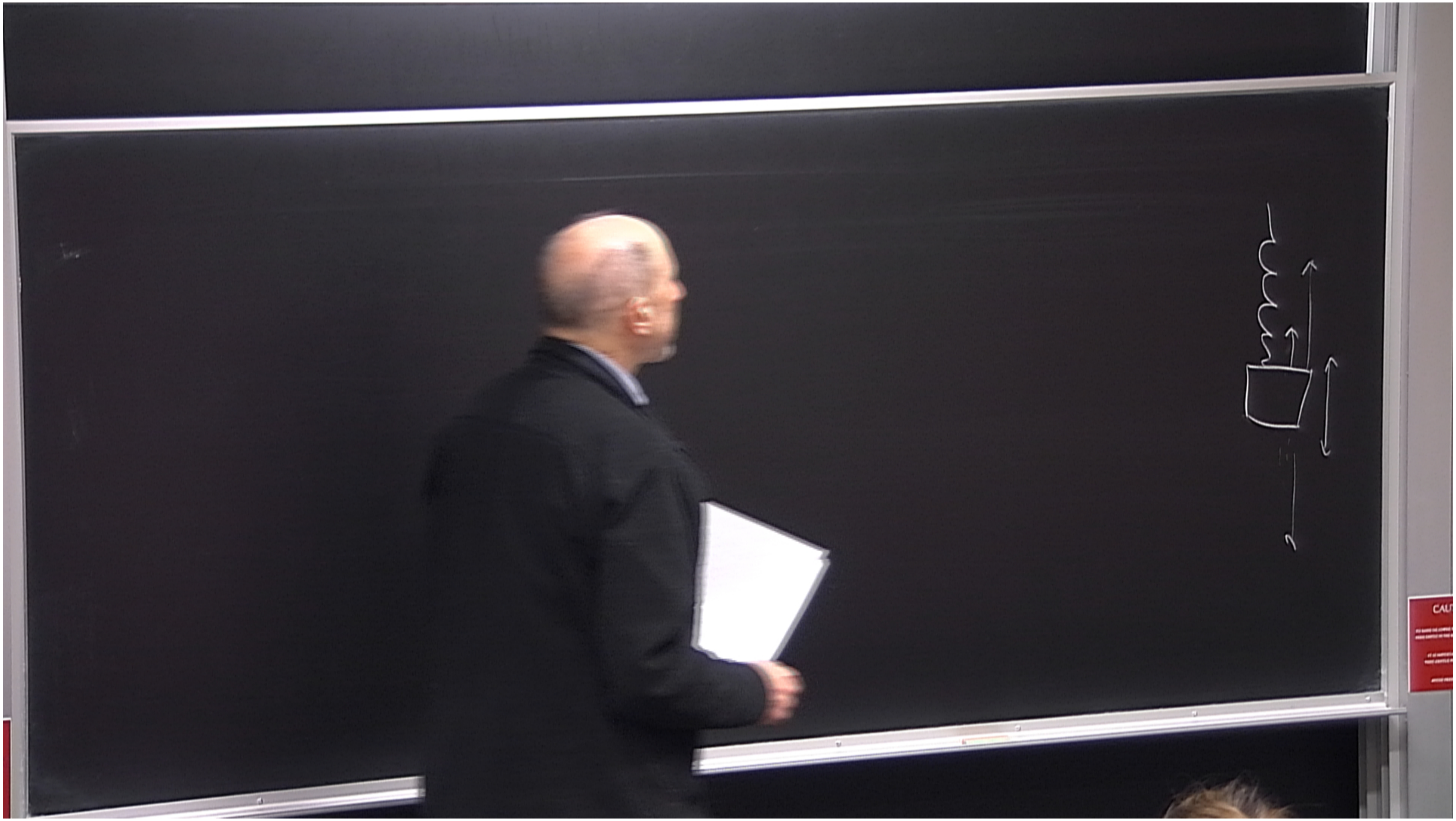
Title: Advanced General Relativity - Lecture 11

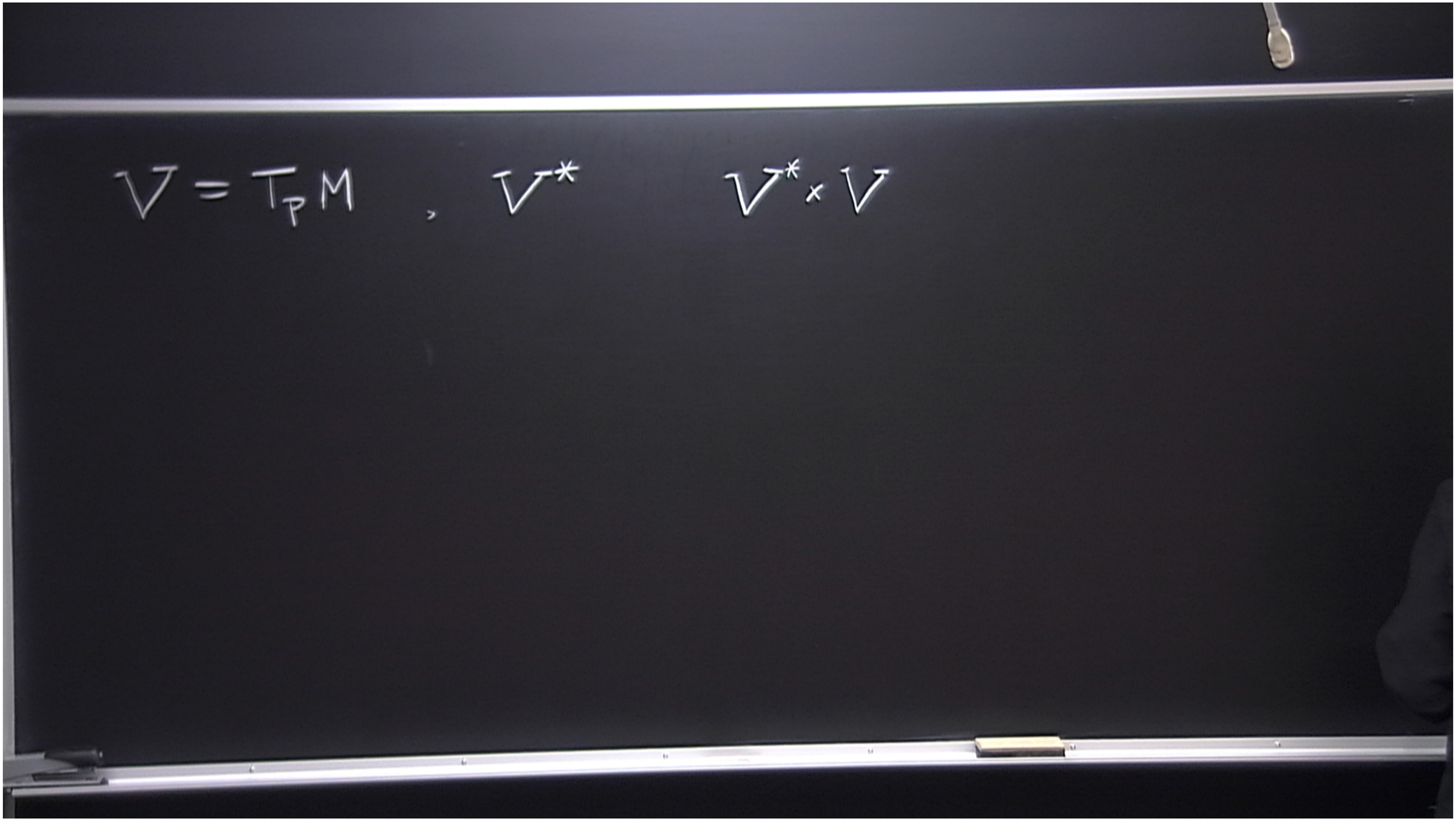
Date: Feb 26, 2013 05:00 PM

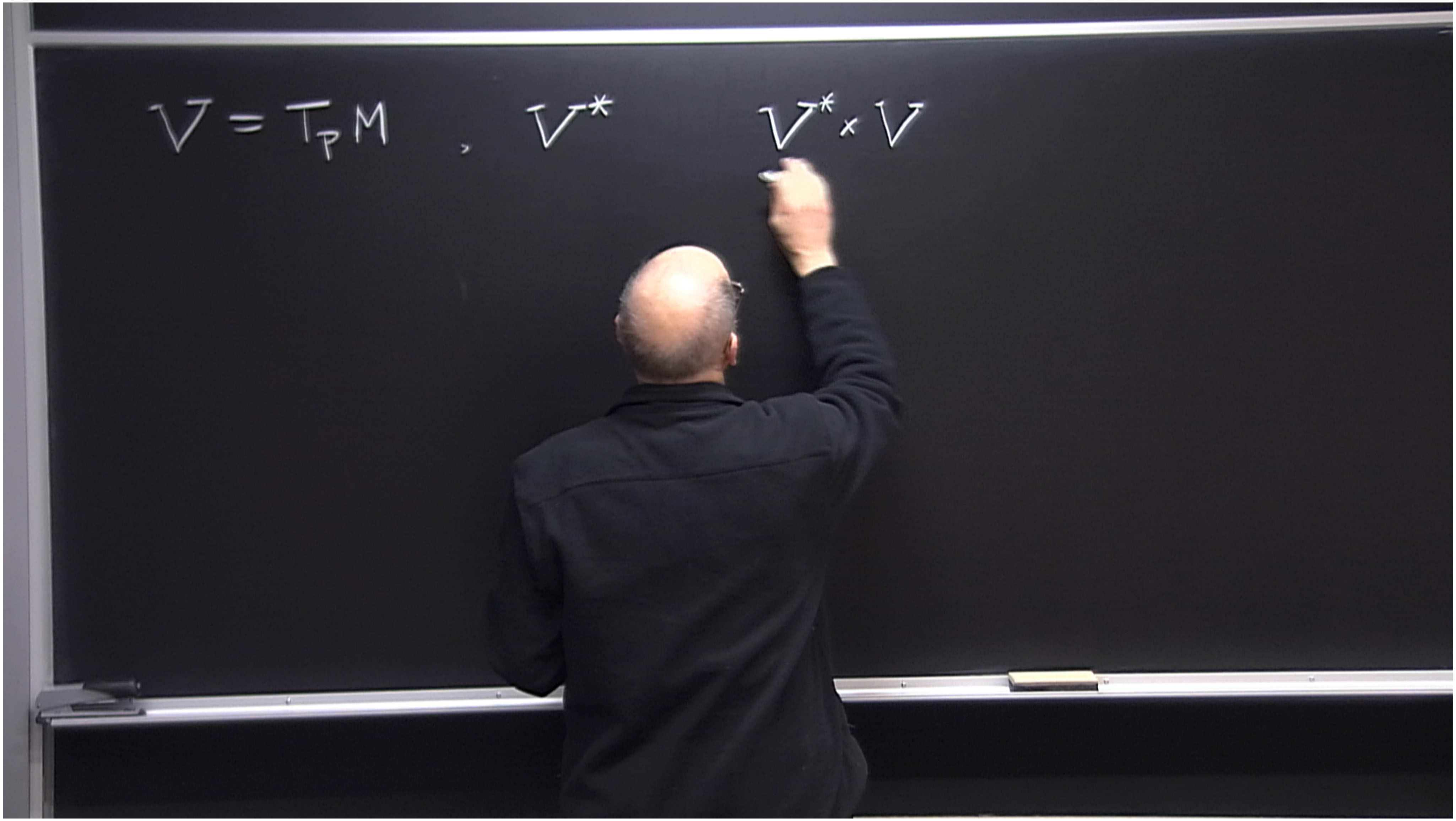
URL: http://pirsa.org/13020026

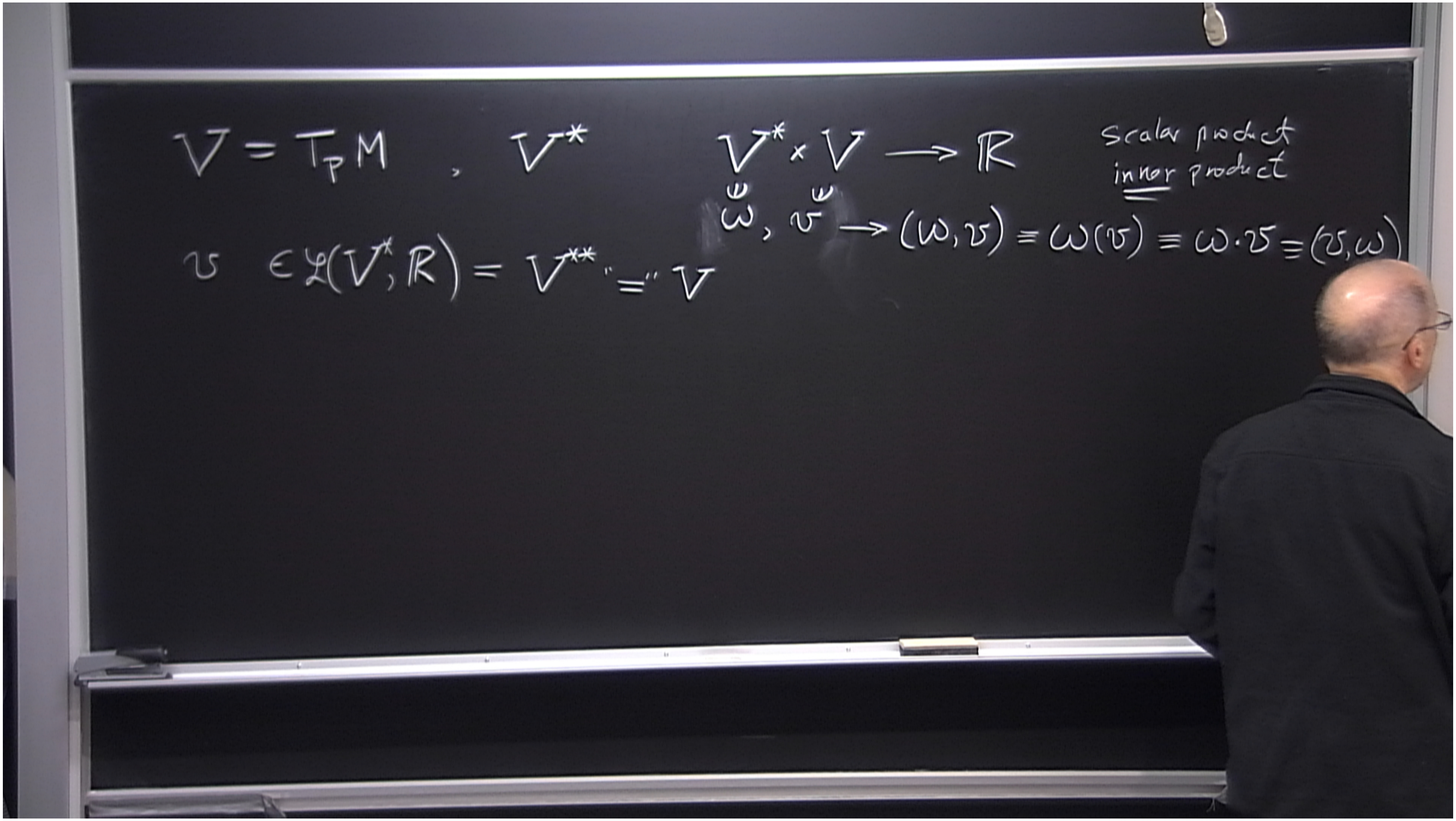
Abstract:

