Title: A monodromy defect in the 3d Ising model.

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Abstract: <span>A conformal defect is
a d-dimensional geometrical object that breaks the SO(D+1,1) symmetry, of a D-dimensional conformal field theory, down to those transformations that leave the defect invariant i.e. SO(D-d) X SO(d+1,1).<br>
We studied the 3D critical Ising model in presence of a special kind of these defects, a monodromy line defect.<br>
In particular we computed, using Montecarlo simulations, the anomalous dimensions of the lowest dimensional operators living at the defect, as well as correlation functions of operators in the bulk with operators at the defect.<br>
We found a good agreement between our results and the expectations of conformal field theory.<br>
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# Motivations

- Conformal field theories are a really powerful tool in quantum field theory.
- They describe a generic quantum field theory at the infrared fixed point of the RG.
- Recently there has been a lot of interest due to some advances in the bootstrap program.

# Motivations

- It makes sense to consider a disturbance of a theory that preserves a big subgroup of the conformal symmetry.
- This is related critical phenomena in presence of an impurity.
- Conformal defects could be used to probe some properties of the conformal field theories.
- The Ising model at criticality is the basic example of a 3d conformal field theory.

- Our main concern is the spectrum of the operators of the "defect conformal field theory".
- This should be similar to a standard conformal field theory with $SO(D-d)$ as a global symmetry group.
- One big difference is that on the "defect conformal field theory" there is no stress tensor with protected anomalous dimension.

- The two point function of this operator is then

$$\langle D^i(x) D^j(0) \rangle = \frac{C_D \delta_{ij}}{|x|^{2d+2}}$$

- Thus the displacement operator has protected anomalous dimension $d + 1$.
- Also the coefficient is fixed by the Ward identity and is an intrinsic property of the defect.

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# Monodromy defect

- In this work we focused on a special class of defect, the monodromy defect.
- A monodromy defect can be defined when a theory has a flavor symmetry group $G$.
- We can insert in the theory a topological $D-1$ dimensional domain-wall, defining all the correlators to be equal to the same correlators as there wasn't the wall but with all the operators on one side of the wall transformed according to an element of the group $G$.

- If the domain wall has a boundary, all the operators charged under $G$ are multivalued around it.
- The boundary on which a domain-wall ends is a monodromy defect.
- The OPE of a bulk operator $\mathcal{O}$ will contain fractional spin operators.
- In $D = 3$ and $G = \mathbb{Z}_2$ which is the subject of this work, we will have, in cylindrical coordinates

$$\mathcal{O}(r, \phi) \sim \sum_{n,a} e^{i\phi(n+\frac{1}{2})} r^{\Delta_a - \Delta_{\mathcal{O}}} \mathfrak{o}_{n+\frac{1}{2},a}$$

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- Besides the usual bulk and defect correlator functions there exists the correlation function between a bulk operator $\mathcal{O}$ and a defect operator $\mathfrak{o}$.

- This correlator can be, at most, a function of the distance between the two operators $|x^\mu|$ and of the distance $|x^i|$ of the point $x$ from the defect line

$$\langle \mathcal{O}(x)\mathfrak{o}(0)\rangle = f(|x^\mu|, |x^i|)$$

- The scale invariance imposes

$$\langle \mathcal{O}(x)\mathfrak{o}(0)\rangle = \frac{1}{|x^\mu|^{\Delta_{\mathcal{O}}+\Delta_{\mathfrak{o}}}} f\left(\frac{|x^i|}{|x^\mu|}\right)$$

- We can also perform a special conformal transformation that preserves the defect, these are parametrized by a vector $b^\mu$ with components different from zero just in the directions parallel to the defect.

