Title: Exotic topological order from quantum fractal code

Date: Jan 14, 2013 04:00 PM

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Abstract: We present a family of three-dimensional local quantum codes with pairs of fractal logical operators. It has two polynomials over finite fields as input parameters which generate fractal shapes of anti-commuting logical operators, and possesses exotic topological order with quantum glassiness which is beyond descriptions of conventional topological field theory. A necessary and sufficient condition for being free from string-like logical operators is obtained under which the model works as marginally self-correcting quantum memory. We also provide theoretical tools to analyze physical and coding properties of translation symmetric stabilizer Hamiltonians.


Exotic topological order from quantum fractal code
Beni Yoshida (Caltech)


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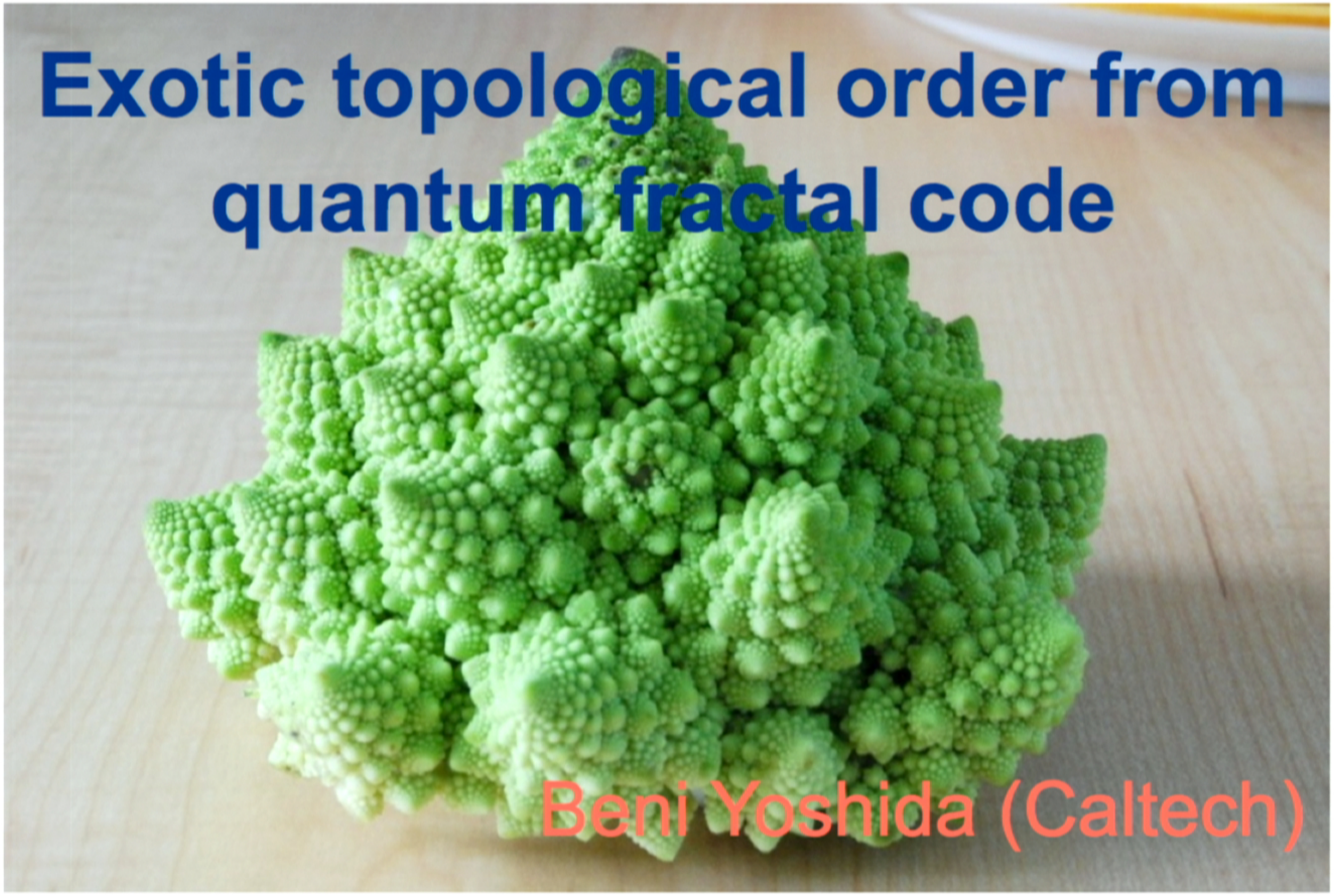
Exotic topological order from
quantum fractal code
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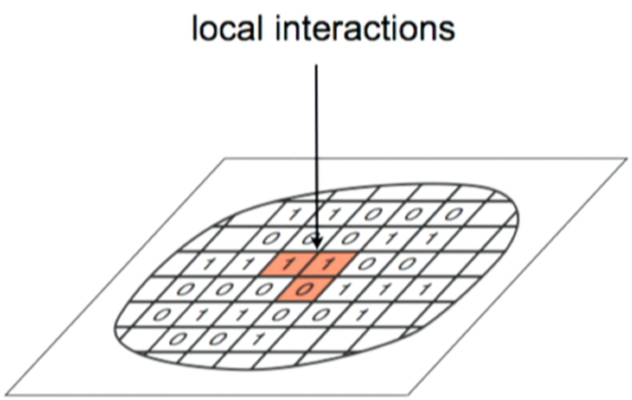
Question:

- *What is the limit on the amount of information that can be reliably stored in a finite physical system?*

Local Code bound

- Encode information into ground states of a gapped local Hamiltonian on a D-dim lattice



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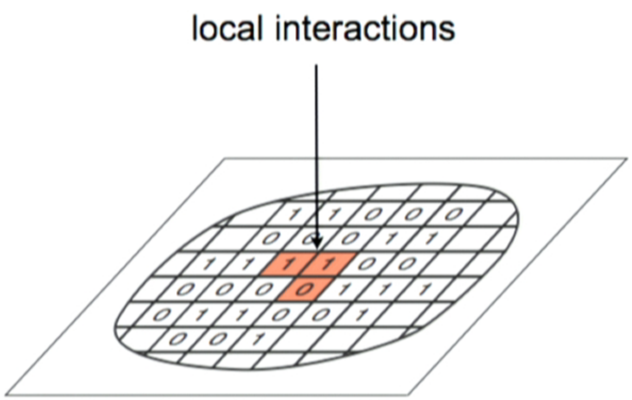


Local Code Bound Bravyi, Terhal and Poulin (2009)

$$kd^{1/D} \leq O(n)$$

k : number of logical bits — Amount

d : code distance — Reliability

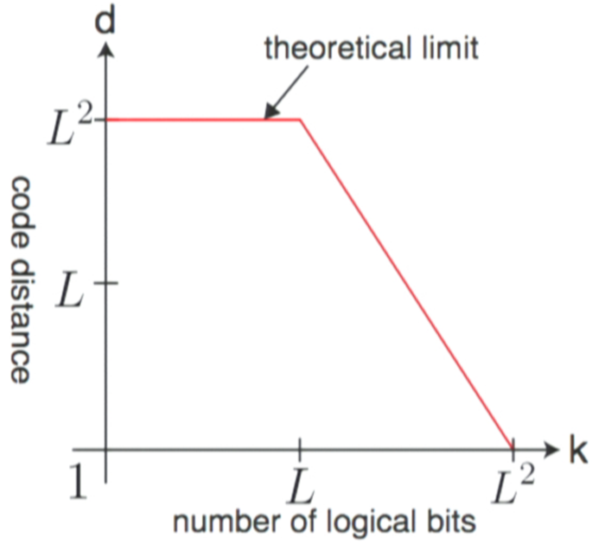
n : total number of spins

Saturation for discrete systems ?