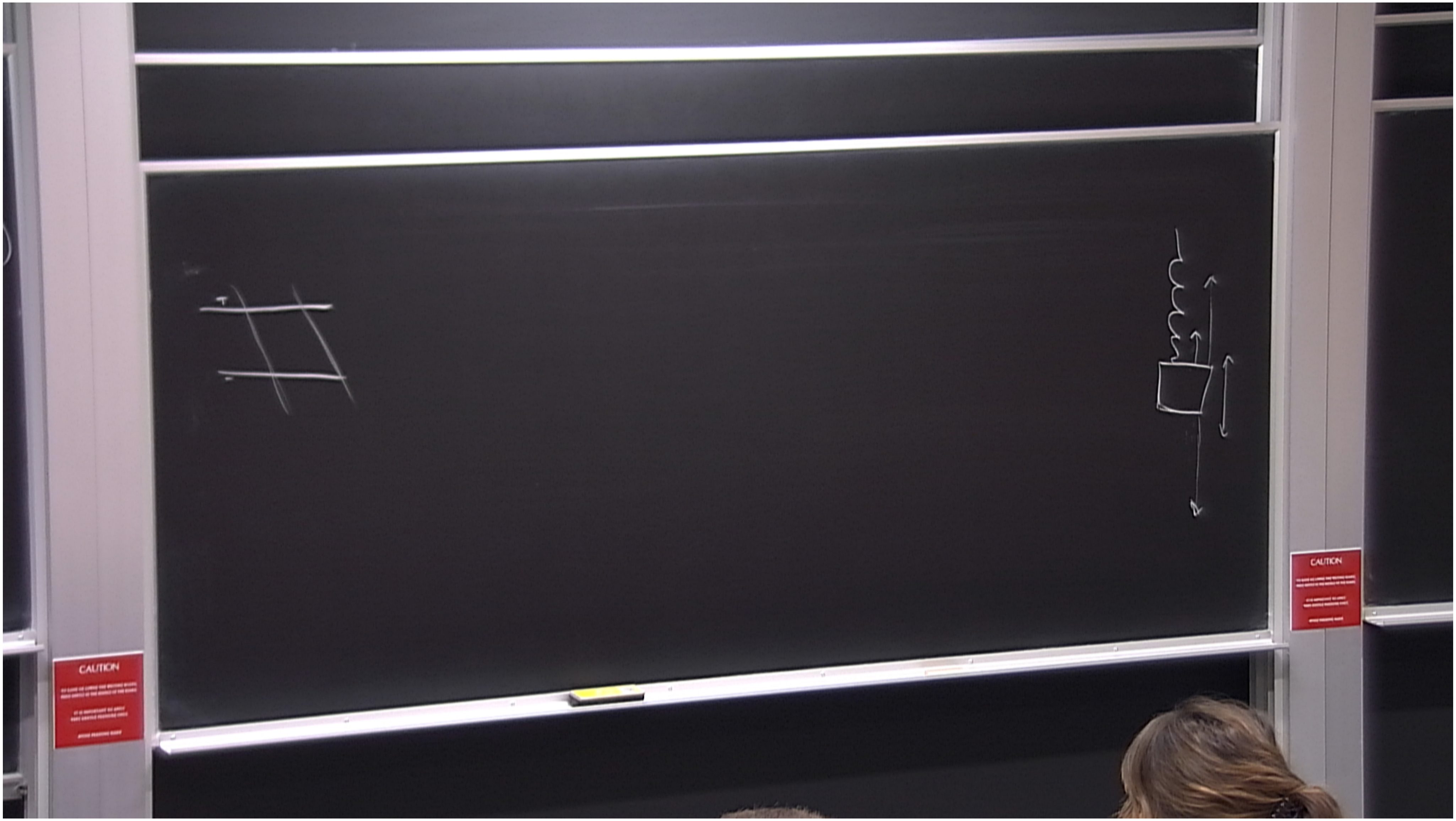


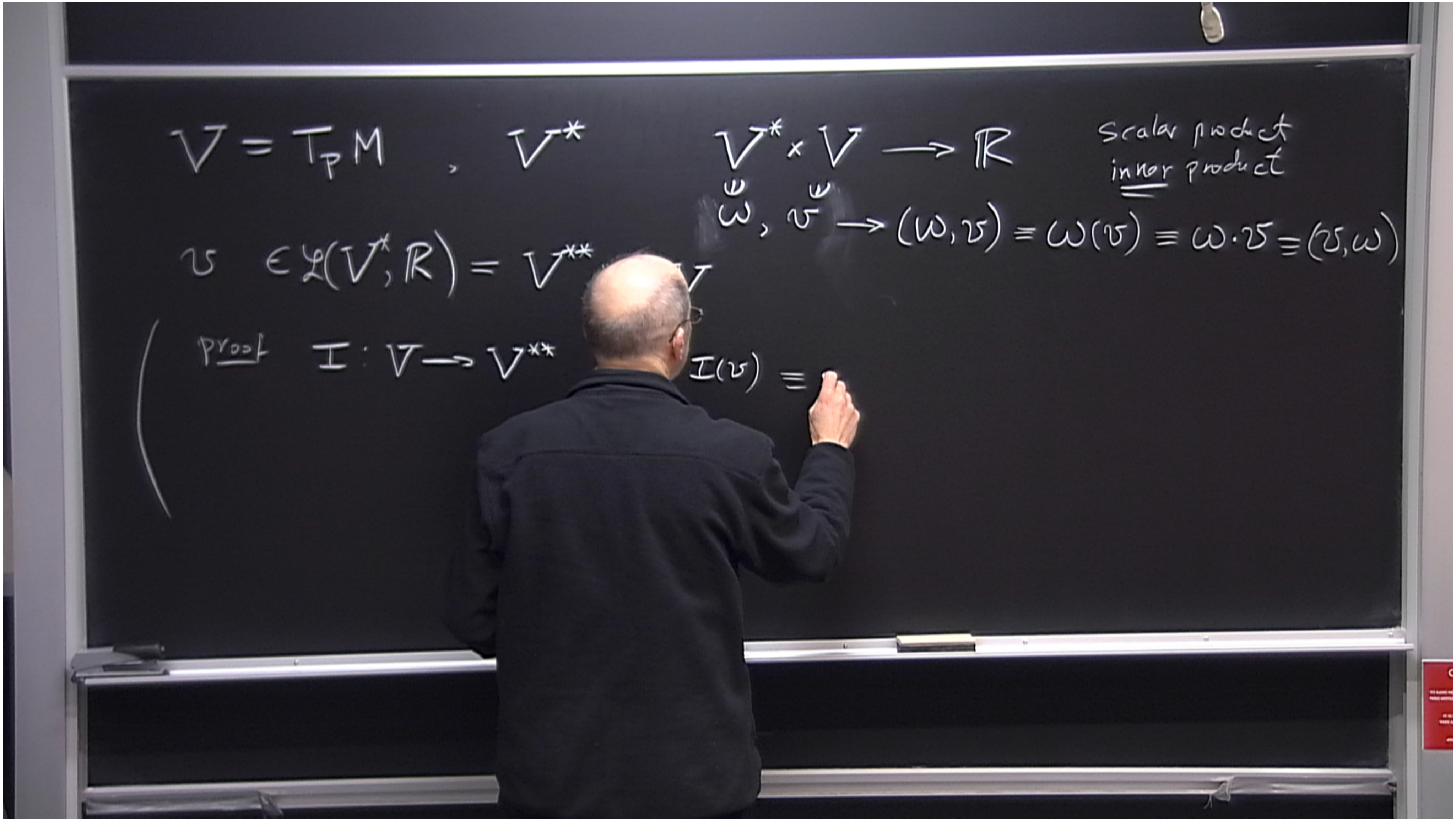


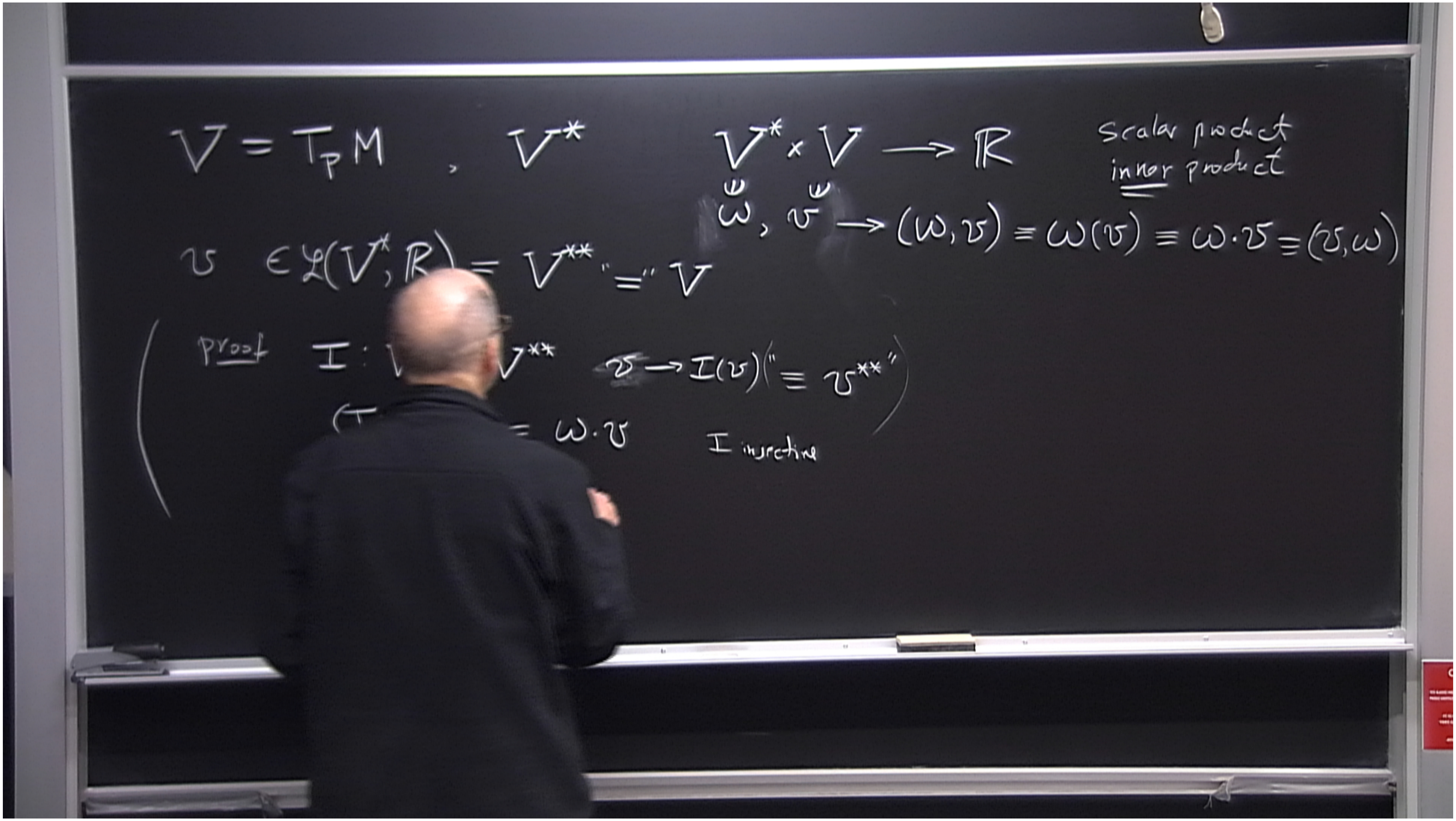

$V = T_p M$, V^*
$V^* \times V$


$V = T_p M$, V^*
$V^* \times V$


$V = T_p M \;,\; V^*$
$v \in \mathcal{L}(V^*; \mathbb{R}) = V^{**} \text{"="} V$
$V^* \times V \longrightarrow \mathbb{R}$
$\omega, v \longrightarrow (\omega, v) \equiv \omega(v) \equiv \omega \cdot v \equiv (v, \omega)$
scalar product
inner product




$V = T_p M$, V^*
$V^* \times V \longrightarrow \mathbb{R}$
scalar product
inner product
$\omega, v \longrightarrow (\omega, v) \equiv \omega(v) \equiv \omega \cdot v \equiv (v, \omega)$
$v \in \mathcal{L}(V^*; \mathbb{R}) = V^{**}$
proof $I : V \longrightarrow V^{**}$
$I(v) \equiv$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$V^* \times V \longrightarrow \mathbb{R}$
scalar product
inner product
$\omega, v \longrightarrow (\omega, v) \equiv \omega(v) \equiv \omega \cdot v \equiv (v, \omega)$
$v \in \mathcal{L}(V^*, \mathbb{R}) = V^{**}$ "=" V
proof
$v \longrightarrow I(v)$ ("$\equiv v^{**}$")
$= \omega \cdot v$
I injective

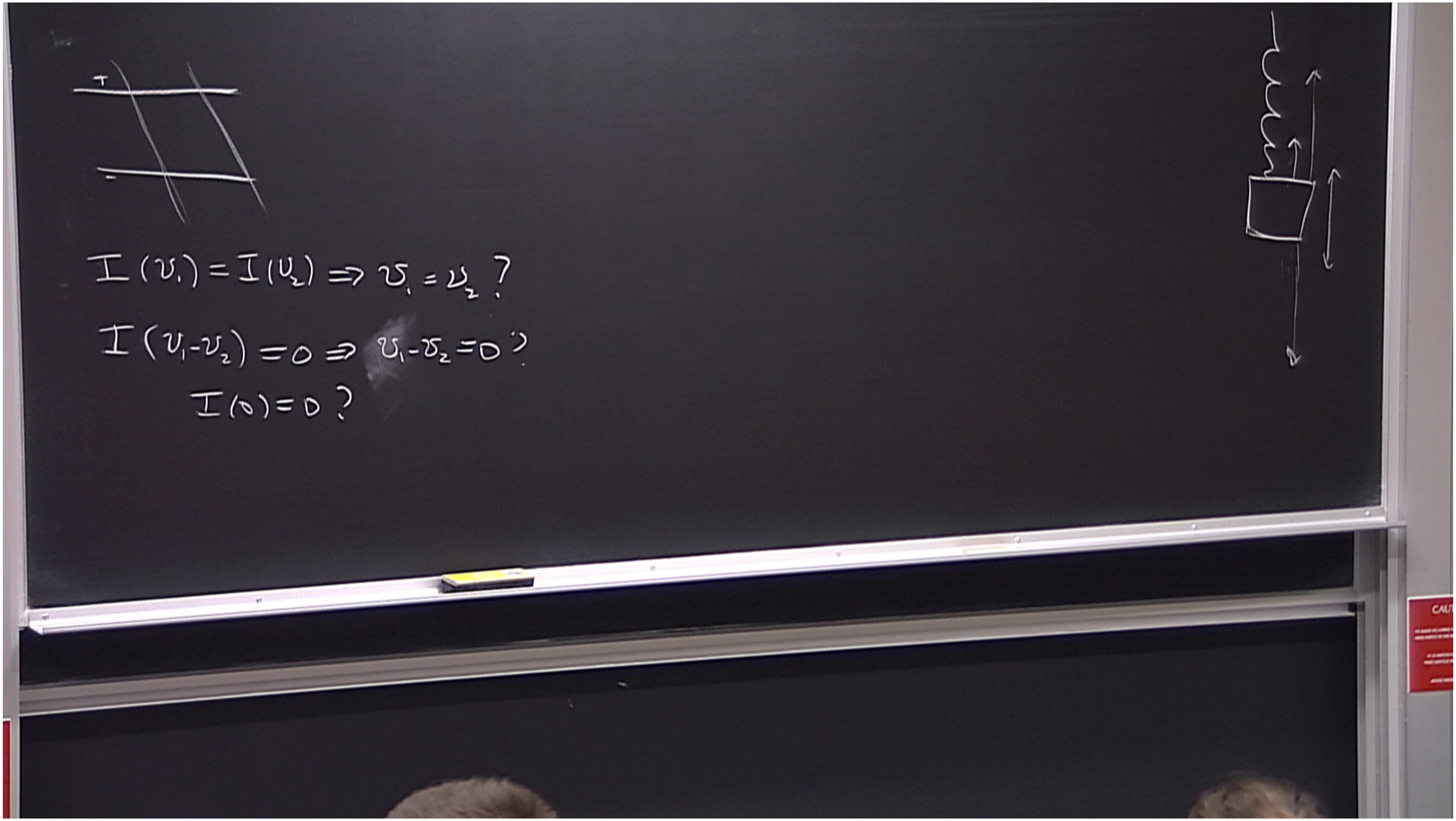
$V = T_pM \quad , \quad V^* \qquad V^* \times V \longrightarrow \mathbb{R}$ scalar product, inner product

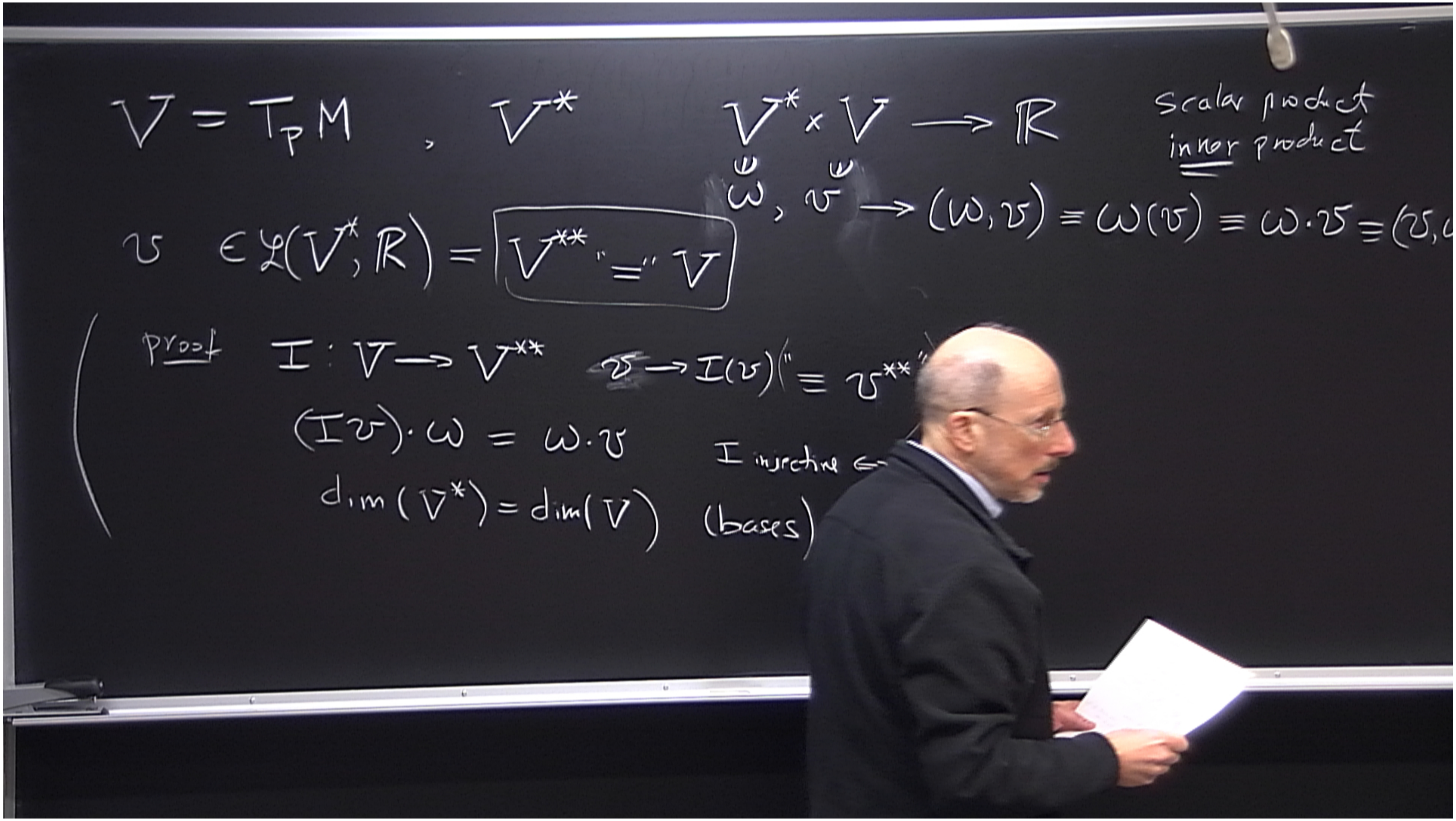
$\underset{\omega}{\Psi}, \underset{v}{\Psi} \longrightarrow (\omega, v) \equiv \omega(v) \equiv \omega \cdot v \equiv (v, \omega)$

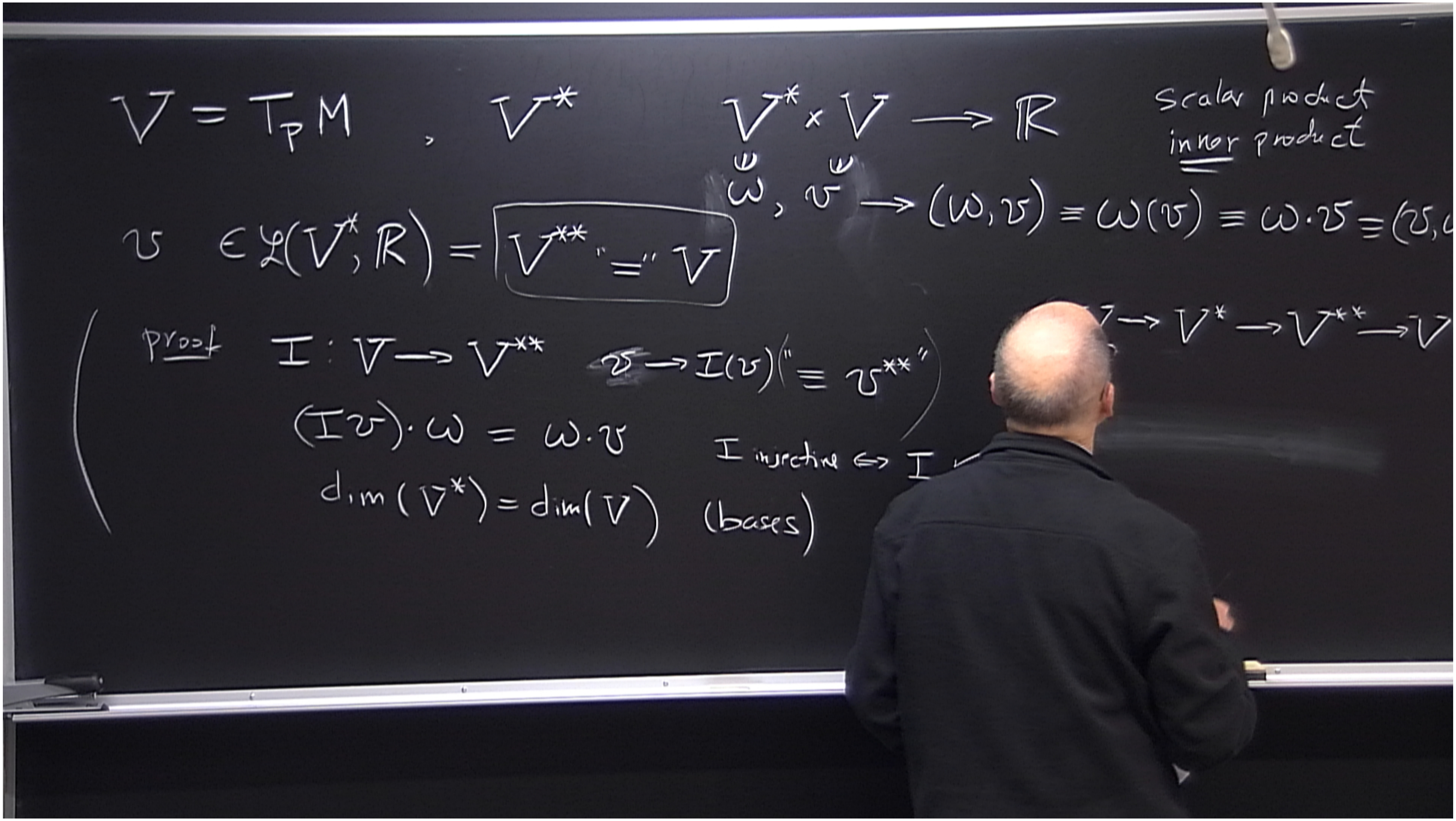
$v \in \mathcal{L}(V^*, \mathbb{R}) = V^{**} \text{ "=" } V$

(proof $I : V \longrightarrow V^{**} \quad v \longrightarrow I(v) \;(\text{"} \equiv v^{**} \text{"})$

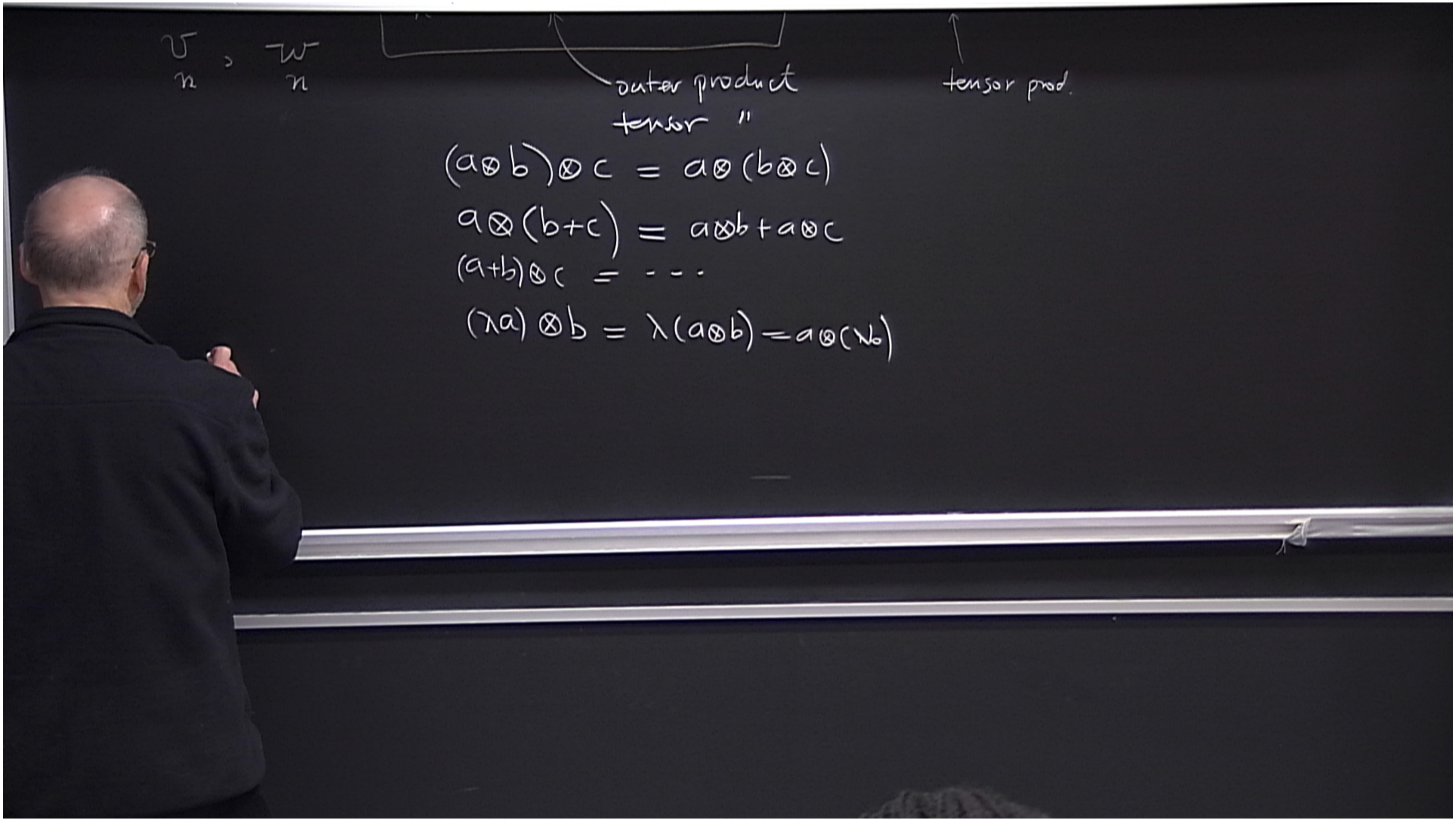
$(Iv) \cdot \omega = \omega \cdot v$ $\quad I$ injective $\Leftrightarrow I$)

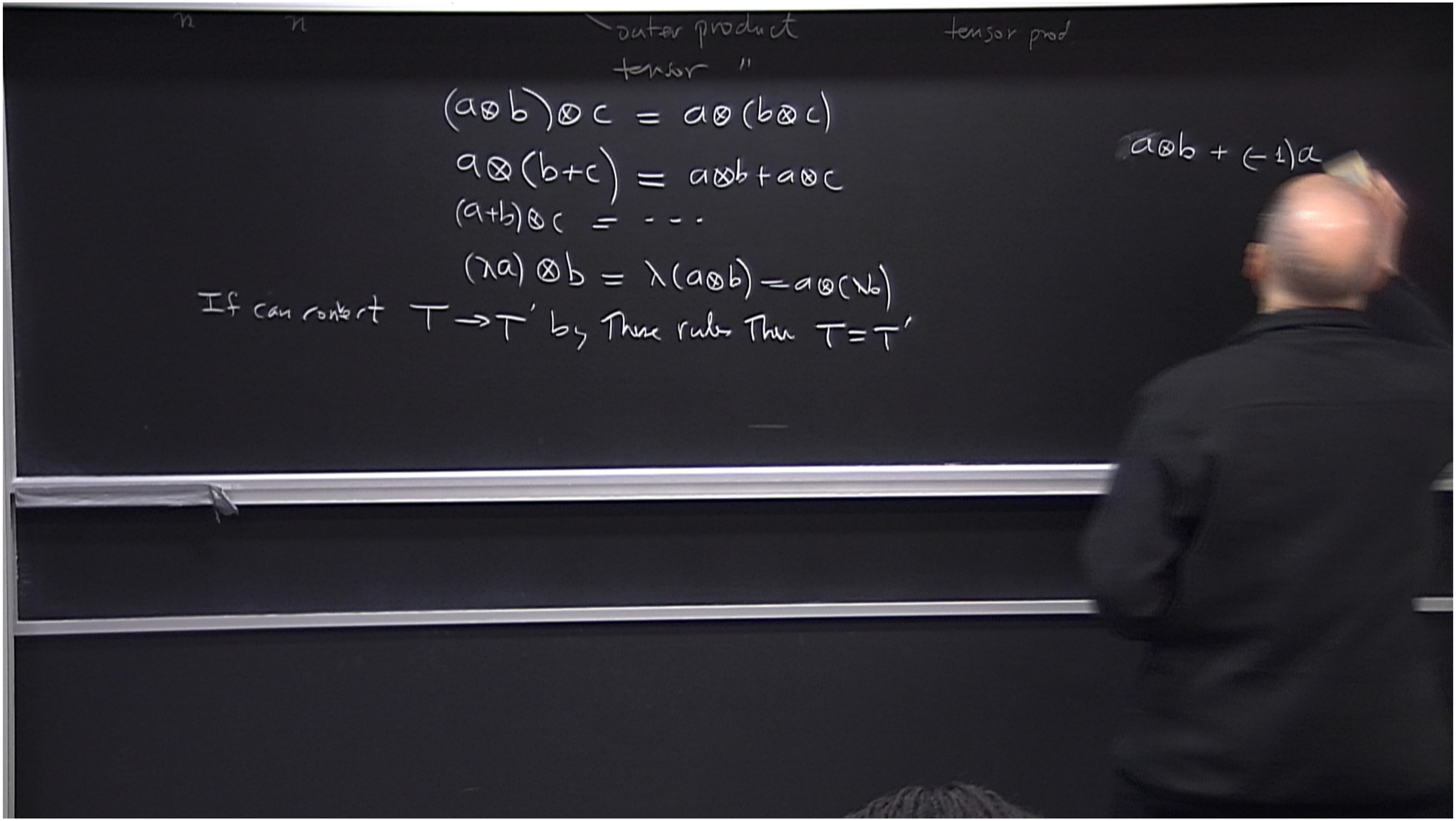

I(v1) = I(v2) ⇒ v1 = v2 ?
I(v1 − v2) = 0 ⇒ v1 − v2 = 0 ?
I(0) = 0 ?