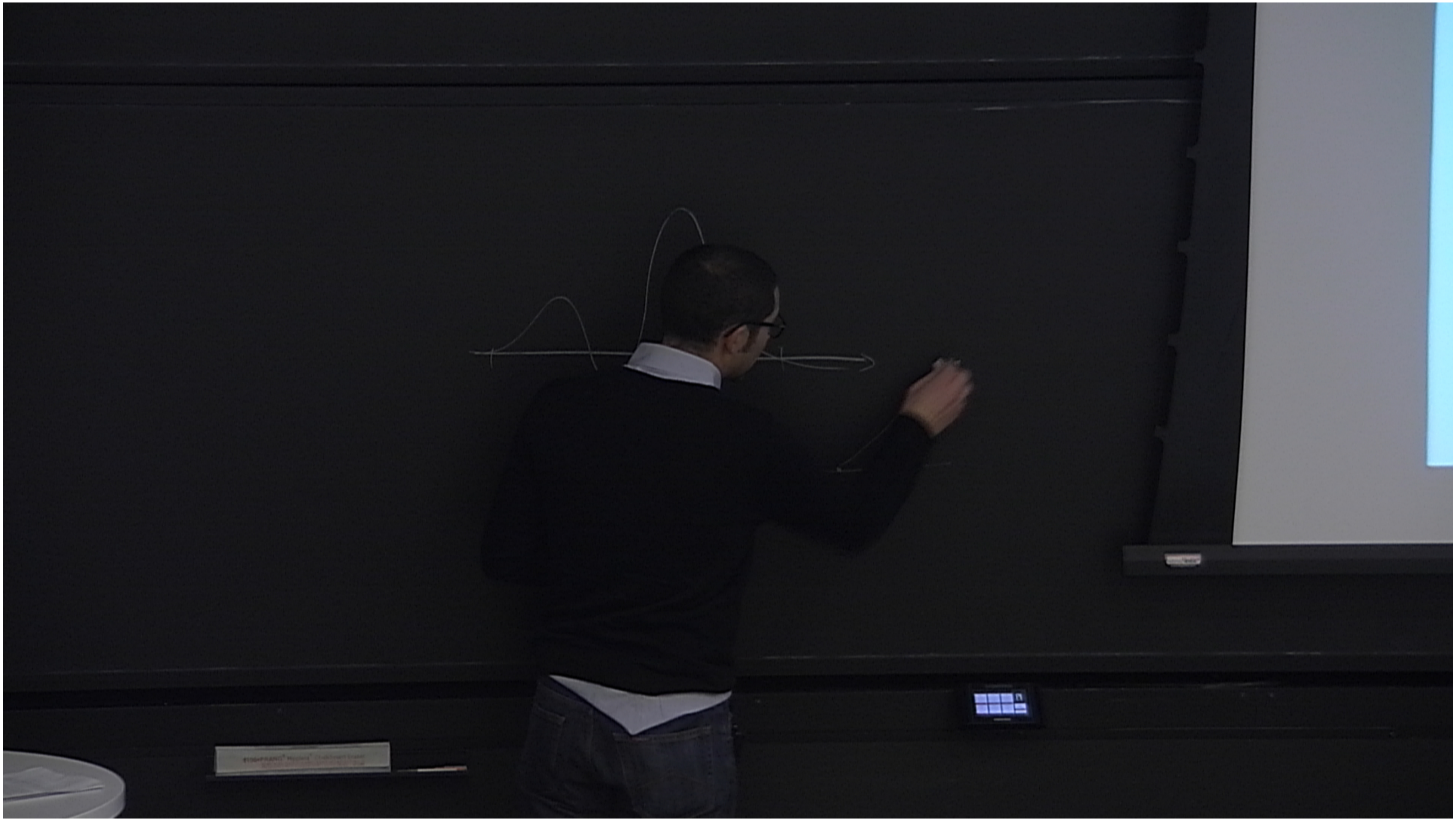
- Under such transformations

$$|x^\mu| \to \frac{|x^\mu|}{\sqrt{\Omega}}, \quad |x^i| \to \frac{|x^i|}{\Omega}$$

with $\Omega = \sqrt{1 + 2b\cdot x + b^2|x^\mu|^2}$

- All toghether we obtain the functional form of the mixed correlator

$$\langle \mathcal{O}(x)\mathfrak{o}(0)\rangle = c_{\mathfrak{o}}^{\mathcal{O}} |x^\mu|^{-2\Delta_{\mathfrak{o}}} |x^i|^{\Delta_{\mathfrak{o}} - \Delta_{\mathcal{O}}}$$