- Previously found systems are far below the bound ...

Bound for D=2

$$kd^{1/2} \leq O(n) \qquad n = L^2$$



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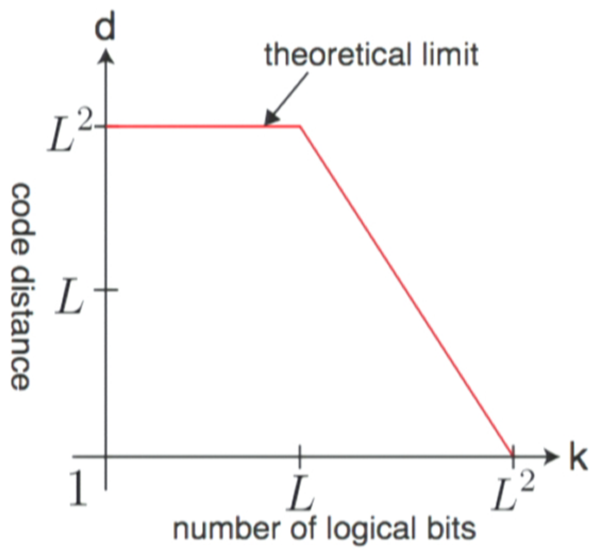

Ex: Repetition code

0 →

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

k = 1

d = n

1 →

1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
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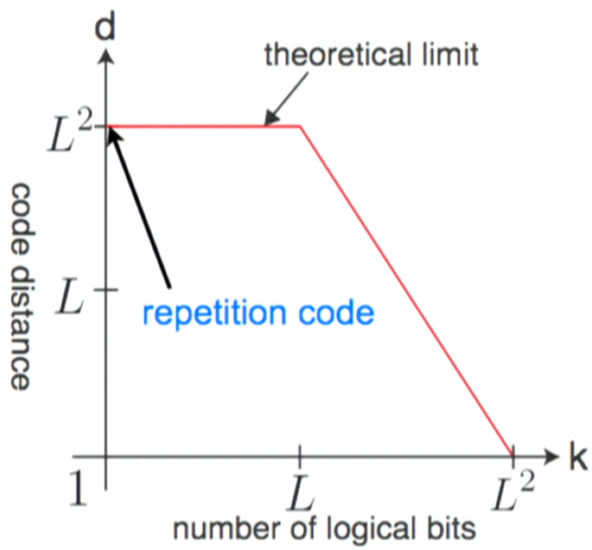

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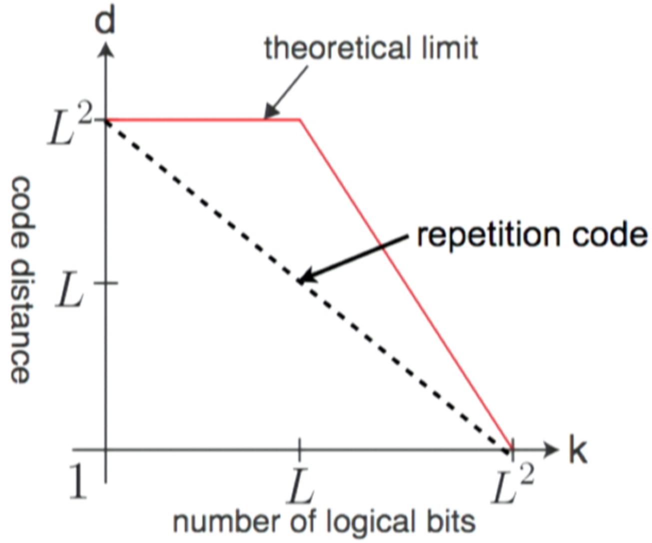
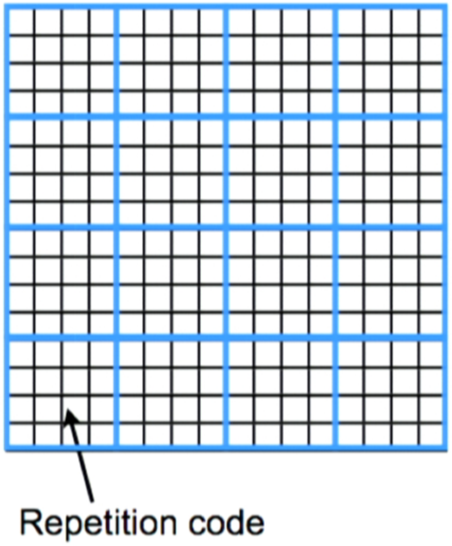


Ex: Copies of repetition codes



Multiple ground states

- Physical realization ? (Newman, Moore 1999)

1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
1	1	1	1	0	0	0	0
1	0	0	0	1	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0
1	1	1	1	1	1	1	1

⋮ ⋱

0	1	0	0	0	0	0	0
0	1	1	0	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	1	0	0	0
0	1	0	0	0	1	0	0
0	1	1	0	0	1	1	0
0	1	0	1	0	1	0	1
0	1	1	1	1	1	1	1

⋮ ⋱

. . .

Strong breaking of translation symmetries

Multiple ground states

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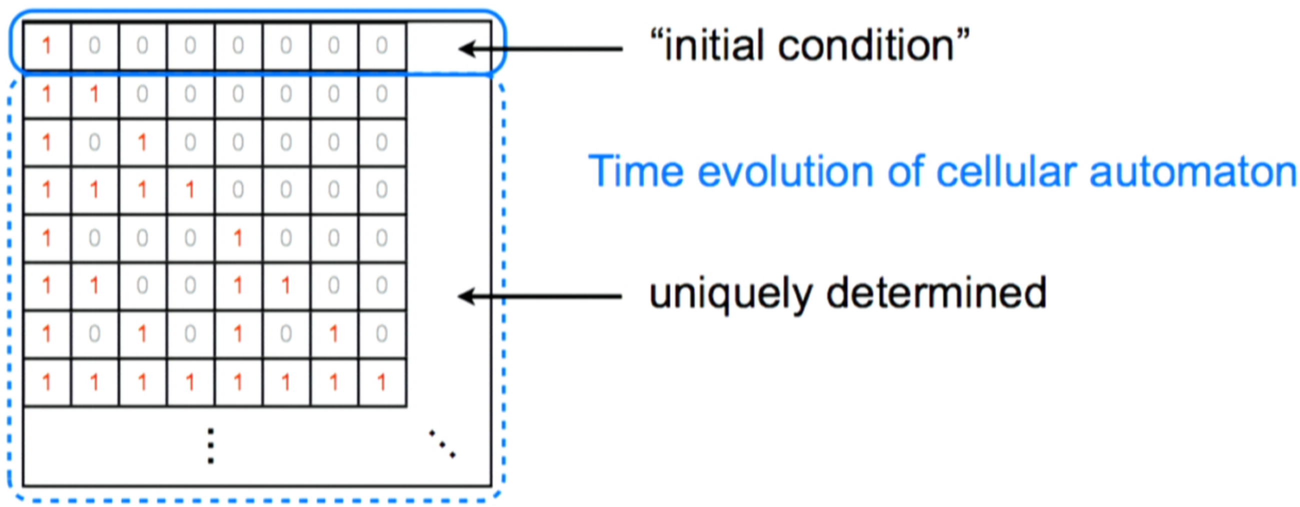
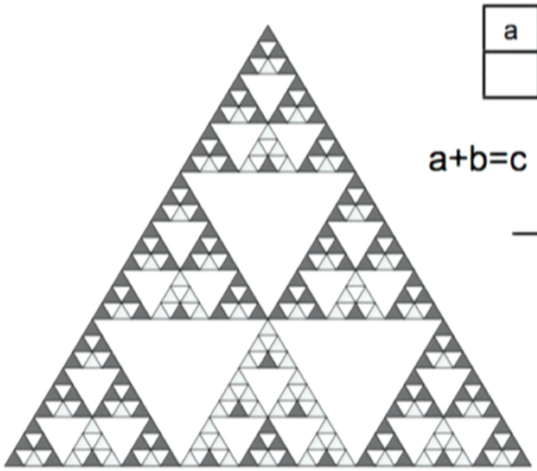