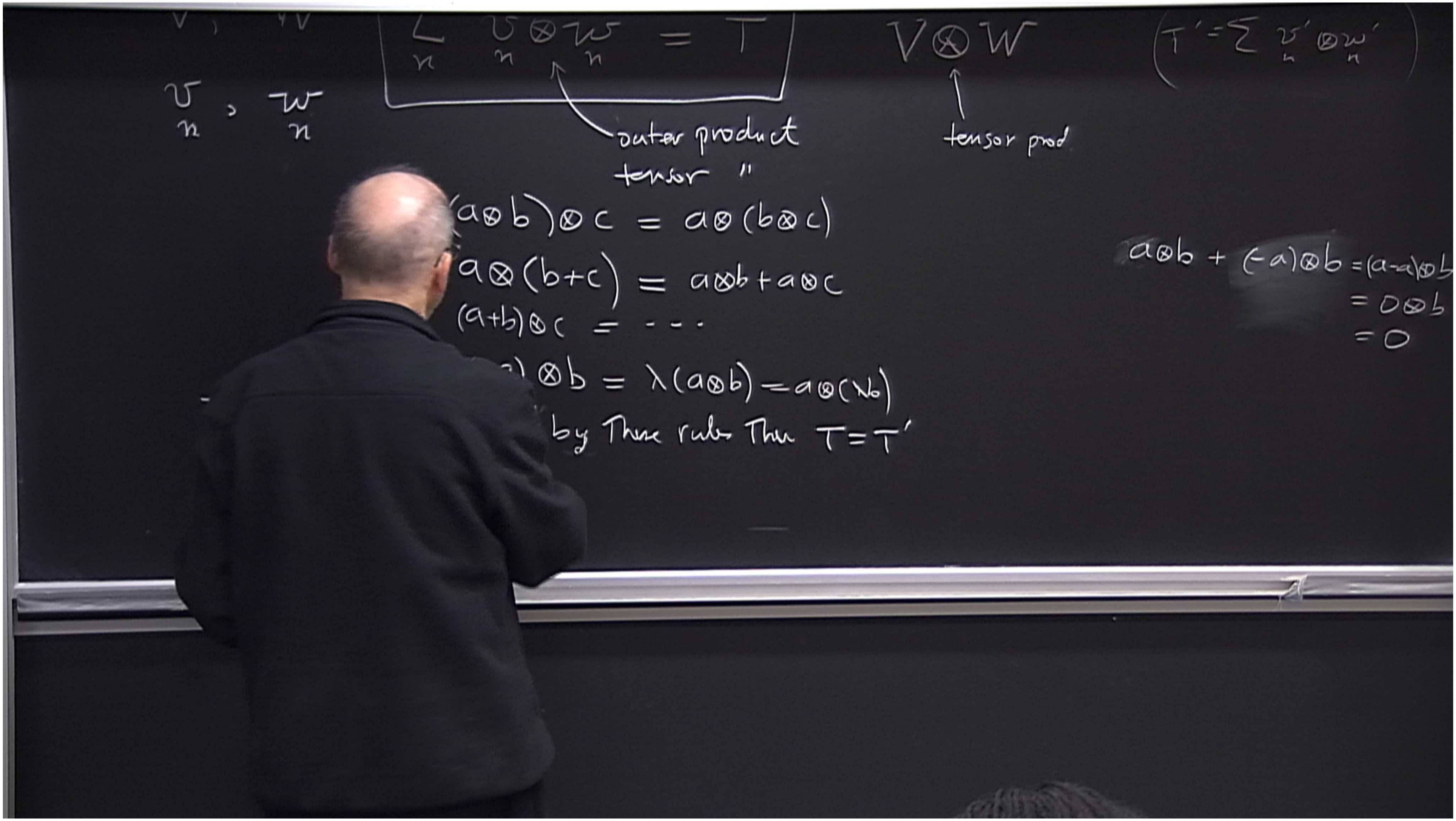

$V = T_p M$, V^*
$V^* \times V \longrightarrow \mathbb{R}$
scalar product
inner product
$\omega , v \longrightarrow (\omega, v) \equiv \omega(v) \equiv \omega \cdot v \equiv (v, \omega$
$v \in \mathcal{L}(V^*; \mathbb{R}) = \boxed{V^{**} \text{"="} V}$
proof $I : V \longrightarrow V^{**}$ $v \longrightarrow I(v)$ "$\equiv v^{**}$"
$(Iv) \cdot \omega = \omega \cdot v$
I injective
$\dim(V^*) = \dim(V)$ (bases)

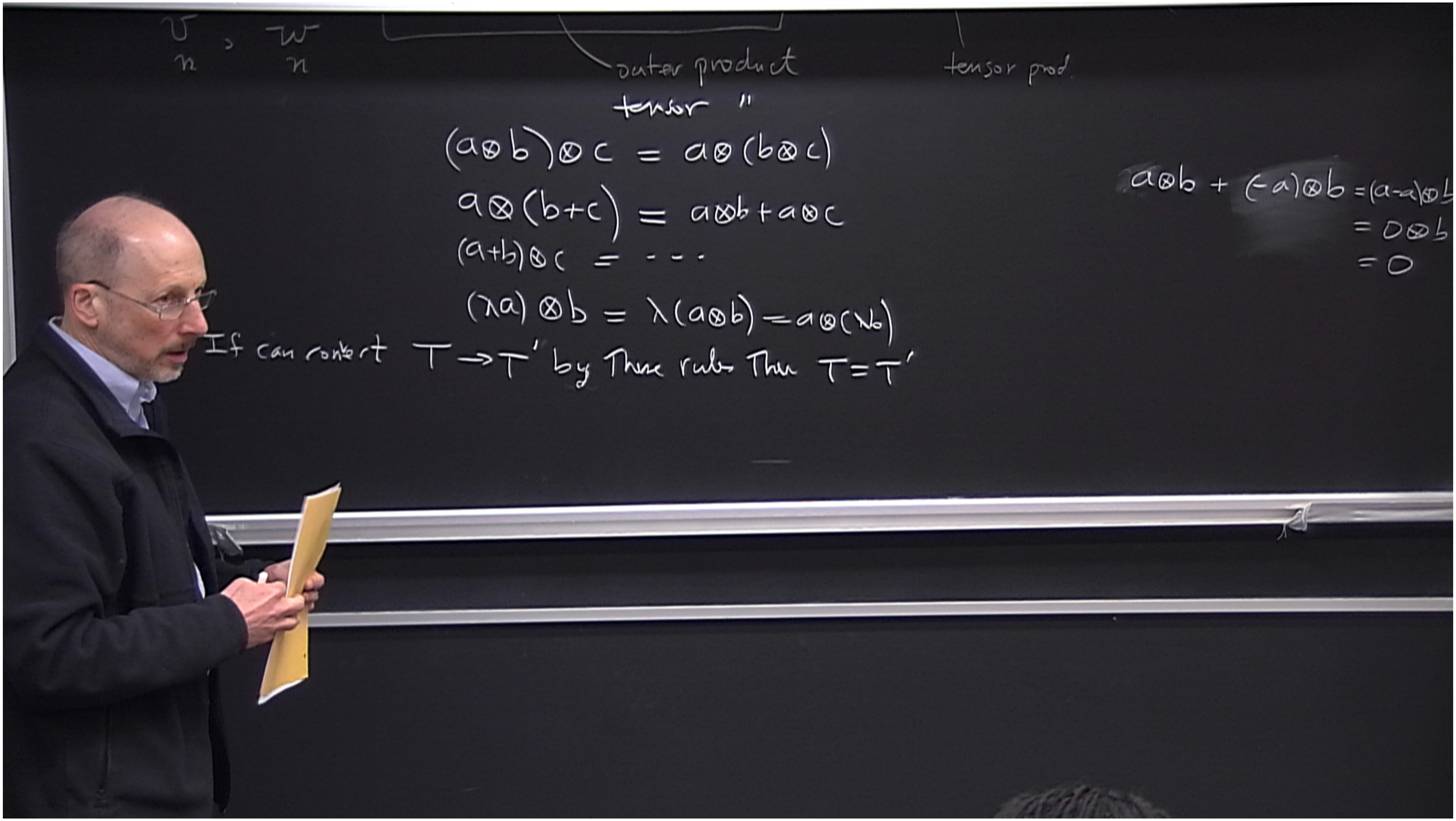


$V = T_p M$, V^*
$V^* \times V \longrightarrow \mathbb{R}$
scalar product
inner product
$\omega, v \longrightarrow (\omega, v) \equiv \omega(v) \equiv \omega \cdot v \equiv (v, \omega$
$v \in \mathcal{L}(V^*, \mathbb{R}) = \boxed{V^{**} \text{"="} V}$
proof $I : V \longrightarrow V^{**}$ $v \longrightarrow I(v)$ ("$\equiv v^{**}$")
$(Iv) \cdot \omega = \omega \cdot v$
I injective ⇔ I
$\dim(V^*) = \dim(V)$ (bases)
$\longrightarrow V^* \longrightarrow V^{**} \longrightarrow V$

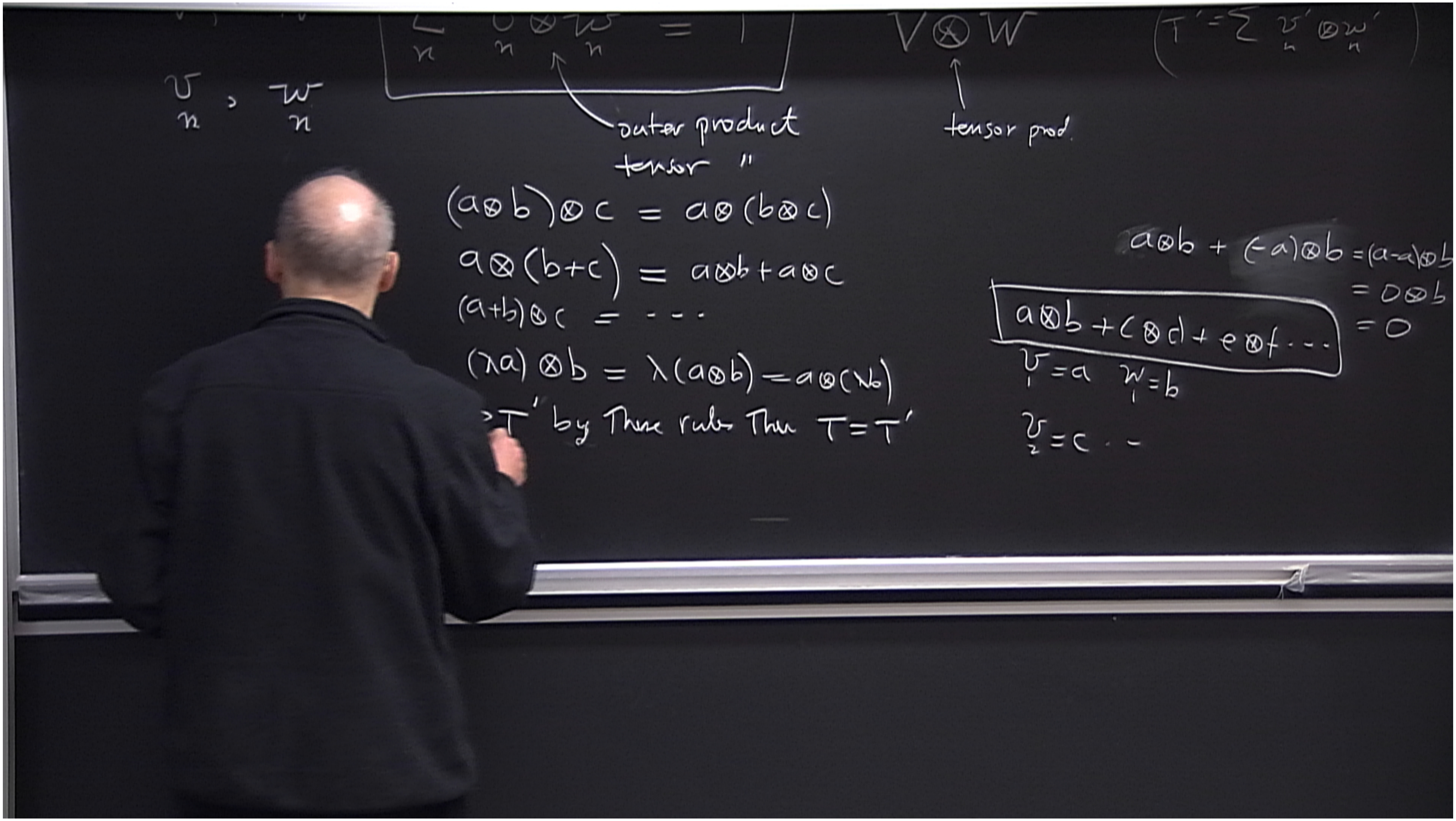

v_n, w_n
outer product
~~tensor~~ "
tensor prod.
$(a\otimes b)\otimes c = a\otimes(b\otimes c)$
$a\otimes(b+c) = a\otimes b + a\otimes c$
$(a+b)\otimes c = \cdots$
$(\lambda a)\otimes b = \lambda(a\otimes b) = a\otimes(\lambda b)$


outer product
tensor "
tensor prod
(a⊗b)⊗c = a⊗(b⊗c)
a⊗(b+c) = a⊗b + a⊗c
(a+b)⊗c = - - -
(λa)⊗b = λ(a⊗b) = a⊗(λb)
If can convert T → T' by these rules then T = T'
a⊗b + (−1)a


V ⊗ W
tensor prod
outer product
tensor "
(a⊗b)⊗c = a⊗(b⊗c)
a⊗(b+c) = a⊗b + a⊗c
(a+b)⊗c = - - -
⊗b = λ(a⊗b) = a⊗(λb)
by these rules then T = T'
a⊗b + (-a)⊗b = (a-a)⊗b
= 0⊗b
= 0



outer product
tensor "
(a⊗b)⊗c = a⊗(b⊗c)
a⊗(b+c) = a⊗b+a⊗c
(a+b)⊗c = ...
(λa)⊗b = λ(a⊗b) = a⊗(λb)
If can convert T → T' by these rules then T = T'
tensor prod.
a⊗b + (-a)⊗b = (a-a)⊗b
= 0⊗b
= 0



outer product
tensor "
V⊗W
tensor prod.
(a⊗b)⊗c = a⊗(b⊗c)
a⊗(b+c) = a⊗b + a⊗c
(a+b)⊗c = ...
(λa)⊗b = λ(a⊗b) = a⊗(λb)
T' by these rules then T = T'
a⊗b + (-a)⊗b = (a-a)⊗b
= 0⊗b
= 0
a⊗b + c⊗d + e⊗f ...
v1 = a w1 = b
v2 = c ...

V, W $\qquad \boxed{\sum_n v_n \otimes w_n = T}$ $\qquad V \otimes W$ $\qquad \left(T' = \sum v'_n \otimes w'_n\right)$

v_n, w_n

outer product
tensor "

tensor prod

$$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$

$$a \otimes (b + c) = a \otimes b + a \otimes c$$

$$(a + b) \otimes c = \cdots$$

$$(\lambda a) \otimes b = \lambda (a \otimes b) = a \otimes (\lambda b)$$

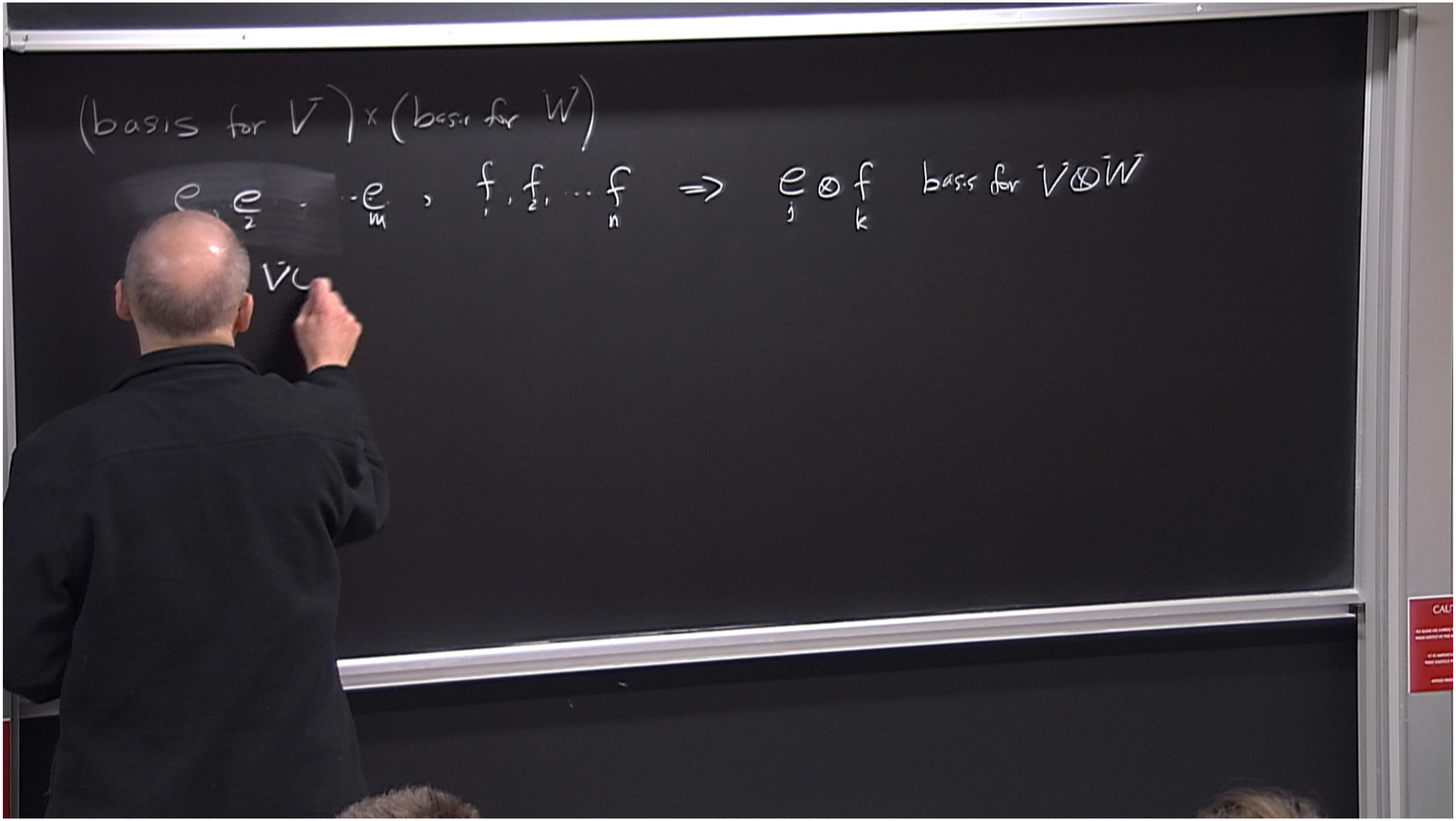
If can convert $T \to T'$ by these rules then $T = T'$

