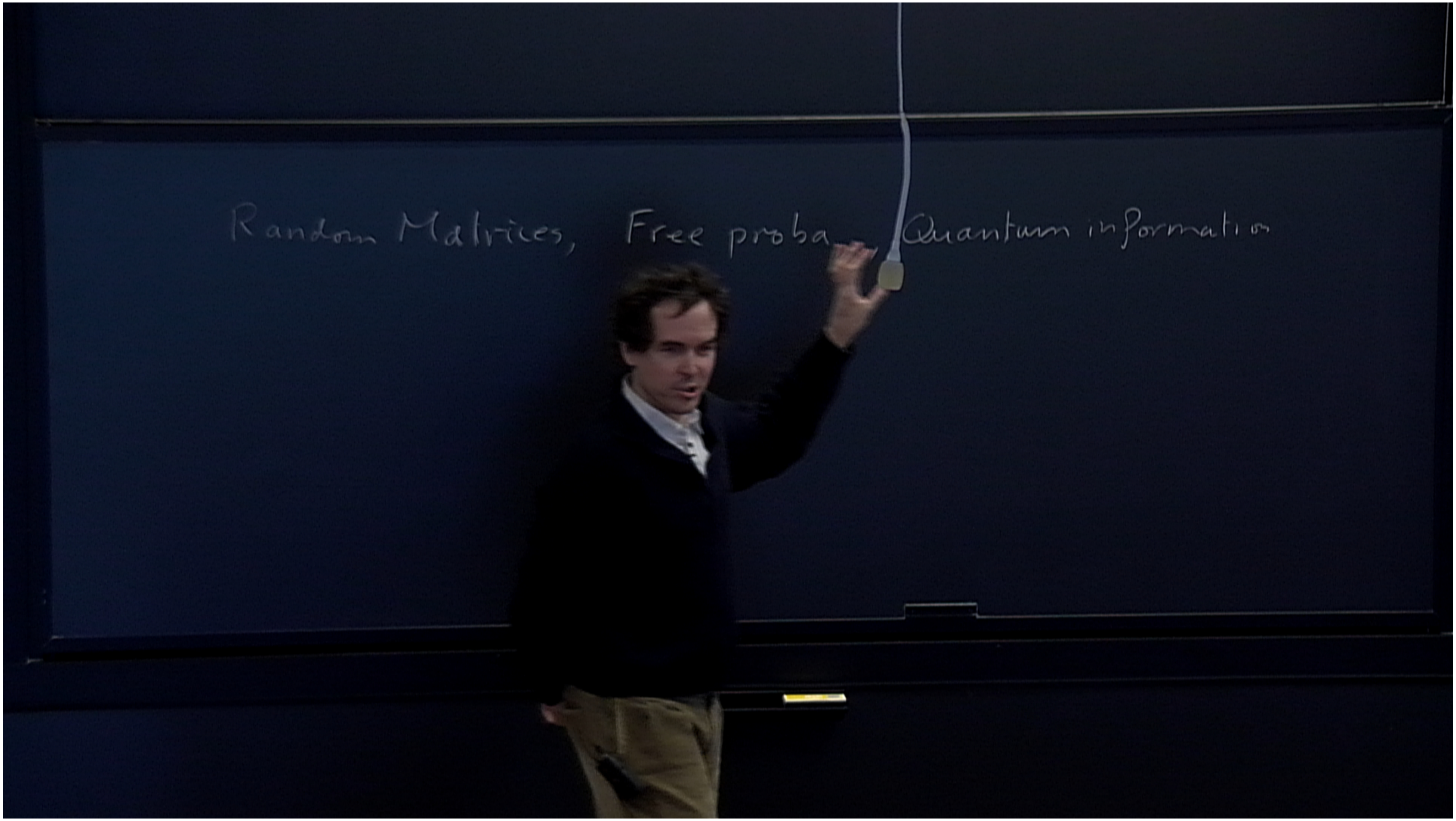
Title: Random matrices, free probability and quantum information theory

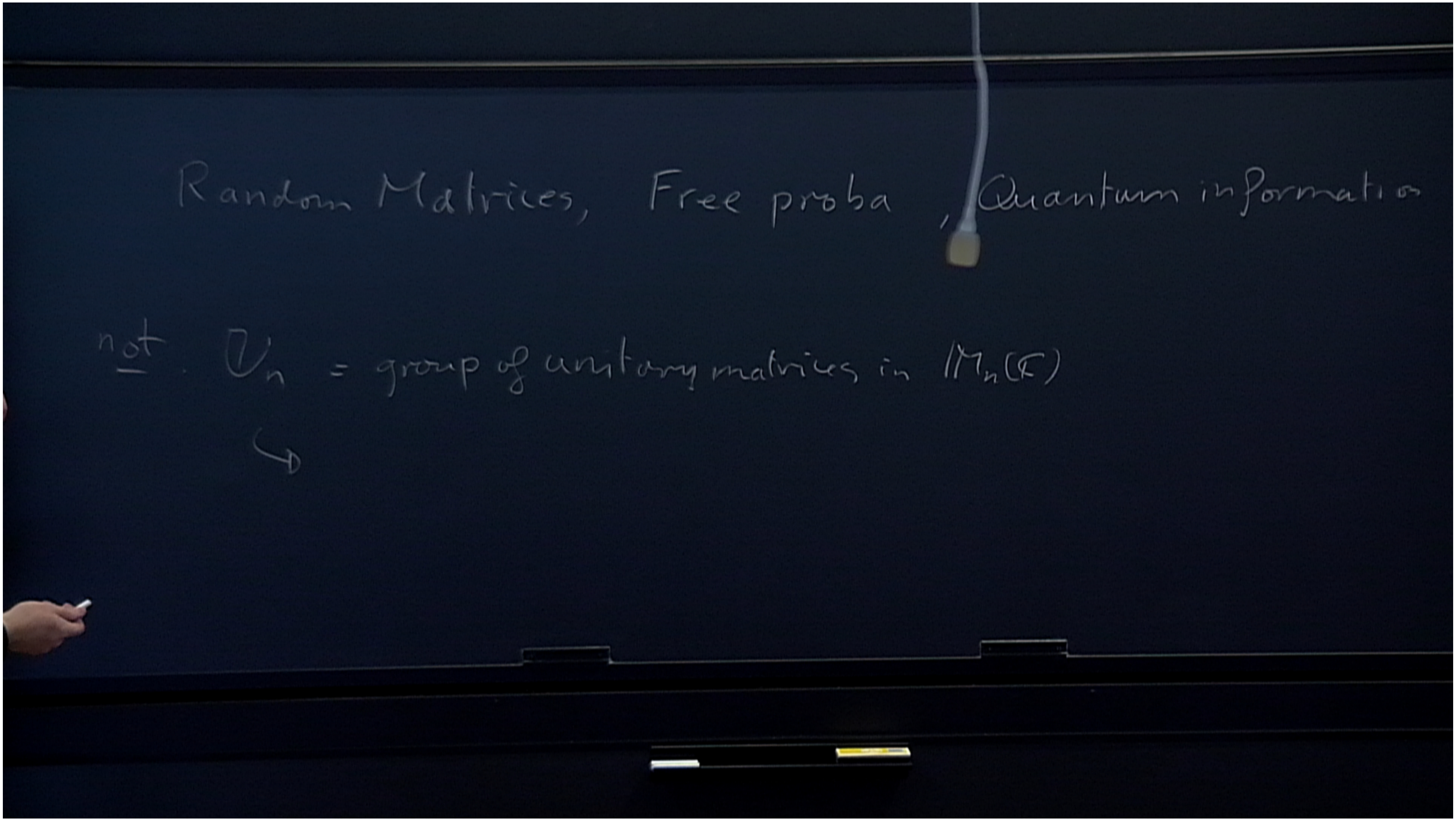
Date: Jan 31, 2013 03:00 PM

URL: http://www.pirsa.org/13010115

Abstract: In
quantum information theory, random techniques have proven to be very useful. For example, many questions related to the problem of the additivity of entropies of quantum channels rely on fine properties of concentration of measure.

In
this talk, I will show that very different techniques of random matrix theory can complement quite efficiently more classical random techniques. I will spend some time on discussing the Weingarten calculus approach, and the operator norm approach. Both techniques have been initially used in free probability theory, and I will give some new applications of these techniques to quantum information theory.