Strong breaking of translation symmetries

The Sierpinski triangle (generalized)

- Physical Realization ?



a	b
	c

$a+b=c \pmod 3$

1								
1	1							
1	2	1						
1			1					
1	1		1	1				
1	2	1	1	2	1			
1			2			1		
1	1		2	2		1	1	
1	2	1	2	1	2	1	2	1

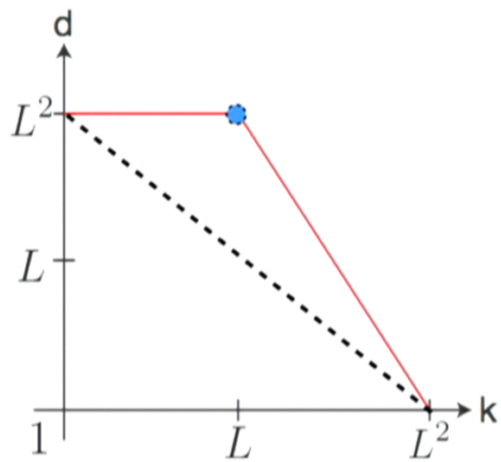
Physically realizable with 3-dimensional spins !

Asymptotic saturation (D=2)

- Sierpinski triangle with p-dim spins

Fractal dimension

$$\frac{\log(\frac{p(p+1)}{2})}{\log p} \to 2 \qquad \text{for} \quad p \to \infty.$$

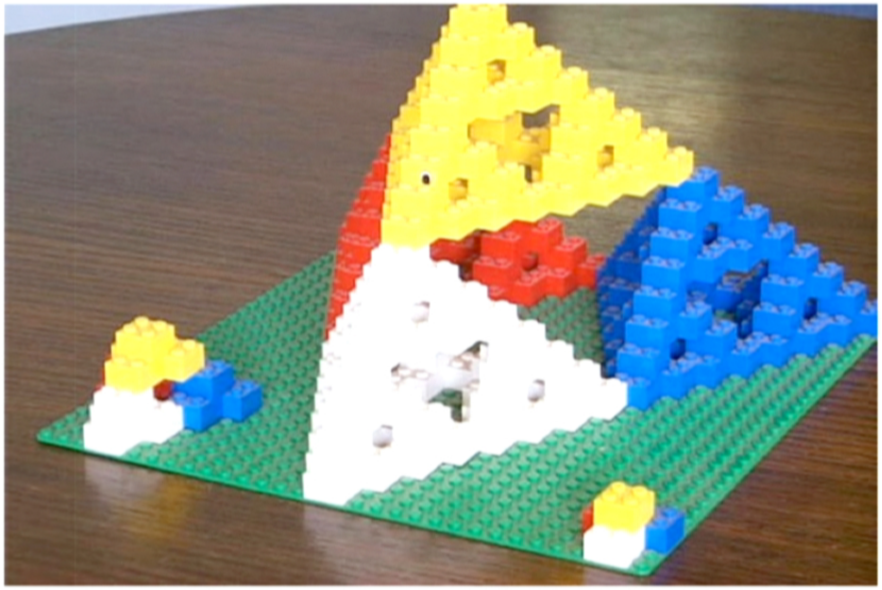


$$k \sim O(L), \qquad d \sim O(L^{2-\epsilon})$$

Asymptotically saturate the bound !

Asymptotic saturation (D>2)

- Higher-dimensions ?



Fractal dimension

$$\frac{\log\left(\frac{p(p+1)\cdots(p+D-1)}{D!}\right)}{\log(p)} \xrightarrow[p\to\infty]{} D$$

Fractal codes saturate the bound for D > 2 too !

$$k \sim O(L^{D-1}), \qquad d \sim O(L^{D-\epsilon}),$$

Area-like   Volume-like

Quantum Fractal Code ?

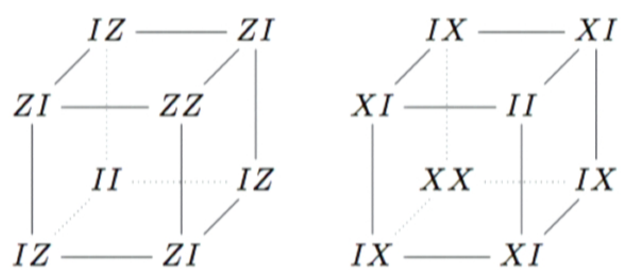
<u>Classical bound</u>

$$kd^{1/D} \leq O(n) \longleftarrow$$ Classical fractal code

<u>Quantum bound</u>

$$kd^{\frac{1}{D-1}} \leq O(n) \longleftarrow$$ Quantum fractal code ?

- The Cubic code has pairs of fractal logical operators. (Haah 2011)



general construction ?

Plan of Part 2

- Translation symmetric stabilizer Hamiltonians
  - via polynomials over finite fields (a new theoretical framework)
- Quantum fractal code
  - fractal logical operators
- Properties of quantum fractal code
  - ground states (exotic topo order, entanglement entropy)
  - quasi-particle excitations
  - code distances (lower bounds)

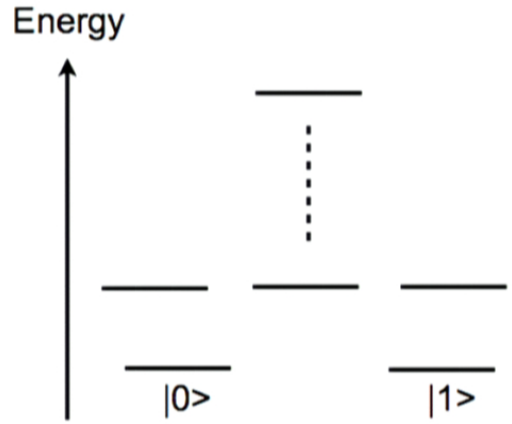
Stabilizer Hamiltonians (Gottesman 1996)

- Encode qubits into degenerate ground states with mass gap

$$H = -\sum_j S_j, \qquad [S_j, S_{j'}] = 0$$

$$S_j|\psi\rangle = |\psi\rangle$$

Pauli Operators



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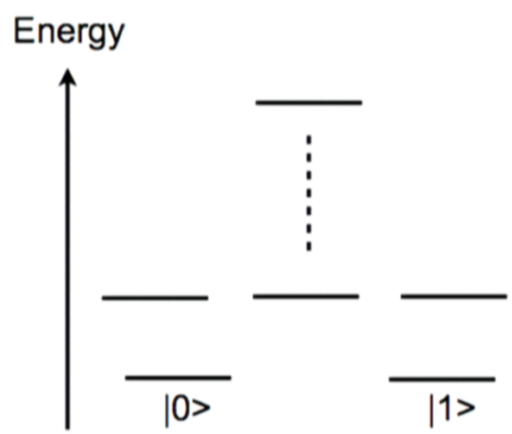