$$a \otimes b + (-a) \otimes b = (a - a) \otimes b = 0 \otimes b = 0$$

$$\boxed{a \otimes b + c \otimes d + e \otimes f \cdots}$$

$v_1 = a \quad w_1 = b$

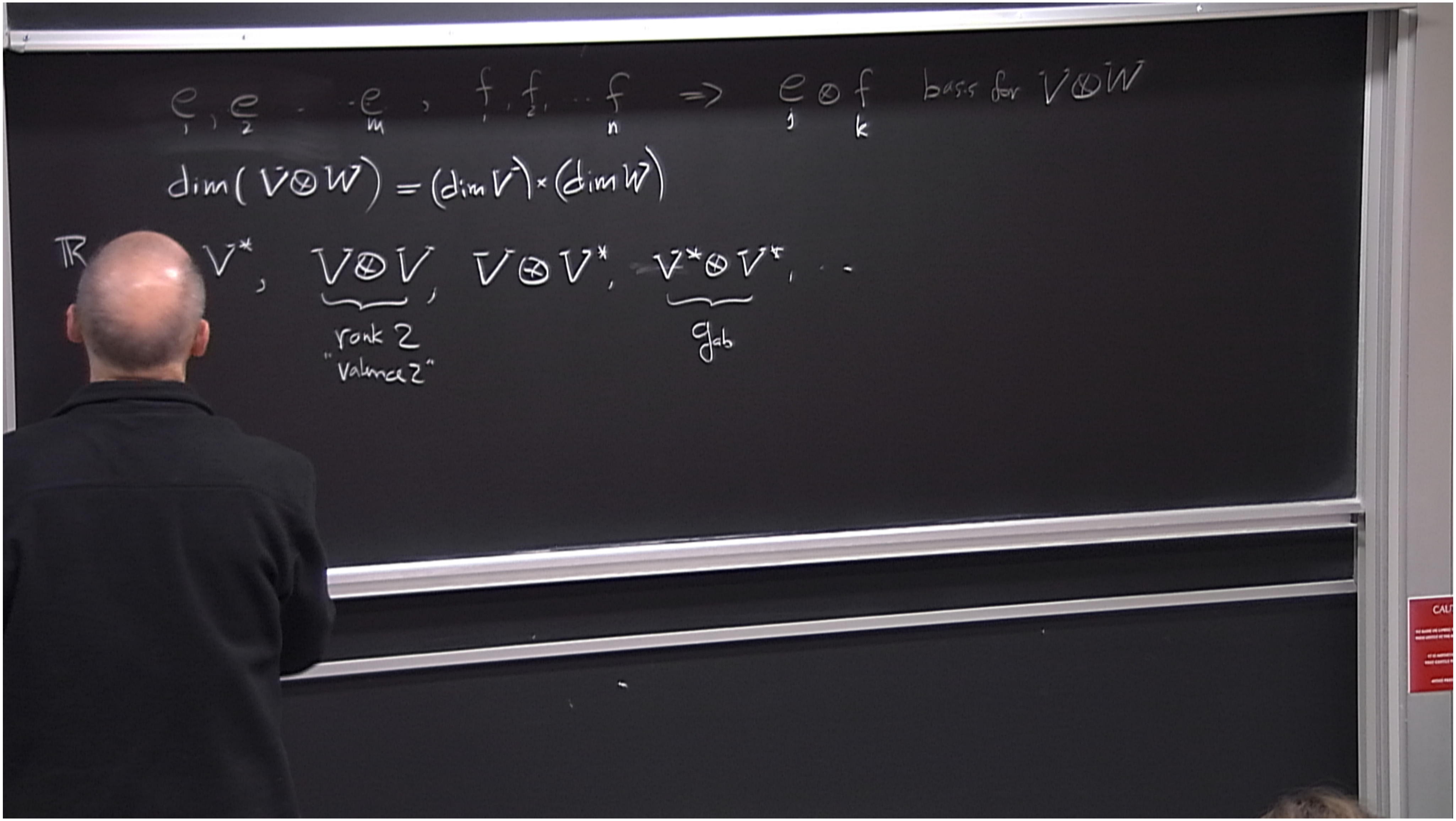
$v_2 = c \ldots$

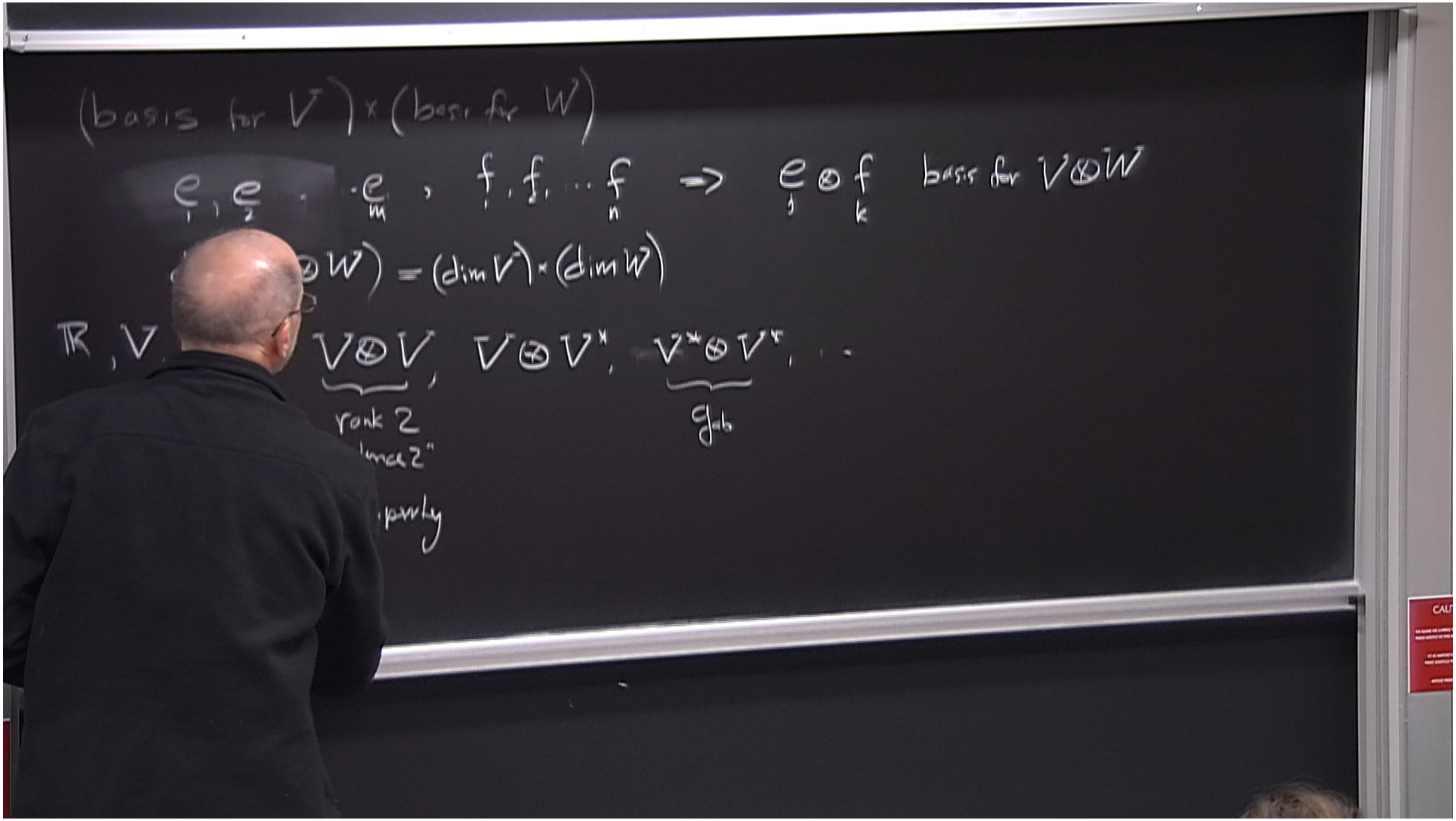

(basis for V) × (basis for W)
$e_1, e_2, \ldots e_m$, $f_1, f_2, \ldots f_n \Rightarrow e_j \otimes f_k$ basis for $V \otimes W$
V

(basis for V) $\times$ (basis for W)

$$e_1, e_2 \cdots e_m, \quad f_1, f_2, \cdots f_n \Rightarrow e_j \otimes f_k \text{ basis for } V \otimes W$$

$$\dim(V \otimes W) = (\dim V) \times (\dim W)$$


$e_1, e_2 \ldots e_m, \quad f_1, f_2, \ldots f_n \Rightarrow e_j \otimes f_k$ basis for $V \otimes W$
$\dim(V \otimes W) = (\dim V) \times (\dim W)$
$\mathbb{R}$, V^*, $\underbrace{V \otimes V}_{\text{rank 2}}$, $V \otimes V^*$, $\underbrace{V^* \otimes V^*}_{g_{ab}}$, ...
"valence 2"

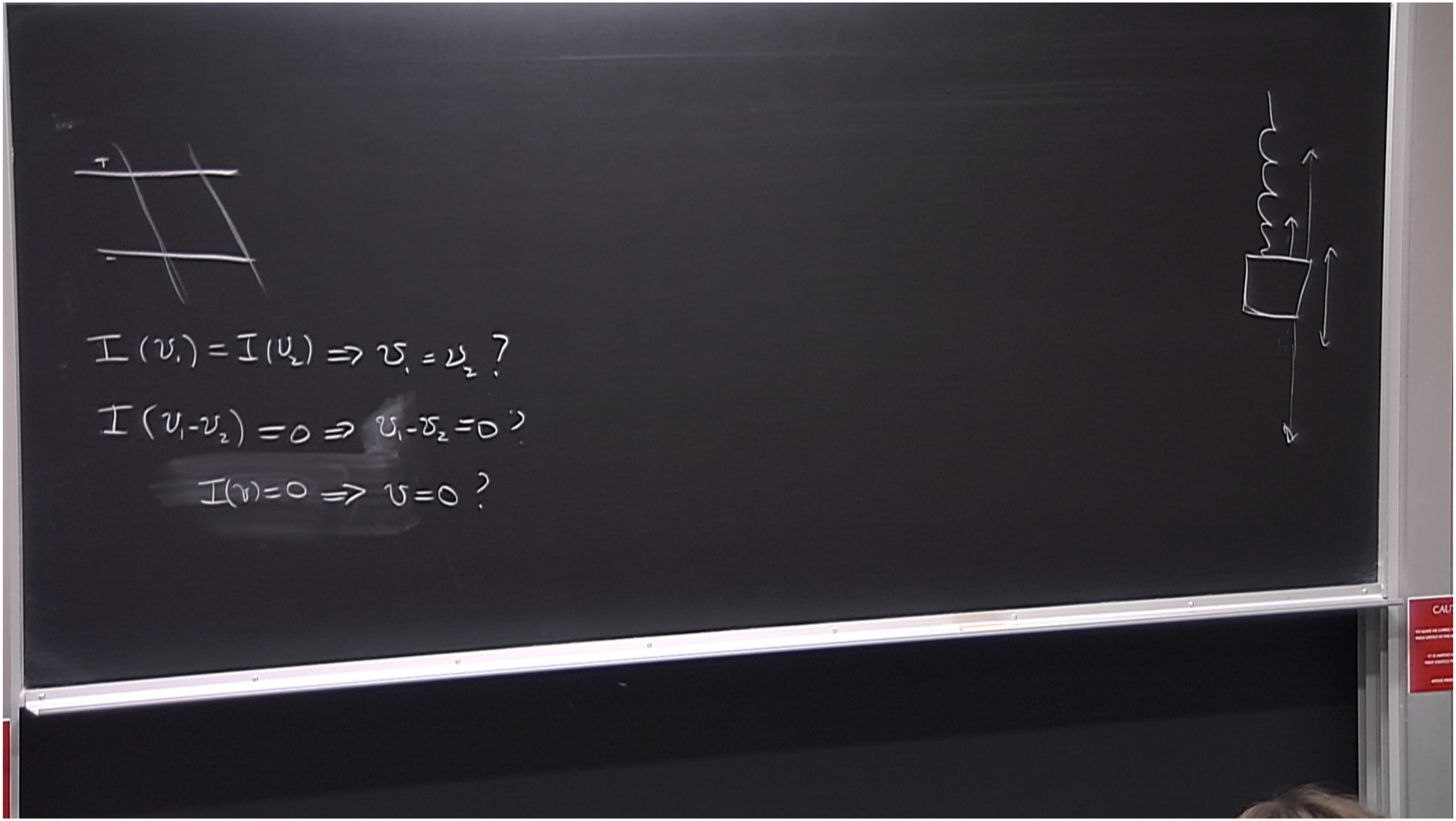

(basis for V) × (basis for W)
e1, e2 … em , f1, f2, … fn ⇒ ej ⊗ fk basis for V⊗W
⊗W) = (dim V) × (dim W)
R, V, V⊗V, V⊗V*, V*⊗V*, …
rank 2
gab

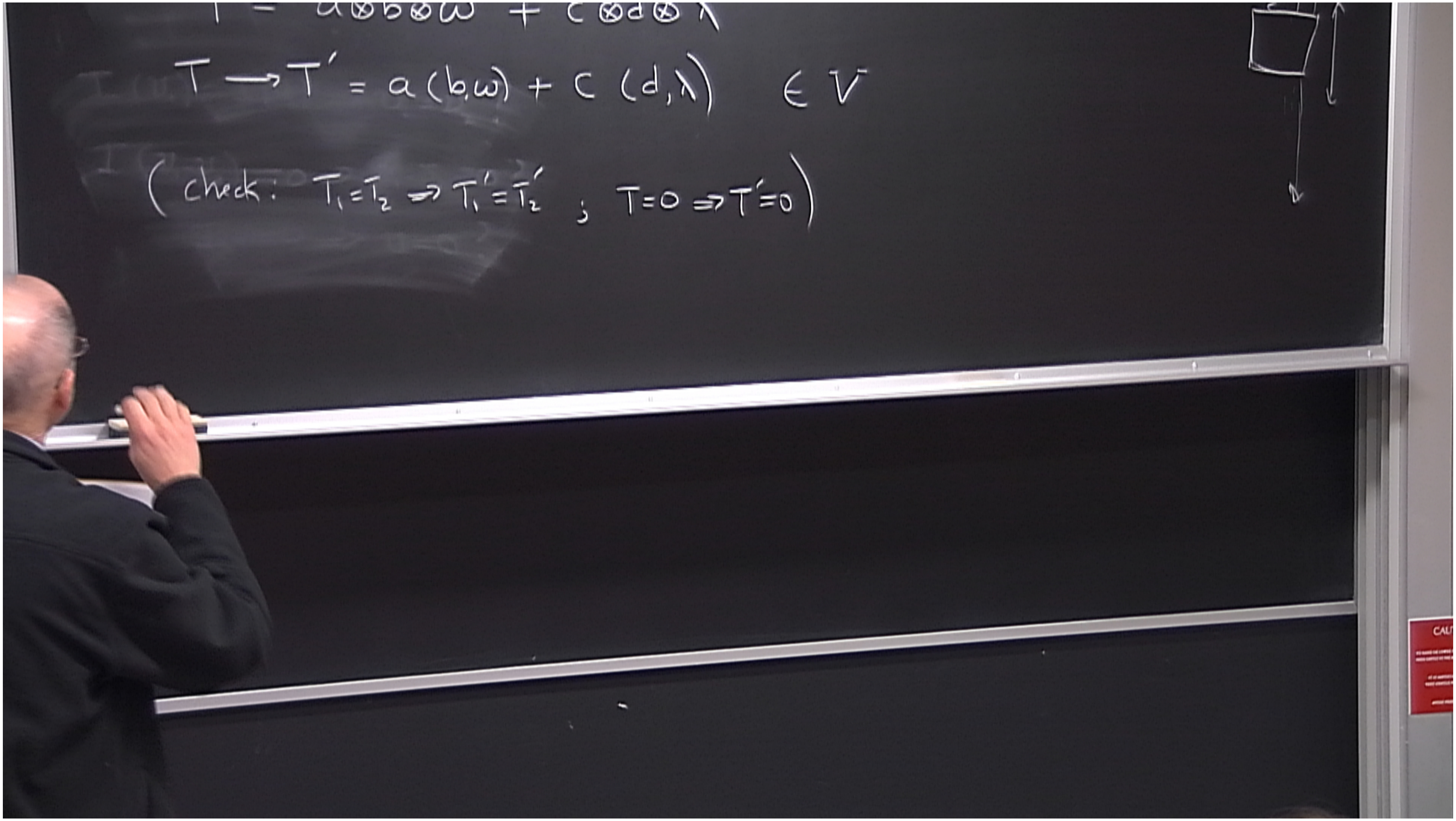
$e_1, e_2 \cdots e_m, \quad f_1, f_2, \cdots f_n \Rightarrow e_j \otimes f_k$ basis for $V \otimes W$

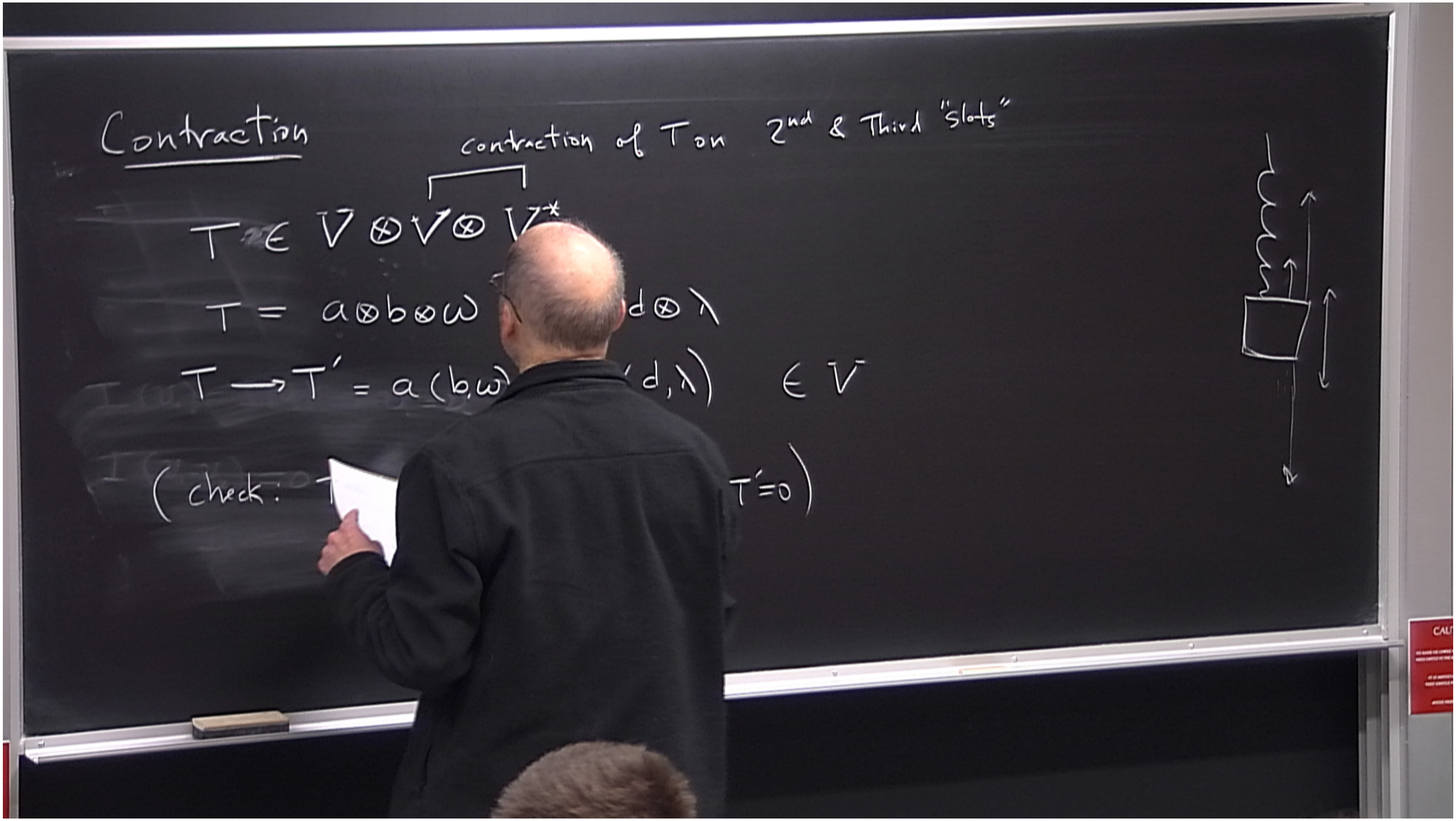
$\dim(V \otimes W) = (\dim V) \times (\dim W)$

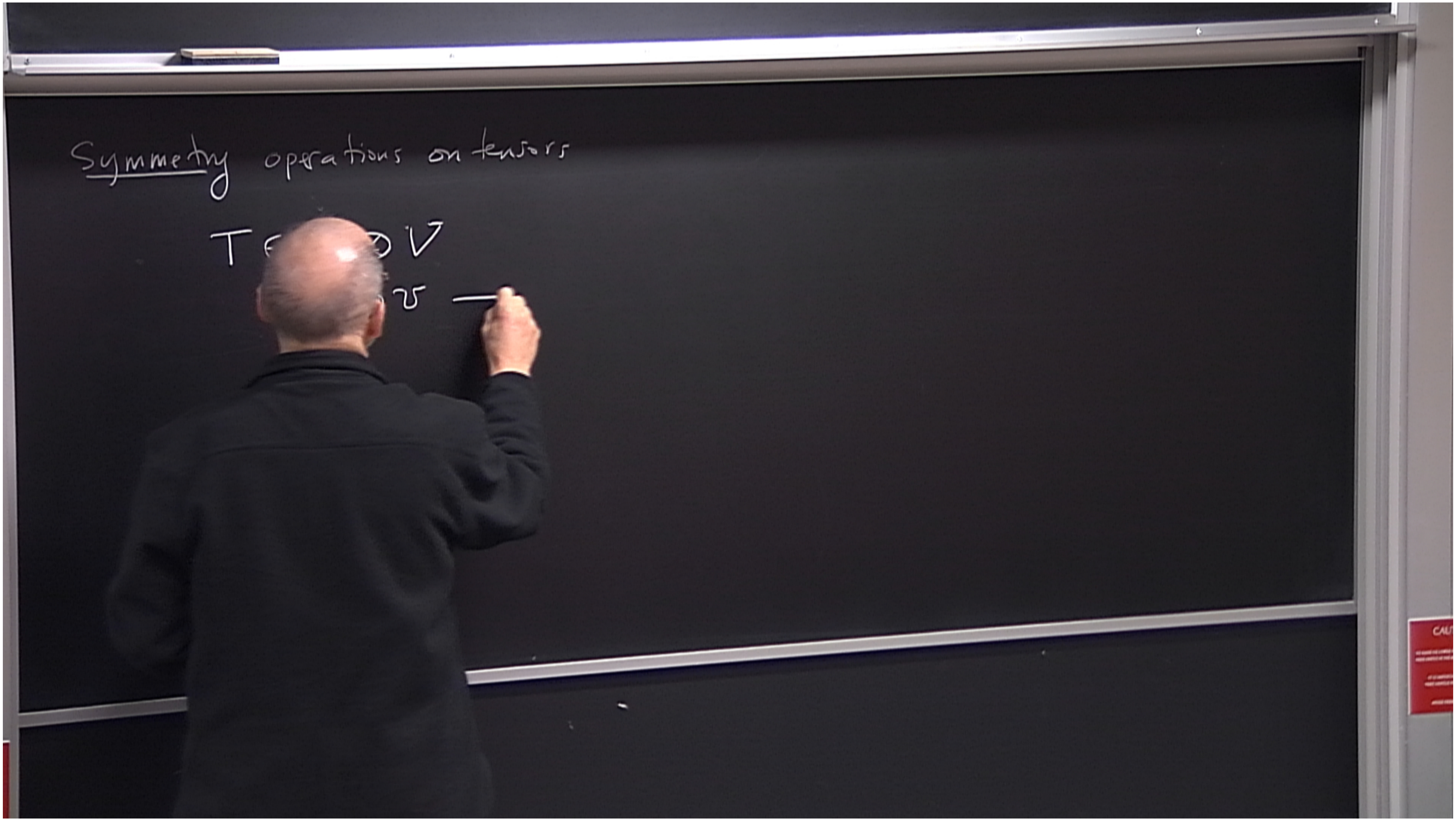
$\mathbb{R}, V, V^*, \underbrace{V \otimes V}_{\substack{\text{rank } 2 \\ \text{"valence 2"}}}, V \otimes V^*, \underbrace{V^* \otimes V^*}_{g_{ab}}, \ldots$

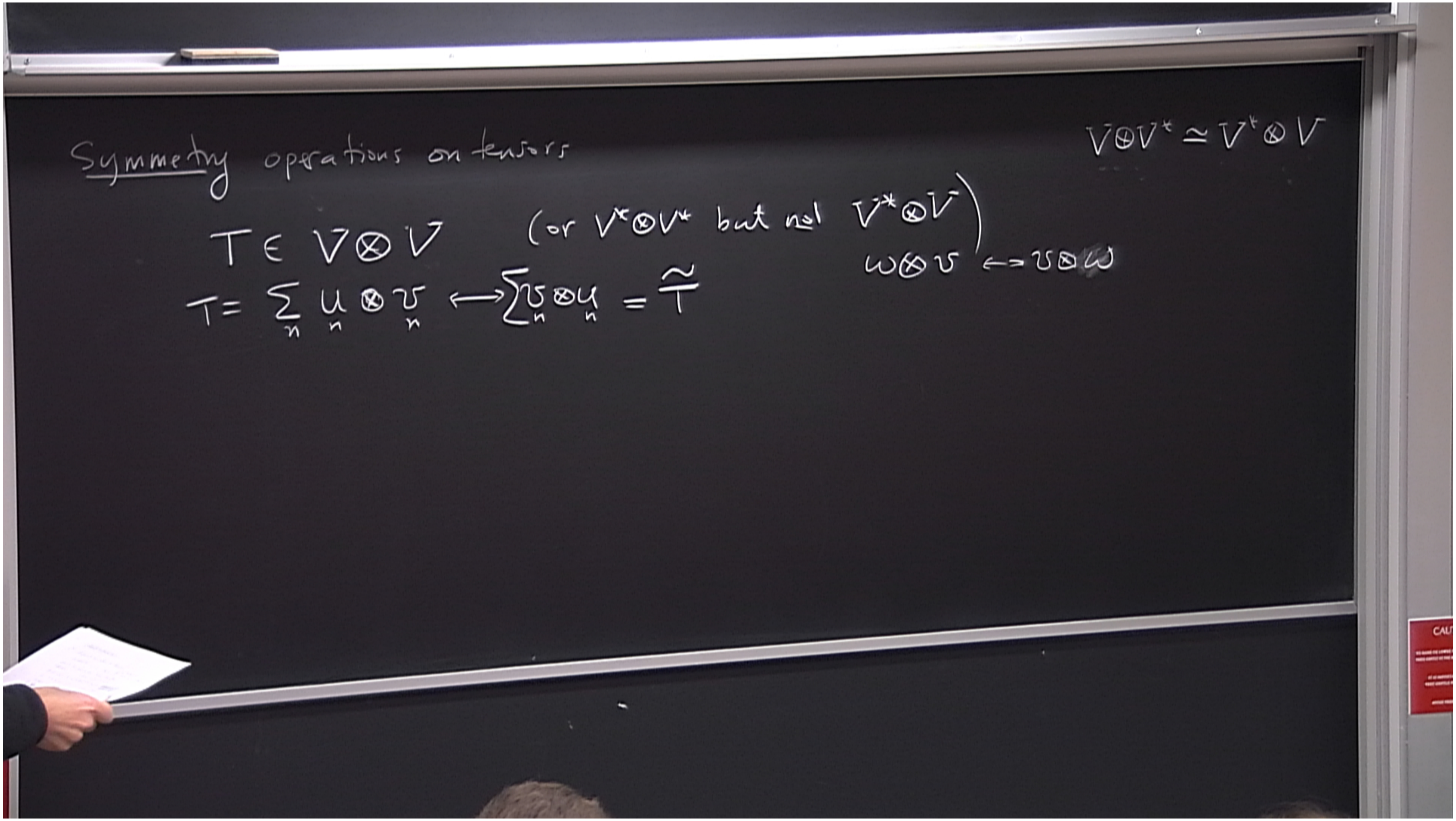
Universal Mapping Property


$I(v_1) = I(v_2) \Rightarrow v_1 = v_2$?
$I(v_1 - v_2) = 0 \Rightarrow v_1 - v_2 = 0$?
$I(v) = 0 \Rightarrow v = 0$?