# The model

- The Ising model is defined by the standard Hamiltonian with $J_{\langle xy \rangle} = 1$.

$$H(\{\sigma_x\}) = -\beta \sum_{\langle xy \rangle} J_{\langle xy \rangle} s_x s_y$$

- In the Ising model a realization of a $\mathbb{Z}_2$ domain-wall is particularly simple: we can flip the sign of the interaction on the bonds that cross a surface $S$.

- If the surface is topologically trivial and has no boundary $\partial S = \emptyset$ it is always possible to deform the surface in any other surface $S^{'}$, $\partial S^{'} = \emptyset$, flipping all the spins in the region between $S$ and $S^{'}$.

# A monodromy defect in the Ising model

- In particular it is always possible to recover the standard Hamiltonian.
- If $S$ has a boundary $\partial S = L$, it is not possible to recover the standard Hamiltonian by flipping the sign of spins.
- The location of $L$ is meaningful, in the present $3d$ case the boundary is a line crossing a set of plaquettes of the lattice.
- Again any surface $S$ with the same boundary defines the same defect.

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# Duality

- The Ising model, on the lattice, is related to a $\mathbb{Z}_2$ gauge model via an exact duality transformation

$$Z_{gauge}(\beta_g) \propto Z_{spin}(\beta), \quad \beta = -\frac{1}{2}\log\tanh\beta_g$$

- The domain-wall is dual to a Polyakov loop correlator

$$\langle P(0)P(R)\rangle_{gauge} = \frac{Z^D_{spin}(R)}{Z^0_{spin}}$$

- This is a very efficient way to compute numerically the static interquark potential in the gauge model.

# Anomalous dimensions

- What can we say about the anomalous dimensions from the conformal invariance?
- As we pointed out before a $\mathbb{Z}_2$ charged operator must contain in the OPE half-integer defect operators, in cylindrical coordinates

$$\mathcal{O}(r,\phi) \sim \sum_{n,a} e^{i\phi(n+\frac{1}{2})} r^{\Delta_a - \Delta_{\mathcal{O}}} \mathfrak{o}_{n+\frac{1}{2},a}$$

- If the theory were a free scalar field theory we could apply the free equation of motion at both side and look at the coefficients of the primary fields

$$(n+\frac{1}{2})^2 = (\Delta_a - \Delta_{\mathcal{O}})^2$$

# Montecarlo method

- The idea is to produce a chain of configurations

$$C_i \to C_{i+1} \to C_{i+2}$$

- Every configuration is obtained by a modification of the previous one and appears in the chain with probability proportional to the boltzmann weight.
- The expectation value of an observable $O$ is then a simple arithmetic mean.

$$\langle O \rangle = \frac{1}{N} \sum_{i=0}^{N} O_i$$

# The choice of the algorithm

- The most important consideration is that we mainly measure operators in the neighbouring of the defect.
- It is a waste of CPU time updating the whole lattice before every measurement.
- We needed a local algorithm to easily increase the number of updates near the defect.
- The Metropolis algorithm is the standard choice.

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# Hierarchical updates

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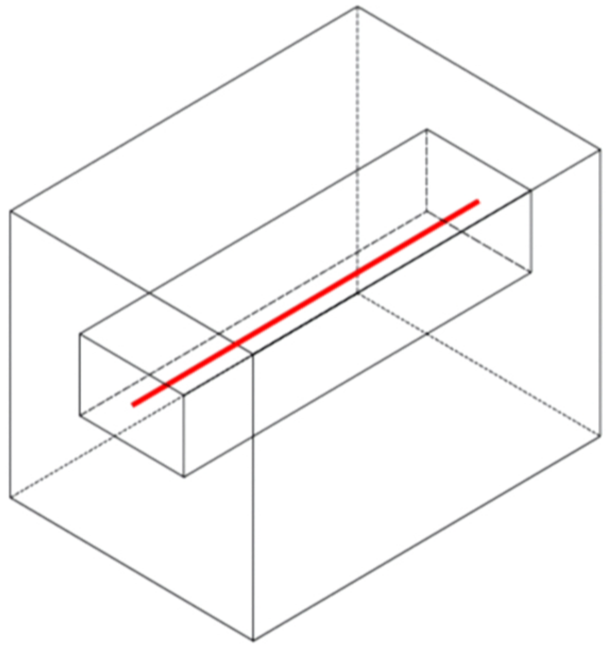