Pauli operators by polynomials over finite fields

Pauli operators by polynomials (1 dim)

$$f = \sum_{j=-\infty}^{\infty} c_j x^j, \qquad c_j = 0, 1.$$

$$Z(f) = \prod_{j=-\infty}^{\infty} Z_j^{c_j}, \qquad X(f) = \prod_{j=-\infty}^{\infty} X_j^{c_j}$$

(eg) $f = 1 + x + x^2$

$$Z(f) = Z_0 Z_1 Z_2$$

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Translations of Pauli operators

$$f = 1 + x + x^2 \quad \rightarrow \quad \boxed{x} f = x + x^2 + x^3$$

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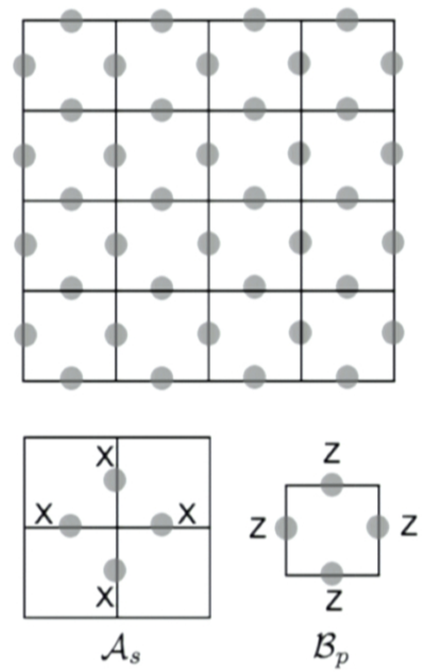
The Toric code via polynomials

The Toric code (qubits on edges)

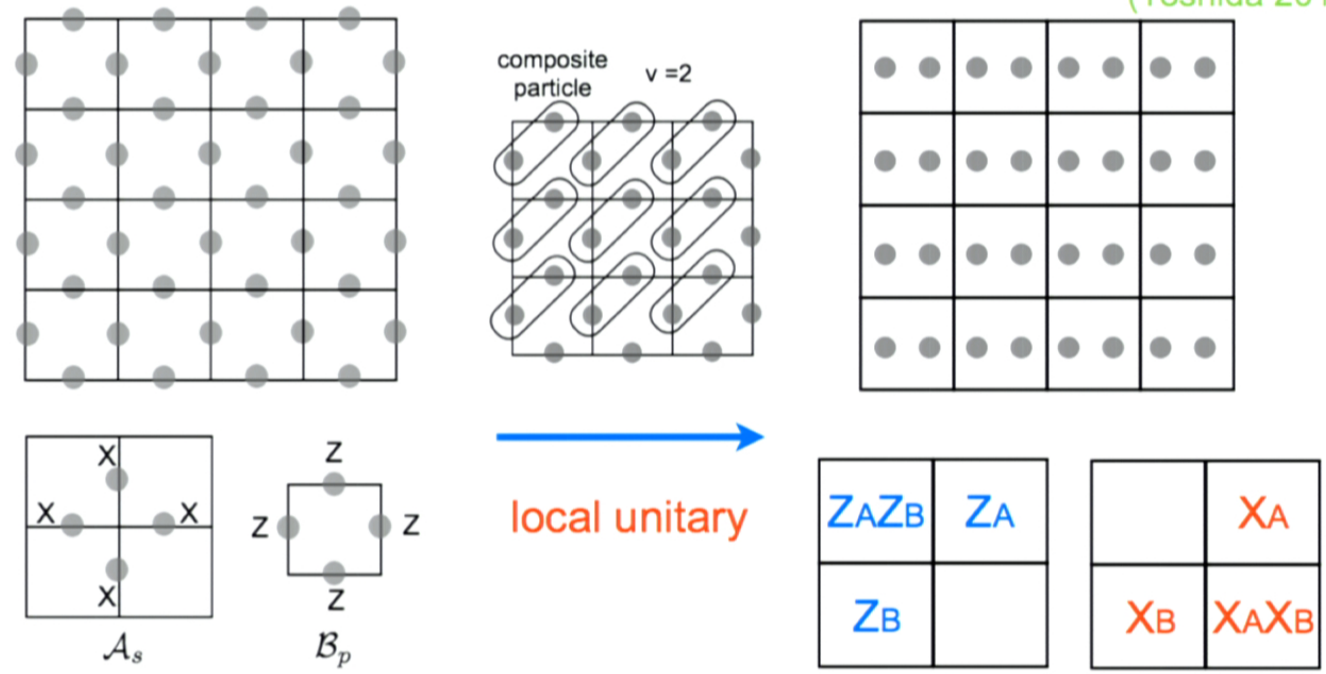


The Toric code via polynomials

The Toric code (qubits on edges)

The Toric code (square lattice)

(Yoshida 2010)



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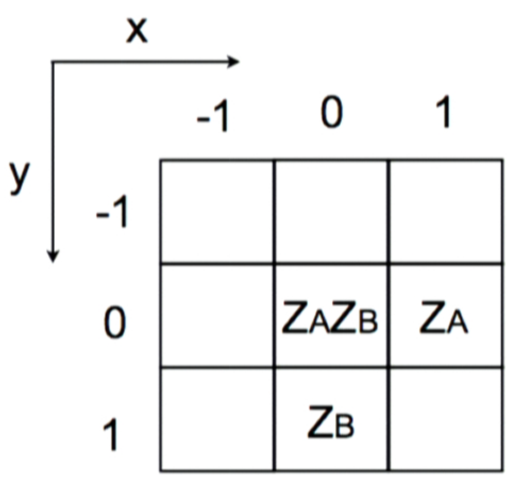

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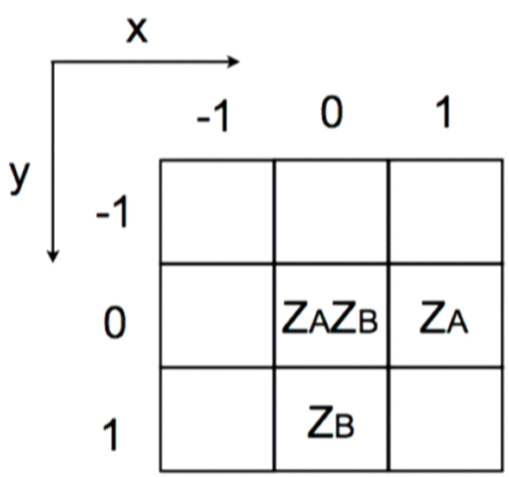

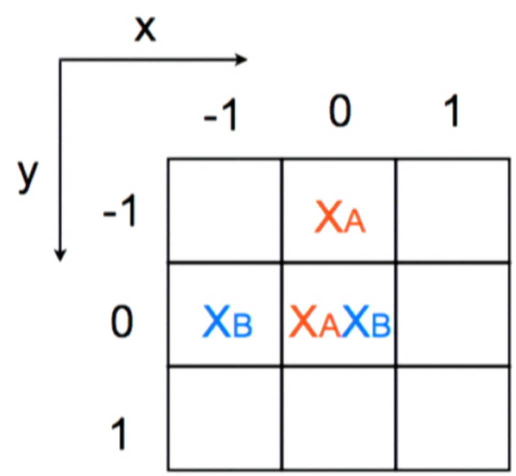

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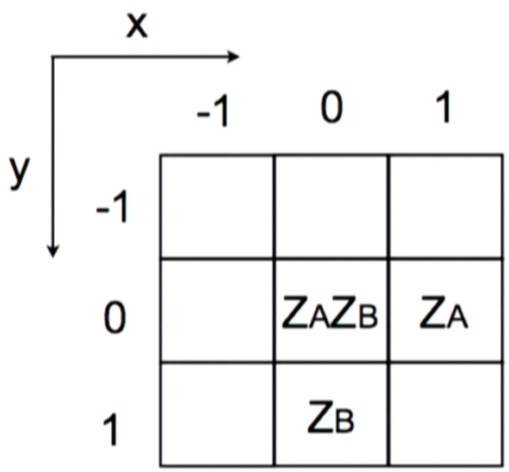