Random Matrices, Free proba, Quantum information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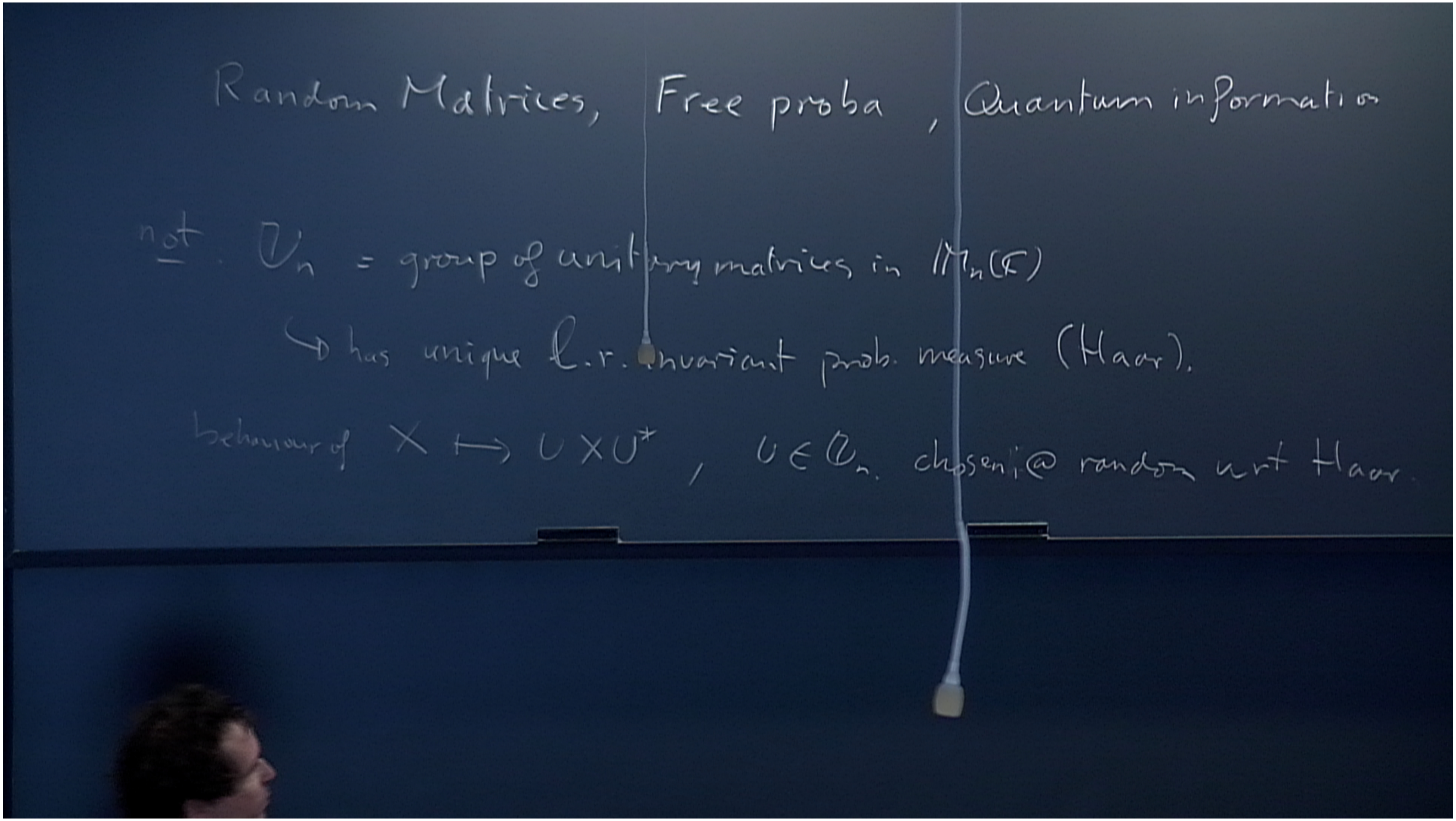
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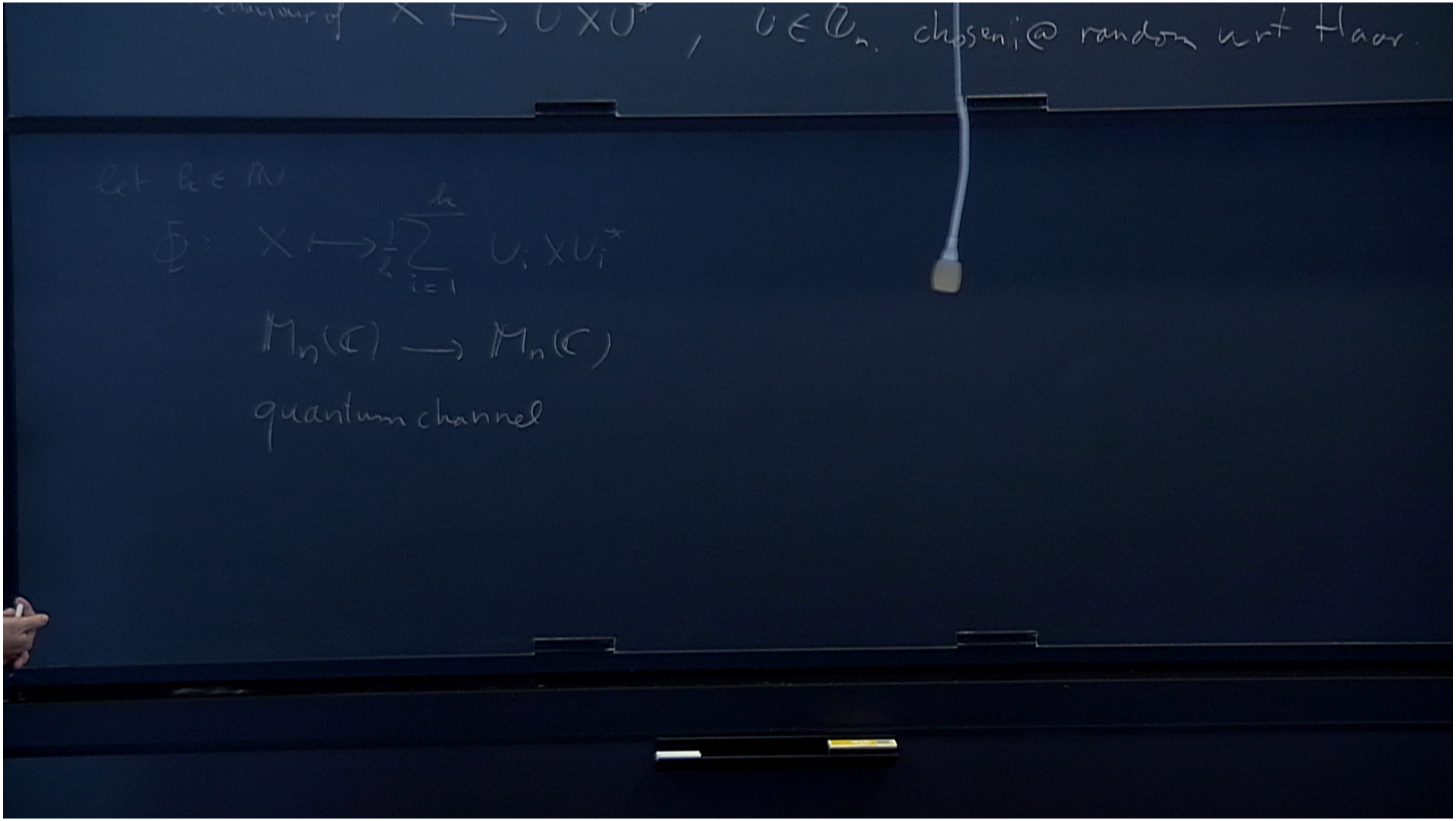
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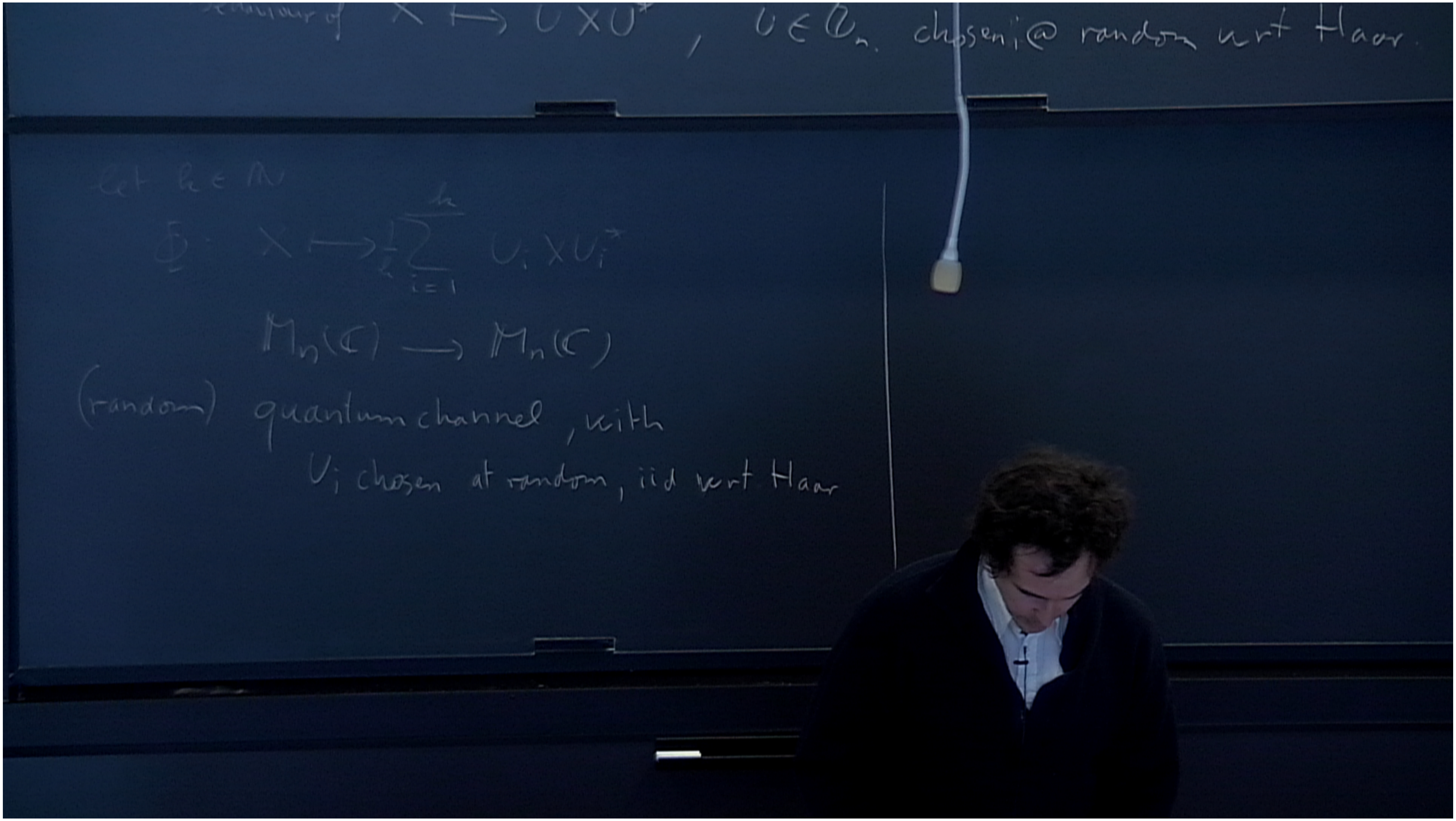
let $k \in \mathbb{N}$
$\Phi : X \longmapsto \frac{1}{k} \sum_{i=1}^{k} U_i X U_i^*$
$M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$
quantum channel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Mn(ℂ) → Mn(ℂ)
(random) quantum channel, with