$T \longrightarrow T' = a(b,\omega) + c(d,\lambda) \in V$
(check: $T_1 = T_2 \Rightarrow T_1' = T_2'$; $T = 0 \Rightarrow T' = 0$)


Contraction
contraction of T on 2nd & Third "slots"
T ∈ V ⊗ V ⊗ V*
T = a⊗b⊗ω
d⊗λ
T → T′ = a(b,ω)
(d,λ) ∈ V
(check:
T′=0)


Symmetry operations on tensors


Symmetry operations on tensors
$V \otimes V^* \simeq V^* \otimes V$
$T \in V \otimes V$ (or $V^* \otimes V^*$ but not $V^* \otimes V$)
$w \otimes v \longleftrightarrow v \otimes w$
$T = \sum_n u_n \otimes v_n \longleftrightarrow \sum_n v_n \otimes u_n = \tilde{T}$

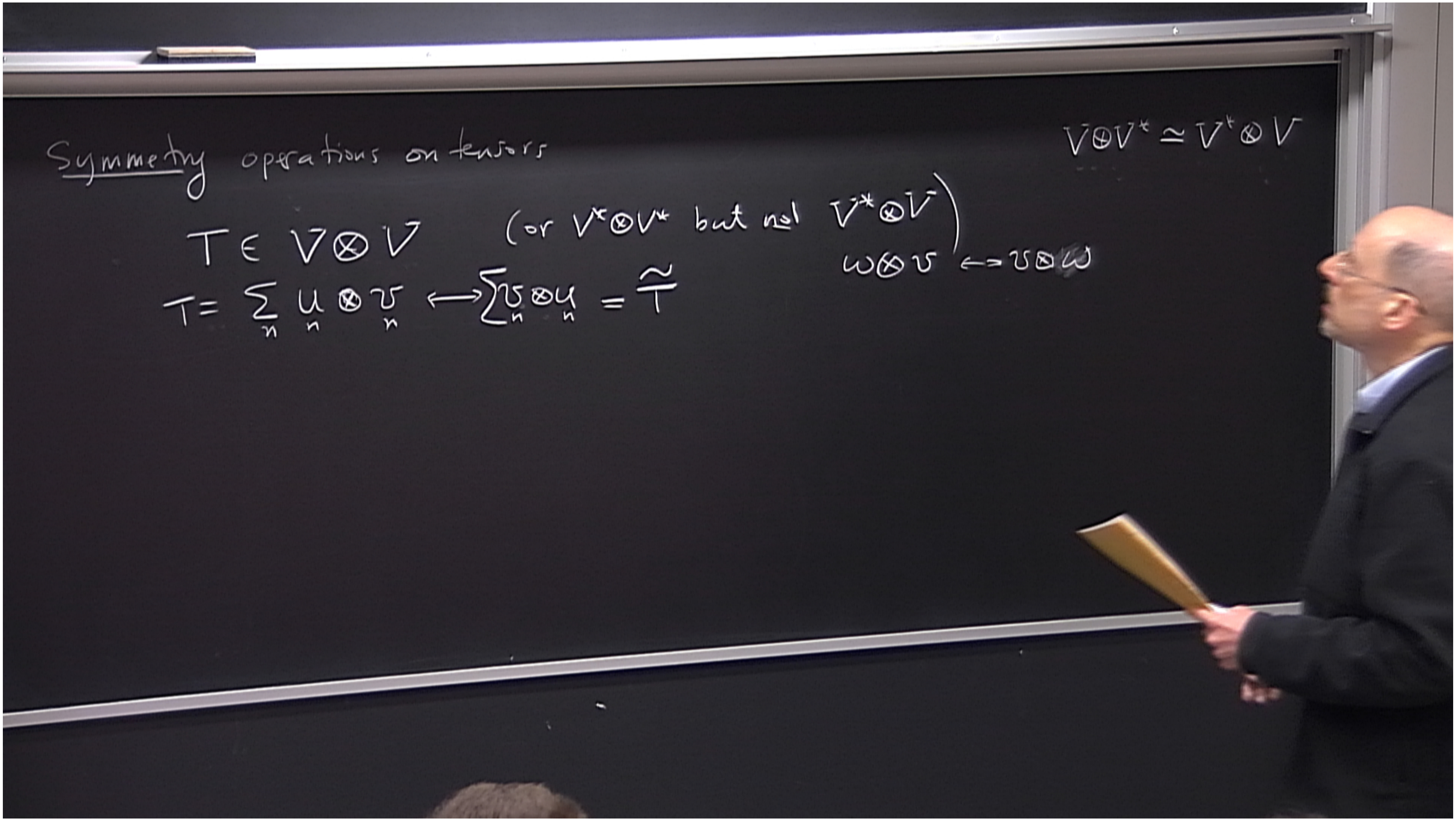
Symmetry operations on tensors

$$V\otimes V^* \simeq V^*\otimes V$$

$T \in V\otimes V$ (or $V^*\otimes V^*$ but not $V^*\otimes V$)

$$\omega\otimes v \longleftrightarrow v\otimes\omega$$

$$T = \sum_n u_n \otimes v_n \longleftrightarrow \sum_n v_n \otimes u_n = \tilde{T}$$


Symmetry operations on tensors
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$T \in V \otimes V$ (or $V^*\otimes V^*$ but not $V^*\otimes V$)
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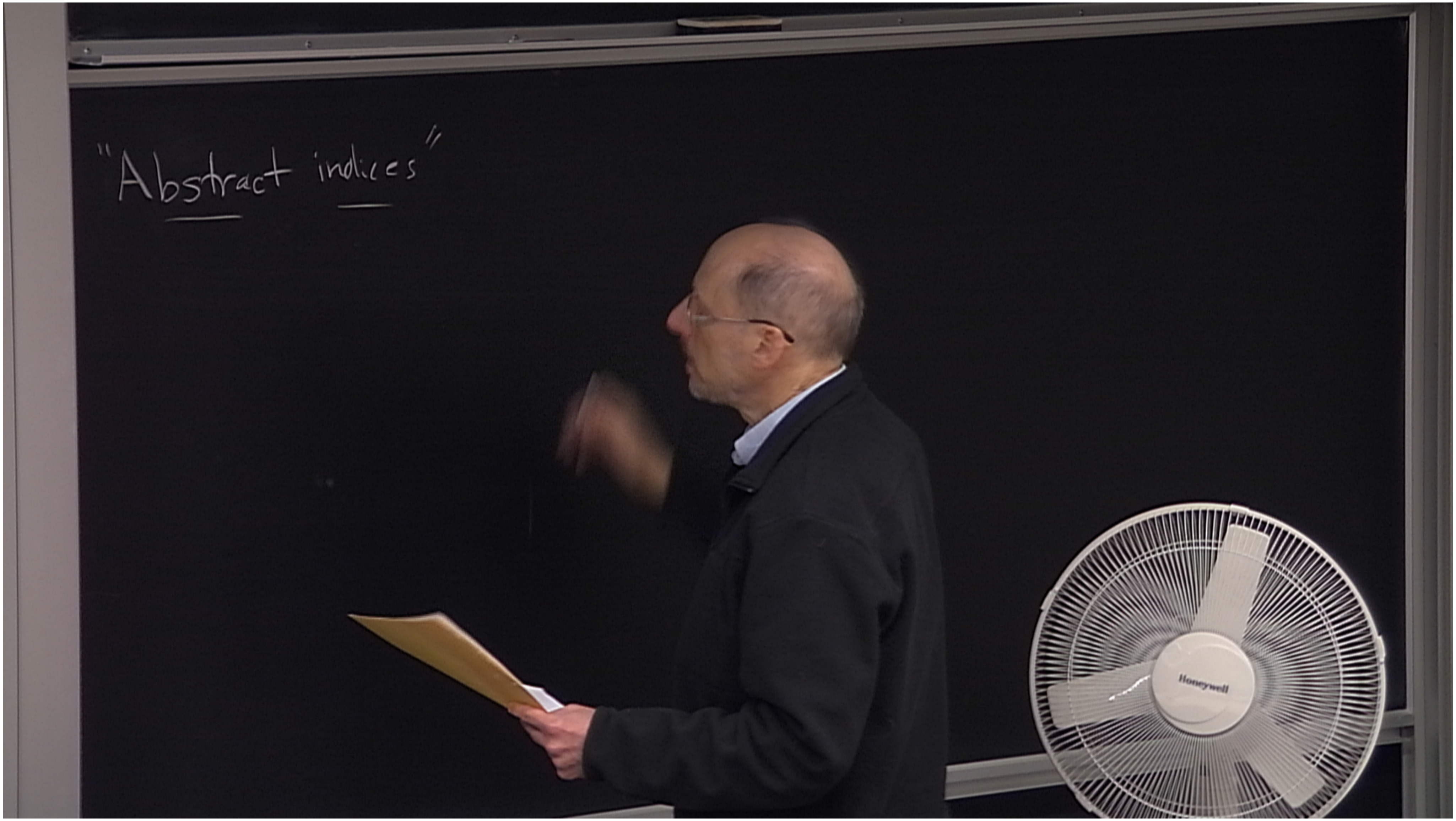
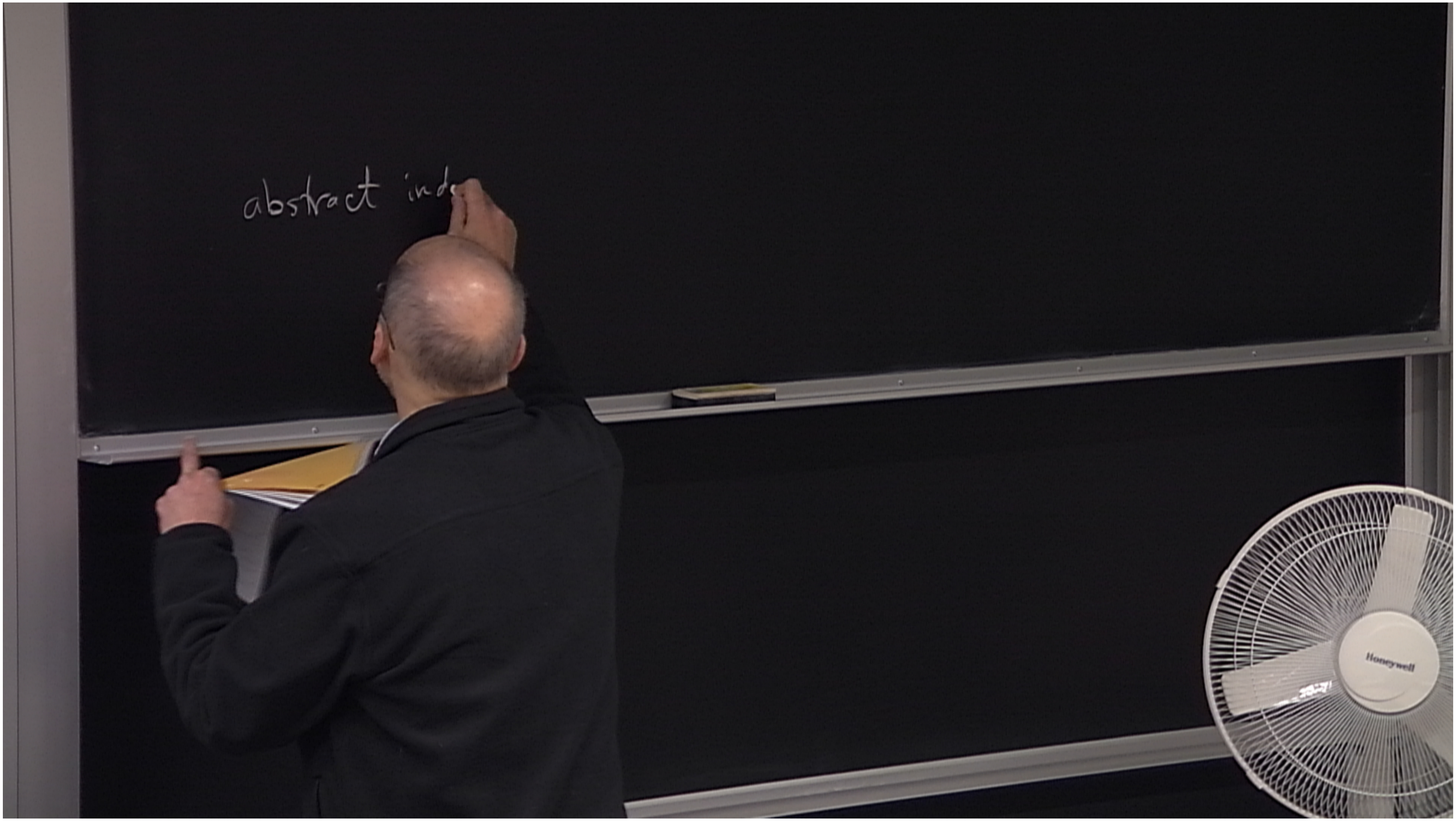
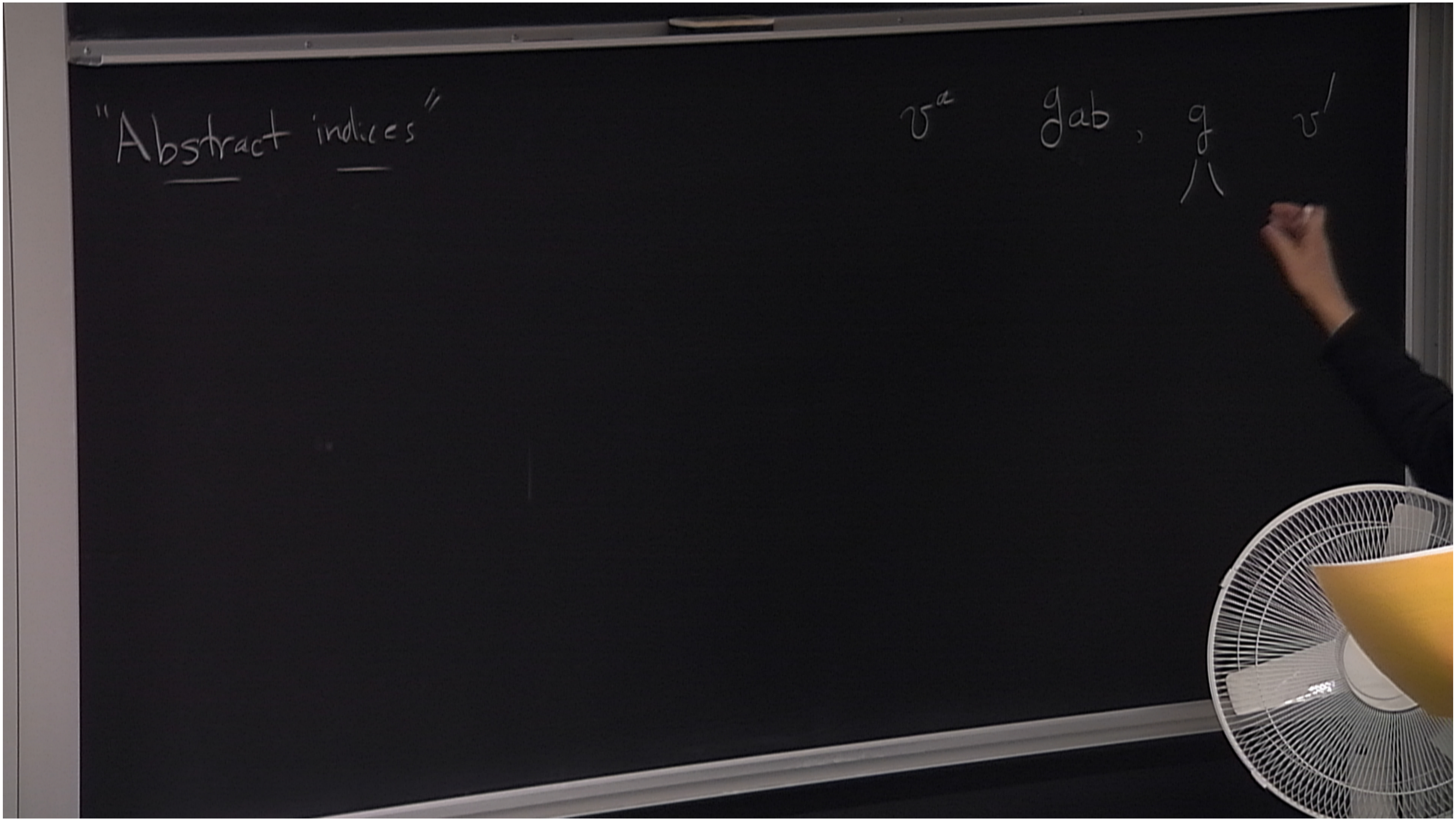
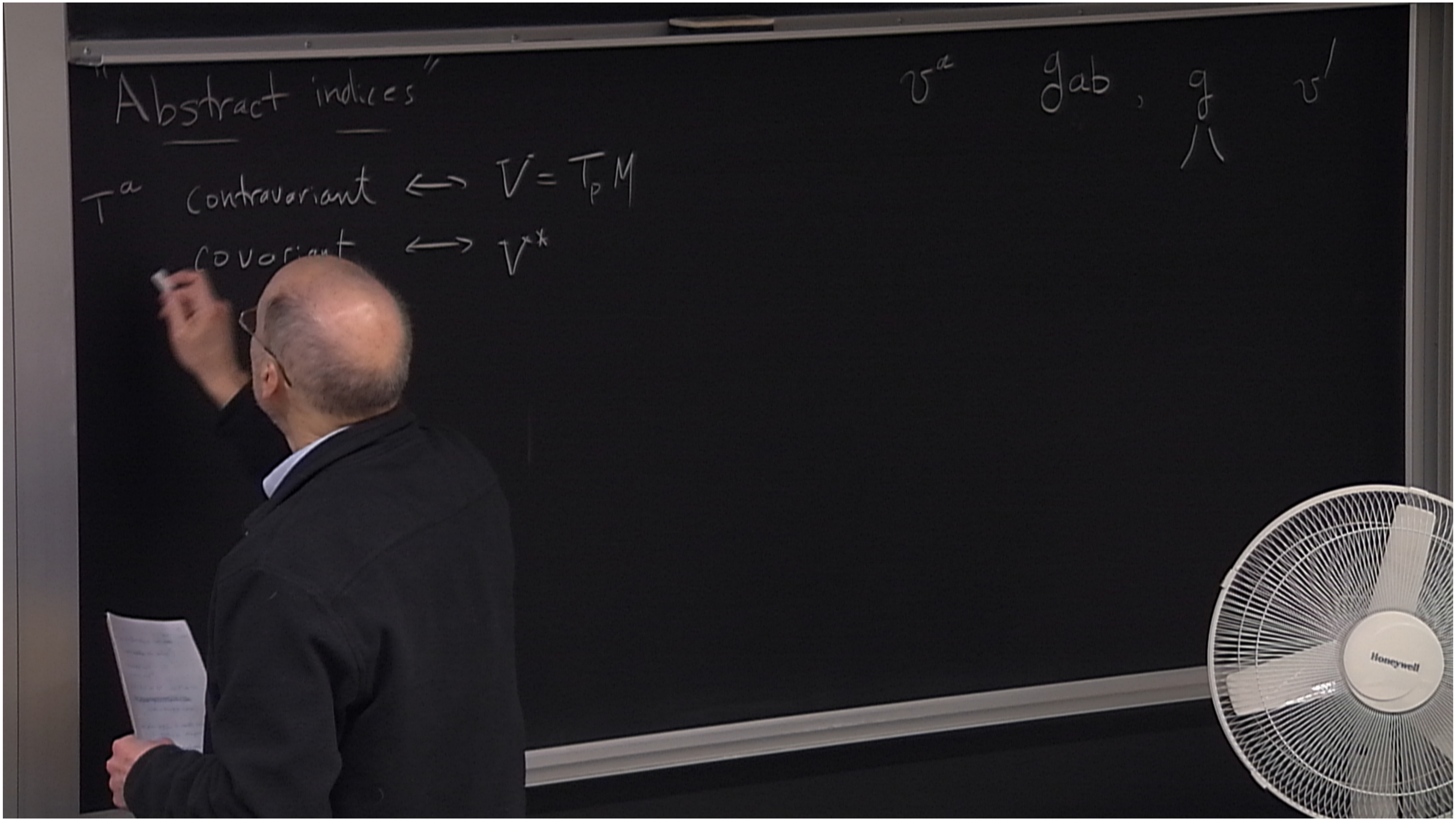
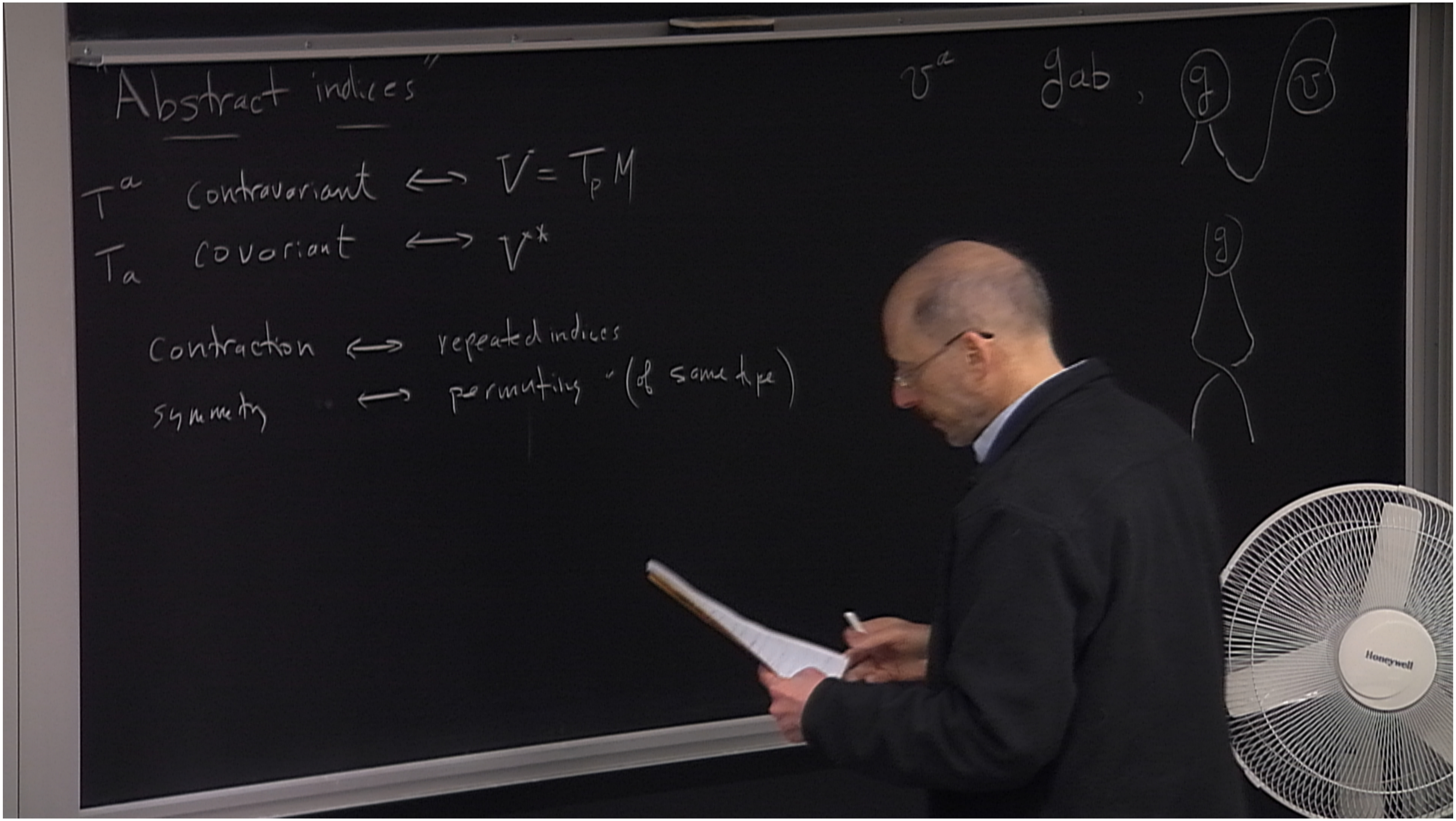
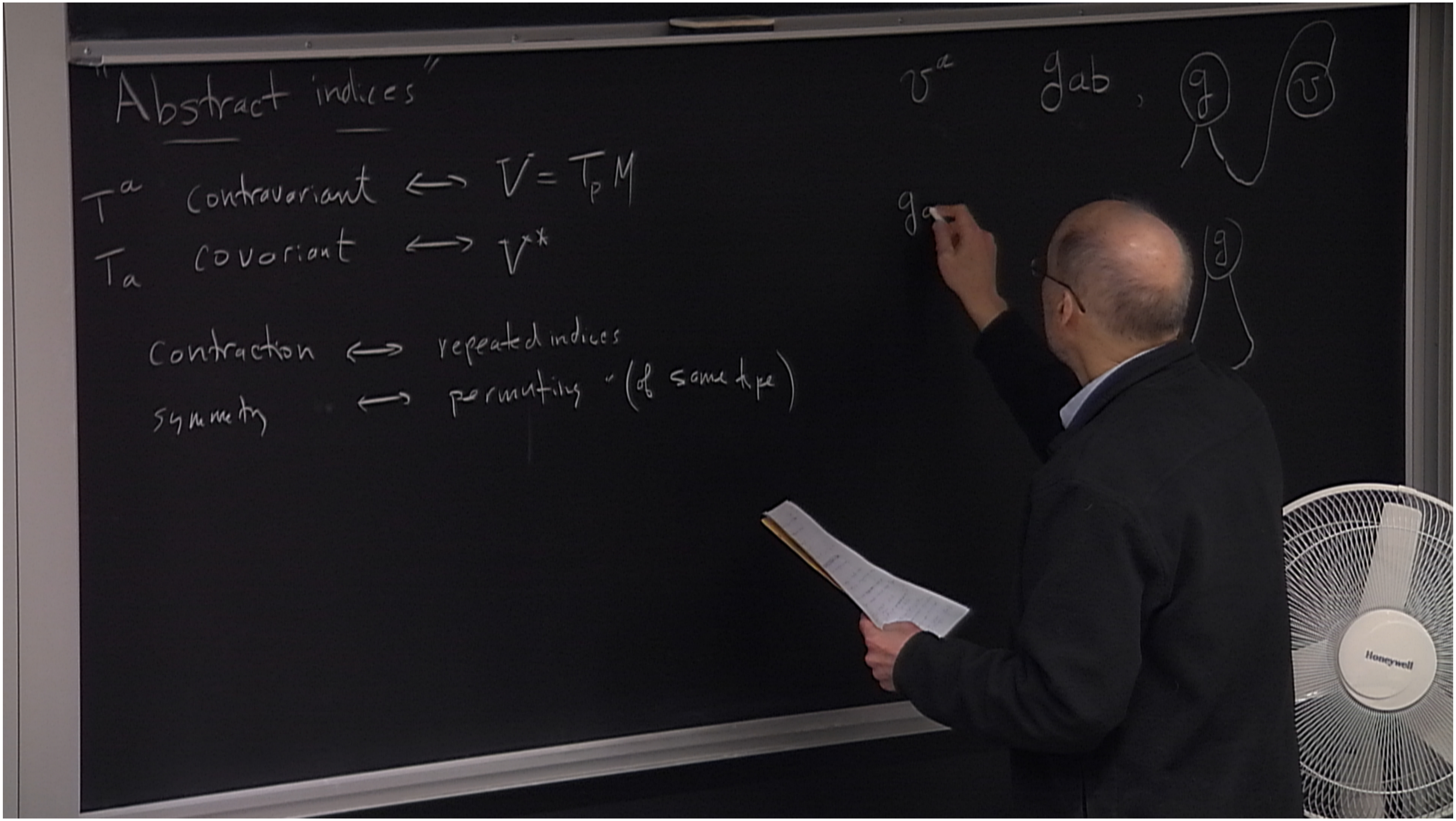