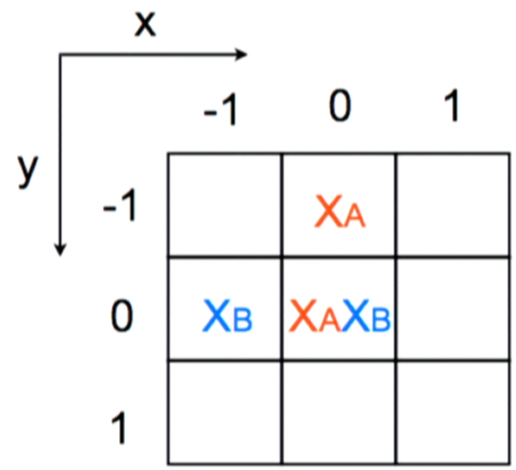

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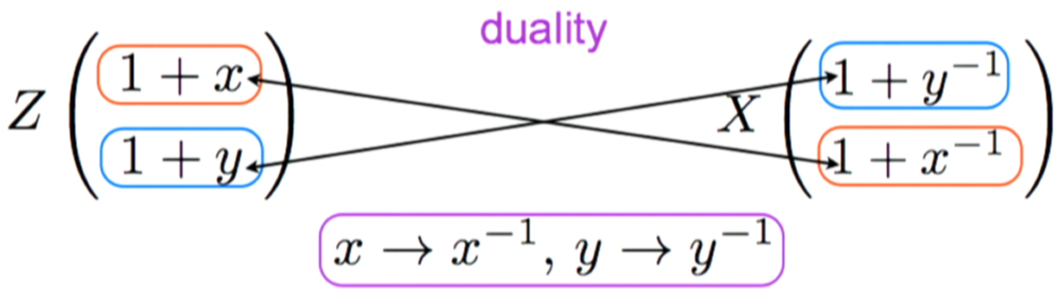
Theory for model with duality



Theory for model with duality

- This duality is required for commutation of stabilizer generators

duality

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$$x \to x^{-1},\ y \to y^{-1}$$

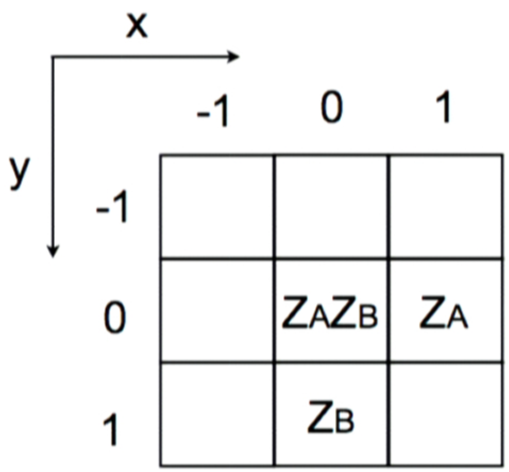
- Model with duality

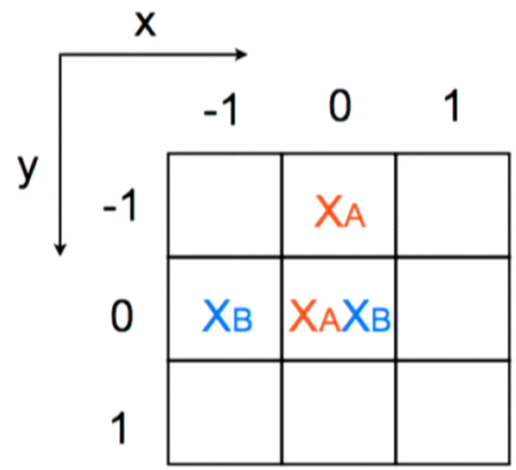
$$Z\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad X\begin{pmatrix} \bar{\beta} \\ \bar{\alpha} \end{pmatrix}$$

"theory" via algebra on polynomials over finite fields

See a forthcoming paper for discussion on codes with duality.

The Toric code via polynomials





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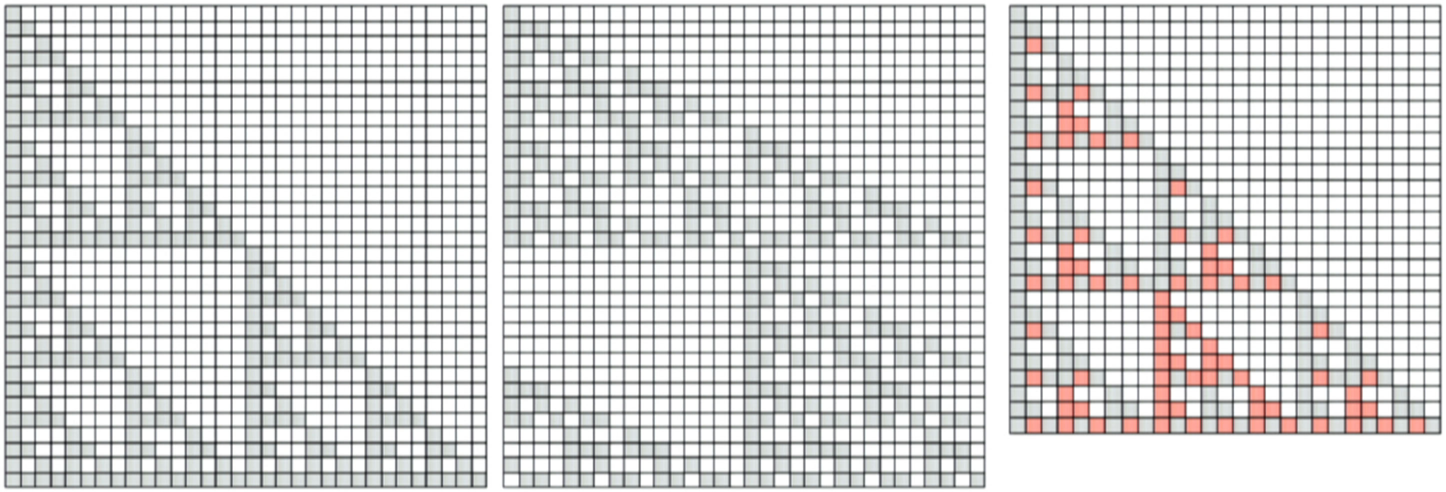
Fractal geometries via polynomials



Fractal geometries via polynomials

- $f = 1 + x$ over $\mathbb{F}_2$

powers of "f"