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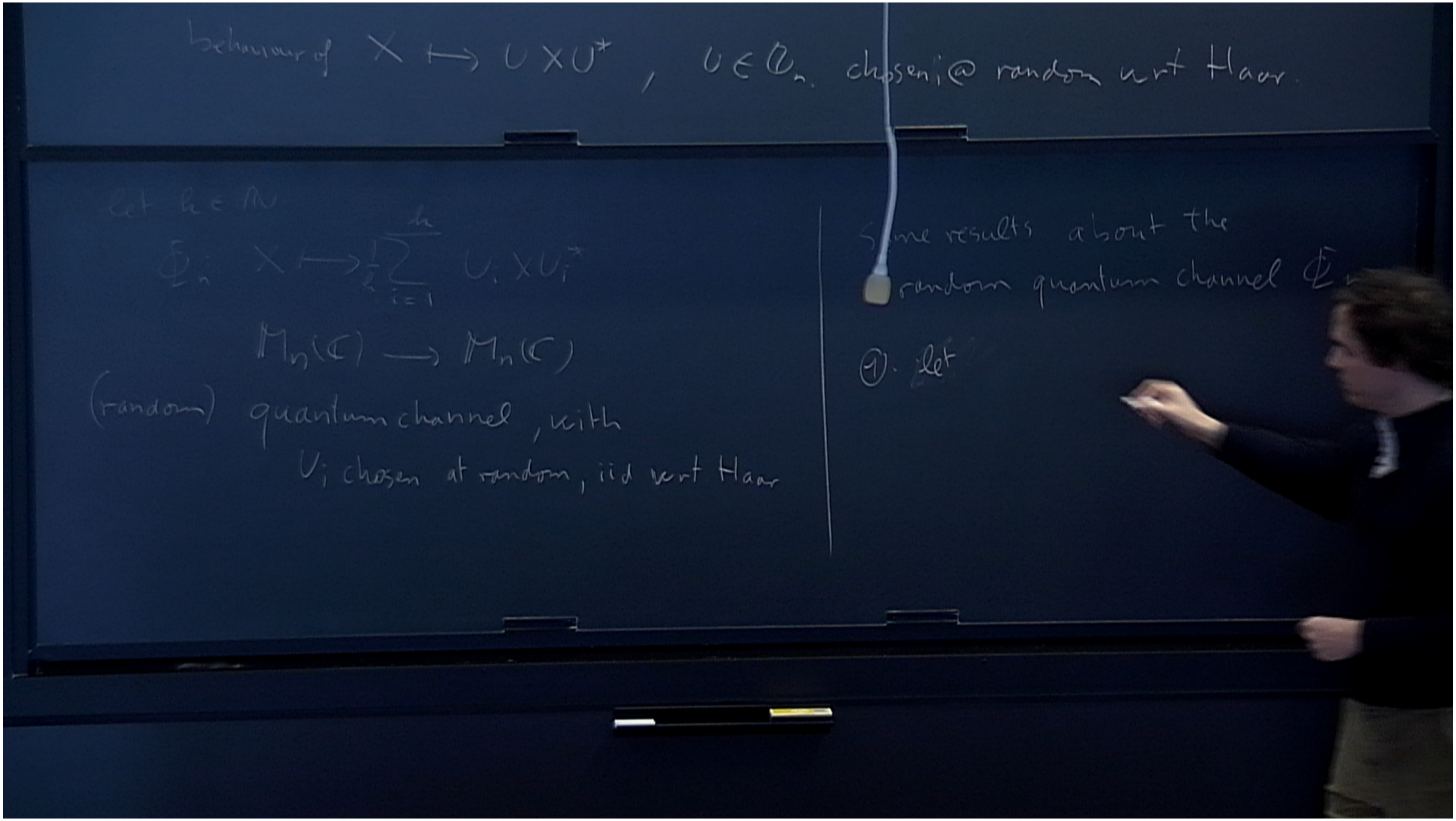
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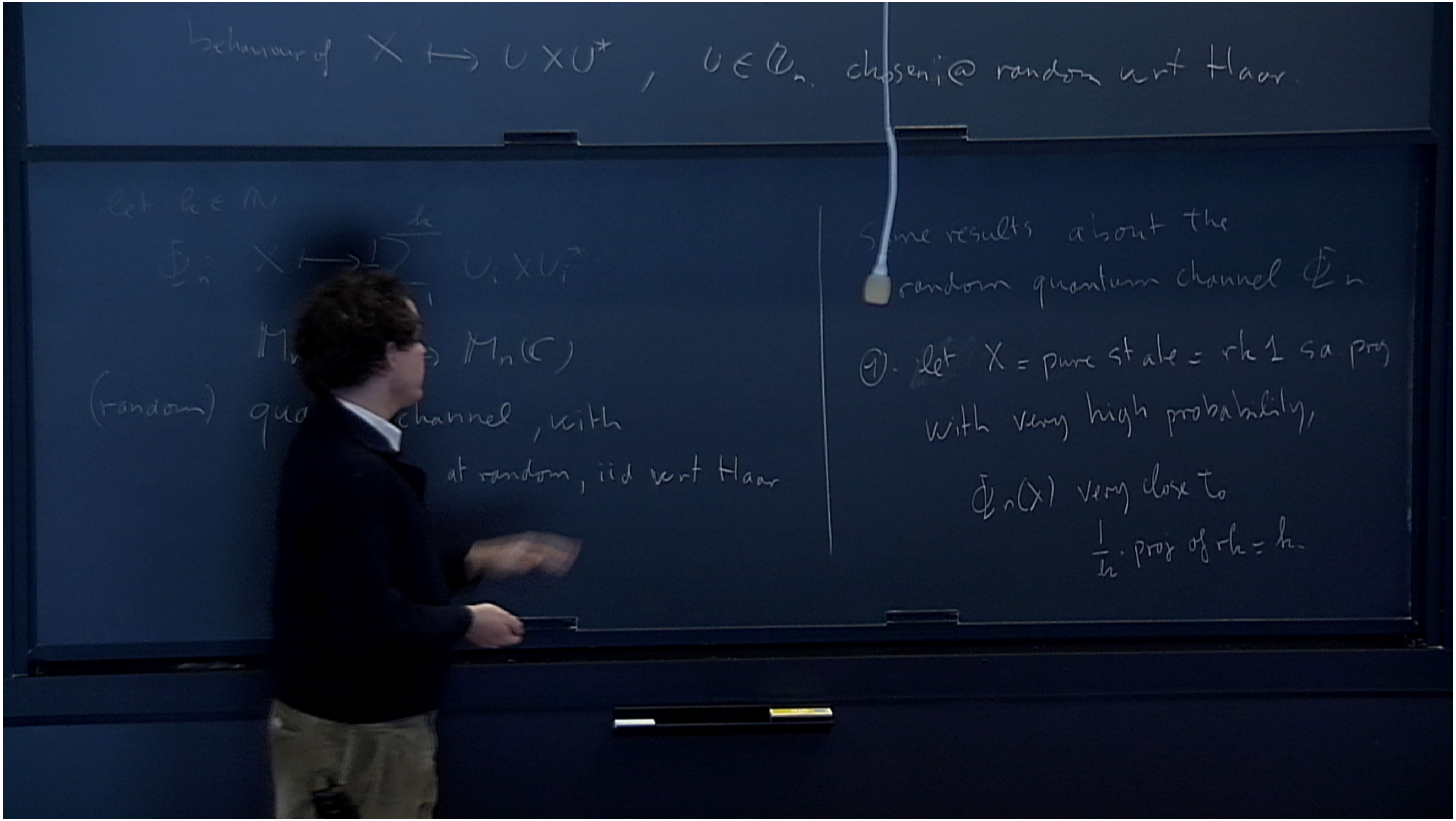
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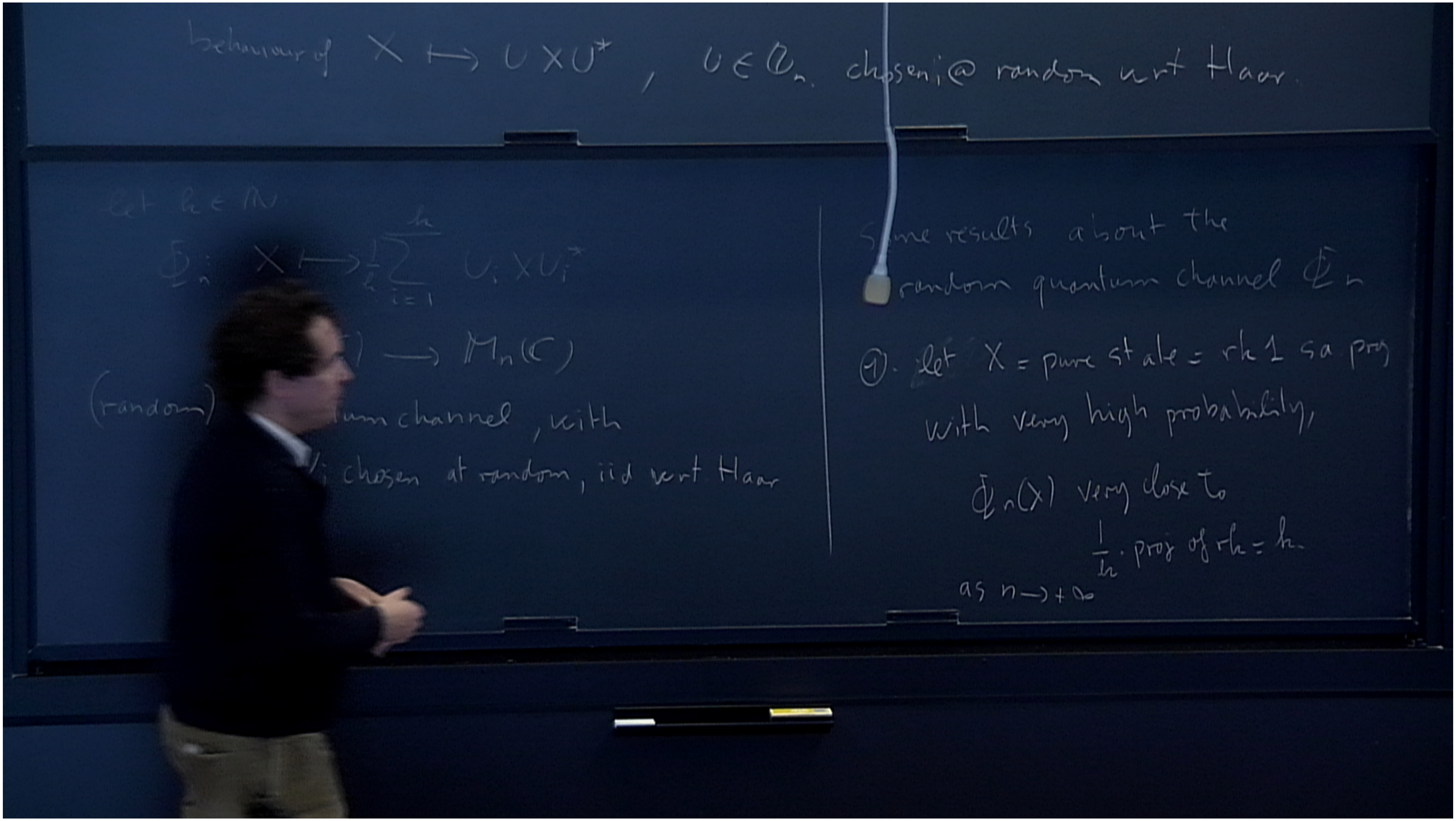
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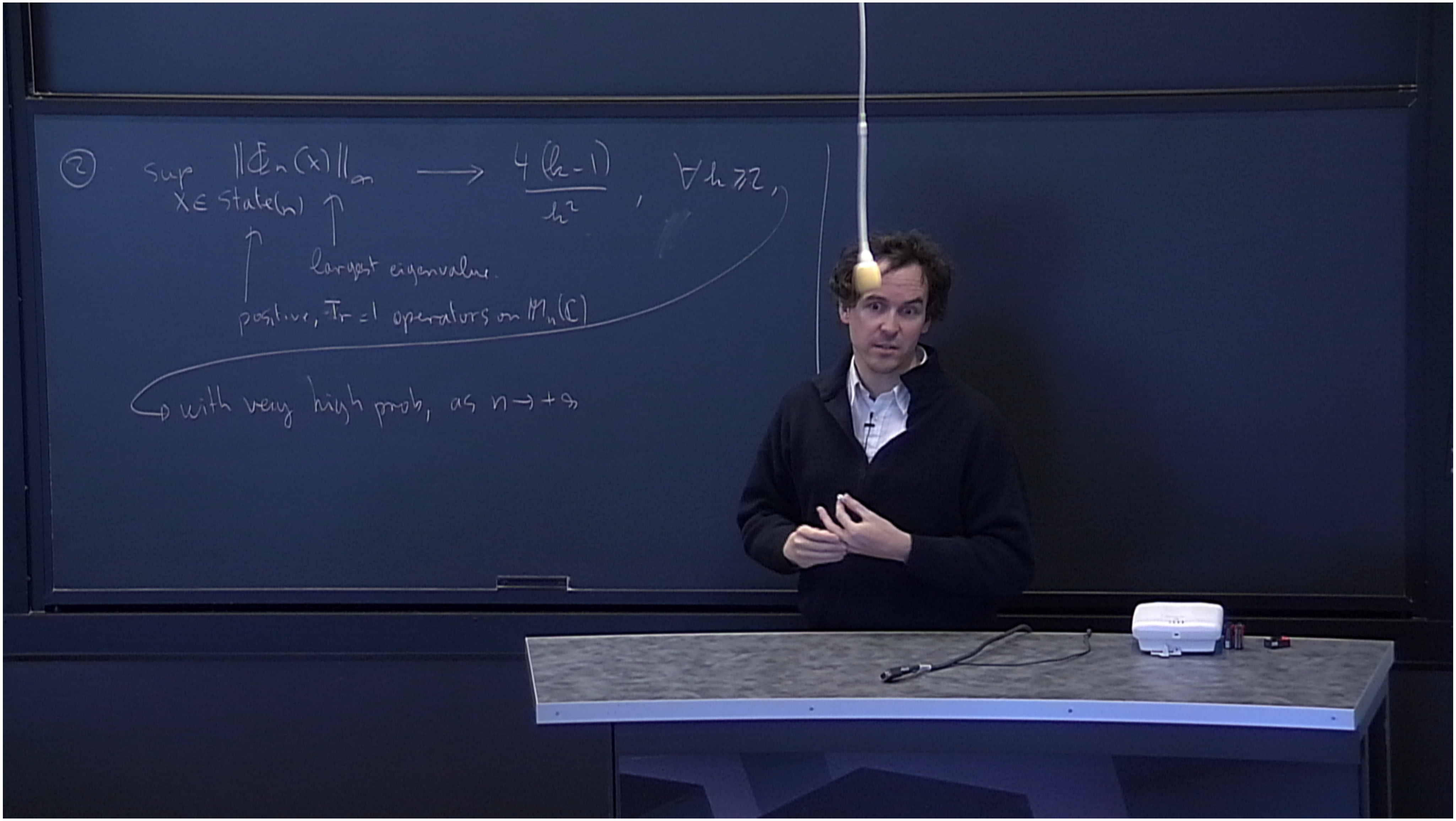
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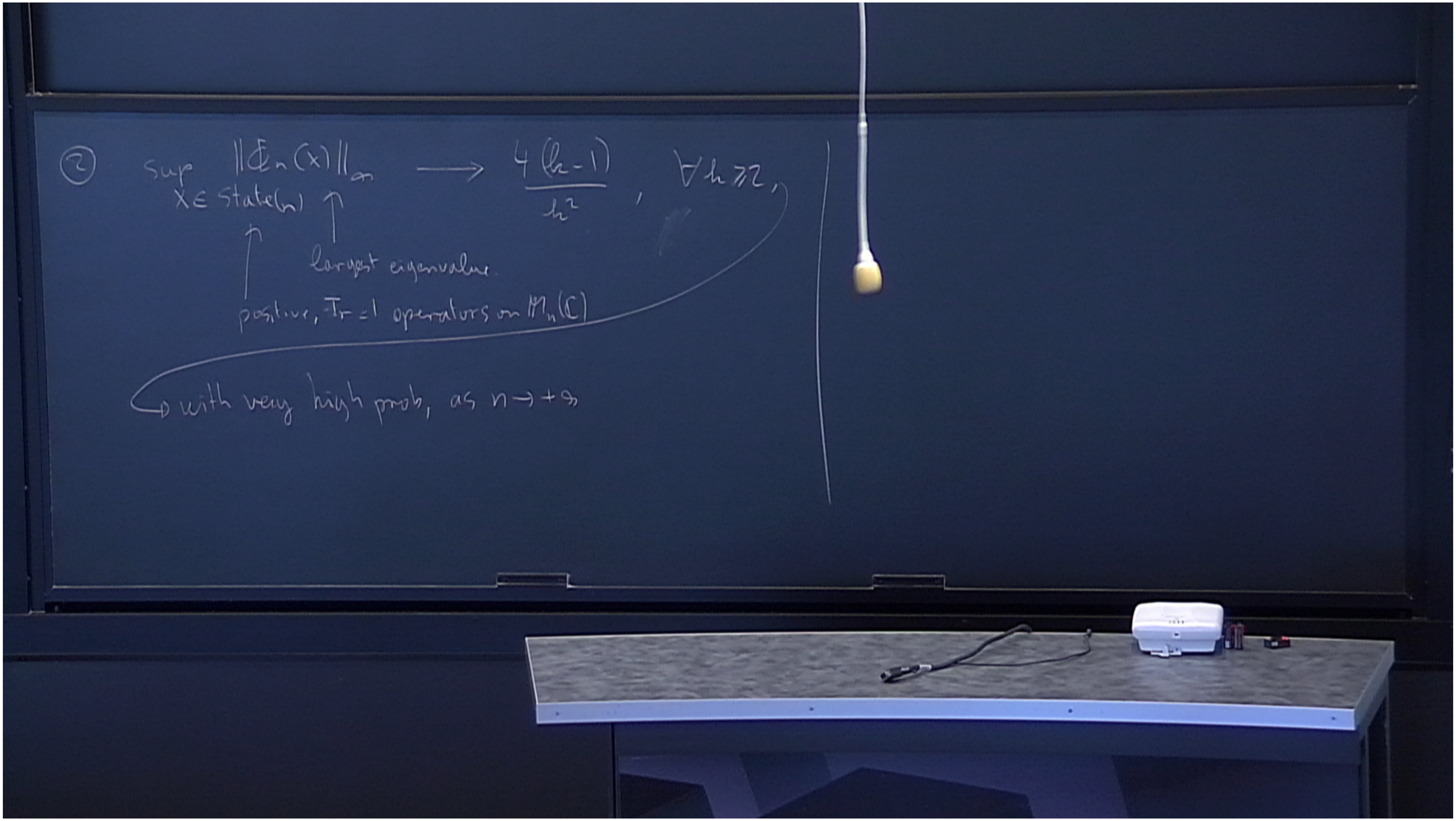
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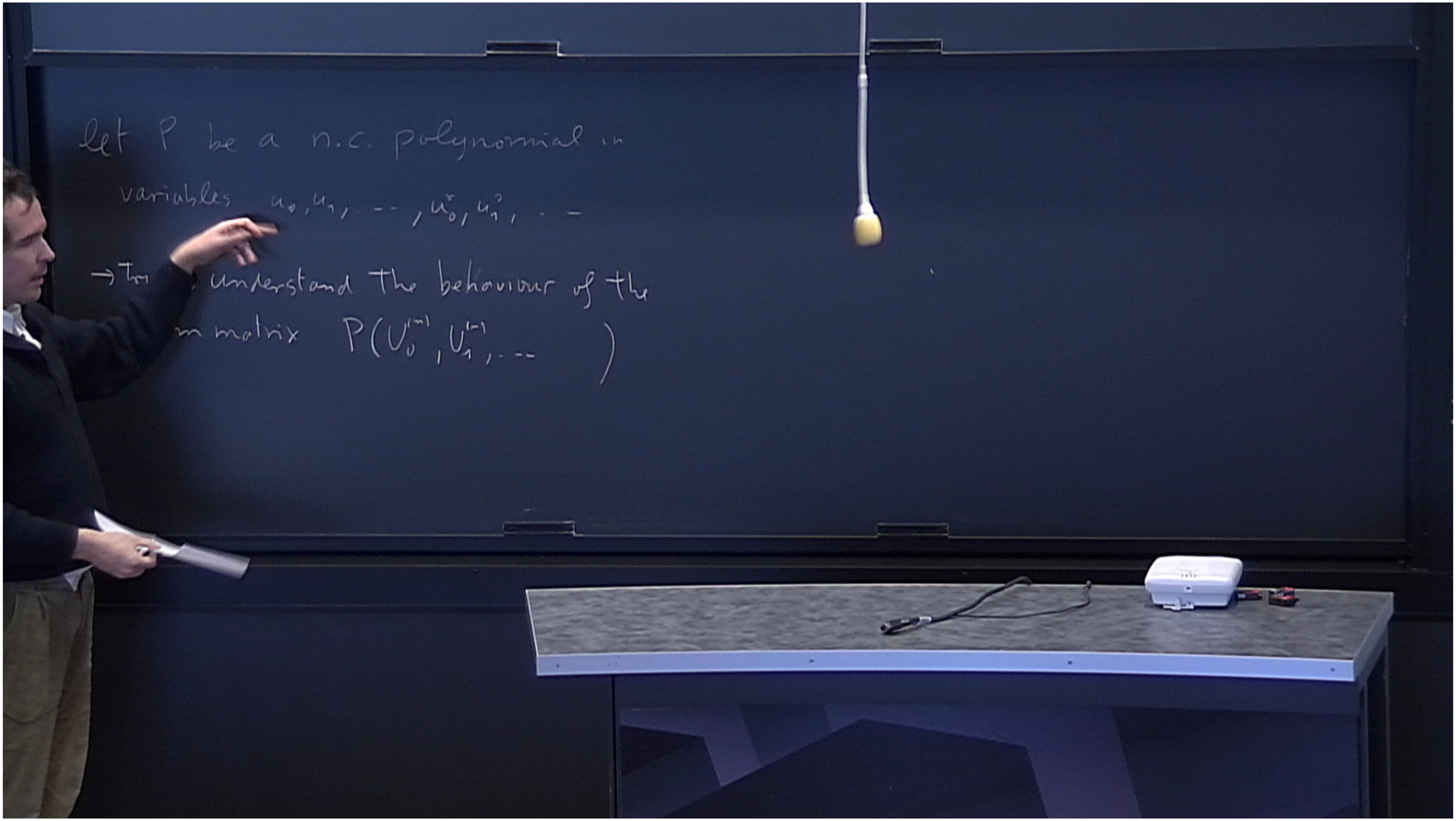
in multi matrix RMT,