"Abstract indices"
Honeywell


abstract ind


"Abstract indices"
v^a
g_{ab} , g
v


"Abstract indices"
T^a contravariant $\longleftrightarrow$ $V = T_pM$
covariant $\longleftrightarrow$ V^*
v^a g_{ab} , g v
Honeywell


"Abstract indices"
T^a contravariant ⟷ V = T_p M
T_a covariant ⟷ V*
contraction ⟷ repeated indices
symmetry ⟷ permuting (of same type)
v^a g_ab ,
g
v
g


"Abstract indices"
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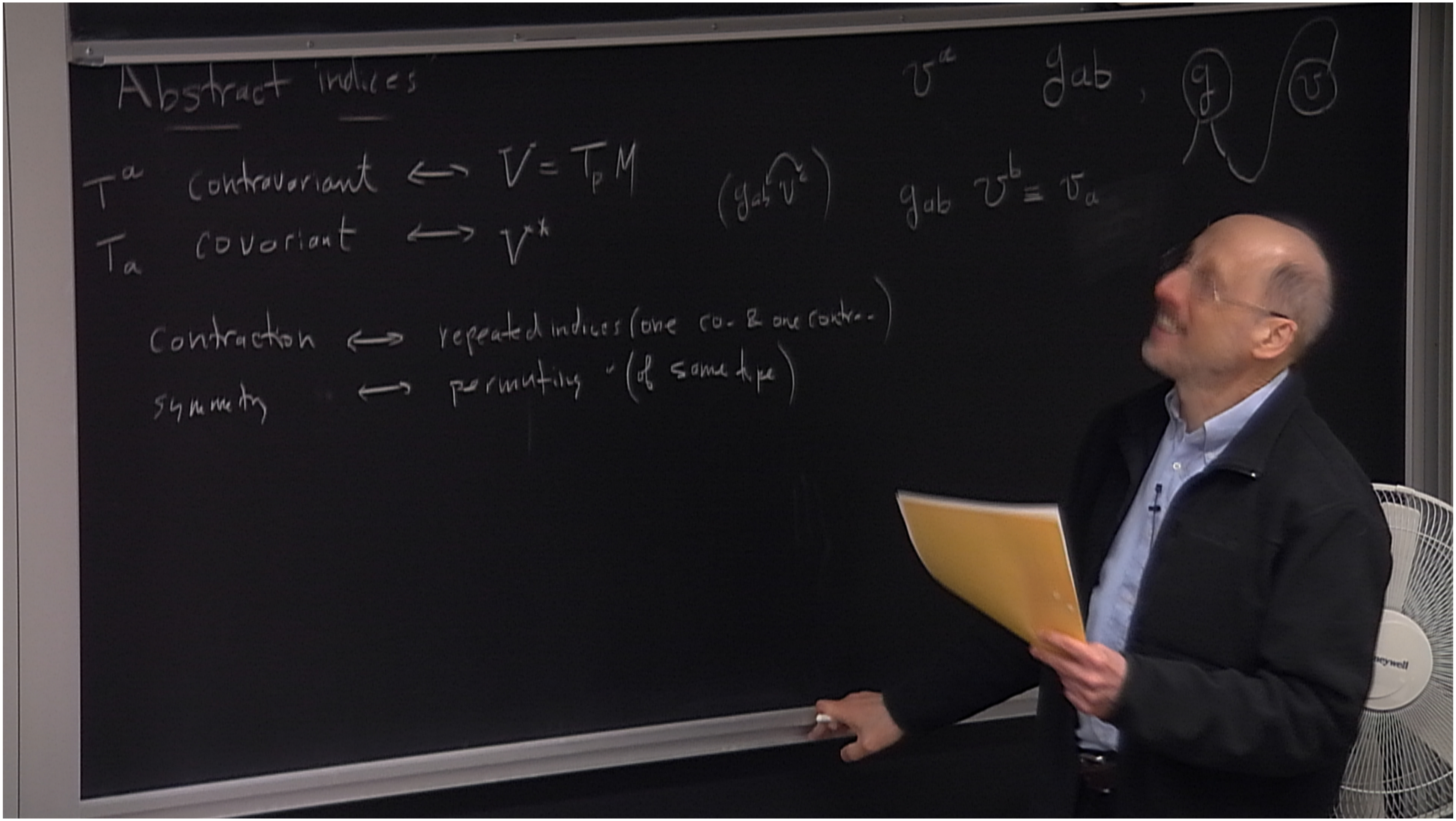
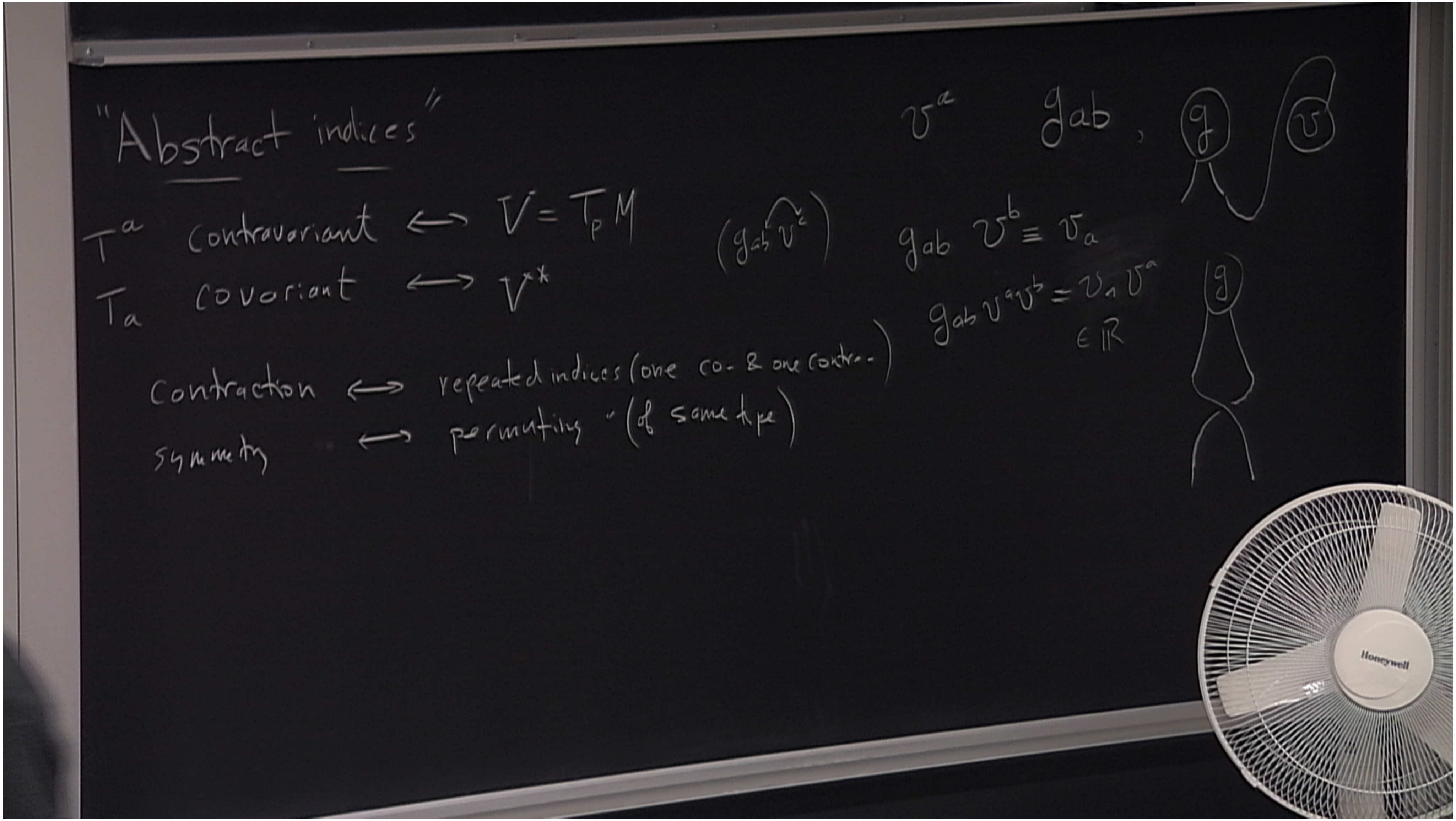
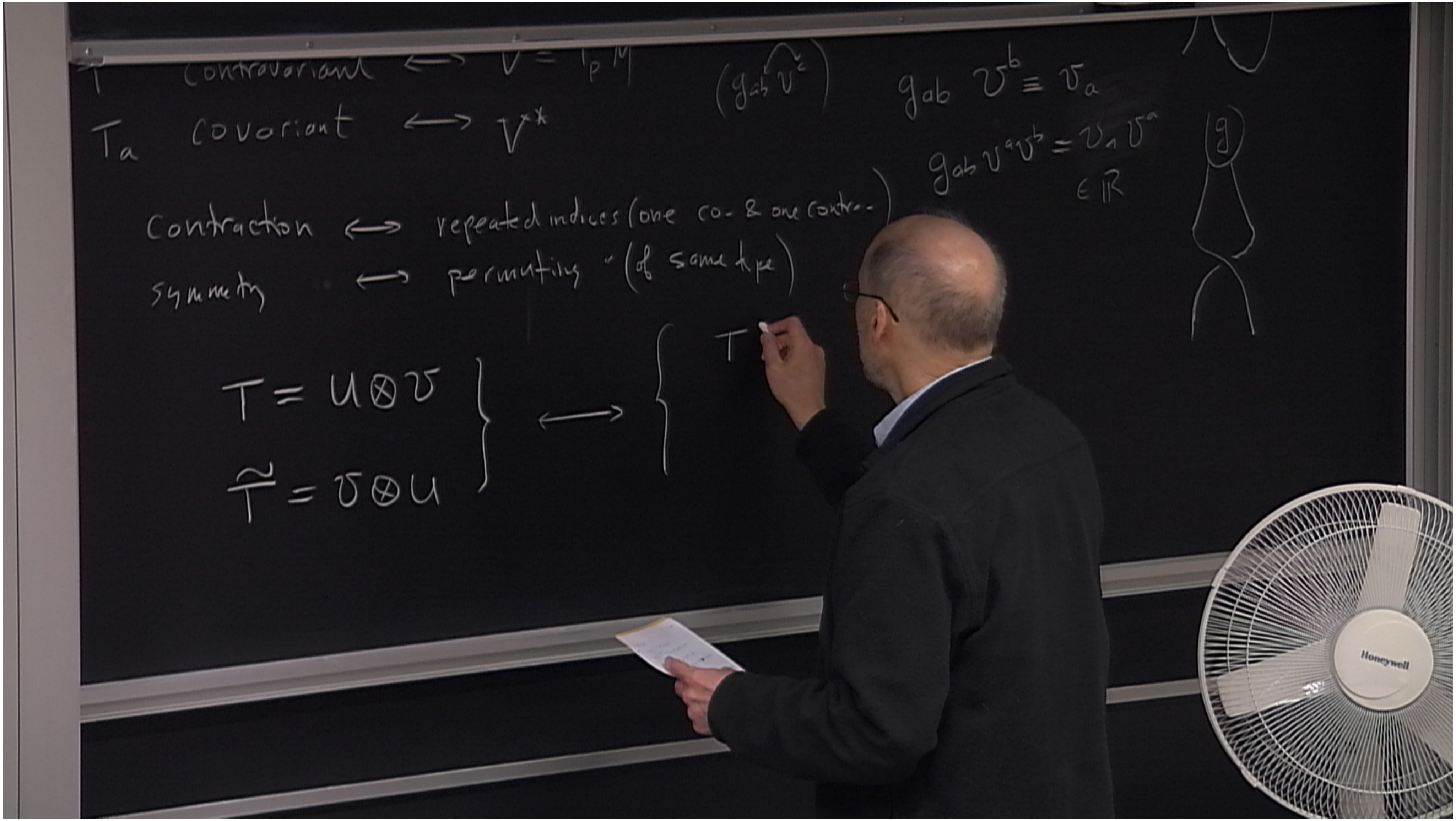
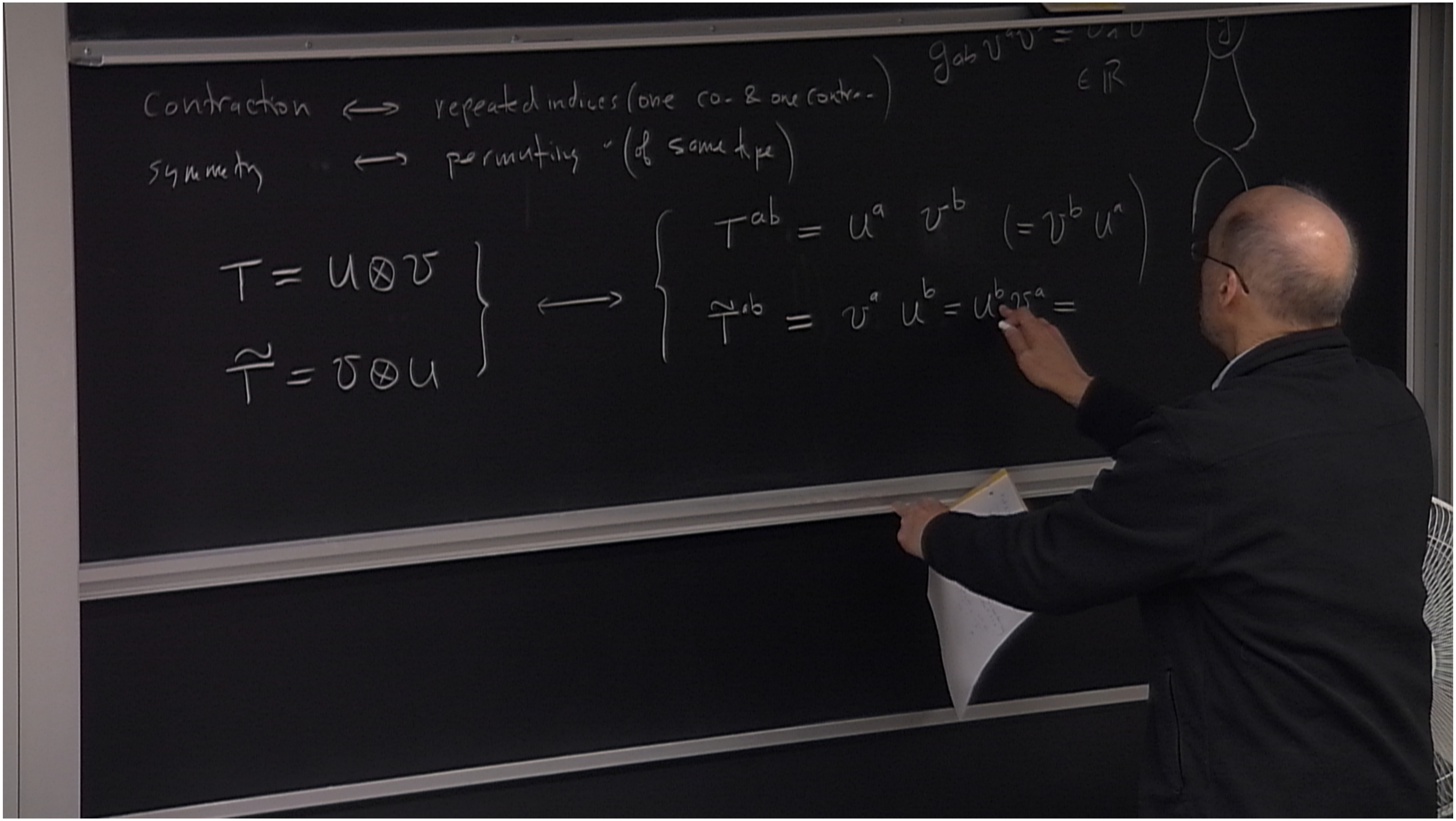
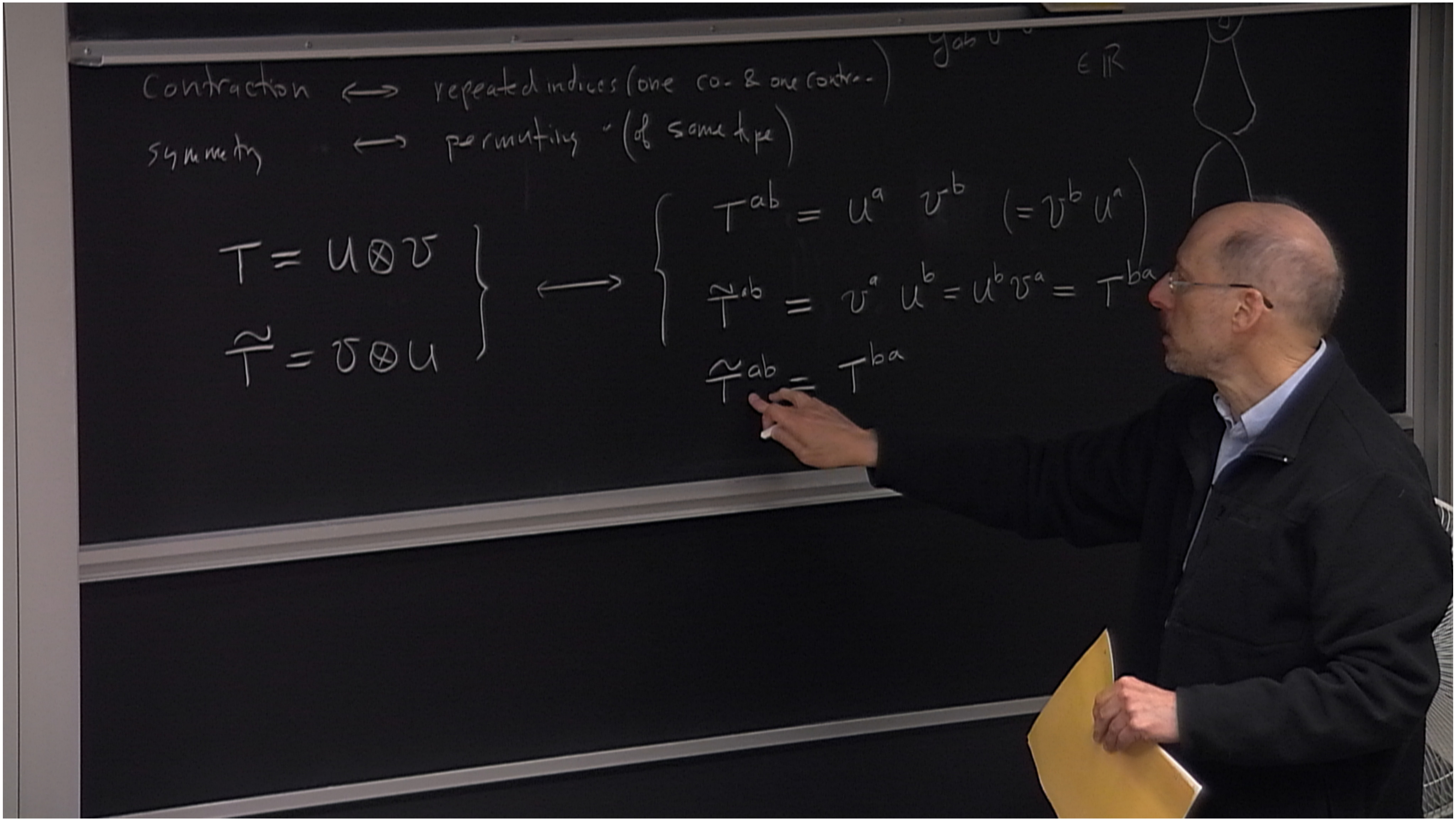
T_a covariant ⟷ V^*
contraction ⟷ repeated indices
symmetry ⟷ permuting (of same type)
v^a g_{ab} ,
g v
g