$f^0 = 1$

$f^1 = 1 + x$

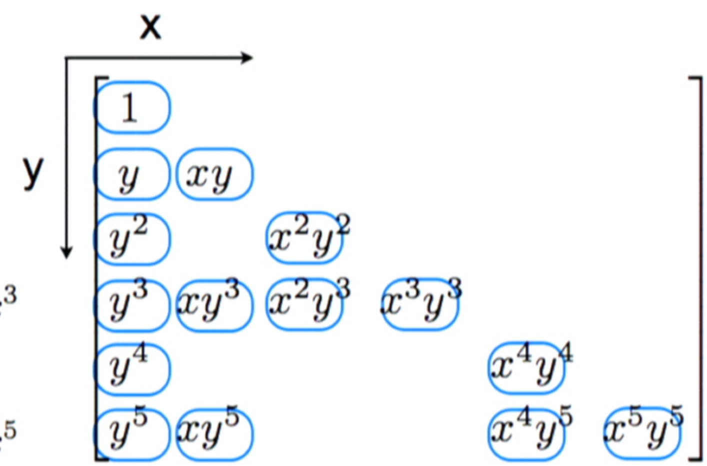
$f^2 = 1 + x^2$

$f^3 = 1 + x + x^2 + x^3$

$f^4 = 1 + x^4$

$f^5 = 1 + x + x^4 + x^5$

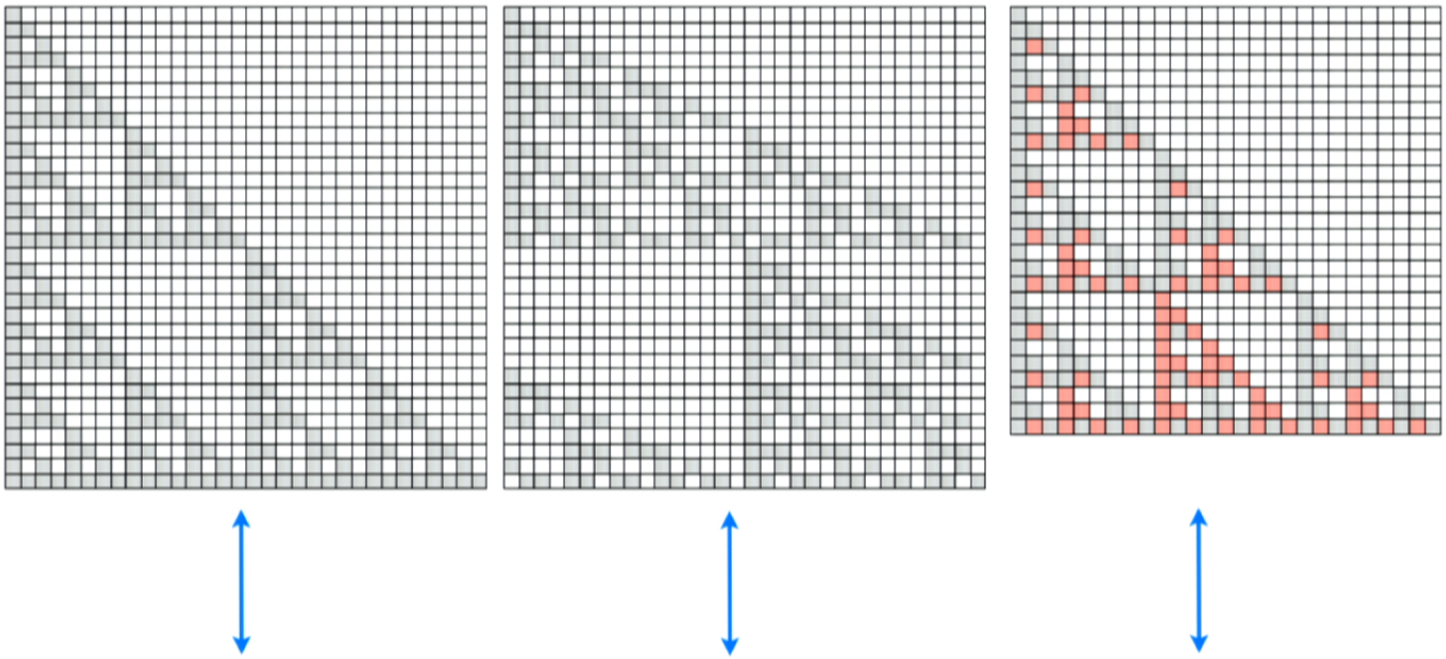
Sierpinski triangle



$$\mathbf{f}(x, y) = 1 + fy + f^2y^2 + f^3y^3 + \cdots$$

Quantum Fractal Code (3 dim)

Fractal geometries via polynomials



$f = 1 + x$ $\quad \mathbb{F}_2$

$f = 1 + x + x^2$ $\quad \mathbb{F}_2$

$f = 1 + x$ $\quad \mathbb{F}_3$

Quantum Fractal Code (3 dim)

- Two parameters $a(x)$ $b(x)$ over $\mathbb{F}_p$
- Interaction terms

$$Z\begin{pmatrix} 1+a(x)y \\ 1+b(x)z \end{pmatrix}, \qquad X\begin{pmatrix} 1+\bar{b}(x)\bar{z} \\ 1+\bar{a}(x)\bar{y} \end{pmatrix}$$

$$x \to x^{-1},\ y \to y^{-1},\ z \to z^{-1}$$

- Fractal logical operators

$$Z\begin{pmatrix} 0 \\ x^i\mathbf{a}(x,y) \end{pmatrix} \qquad Z\begin{pmatrix} x^i\mathbf{b}(x,z) \\ 0 \end{pmatrix}$$

$$X\begin{pmatrix} x^i\bar{\mathbf{a}}(x,y) \\ 0 \end{pmatrix} \qquad X\begin{pmatrix} 0 \\ x^i\bar{\mathbf{b}}(x,z) \end{pmatrix}$$

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- Fractal logical operators

$$Z\begin{pmatrix} 0 \\ x^i\mathbf{a}(x,y) \end{pmatrix} \leftarrow a(x) \qquad Z\begin{pmatrix} x^i\mathbf{b}(x,z) \\ 0 \end{pmatrix} \leftarrow b(x)$$

$$X\begin{pmatrix} x^i\bar{\mathbf{a}}(x,y) \\ 0 \end{pmatrix} \leftarrow \bar{a}(x) \qquad X\begin{pmatrix} 0 \\ x^i\bar{\mathbf{b}}(x,z) \end{pmatrix} \leftarrow \bar{b}(x)$$

Plan of Part 2

- Translation symmetric stabilizer Hamiltonians

  - via polynomials over finite fields (a new theoretical framework)