similar problem:

let $U_0^{(n)}, U_1^{(n)}, U_2^{(n)}, \dots$ be iid Haar unitary matrices

let P_n be a n.c.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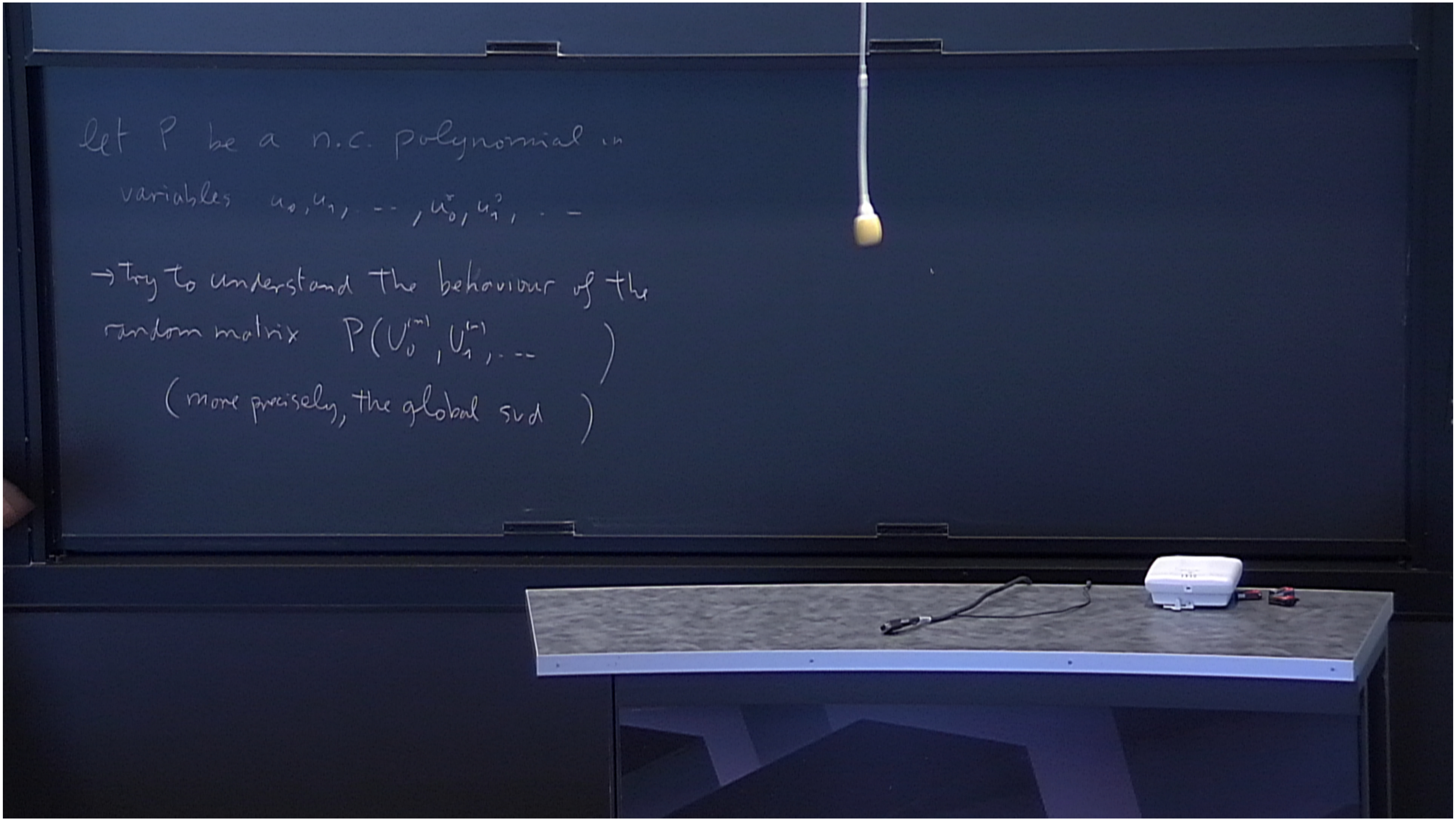
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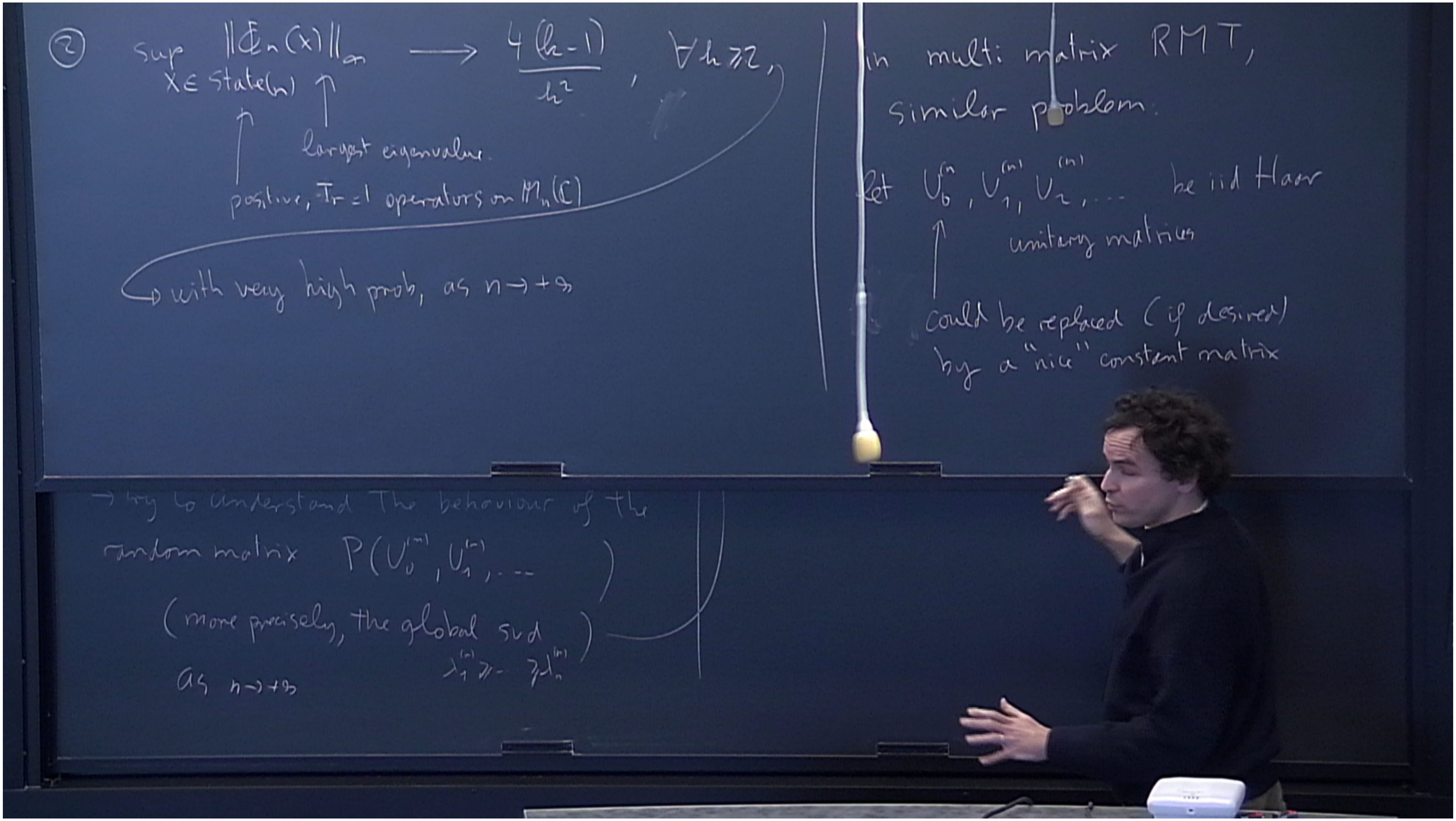
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let U0(n), U1(n), U2(n), ... be iid Haar
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could be replaced (if desired)
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λ1(n) ≥ ... ≥ λn(n)
as n → +∞

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positive, $\mathrm{Tr}=1$ operators on $M_n(\mathbb{C})$

$\hookrightarrow$ with very high prob, as $n \to +\infty$

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$\to$ Try to understand the beha... th...

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as $n \to +\infty$

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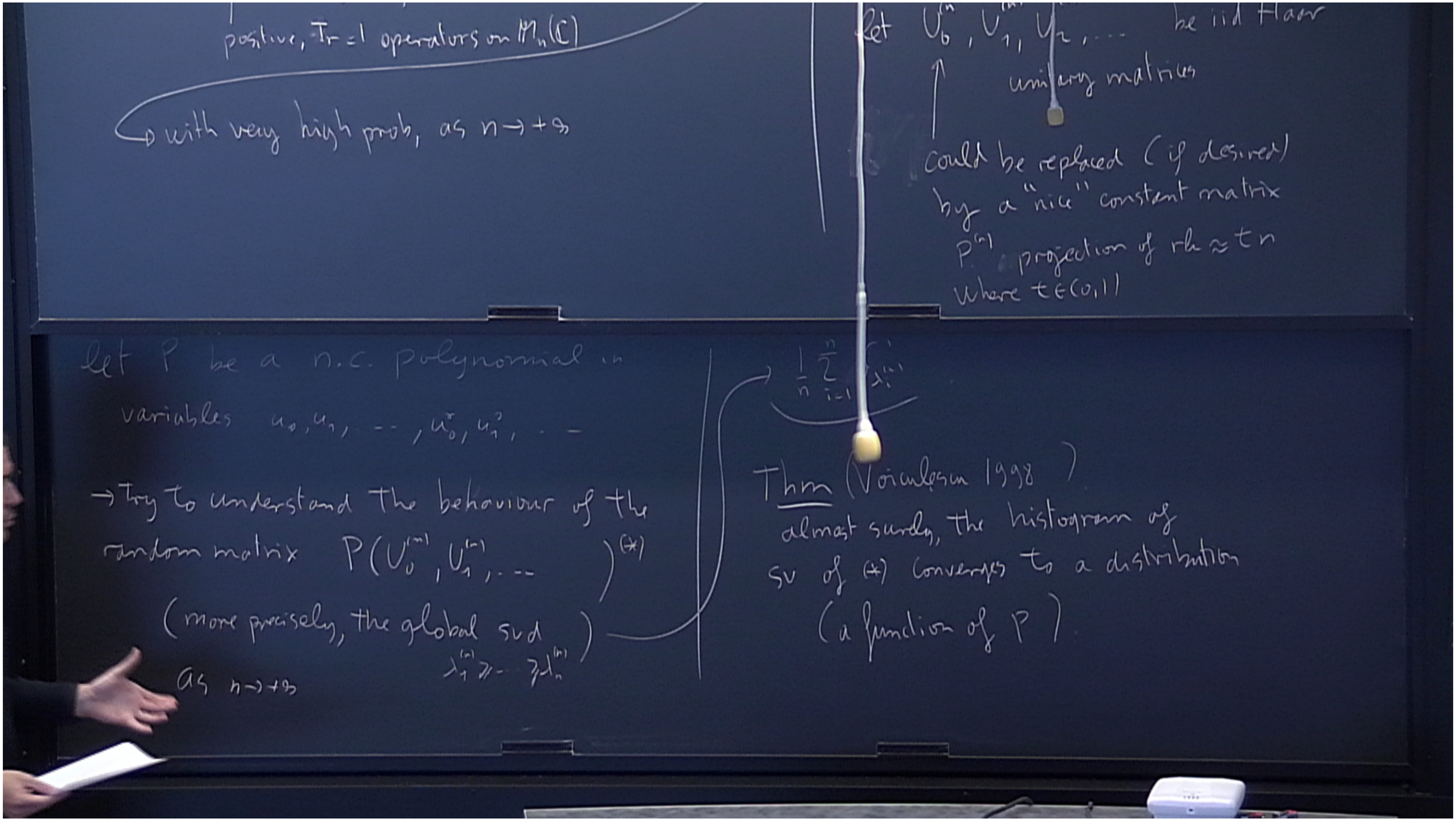
$\frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i^{(n)}}$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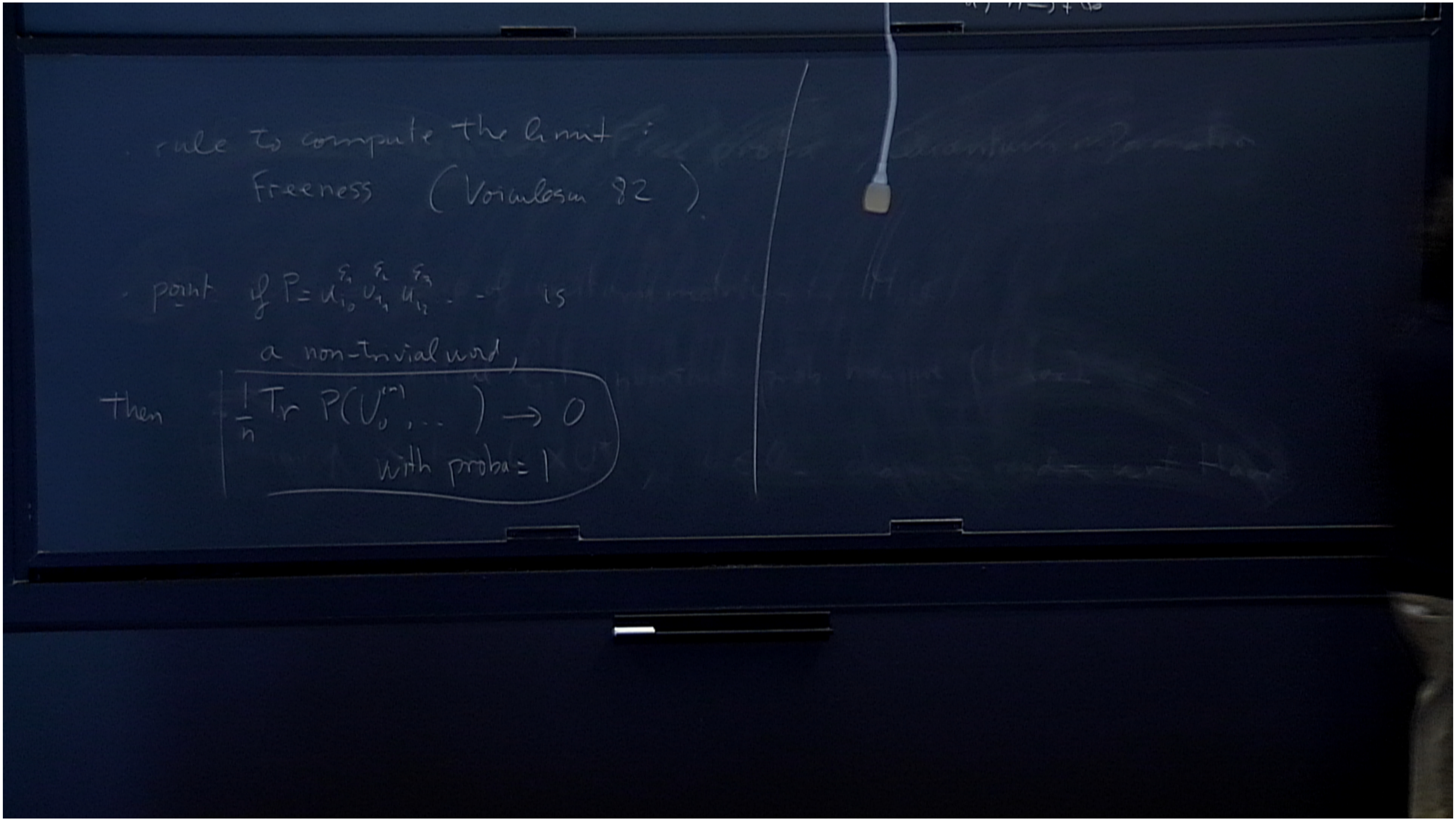
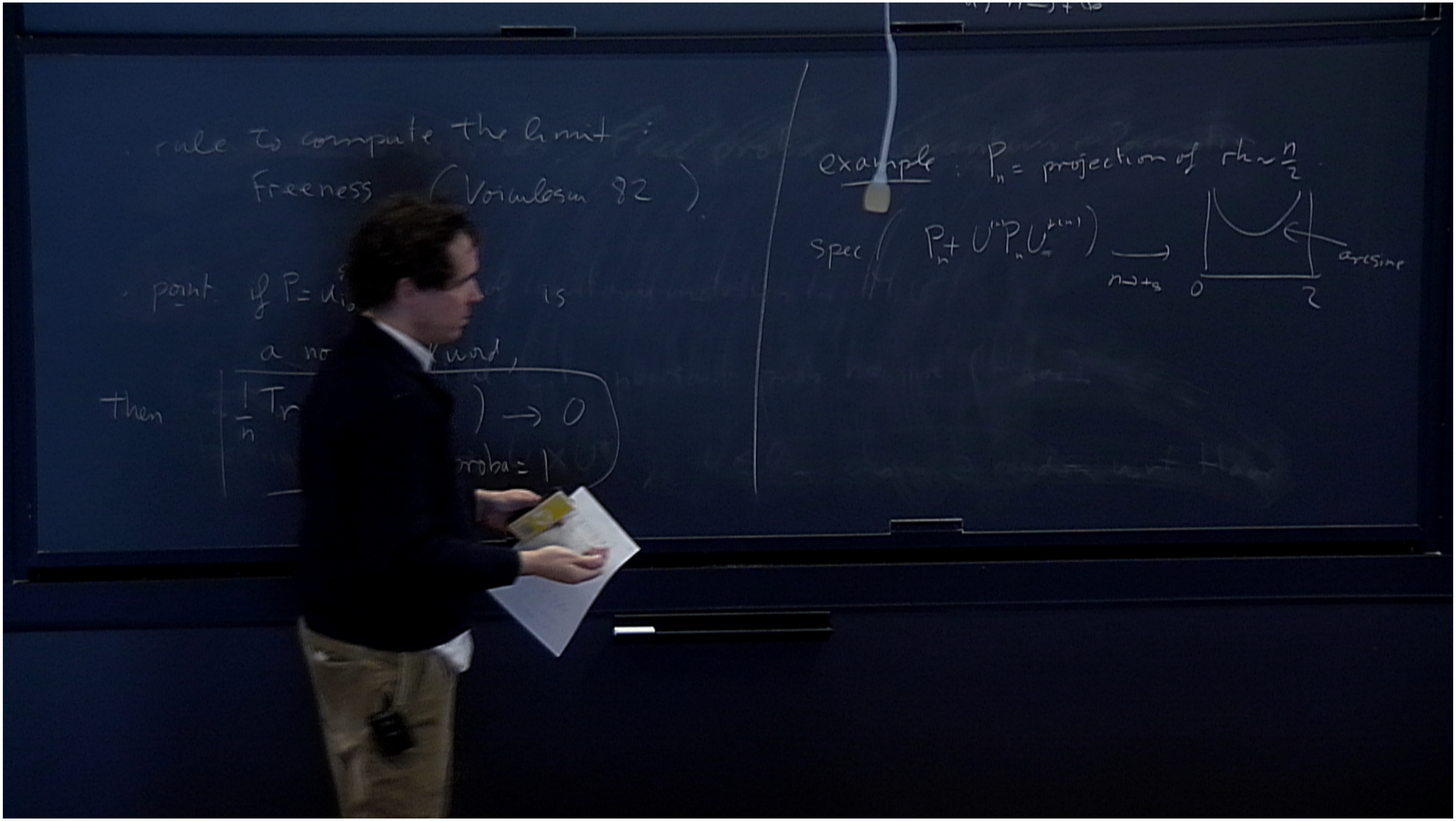