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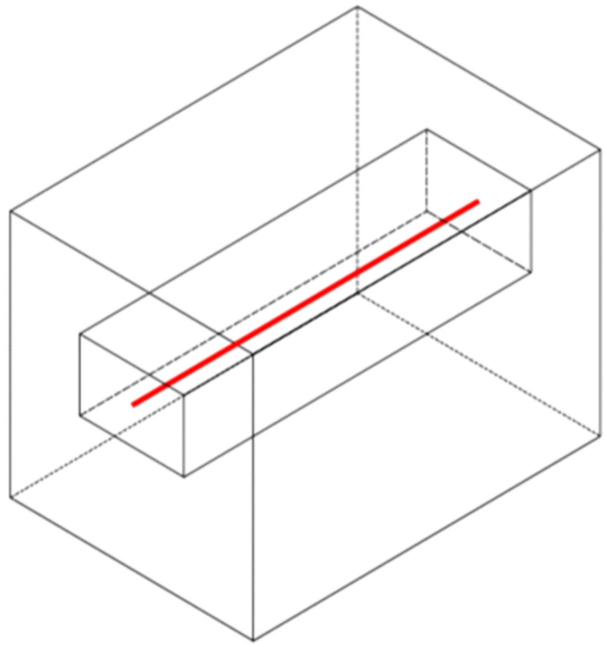


# Multispin tecnique

- Present desktop machines can process 64 bits in parallel.
- The Ising model is essentially a boolean model.
- It is quite simple to code a fully vectorized version of the Metropolis algorithm.

# Multispin tecnique in 1d Ising model

- In the 1d ising model for example if we take the exclusive-OR of two nearest neighbour spin variable $\sigma_i \oplus \sigma_{i+1}$ we obtain 1 only if the two are pointing in different directions.
- The expression $(\sigma_i \oplus \sigma_{i-1}) \vee (\sigma_i \oplus \sigma_{i+1})$ is 1 if either one or both of the neighbours of $j$ are pointing in the opposite direction.
- If we could generate a random word $w$ in which the bits are 1 with probability $\exp(-4\beta)$ then the vectorized metropolis update would be

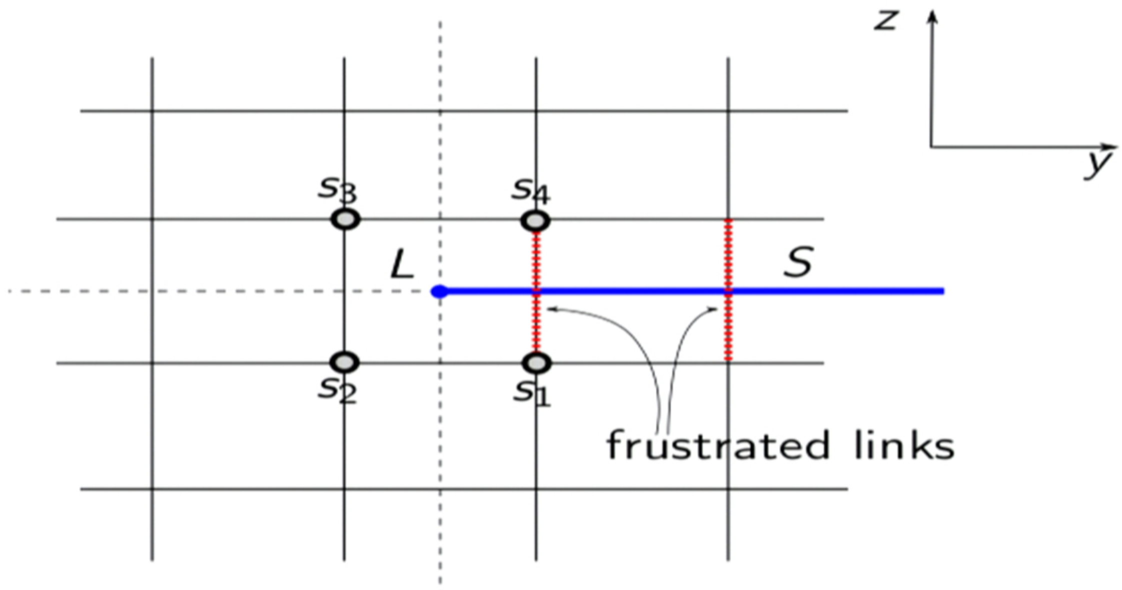
$$\sigma_j = \sigma_j \oplus \{(\sigma_i \oplus \sigma_{i-1}) \vee (\sigma_i \oplus \sigma_{i+1}) \vee w\}$$