- Quantum fractal code

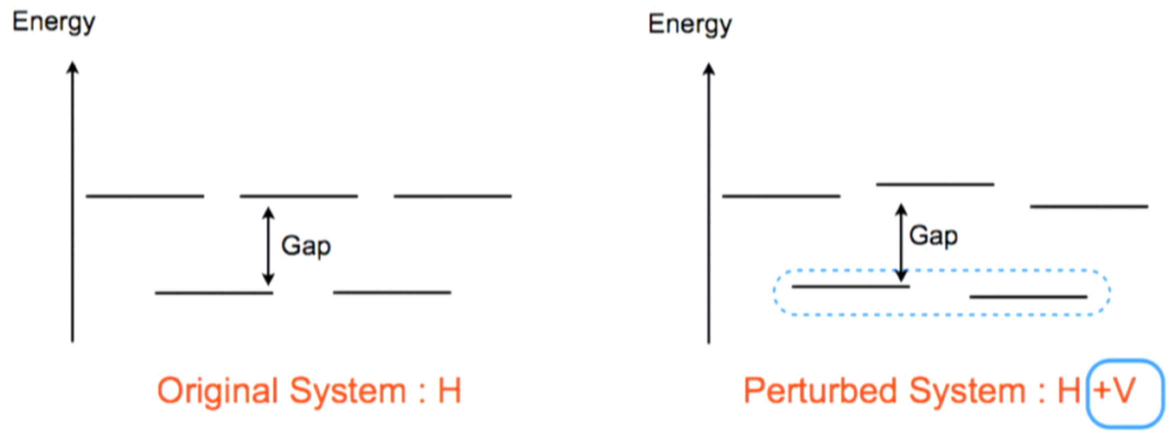
  - fractal logical operators

- Properties of quantum fractal code

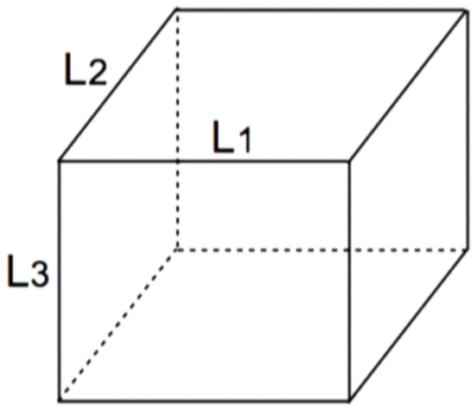
  - ground states (exotic topo order, entanglement entropy)
  - quasi-particle excitations
  - code distances (lower bounds)

Topological Order (Stability against perturbation)

Def: stability against perturbations



Number of ground states

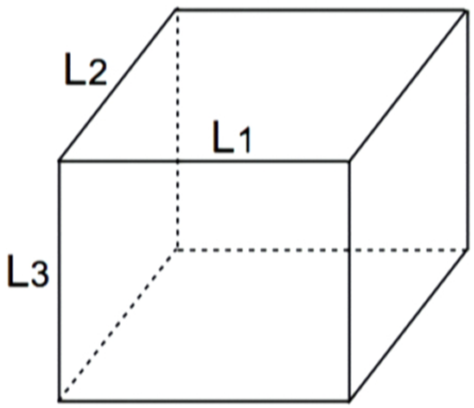


Number of solutions to

$$a(x)^{L_2}\gamma(x) = b(x)^{L_3}\gamma(x) = \gamma(x), \qquad x^{L_1} = 1$$

$= \frac{1}{2}$ Number of ground states

Number of ground states



Number of solutions to

$$a(x)^{L_2}\gamma(x) = b(x)^{L_3}\gamma(x) = \gamma(x), \qquad x^{L_1} = 1$$

$$= \frac{1}{2} \text{ Number of ground states}$$

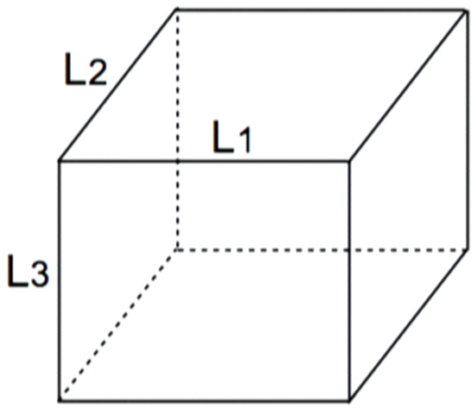
Discrete scale symmetries

$$k(pL_1, pL_2, pL_3) = pk(L_1, L_2, L_3)$$

⟶ topological order with extensive degeneracy

⟶ beyond topological field theory

Number of ground states



Number of solutions to

$$a(x)^{L_2}\gamma(x) = b(x)^{L_3}\gamma(x) = \gamma(x), \qquad x^{L_1} = 1$$

$$= \frac{1}{2} \text{ Number of ground states}$$

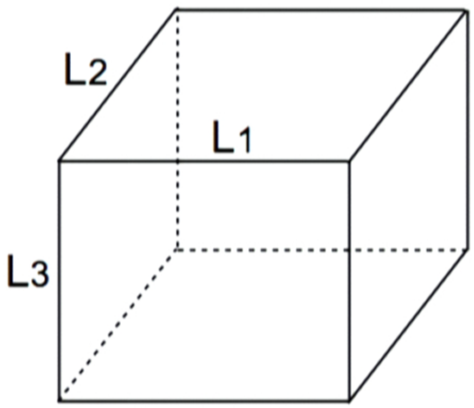
Discrete scale symmetries

$$k(pL_1, pL_2, pL_3) = pk(L_1, L_2, L_3)$$

⟶ topological order with extensive degeneracy

⟶ beyond topological field theory

Number of ground states



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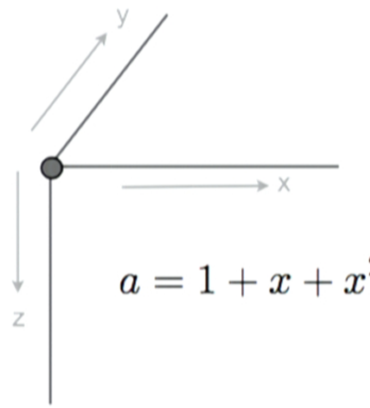
Quasi-particle excitations in quantum fractals

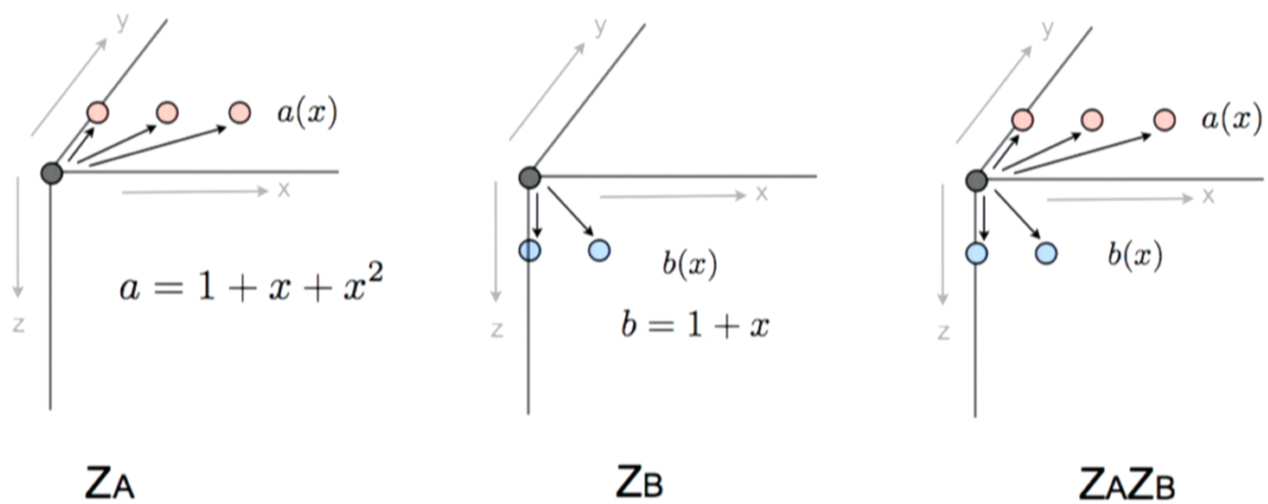


$a = 1 + x + x^2$

$b = 1 + x$

Z_A

Quasi-particle excitations in quantum fractals



- Propagations of quasi-particles are highly constrained (not free).
- Propagations can be discussed concisely via polynomials.