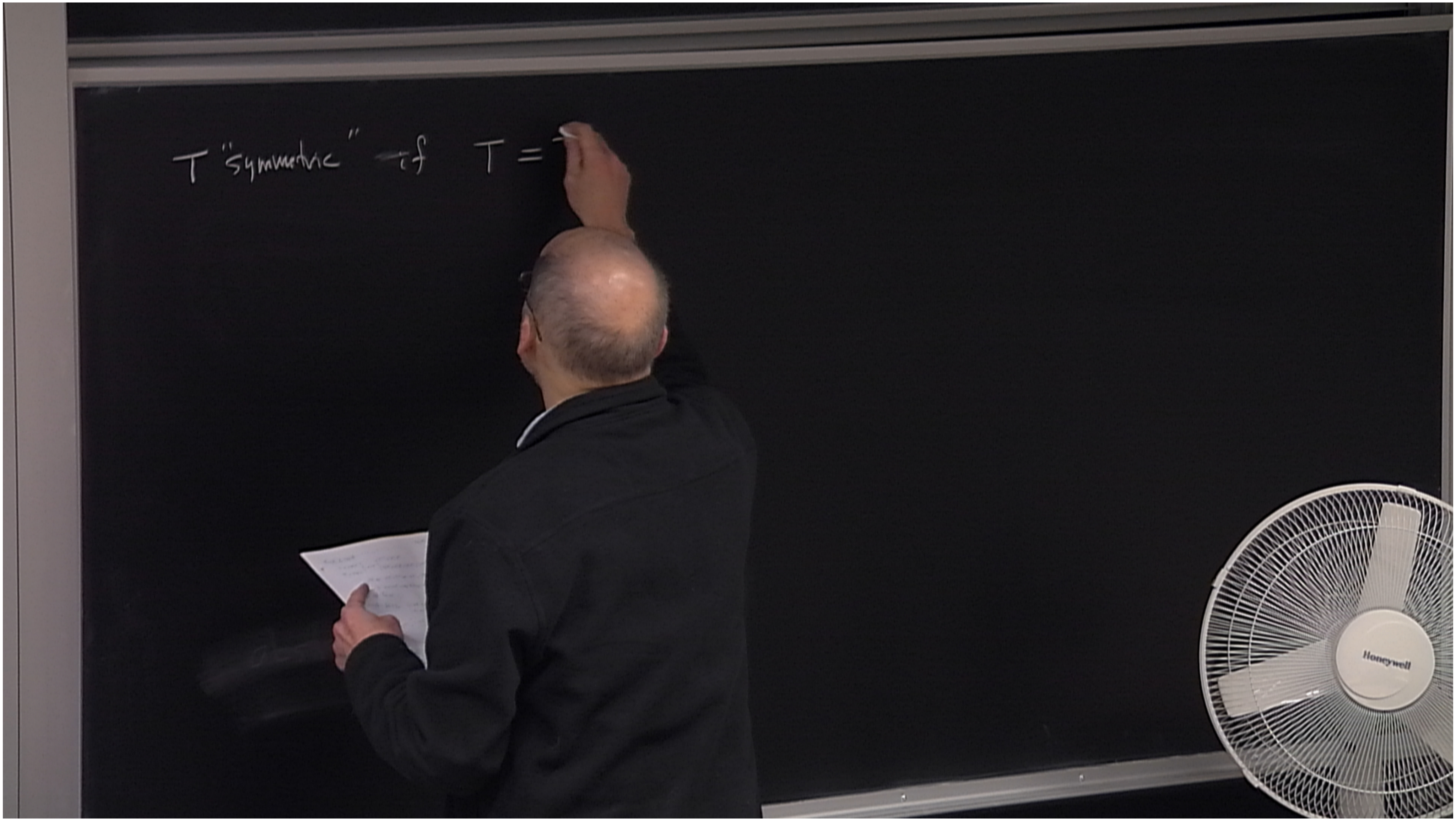

Abstract indices
T^a contravariant ⟷ $V = T_p M$
T_a covariant ⟷ V^*
$(g_{ab} v^c)$
v^a g_{ab},
$g_{ab} v^b \equiv v_a$
contraction ⟷ repeated indices (one co- & one contra-)
symmetry ⟷ permuting (of some type)


"Abstract indices"
T^a contravariant $\longleftrightarrow$ $V = T_p M$
T_a covariant $\longleftrightarrow$ V^*
$(g_{ab} v^c)$
v^a g_{ab} , g v
$g_{ab} v^b \equiv v_a$
$g_{ab} v^a v^b = v_a v^a \in \mathbb{R}$
g
Contraction $\longleftrightarrow$ repeated indices (one co- & one contra-)
symmetry $\longleftrightarrow$ permuting " (of same type)
Honeywell


contravariant ⟷ V = T_p M
T_a covariant ⟷ V*
g_ab v^b ≡ v_a
g_ab v^a v^b = v_a v^a ∈ ℝ
contraction ⟷ repeated indices (one co- & one contra-)
symmetry ⟷ permuting (of same type)
T = u ⊗ v
T̃ = v ⊗ u
Honeywell

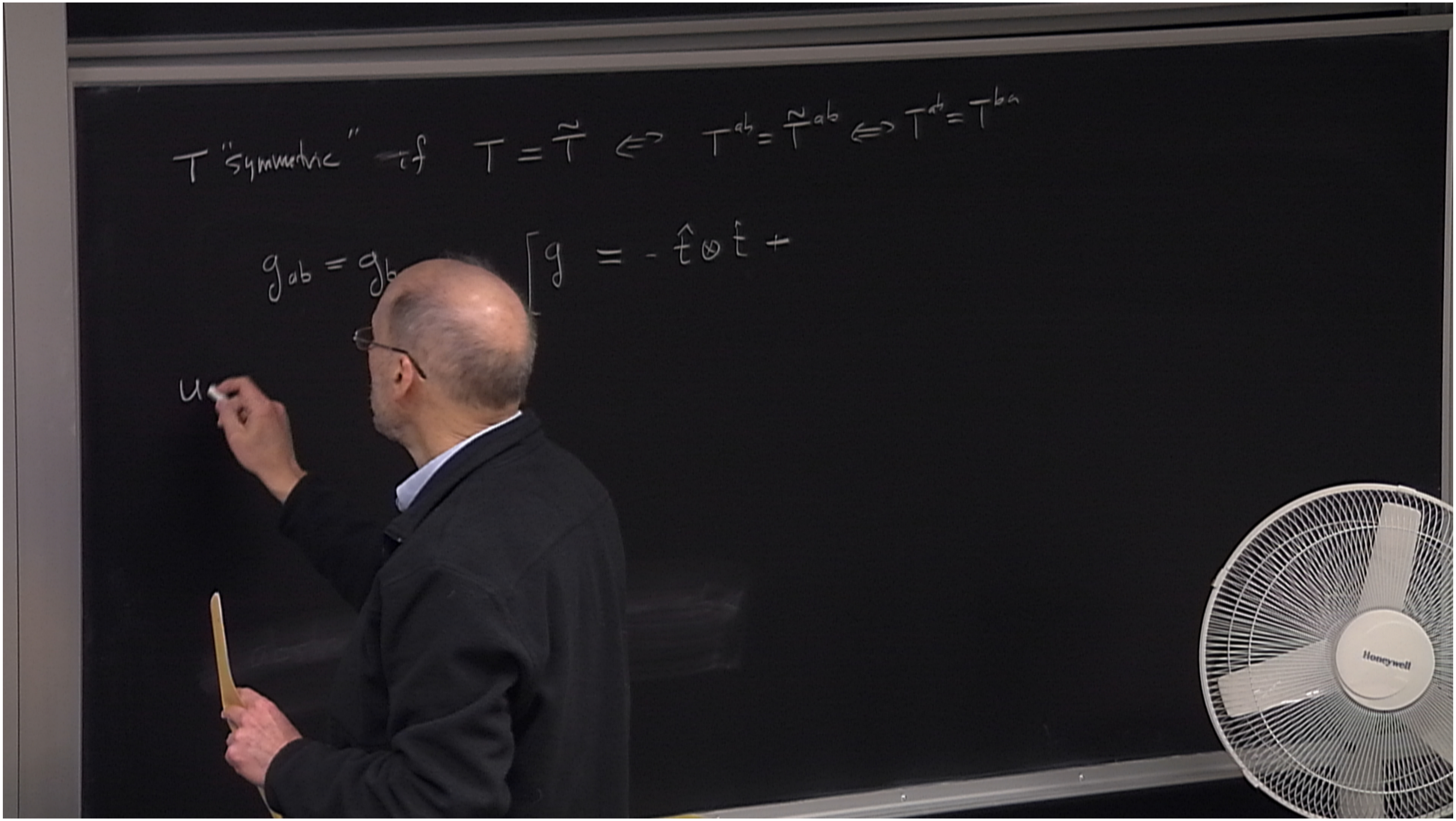

contraction ⟷ repeated indices (one co- & one contra-)
symmetry ⟷ permuting (of same type)
$g_{ab} v^a v^b$ =
∈ ℝ
$T = u \otimes v$
$\tilde{T} = v \otimes u$
$T^{ab} = u^a v^b \; (= v^b u^a)$
$\tilde{T}^{ab} = v^a u^b = u^b v^a =$

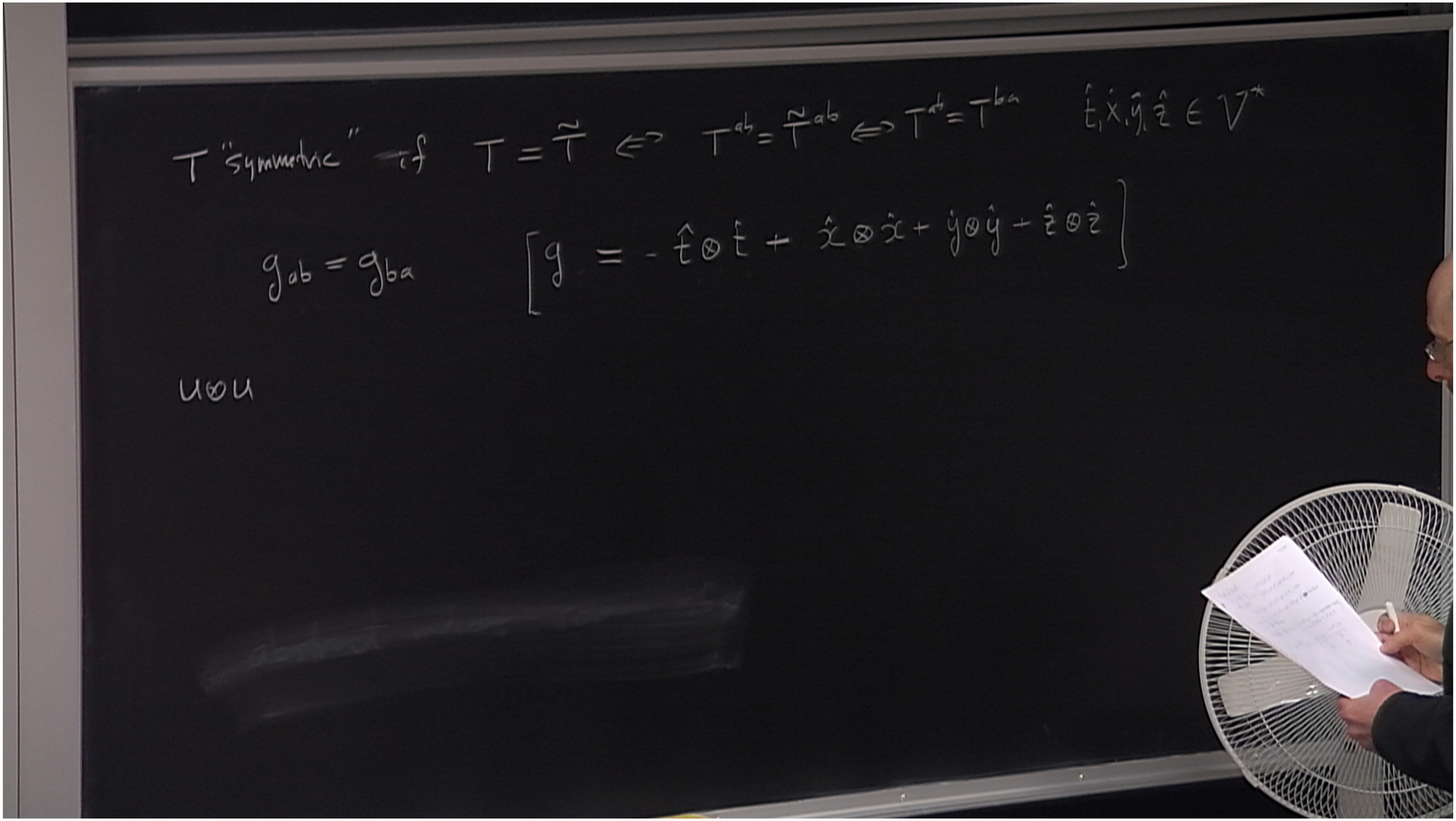

Contraction $\longleftrightarrow$ repeated indices (one co- & one contra-)
symmetry $\longleftrightarrow$ permuting (of same type)
$T = u \otimes v$
$\tilde{T} = v \otimes u$
$T^{ab} = u^a v^b \;(= v^b u^a)$
$\tilde{T}^{ab} = v^a u^b = u^b v^a = T^{ba}$
$\tilde{T}^{ab} = T^{ba}$
$\in \mathbb{R}$


T "symmetric" if T =

T "symmetric" if $T = \tilde{T} \iff T^{ab} = \tilde{T}^{ab} \iff T^{ab} = T^{ba}$

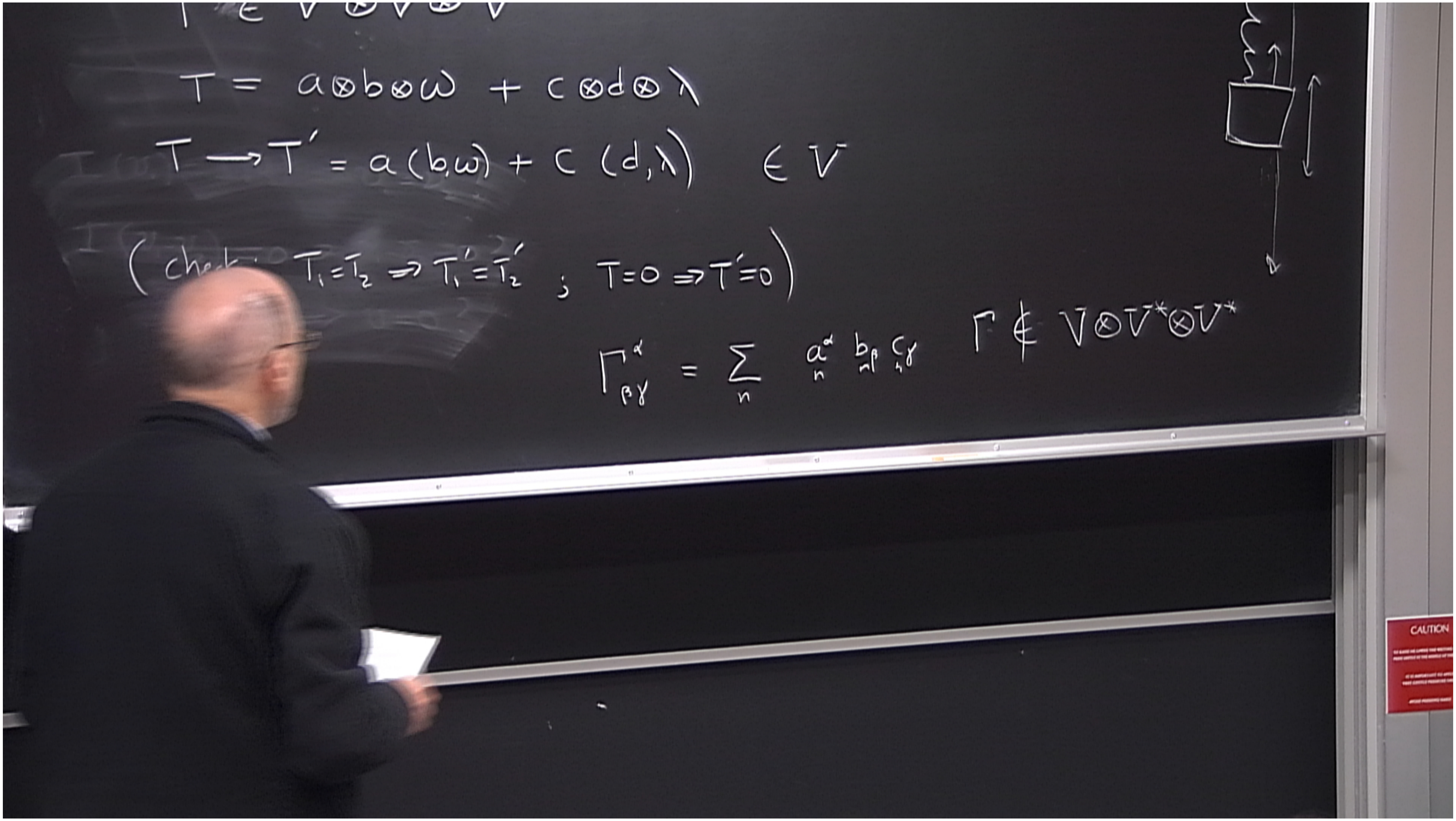
$g_{ab} = g_{ba}$


T "symmetric" if $T = \tilde{T} \iff T^{ab} = \tilde{T}^{ab} \iff T^{ab} = T^{ba}$
$g_{ab} = g_{b}$
$[g = -\hat{t} \otimes \hat{t} +$



T "symmetric" if $T = \tilde{T} \iff T^{ab} = \tilde{T}^{ab} \iff T^{ab} = T^{ba}$
$\hat{t}, \hat{x}, \hat{y}, \hat{z} \in V^*$
$g_{ab} = g_{ba}$
$\left[g = -\hat{t} \otimes \hat{t} + \hat{x} \otimes \hat{x} + \hat{y} \otimes \hat{y} + \hat{z} \otimes \hat{z} \right]$
$u \otimes u$