# Rotational symmetry on the lattice

- We will realize some defect local operators on the lattice in term of spin variables sitting close to the monodromy line.
- In the continuum defect operators are defined by the scaling dimension and the spin corresponding to the $SO(2)$ rotational symmetry around the line.
- Without the defect the rotational symmetry group would be broken down, by the lattice, to the symmetry of the square $D_4$.

# Rotational symmetry on the lattice

- Consider the four spins around the defect (as depicted in figure)




- After a rotations around the defect we need to perform a "gauge" transformation to bring the $\mathbb{Z}_2$ domain wall back to the original position.

# The Dihedral group $D_8$

- $D_8$ is of order 16 and has got 7 irrepses. 4 of them are one dimensional and 3 two dimensional.
- The four dimensional representation acting on the spin is a direct sum of two bidimensional representation $H_{\frac{1}{2}}$ and $H_{\frac{3}{2}}$.
- A basis for $H_{\frac{1}{2}}$ is given by $(\psi, \psi^*)$ with

$$\psi = s_1 + \omega s_2 + \omega^2 s_3 + \omega^3 s_4$$

with $\omega = \exp\{i\frac{\pi}{4}\}$

- We have

$$a\begin{pmatrix}\psi\\ \psi^*\end{pmatrix} = \begin{pmatrix}\omega^{-1} & 0\\ 0 & \omega\end{pmatrix}\begin{pmatrix}\psi\\ \psi^*\end{pmatrix}, \quad b\begin{pmatrix}\psi\\ \psi^*\end{pmatrix} = \begin{pmatrix}0 & \omega^3\\ \omega^{-3} & 0\end{pmatrix}\begin{pmatrix}\psi\\ \psi^*\end{pmatrix}.$$

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- With our conventions, a field $\varphi$ transforming under $a$ as $a\varphi = e^{-Ji\frac{\pi}{2}}\varphi$ has spin $J$, so $\psi$ carries spin $J = \frac{1}{2}$.

- In a similar way we can give a basis for the irrep $H_{\frac{3}{2}}$

$$\psi_{\frac{3}{2}} = s_1 + \omega^3 s_2 + \omega^6 s_3 + \omega s_4$$

- It can be shown that this irrep carries spin $J = \frac{3}{2}$

- Both $H_{\frac{1}{2}}$ and $H_{\frac{3}{2}}$ are $\mathbb{Z}_2$ odd, all the other irrepses are even.

# Results

- In a first set of simulations we measured the correlators between operators built from links.
- The expected power-law behaviour is

$$\langle s(0)s(x)\rangle = a_s x^{-2\Delta_s} + b\,,\ \langle D(0)D(x)\rangle = a_D x^{-2\Delta_D}\,,\ \langle t^+(0)t^+(x)\rangle = a_t x^{-2\Delta_t}\,.$$

- We checked this behaviour in all these cases.
- The anomalous dimensions turned out to be quite large $\Delta_i \geq 2$ and after few lattice spaces the signal was too small to be measured.