rule to compute the limit :
Freeness (Voiculescu 82)
point: if $P = u_{i_0}^{\varepsilon_1} u_{i_1}^{\varepsilon_2} u_{i_2}^{\varepsilon_3} \cdots$ is
a non-trivial word,
Then $\frac{1}{n} \mathrm{Tr}\, P(U_0^{(n)}, \ldots) \to 0$
with proba = 1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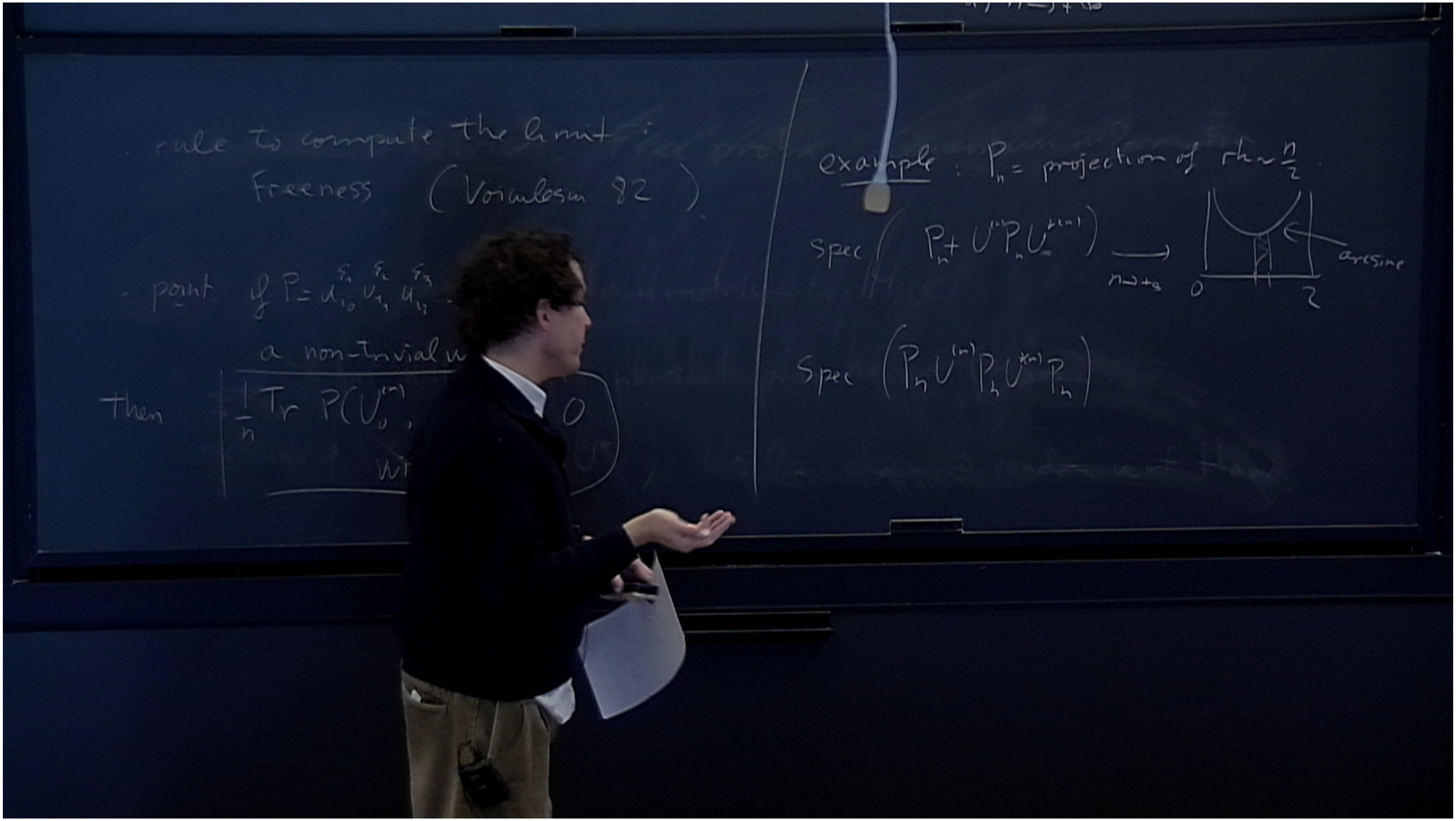
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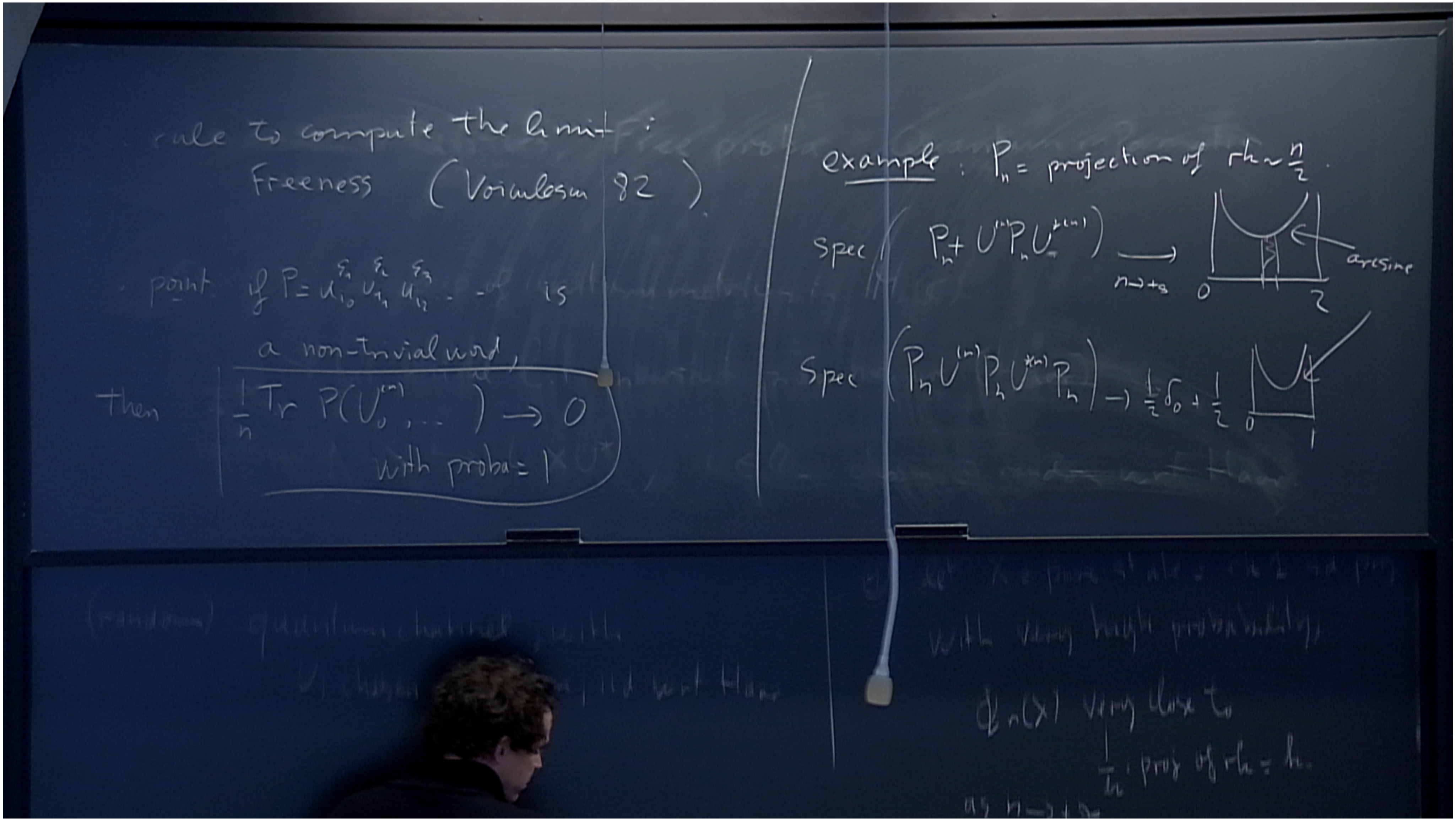
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arcsine