No string logical operators

No string logical operators

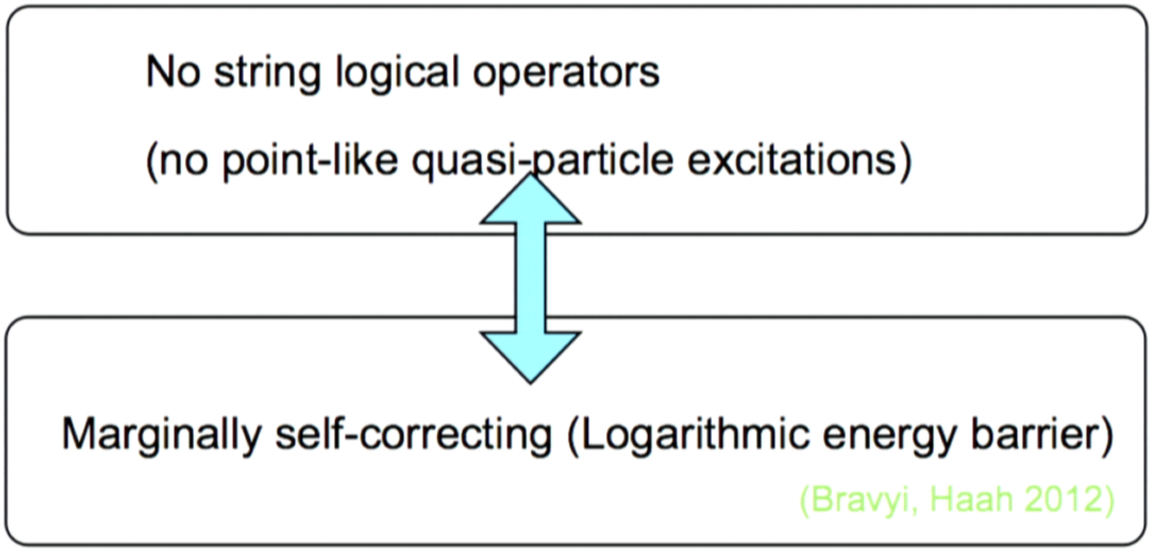
(no point-like quasi-particle excitations)

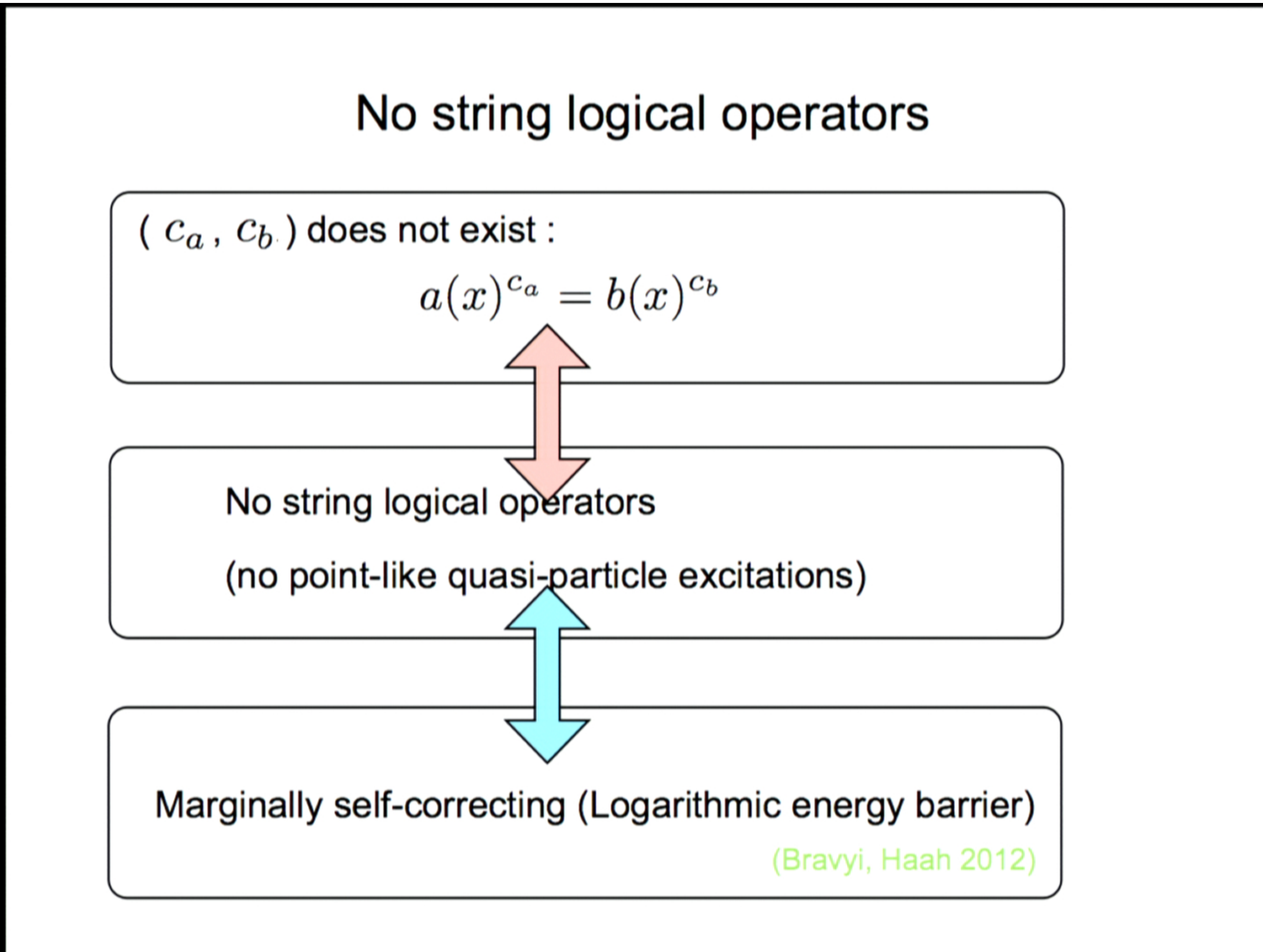
↕

Marginally self-correcting (Logarithmic energy barrier)

(Bravyi, Haah 2012)

No string logical operators

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(c_a , c_b) does not exist :
$a(x)^{c_a} = b(x)^{c_b}$
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(no point-like quasi-particle excitations)
Marginally self-correcting (Logarithmic energy barrier)
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(Bravyi, Haah 2012)

Plan of Part 2

- Translation symmetric stabilizer Hamiltonians
  - via polynomials over finite fields (a new theoretical framework)
- Quantum fractal code
  - fractal logical operators
- Properties of quantum fractal code
  - ground states (exotic topo order, entanglement entropy)
  - quasi-particle excitations
  - code distances (lower bounds)

Polynomial decomposition problem

Theorem (lower bound)

Consider all the tensors $C_{ij}^{(\ell)}$ satisfying the following equation:

$$\gamma b^\ell = \sum_{ij} x^i a^j C_{ij}^{(\ell)}, \qquad \text{for all} \quad \ell.$$

and consider all the tensors $D_{i\ell}^{(j)}$ satisfying the following equation:

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$$\text{Weight}(C^{(\ell)}{}_{ij}) \geq W \qquad \text{Weight}(D^{(\ell)}{}_{ij}) \geq W \qquad \text{for } \gamma \neq 0 \qquad \Rightarrow \qquad d \geq W.$$

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- Reduced to some mathematical problem
- Still difficult to handle this problem

Summary

Quantum fractal code

$$Z\begin{pmatrix} 1+a(x)y \\ 1+b(x)z \end{pmatrix}, \qquad X\begin{pmatrix} 1+\bar{b}(x)\bar{z} \\ 1+\bar{a}(x)\bar{y} \end{pmatrix}$$

$x \to x^{-1}$, $y \to y^{-1}$, $z \to z^{-1}$

- Is this universal ?
- Cubic code as quantum fractal code ?

Coding properties ?

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