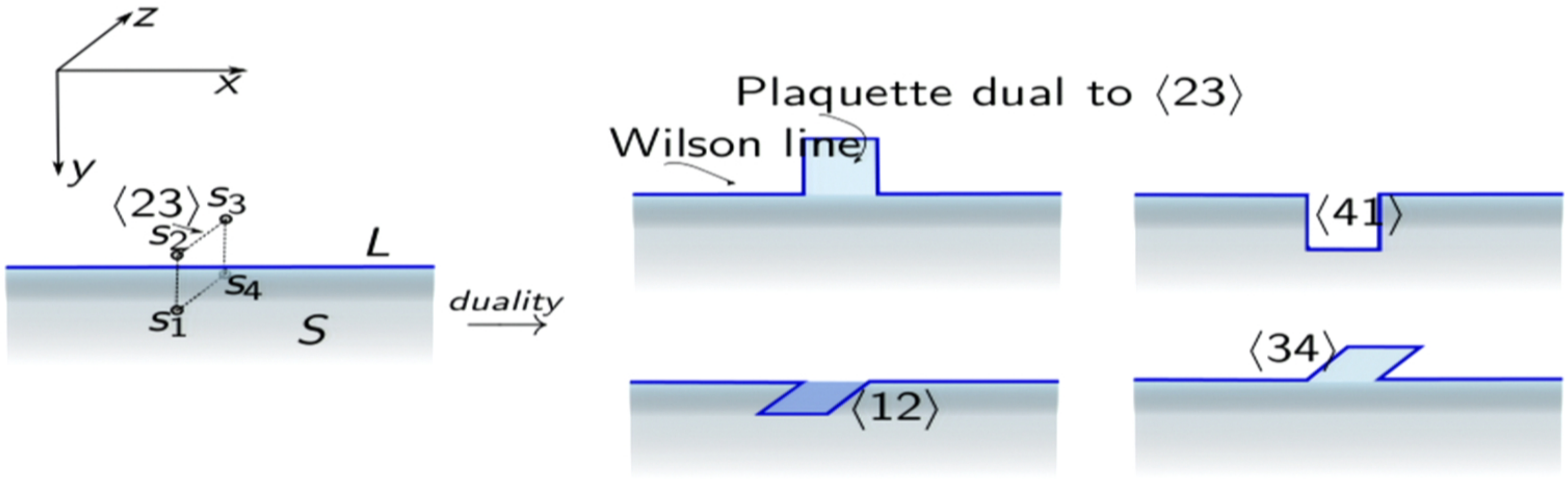



Figure: Using Kramers-Wannier duality one can transform the four link variables into staples which deform the line defect and can be used to build the displacement operator.