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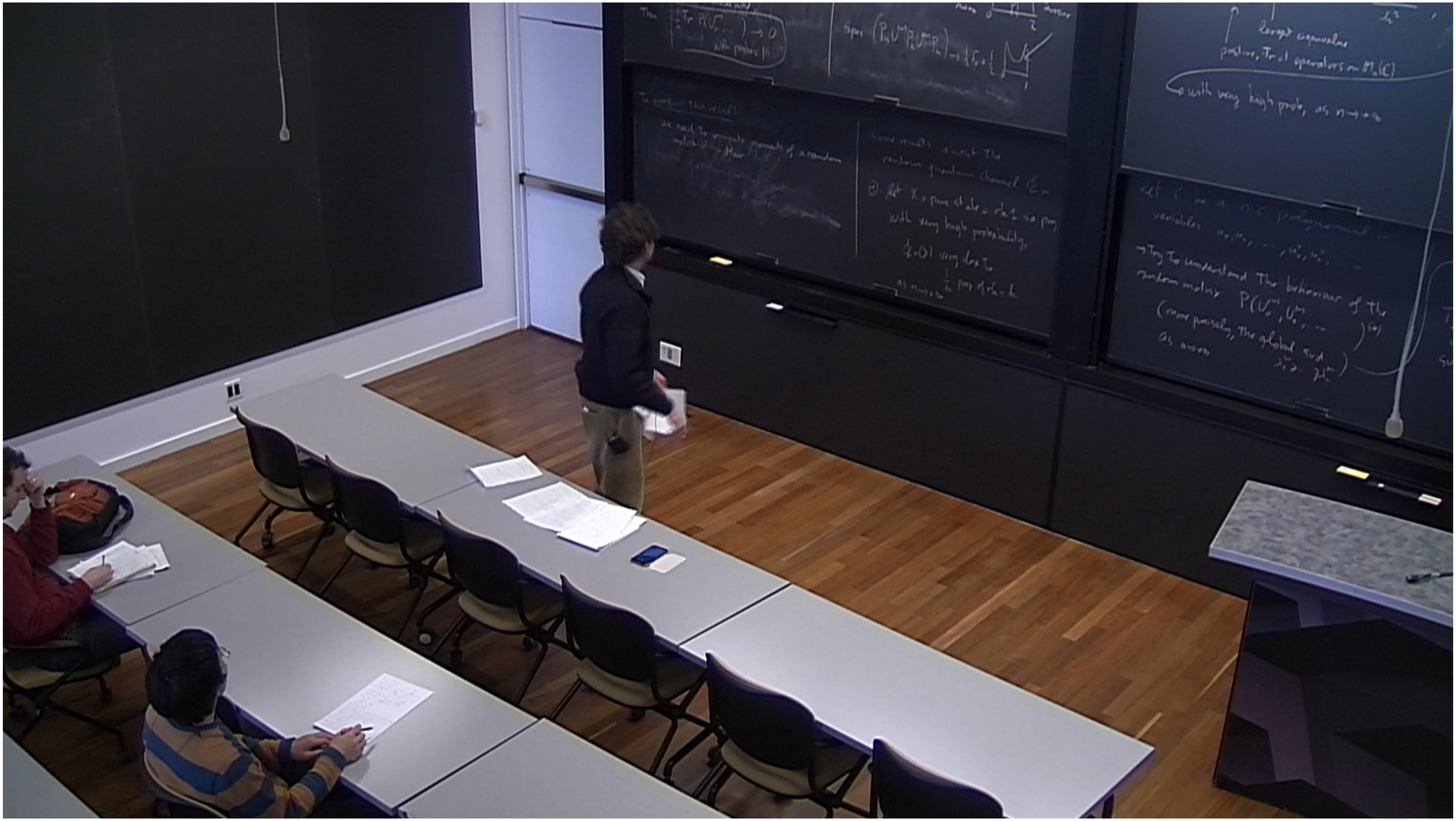
0
1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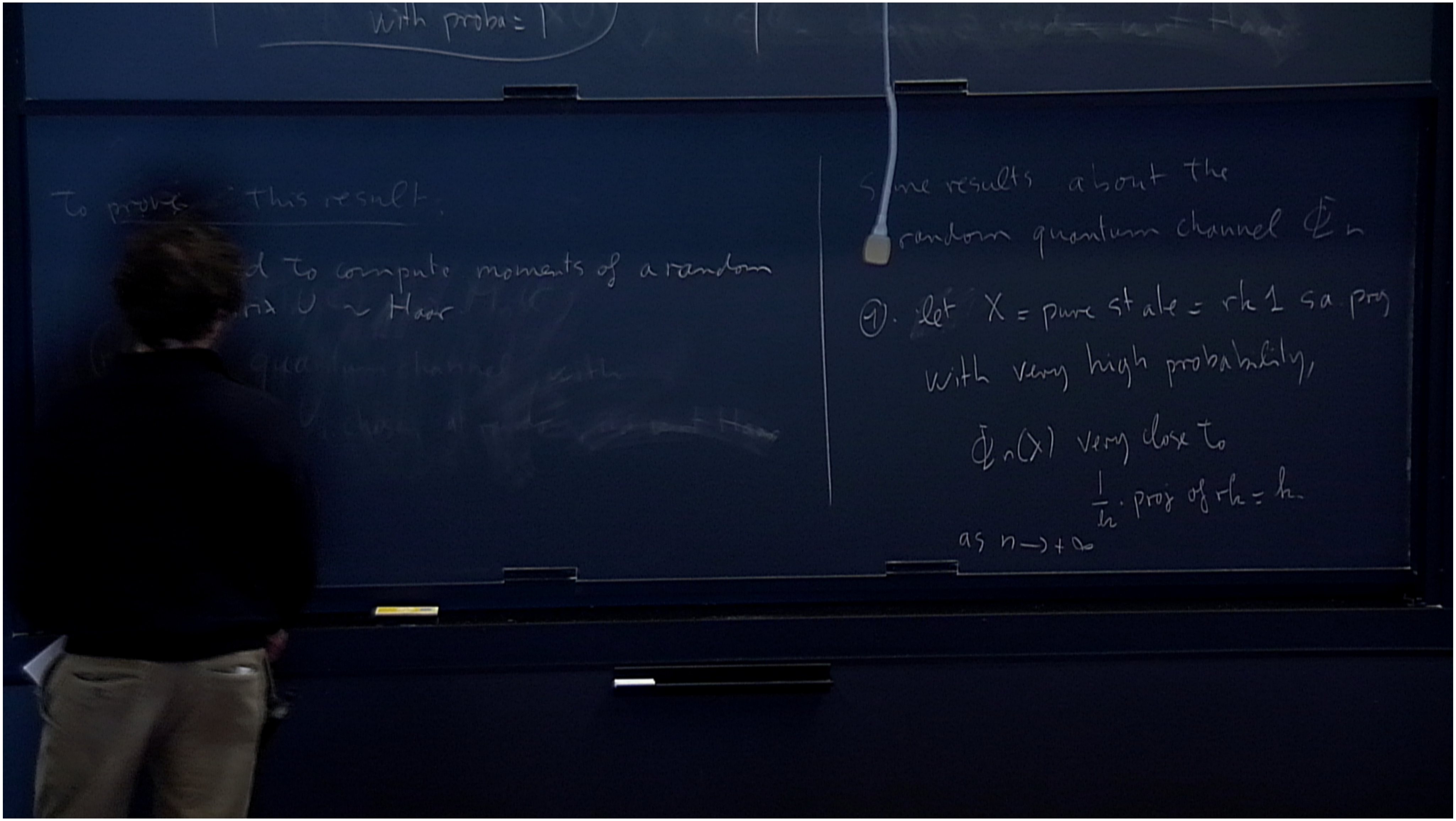
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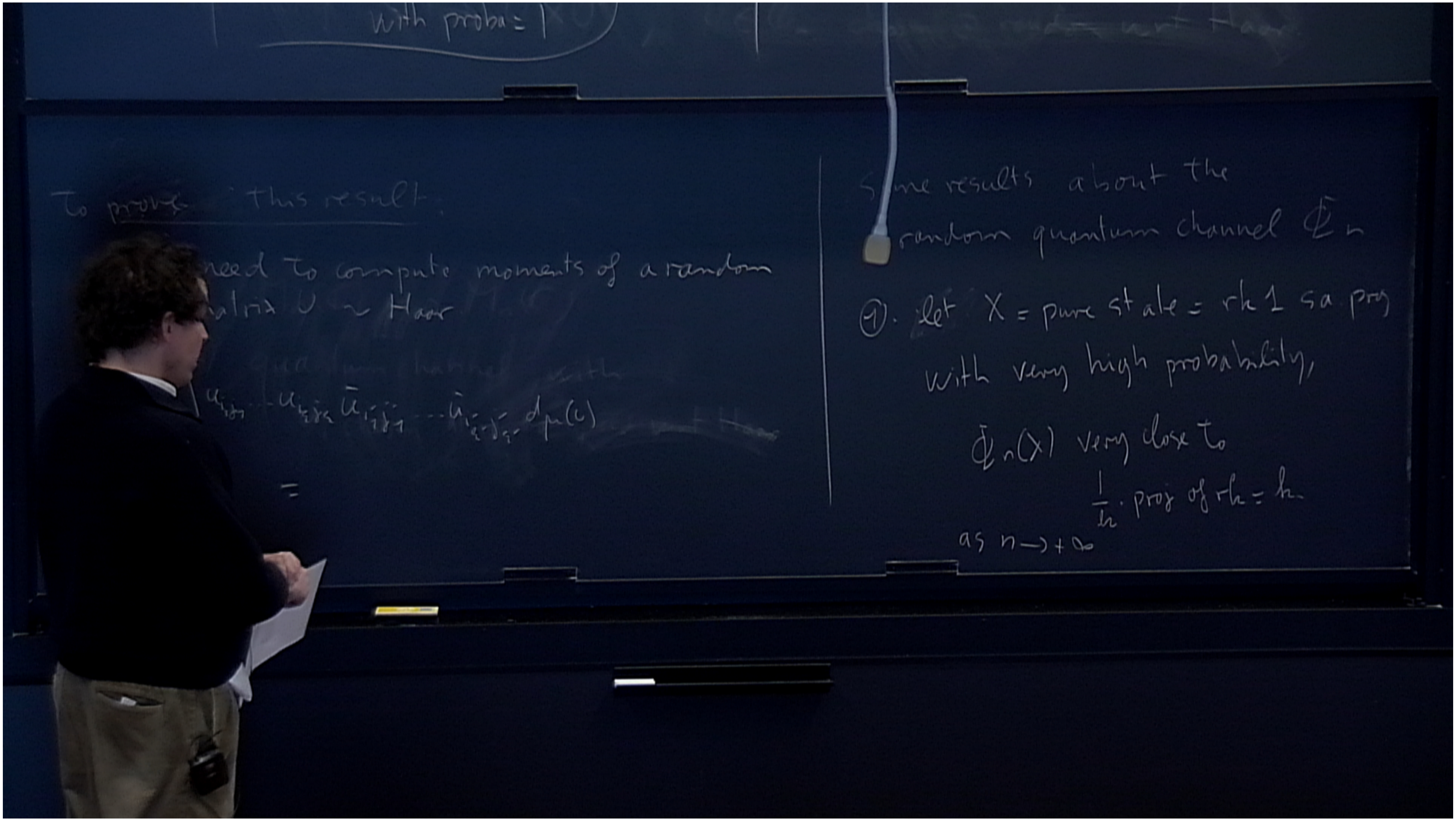
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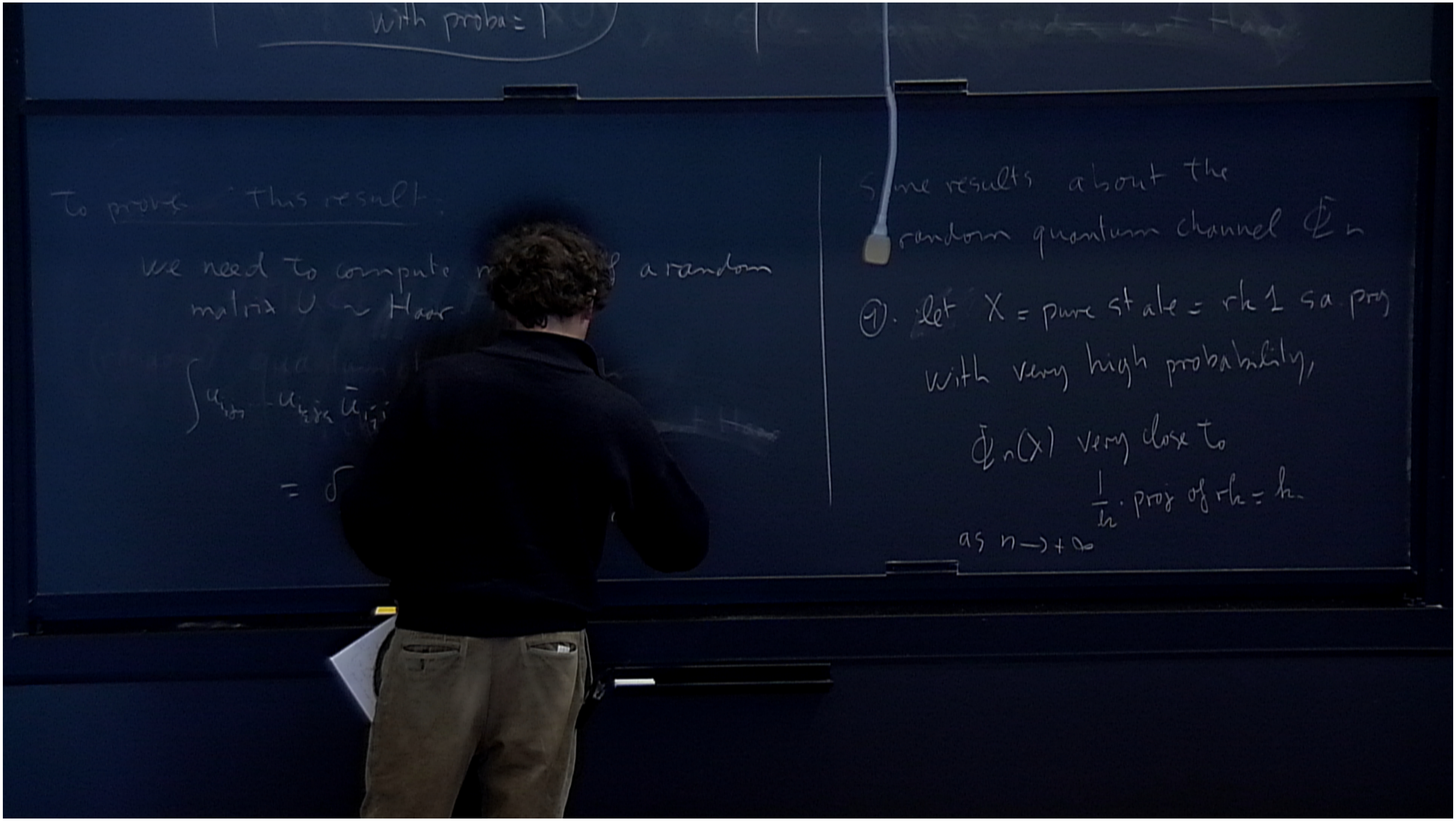
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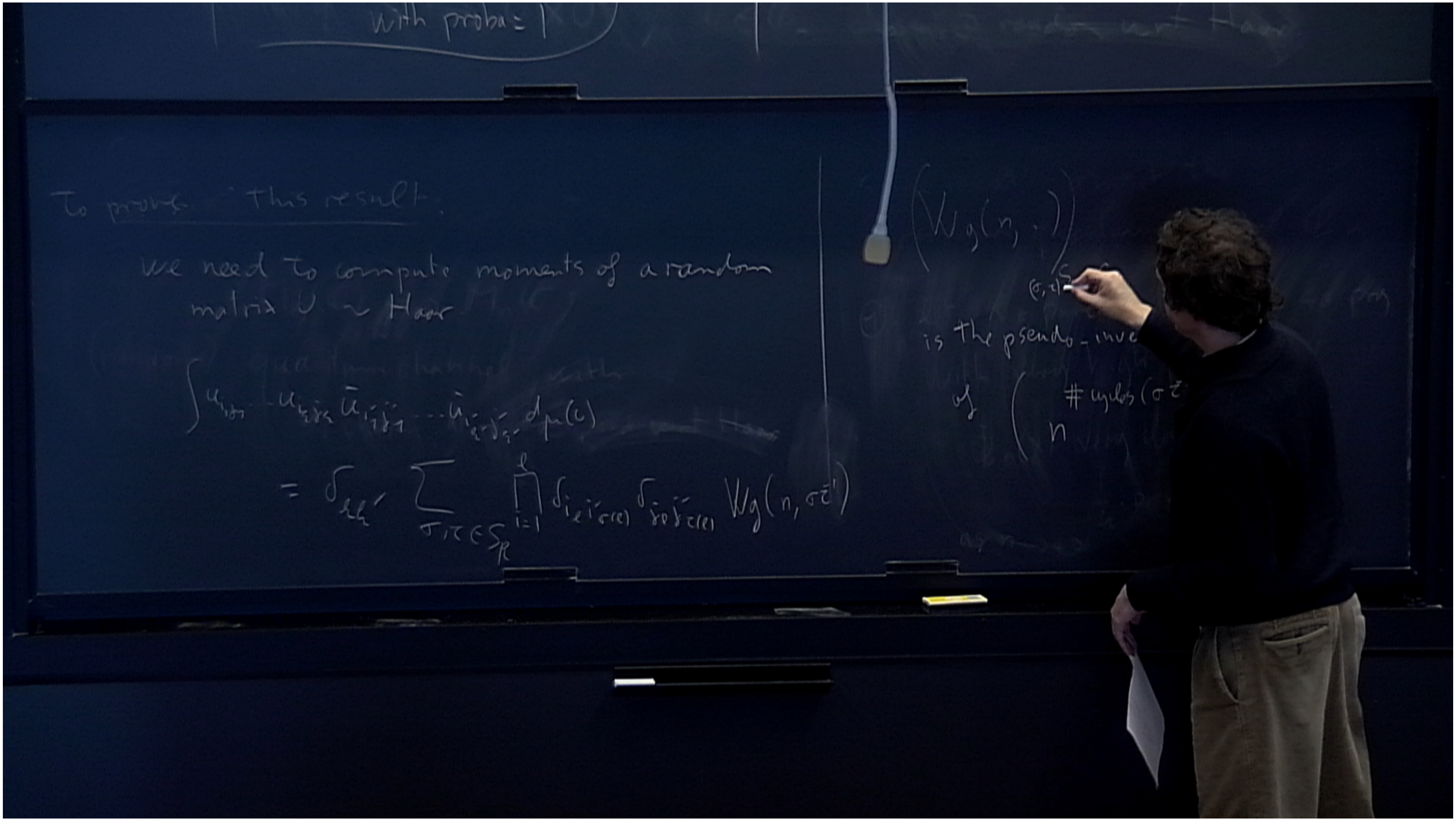
$$\int u_{i_1 j_1} \cdots u_{i_k j_k} \bar{u}_{i'_1 j'_1} \cdots \bar{u}_{i'_{k'} j'_{k'}} \, d\mu(U)$$

