

Title: 12/13 PSI - Condensed Matter Review Lecture 4

Date: Jan 31, 2013 11:30 AM

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Abstract:

- Exact spectrum of quantum Ising model H_I
- exact equal-time correlation functions
- some of the features of finite T crossovers

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- exact equal-time correlation functions
- some of the features of finite T crossovers

$$H_I = -g$$

g model H_I

ions

T crossovers

$$H_I = -gJ \sum_i \sigma_i^x - J \sum_i \sigma_i^z \sigma_{i+1}^z$$

g model H_I

ions

T crossovers

$$H_I = -gJ \sum_i^M \sigma_i^x - J \sum_i^M \sigma_i^z \sigma_{i+1}^z$$

g model H_I

ions

T crossovers

$$H_I = -gJ \sum_i \hat{\sigma}_i^x - J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$\hat{\sigma}_i^{\pm} = \frac{\hat{\sigma}_i^x \pm i \hat{\sigma}_i^y}{2}$$

Jordan-Wigner transformation

$$H_I = -gJ \sum_i M \hat{\sigma}_i^x - J \sum_i M \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$\hat{\sigma}_i^{\pm} = \frac{\hat{\sigma}_i^x \pm i \hat{\sigma}_i^y}{2}$$

Jordan-Wigner transformation : Define a string of

$$H_I = -gJ \sum_i \hat{\sigma}_i^x - J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$\hat{\sigma}_i^{\pm} = \frac{\hat{\sigma}_i^x \pm i \hat{\sigma}_i^y}{2}$$

Jordan-Wigner transformation : Define a string operator

$$[C_j^{\dagger}, \hat{\sigma}_i^{\pm}] = \prod_{j < i} (1 - 2C_j^{\dagger} C_j) C_i$$

fermionic operators

$$C_i = \left(\prod_{j < i} \frac{1 - \hat{\sigma}_j^z}{2} \right) \hat{\sigma}_i^+$$

$$H_I = -gJ \sum_i \hat{\sigma}_i^x - J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$\hat{\sigma}_i^{\pm} = \frac{\hat{\sigma}_i^x \pm i \hat{\sigma}_i^y}{2}$$

Jordan-Wigner transformation : Define a string operator

$$c_i^{\dagger}, c_j^{\dagger} : \hat{\sigma}_i^{\pm} = \prod_{j < i} (1 - 2c_j^{\dagger} c_j) c_i^{\pm}$$

$$\{c_i^{\dagger}, c_j\} = \delta_{ij}$$

feynmanic
operators

$$c_i = \left(\prod_{j < i} \hat{\sigma}_j^z \right) \hat{\sigma}_i^+$$

$$H_I = -J \sum_i \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i \right)$$

$$H_I = -J \sum_i \left(\underbrace{c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i}_{\text{hopping}} + c_{i+1}^\dagger c_{i+1}^\dagger + c_{i+1} c_i - 2g c_i^\dagger c_i + g \right)$$

$$H_I = -J \sum_i \left(\underbrace{c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i}_{\text{hopping}} + c_{i+1}^\dagger c_i^\dagger + c_{i+1} c_i - 2g c_i^\dagger c_i + g \right)$$

$$H_I = -J \sum_i \left(\underbrace{C_i^\dagger C_{i+1} + C_{i+1}^\dagger C_i}_{\text{hopping}} + C_i^\dagger C_{i+1}^\dagger + C_{i+1} C_i - 2g C_i^\dagger C_i + g \right)$$

2nd Step. Go to k space by Fourier transform $\rightarrow H_I = J \sum_k \left(2[1 - \cos(ka)] C_k^\dagger C_k + i \sin(ka) [C_{-k}^\dagger C_k^\dagger + C_{-k} C_k] - g \right)$

$$C_k = \frac{1}{\sqrt{N}} \sum_j e^{-ikr_j}$$

$$k = 0, 1, \dots$$

$$H_I = -J \sum_i \left(\underbrace{C_i^\dagger C_{i+1} + C_{i+1}^\dagger C_i}_{\text{hopping}} + C_i^\dagger C_{i+1}^\dagger + C_{i+1} C_i - 2g C_i^\dagger C_i + g \right)$$

2nd Step. Go to k space by Fourier transform $\rightarrow H_I = J \sum_k \left(2[g - \cos(ka)] C_k^\dagger C_k + i \sin(ka) [C_{-k}^\dagger C_k - g] \right)$

$$C_k = \frac{1}{\sqrt{M}} \sum_j e^{-ikr_j}$$

$k=0, 1, \dots$

3rd Step Bogoliubov Transformation

$$\begin{cases} \gamma_k = u_k C_k - i v_k C_{-k}^\dagger \\ C_k = u_k \gamma_k + i v_k \gamma_{-k}^\dagger \end{cases}$$

$$H_I = -J \sum_i \left(\underbrace{C_i^\dagger C_{i+1} + C_{i+1}^\dagger C_i}_{\text{hopping}} + C_i^\dagger C_{i+1}^\dagger + C_{i+1} C_i - 2g C_i^\dagger C_i + g \right)$$