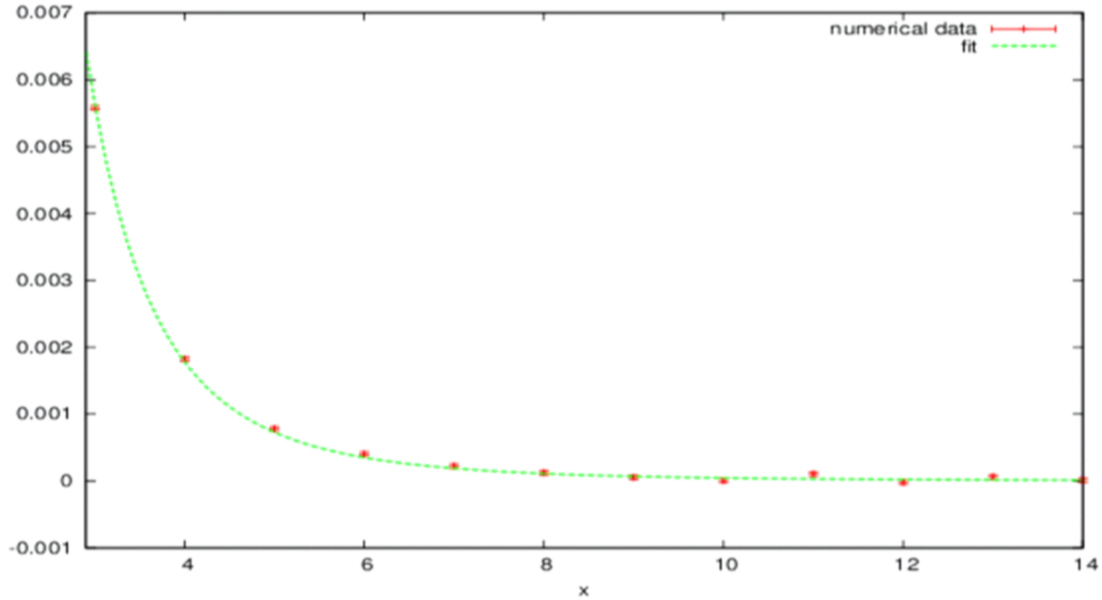
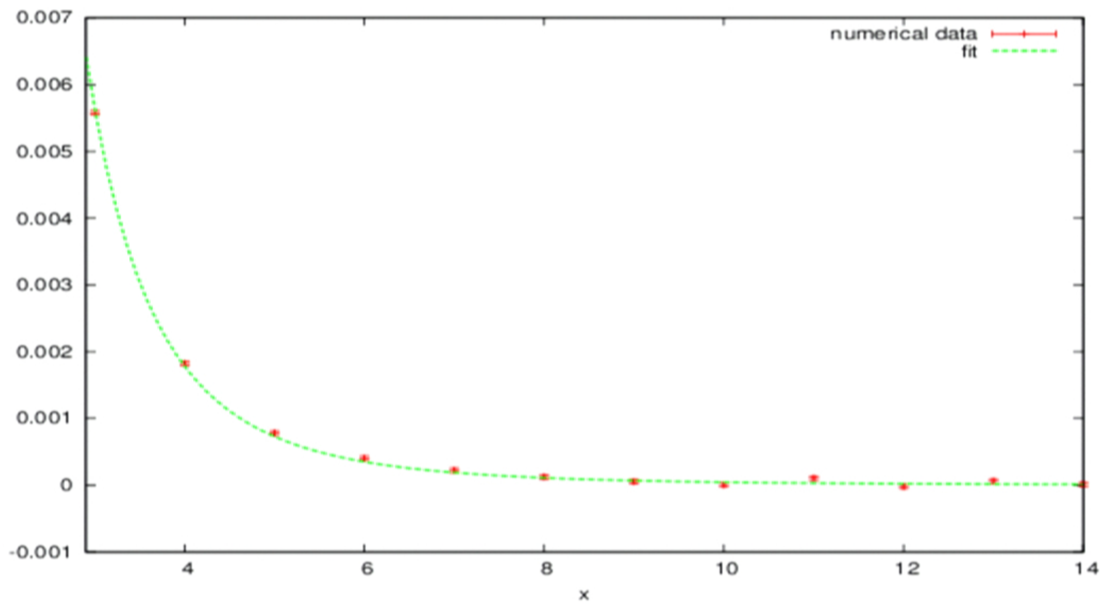



Figure: The correlation function of the displacement operator. The curve is the one-parameter fit to $\frac{a}{x^4}$.