Riem ⟷
Honeywell



$T = a\otimes b\otimes \omega + c\otimes d\otimes \lambda$
$T \longrightarrow T' = a(b,\omega) + c(d,\lambda) \in V$
$T_1 = T_2 \Rightarrow T_1' = T_2'$; $T=0 \Rightarrow T'=0$)
$\Gamma^{\alpha}_{\beta\gamma} = \sum_n a^{\alpha}_{n} b_{n\beta} c_{n\gamma}$ $\Gamma \notin V\otimes V^*\otimes V^*$
CAUTION

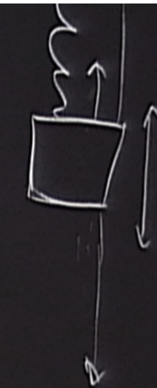
$$\Gamma \in V \otimes V \otimes V$$

$$T = a \otimes b \otimes \omega + c \otimes d \otimes \lambda$$

$$T \longrightarrow T' = a\,(b,\omega) + c\,(d,\lambda) \quad \in V$$

$$\left(\text{check: } T_1 = T_2 \Rightarrow T_1' = T_2' \;;\; T = 0 \Rightarrow T' = 0 \right)$$

$$\Gamma^{\alpha}{}_{\beta\gamma} = \sum_n a^{\alpha}_{n}\, b_{n\beta}\, c_{n\gamma} \qquad \Gamma \notin V \otimes V^* \otimes V^*$$



$T \in V \otimes V \otimes V$

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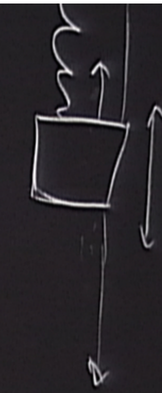


Tensors as linear mapping $\quad T^a{}_{bc}$ contract S^{bc} $\quad T^a{}_{bc} S^{bc}$ dummy indices

$V \otimes V^* \otimes V^* \qquad V \otimes V \qquad V$

$$T \in V \otimes V \otimes V$$

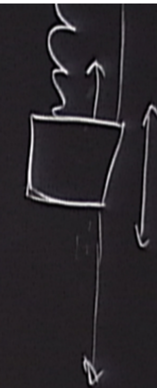
$$T = a \otimes b \otimes \omega + c \otimes d \otimes \lambda$$

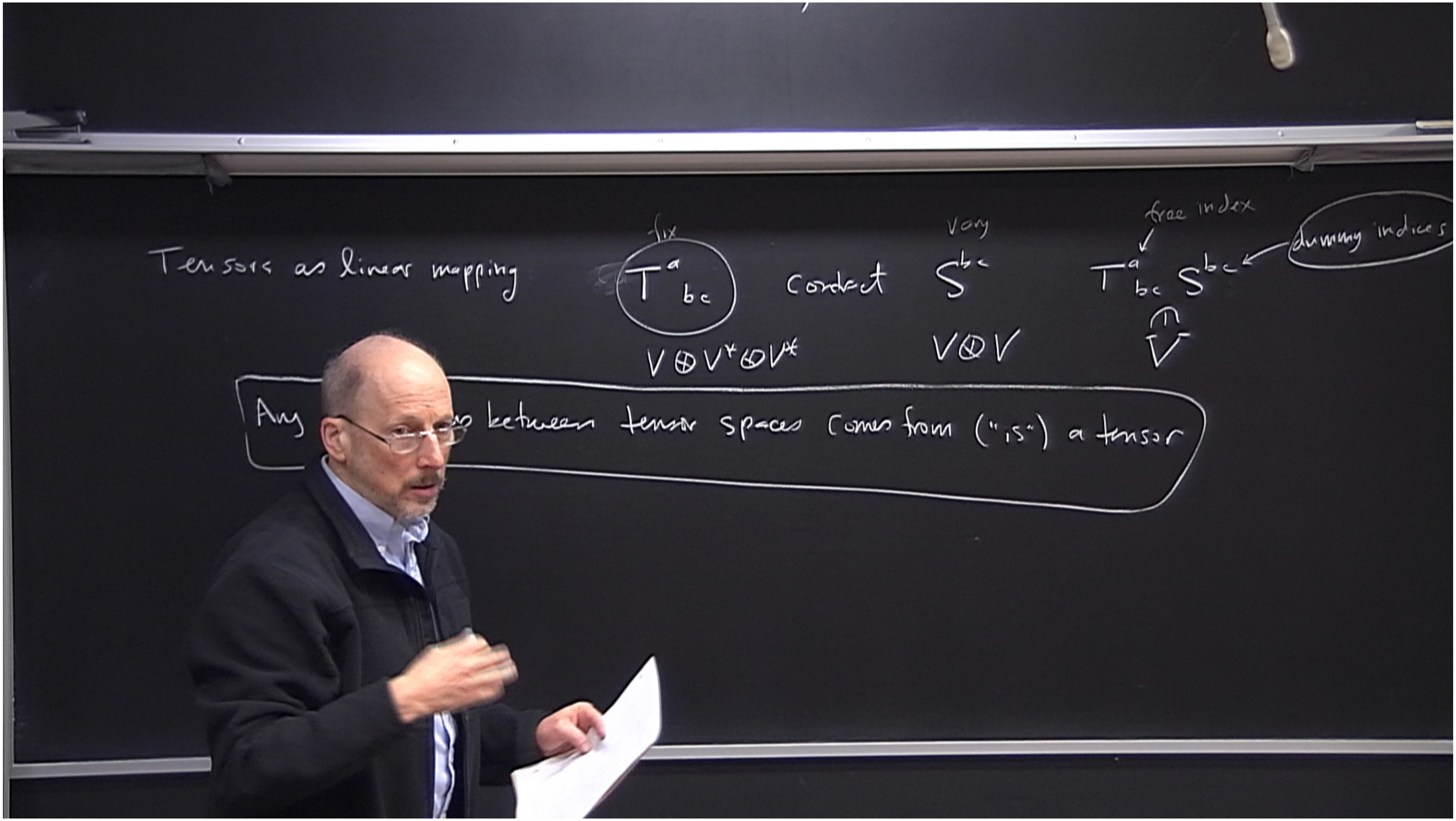
$$T \longrightarrow T' = a(b,\omega) + c(d,\lambda) \quad \in V$$

$$(\text{check: } T_1 = T_2 \Rightarrow T_1' = T_2' \; ; \; T = 0 \Rightarrow T' = 0)$$

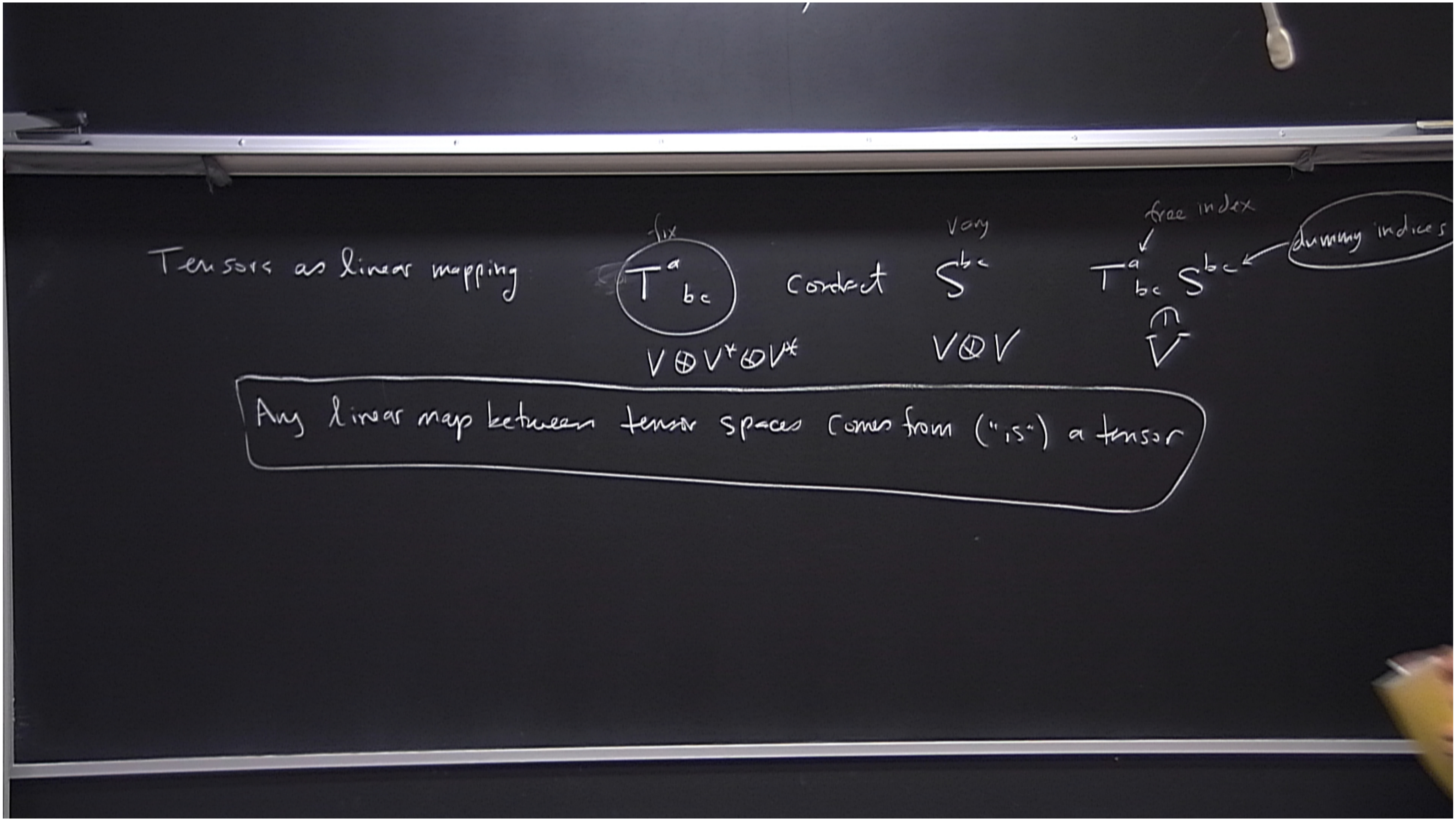
$$v^a w_a = v^b w_b$$

$$\Gamma^{\alpha}{}_{\beta\gamma} = \sum_n a^{\alpha}_{n} \, b_{n\beta} \, c_{n\gamma} \qquad \Gamma \notin V \otimes V^* \otimes V^*$$





Tensors as linear mapping
fix
$T^{a}{}_{bc}$
contract
vary
S^{bc}
free index
dummy indices
$T^{a}{}_{bc} S^{bc}$
$V \otimes V^* \otimes V^*$
$V \otimes V$
V
Any
between tensor spaces comes from ("is") a tensor



Tensors as linear mapping
fix
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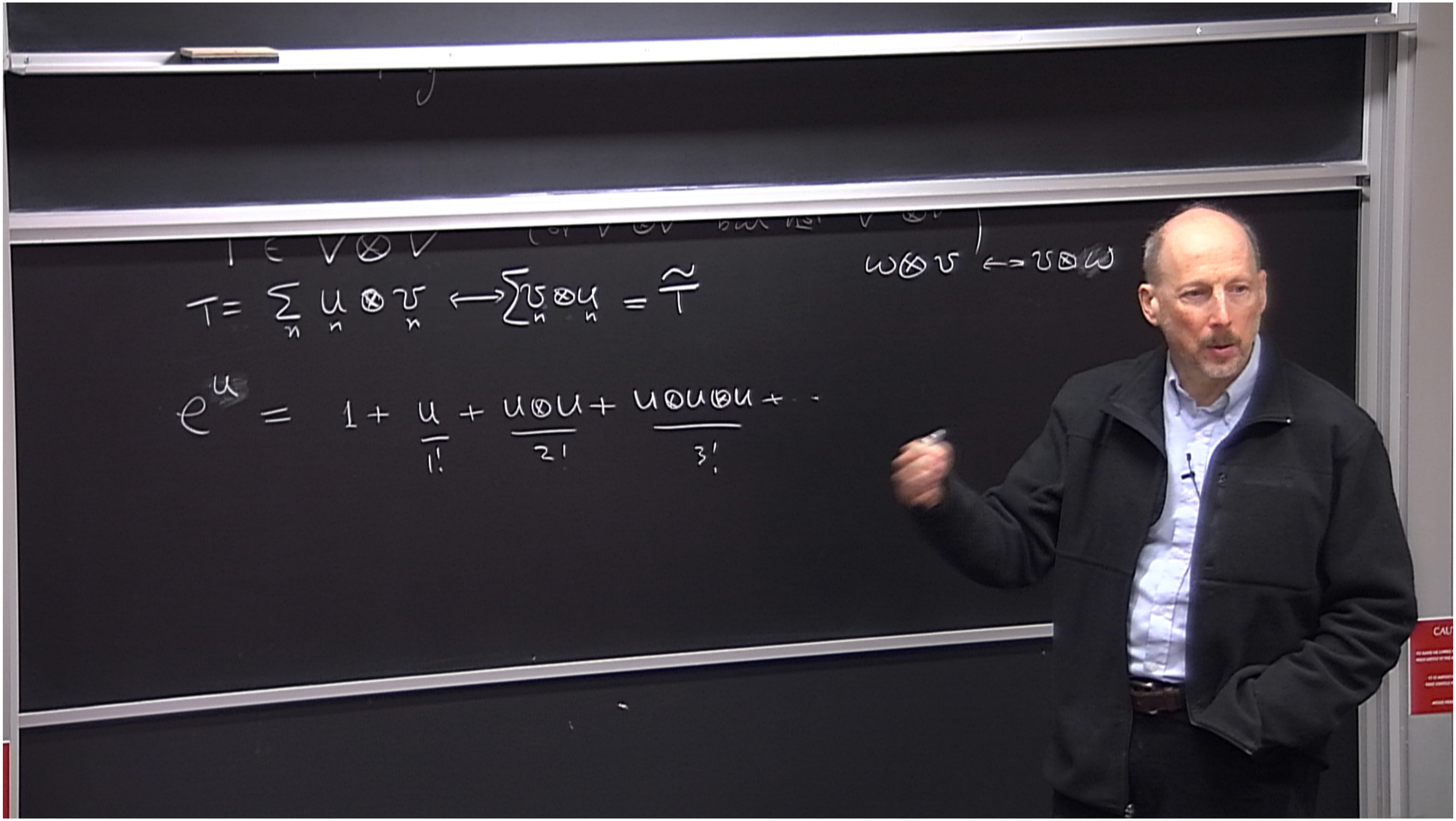
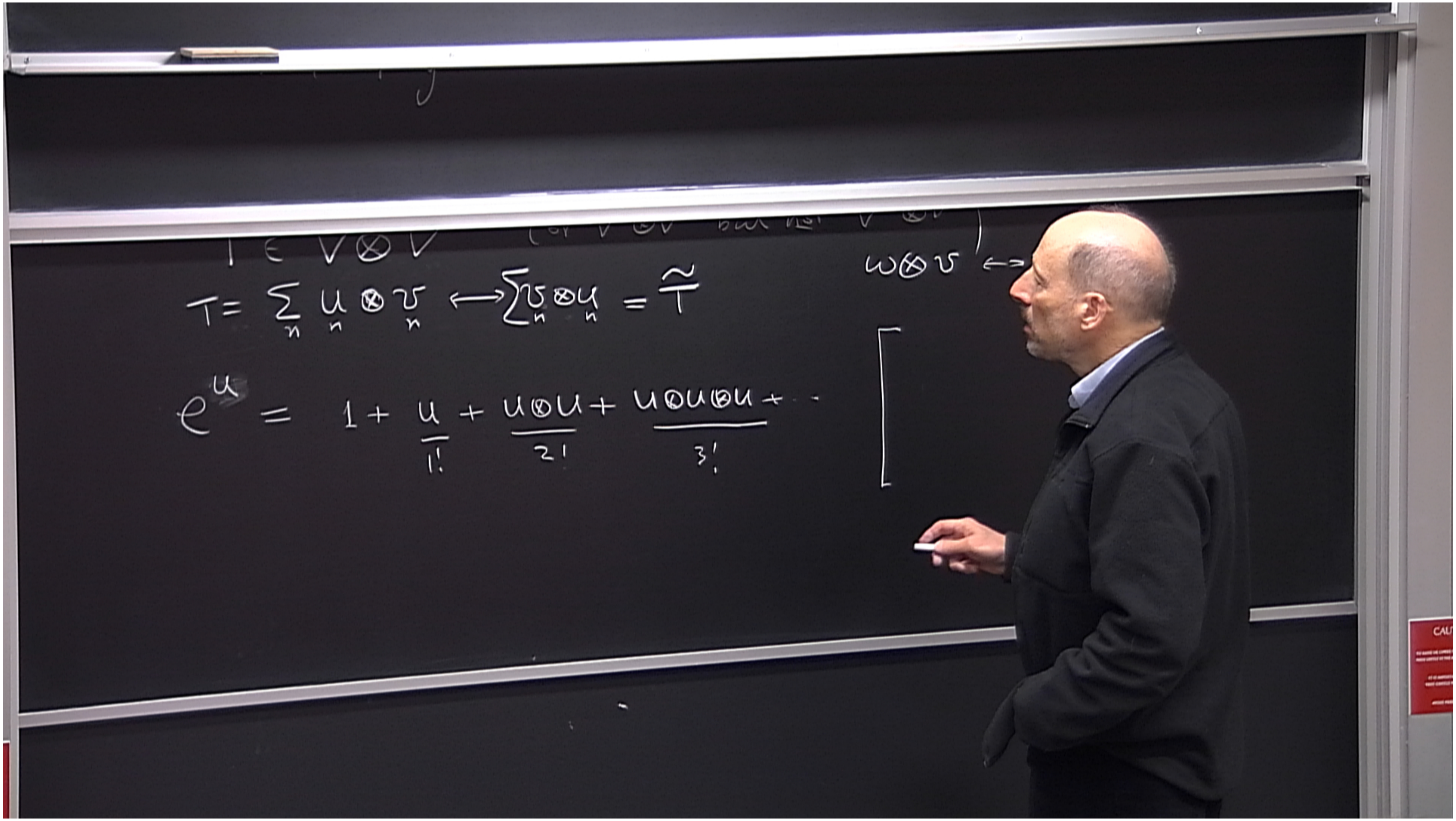
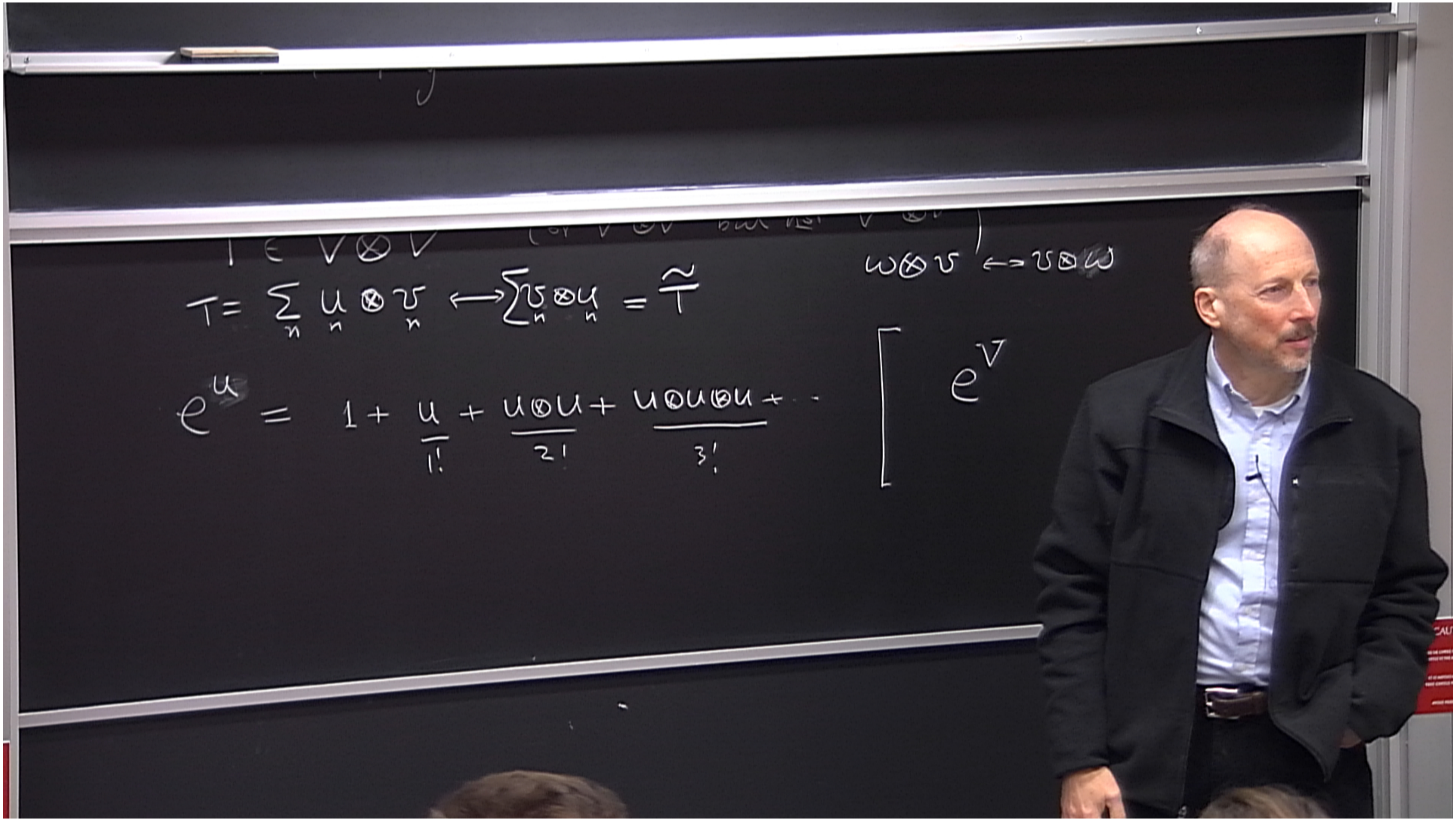
$V \otimes V$
V
Any linear map between tensor spaces comes from ("is") a tensor


$T \in V \otimes V$
$\omega \otimes v \longleftrightarrow v \otimes \omega$
$T = \sum_n u_n \otimes v_n \longleftrightarrow \sum v_n \otimes u_n = \tilde{T}$
$e^u = 1 + \frac{u}{1!} + \frac{u \otimes u}{2!} + \frac{u \otimes u \otimes u}{3!} + \cdots$


$T \in V \otimes V$
$T = \sum_n u_n \otimes v_n \longleftrightarrow \sum v_n \otimes u_n = \tilde{T}$
$w \otimes v \longleftrightarrow$
$e^u = 1 + \frac{u}{1!} + \frac{u \otimes u}{2!} + \frac{u \otimes u \otimes u}{3!} + \cdots$


$T \in V \otimes V$
$w \otimes v \leftrightarrow v \otimes w$
$T = \sum_n u_n \otimes v_n \longleftrightarrow \sum v_n \otimes u_n = \tilde{T}$
$e^u = 1 + \frac{u}{1!} + \frac{u \otimes u}{2!} + \frac{u \otimes u \otimes u}{3!} + \cdots$
e^V