$$= \delta_{k k'} \sum_{\sigma, \tau \in S_k} \prod_{l=1}^{k} \delta_{i_l i'_{\sigma(l)}} \delta_{j_l j'_{\tau(l)}} \, Wg(n, \sigma\tau^{-1})$$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$$\left(Wg(n, \sigma\tau^{-1}) \right)_{(\sigma,\tau) \in S_\ell \times S_\ell}$$

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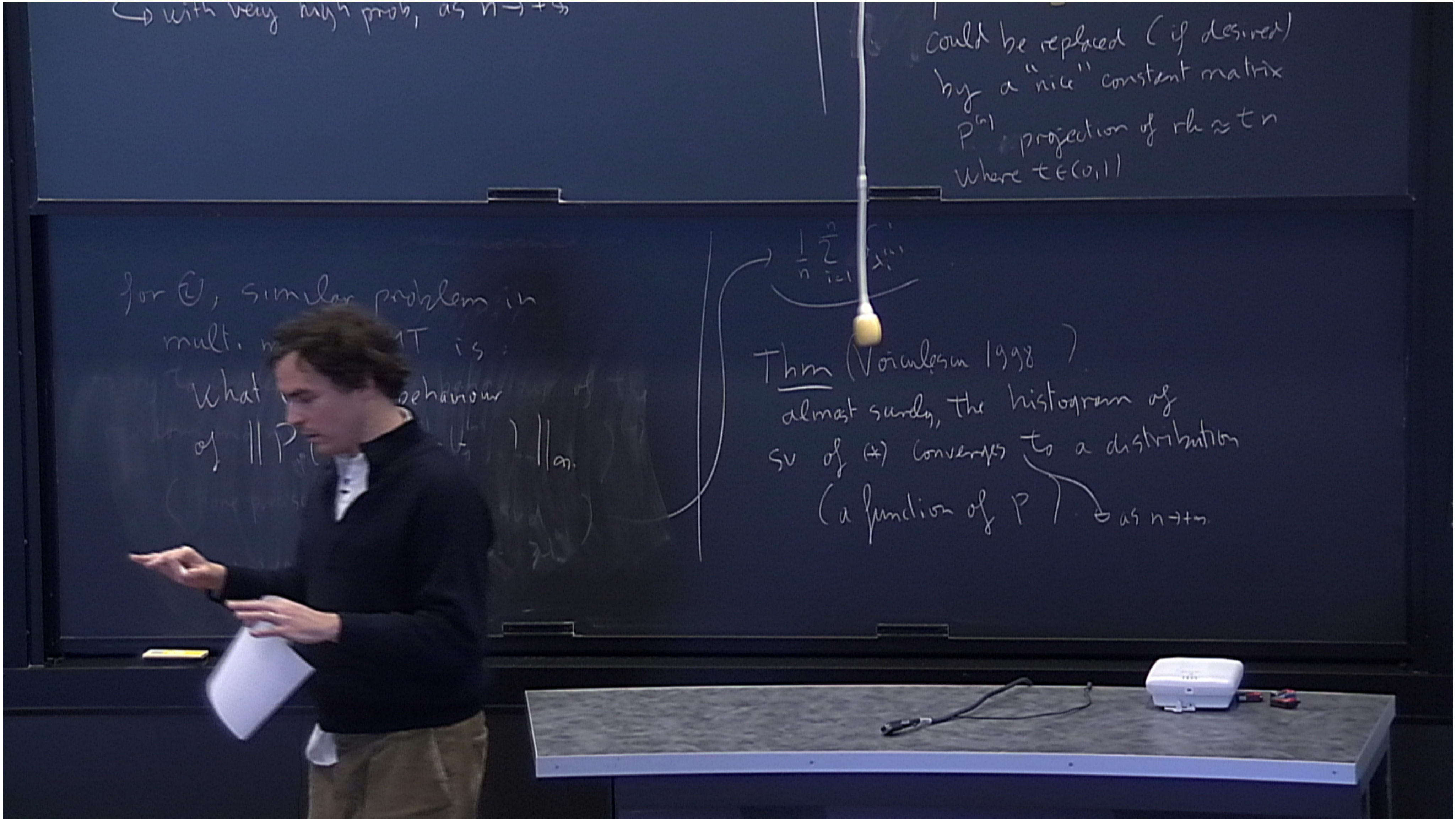
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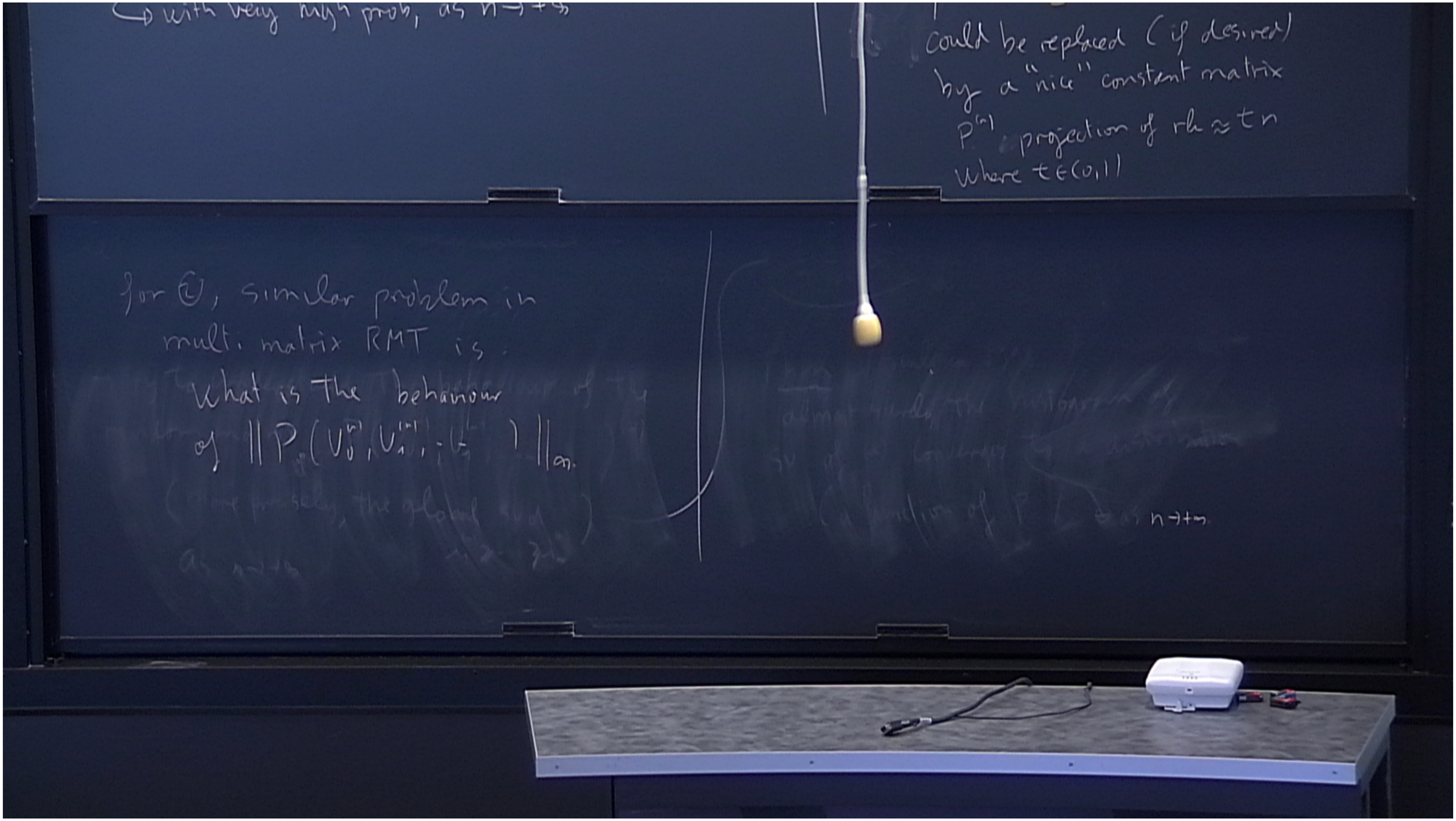
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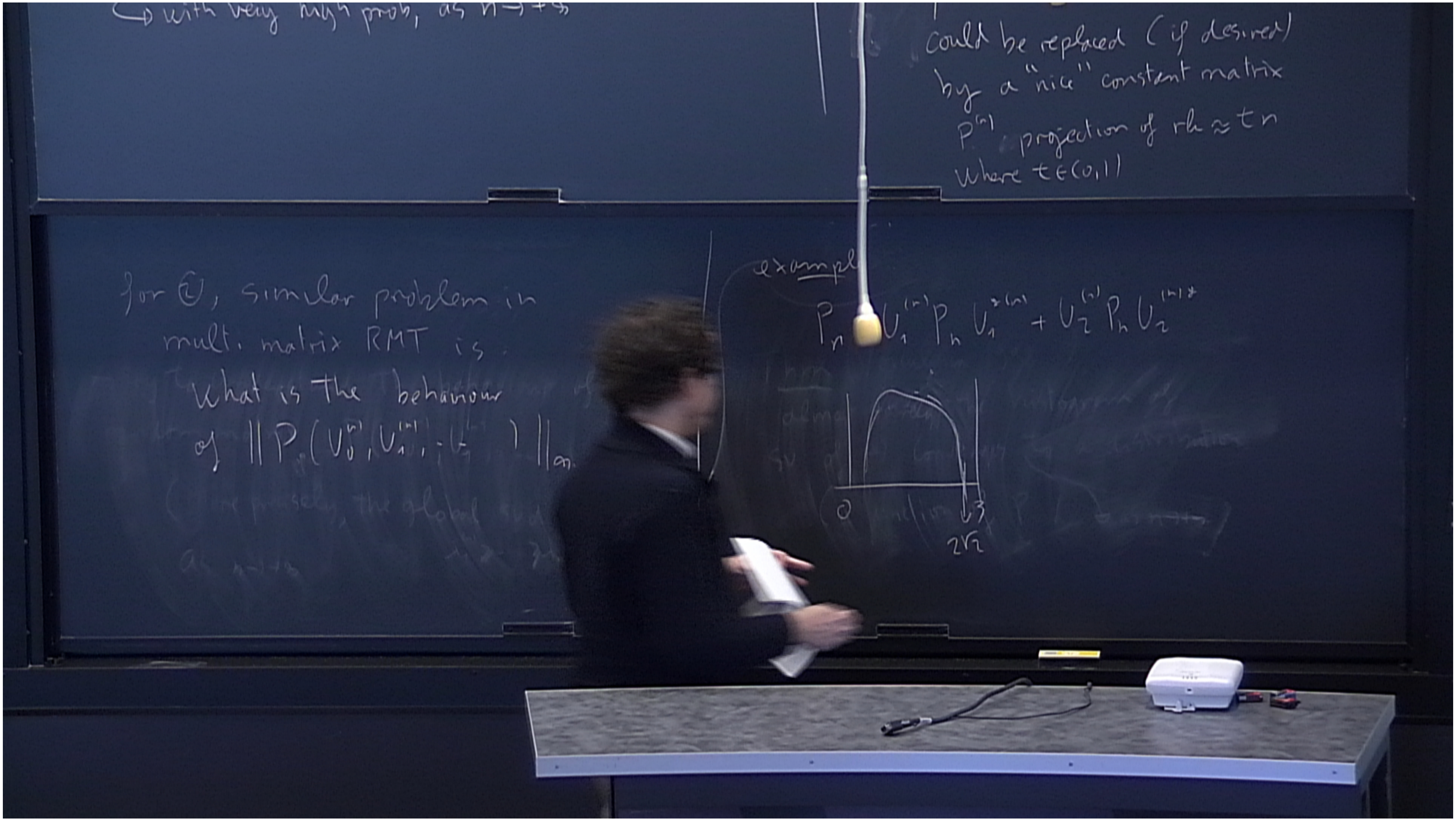
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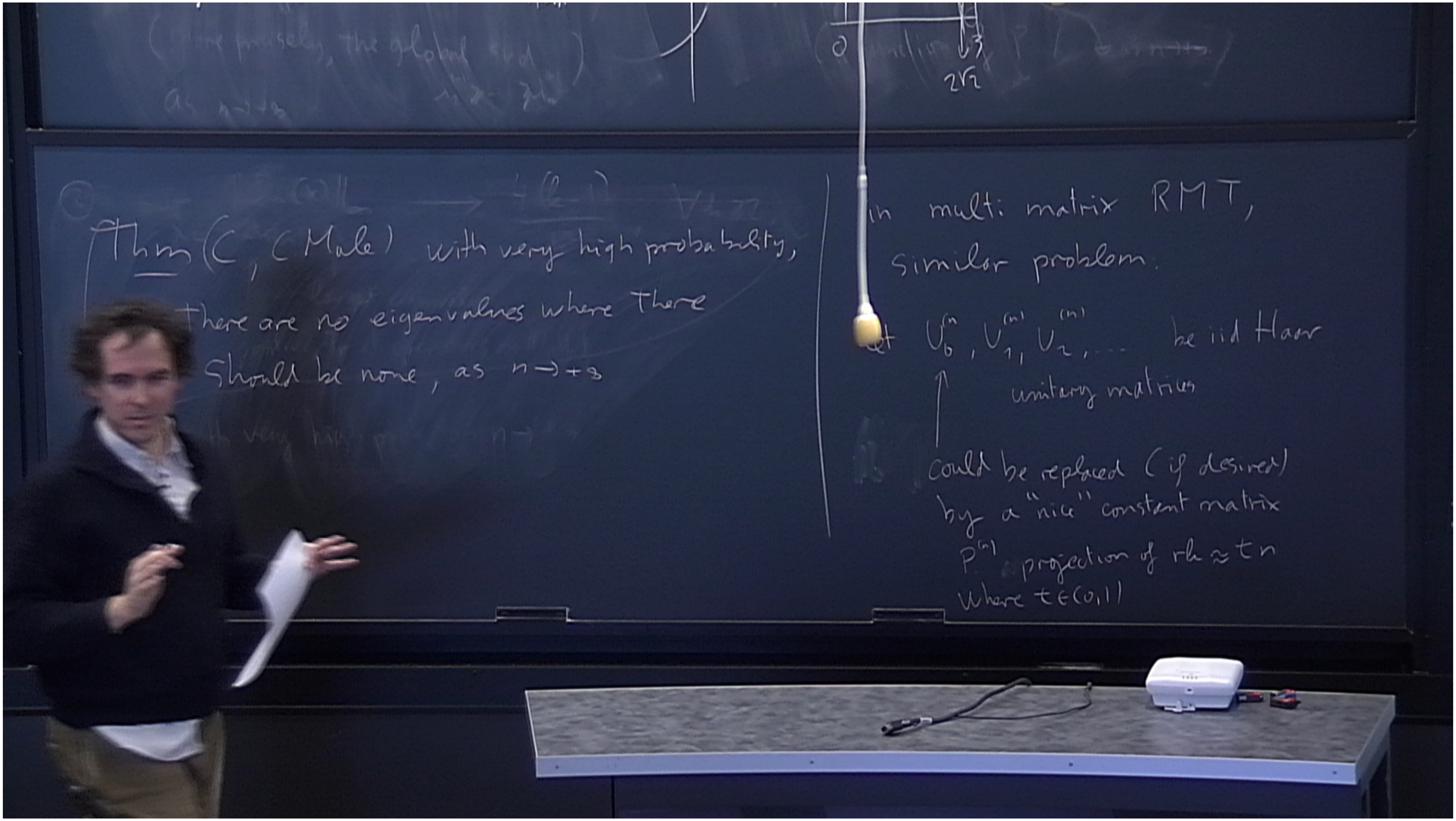
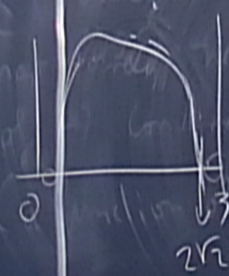
mult. matrix RMT is:
What is the behaviour
of $\| P_n(U_0^{(n)}, U_1^{(n)}, \ldots) \|_\infty$
example
$P_n U_1^{(n)} P_n U_1^{*(n)} + U_2^{(n)} P_n U_2^{(n)*}$
0
3
$2\sqrt{2}$