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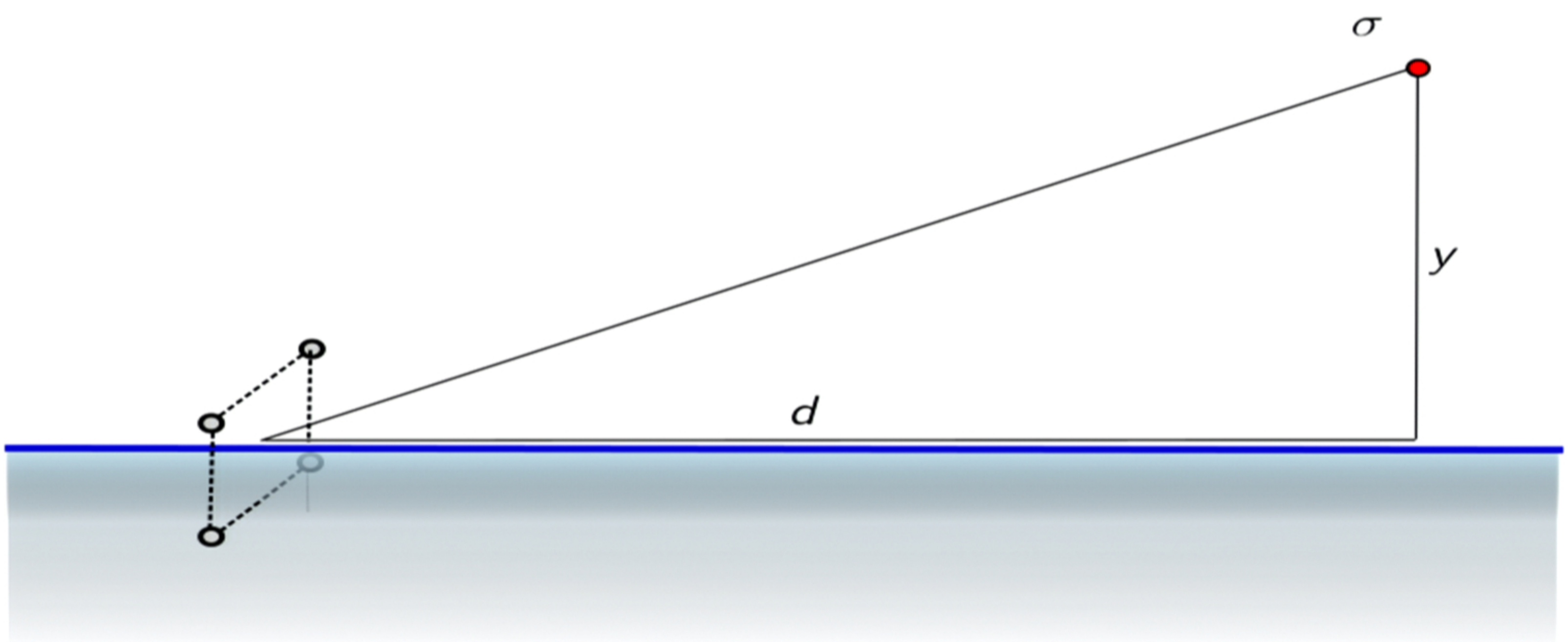
# Conformality test

- We also considered mixed correlation function of local operators in the bulk and on the defect.



# Summary

- We studied numerically the "defect conformal field theory" living on a monodromy line in the Ising model.
- We tested with very high precision the functional form of the mixed correlator fixed by conformal invariance.
- We found the anomalous dimensions of some operators of the spectrum of this theory.
- We hope this results will help to achieve a better theoretical understanding of conformal field theories.

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# Anomalous dimensions

| $\mathfrak{o}$ | D8 irrep | $\mathbb{Z}_2$ parity | $O(2)$ spin | $\Delta$ |
|---|---|---|---|---|
| $s$ | $S$ | $+$ | $0^+$ | 2.2(1) |
| $p$ | $P$ | $+$ | $0^-$ | $\geq$2.8(1) |
| $\psi$ | $H_{\frac{1}{2}}$ | $-$ | $\frac{1}{2}$ | 0.9187(7) |
| $D$ | $V$ | $+$ | 1 | 2 |
| $\psi_{\frac{3}{2}}$ | $H_{\frac{3}{2}}$ | $-$ | $\frac{3}{2}$ | 2.265(11) |
| $t^+$ | $T^+$ | $+$ | 2 | 2.61(7) $\wr$ 3.1(5) |
| $t^-$ | $T^-$ | $+$ | 2 | 2.5(1) |