Thm (C, C Male) with very high probability,
There are no eigenvalues where there
should be none, as n → +∞
in multi matrix RMT,
similar problem:
Let U_0^(n), U_1^(n), U_2^(n), ... be iid Haar
unitary matrices
could be replaced (if desired)
by a "nice" constant matrix
P^(n) projection of rk ≈ tn
where t ∈ (0,1)

multi matrix RMT is:

What is the behaviour

of $\| P(U_0^{(n)}, U_1^{(n)}, \dots) \|_{op}$



Thm (C, C Male) with very high probability,

There are no eigenvalues where there should be none, as $n \to +\infty$

(true $\forall P$)

in multi matrix RMT,

similar problem:

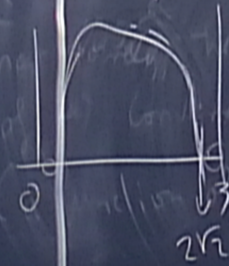
Let $U_0^{(n)}, U_1^{(n)}, U_2^{(n)}, \dots$ be iid Haar unitary matrices

could be replaced (if desired) by a "nice" constant matrix $P^{(n)}$ projection of rk $\approx tn$ where $t \in (0,1)$

multi matrix RMT is:

What is the behaviour

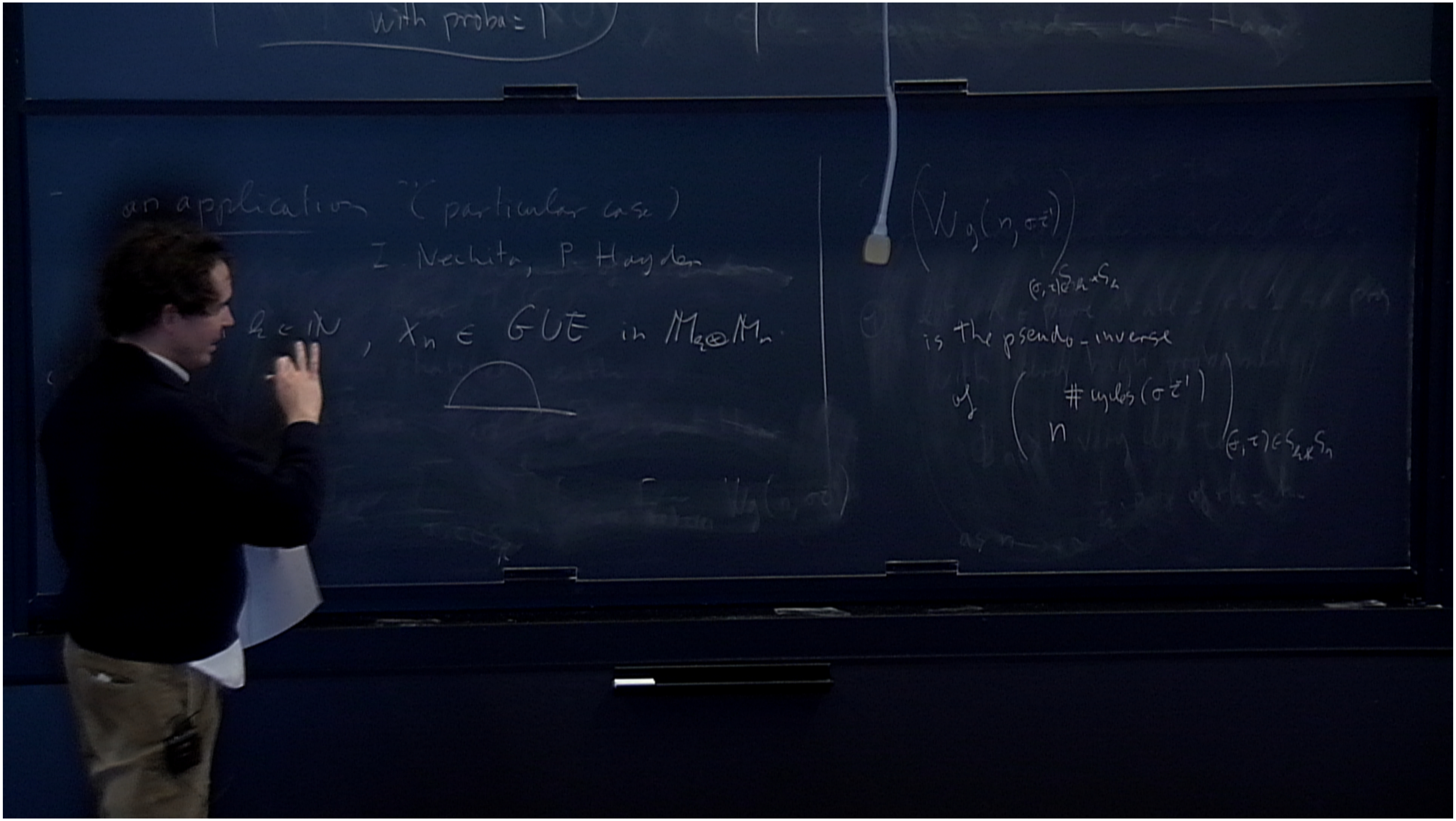
of $\| P(U_0^{(n)}, U_1^{(n)}, \dots) \|_{op}$



Thm (C., C. Male) with very high probability,

There are no eigenvalues where there should be none, as $n \to +\infty$

(true $\forall P$)



with proba = 1
an application (particular case)
I. Nechita, P. Hayden
k ∈ ℕ, X_n ∈ GUE in M_k ⊗ M_n
Wg(n, στ⁻¹)
is the pseudo-inverse
of
n # cycles(στ⁻¹)

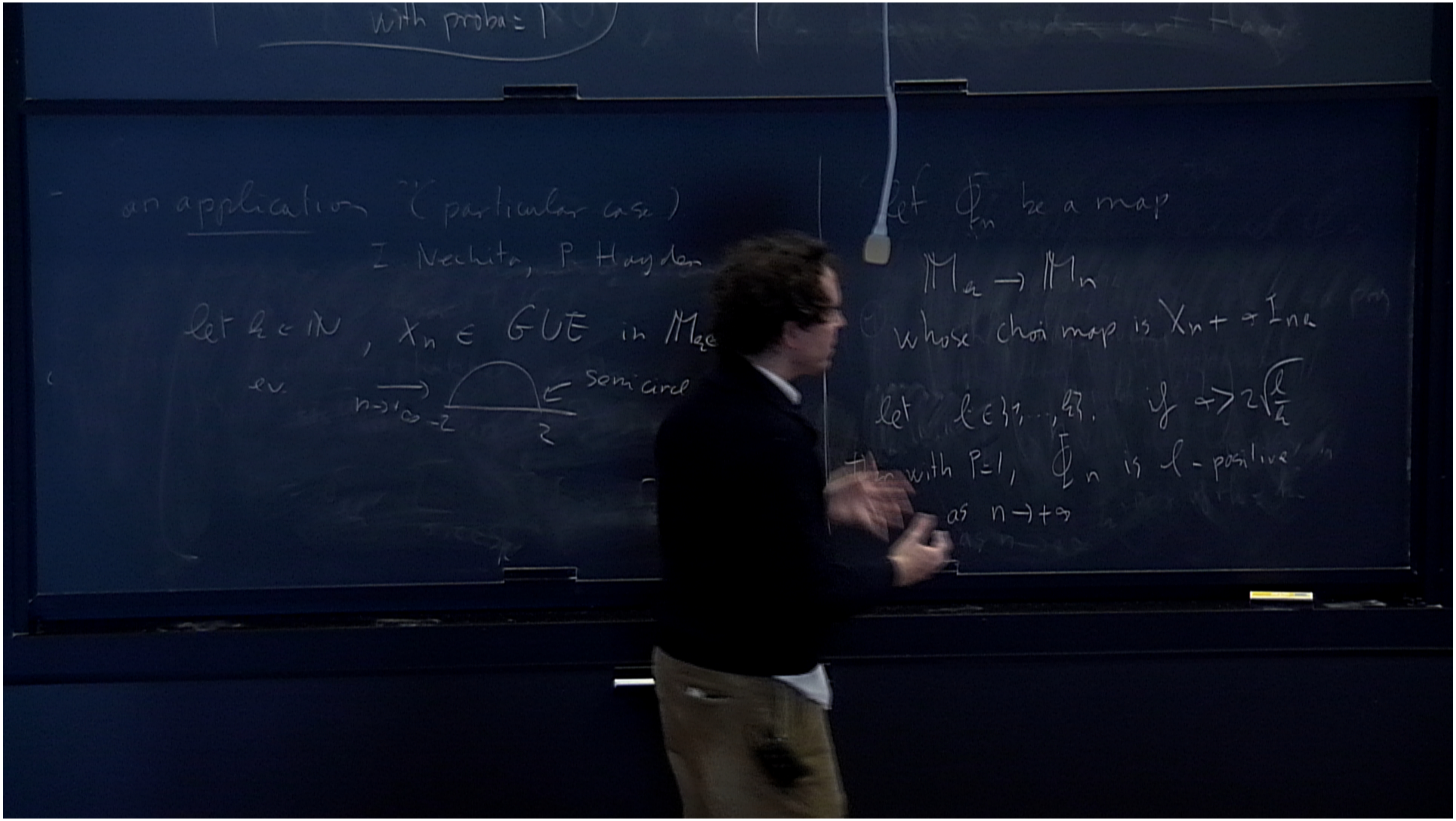
with proba = 1

an application (particular case)

I. Nechita, P. Hayden

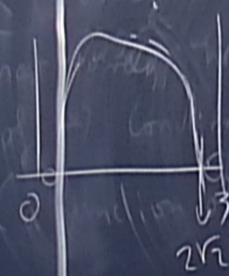
let $k \in \mathbb{N}$, $X_n \in GUE$ in $M_k \otimes M_n$.

ev. $\xrightarrow[n\to\infty]{}$ -2 2 $\leftarrow$ semicircle


with proba = 1
an application (particular case)
I. Nechita, P. Hayden
let k ∈ ℕ, X_n ∈ GUE in M_kn
ev. semicirc
n→+∞ -2 2
let Φ_n be a map
M_k → M_n
whose choi map is X_n + α I_nk
let l ∈ {1,...,k}. if α > 2√(l/k)
with P=1, Φ_n is l-positive
as n→+∞

multi matrix RMT is:

What is the behaviour of $\| P(U_0^{(n)}, U_1^{(n)}, \dots) \|_{op}$



Thm (C, C Male) with very high probability,

There are no eigenvalues where there should be none, as $n \to +\infty$

(true $\forall P$)