2nd Step. Go to k space by Fourier transform $\rightarrow H_I = J \sum_k \left(2 \cos(ka) C_k^\dagger C_k + i \sin(ka) [C_{-k}^\dagger C_k^\dagger + C_{-k} C_k] - g \right)$

$$C_k = \frac{1}{\sqrt{N}} \sum_j e^{-ikr_j} c_j$$

$k=0, 1, \dots$

3rd Step Bogoliubov Transformation

$$\begin{cases} u_k C_k - i v_k C_{-k}^\dagger \\ v_k C_k + i u_k C_{-k}^\dagger \end{cases}$$

$$u_k^2 + v_k^2 = 1$$

$$u_{-k} = u_k$$

$$v_{-k} = -v_k$$

$$k=0, 1, \dots$$

transformation

$$c_k = u_k \gamma_k + i v_k \gamma_{-k}^\dagger$$

$$v_{-k} = -v_k$$

$$\begin{cases} u_k = \cos \frac{\theta_k}{2} \\ v_k = \frac{\sin \theta_k}{2} \end{cases}$$

$$+ a \frac{\sin(ka)}{q - \cos(ka)}$$

$$S_{kk'}$$

$$H_I = \sum_k \lambda_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right)$$

energies

$$k=0, 1, \dots$$

Transformation

$$c_k = u_k \gamma_k + i v_k \gamma_{-k}^\dagger$$

$$v_{-k} = -v_k$$

$$\begin{cases} u_k = \cos \frac{\theta_k}{2} \\ v_k = \sin \frac{\theta_k}{2} \end{cases}$$

$$\tan \theta = \frac{m \sin(ka)}{g - \cos(ka)}$$

$$\{\gamma_k, \gamma_{k'}^\dagger\} = \delta_{kk'}$$

$$H_I = \sum_k \Lambda_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right)$$

$$\Lambda_k = 2J \sqrt{1 + g^2 - 2g \cos k}$$

$$\Lambda_k > 0$$

$$\gamma_k |0\rangle = 0 \quad \forall k \quad \text{und stat}$$

$$\Pi_i \gamma_{k_i}^\dagger |0\rangle$$

$$k=0, 1, \dots$$

transformation

$$c_k = u_k \gamma_k + i v_k \gamma_{-k}^\dagger$$

$$v_{-k} = -v_k$$

$$\begin{cases} u_k = \cos \frac{\theta_k}{2} \\ v_k = \sin \frac{\theta_k}{2} \end{cases}$$

$$\tan \theta = \frac{m(ka)}{g - \omega(ka)}$$

$$\{\gamma_k, \gamma_{k'}^\dagger\} = \delta_{kk'}$$

$$H_I = \sum_k \Lambda_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right)$$

$$\Lambda_k = 2J \sqrt{1 + g^2 - 2g \cos k}$$

$$\Lambda_k > 0$$

$$\gamma_k |0\rangle = 0 \quad \forall k \text{ ground state}$$

$$\prod_i \gamma_{k_i}^\dagger |0\rangle$$

$$\Delta(g) = 2J |1 - g| \quad g_c = 1$$

$$\Delta = 2J(1 - g) \quad \text{gap} = |\Delta|$$

$$\psi(x_i) = \frac{1}{\sqrt{N}} c_i$$

$$\kappa - \frac{1}{2})$$

$$\Delta(g) = 2J |1-g|$$

$$g_c = 1$$

$$\Delta = 2J(1-g)$$

$$\text{gap} = |\Delta|$$

ψ κ ground state

$$\psi(x_i) = \frac{1}{\sqrt{N}} c_i \quad \text{in the continuous limit}$$

$$H_I = \int dx \quad \frac{c}{2} \left(\psi^\dagger \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^\dagger}{\partial x} \right) +$$

$$\kappa - \frac{1}{2})$$

$$\Delta(g) = 2J |1-g|$$

$$g_c = 1$$

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$$\psi(x_i) = \frac{1}{\sqrt{N}} c_i \quad \text{in the continuous limit}$$

$$H_I = \int dx \left(\frac{c}{2} \left(\psi^\dagger \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^\dagger}{\partial x} \right) + \Delta \psi^\dagger \psi + \dots \right)$$

$$c = 2Ja$$

Quantum XY model

$$H_{XY} = \sum_i \frac{1-\gamma}{2} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \frac{1+\gamma}{2} \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y + h \hat{\sigma}_i^z$$

γ, h

Quantum XY model

$$H_{XY} = \sum_i \frac{1+\gamma}{2} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \frac{1-\gamma}{2} \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y + h \hat{\sigma}_i^z$$

$\gamma, h \quad \gamma=1 \rightarrow H_I$ can be diagonalized by the same procedure

Quantum XY model

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Quantum XY model $H_{XY} = \sum_i \frac{1+\gamma}{2} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \frac{1-\gamma}{2} \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y + h \hat{\sigma}_i^z$

$\gamma, h \quad \gamma=1 \rightarrow H_I$ can be diagonalized by the same procedure

$$\langle \dots \rangle = \sum T^{1/4} G_I \left(\frac{\Lambda}{T} \right)$$

Quantum XY model $H_{XY} = \sum_i \frac{1+\gamma}{2} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \frac{1-\gamma}{2} \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y + h \hat{\sigma}_i^z$

$\gamma, h \quad \gamma=1 \rightarrow H_I$ can be diagonalized by the same procedure

$\langle \hat{\sigma}_i^z \hat{\sigma}_{i+n}^z \rangle_x = \sum T^{1/4} G_I\left(\frac{\Lambda}{T}\right) \exp\left(-\frac{T}{c} F_I\left(\frac{\Lambda}{T}\right)\right)$

$F_I(s) = |s| \theta(-s) + \frac{1}{\pi}$

$G_I(s) = \int \frac{dy}{y} \left(\frac{dF_I}{dy}\right)^2$

$\sim \coth \frac{\sqrt{y^2 + s^2}}{2}$

$\sim \frac{1}{10}$

large n

Quantum XY model $H_{XY} = \sum_i \frac{1+\gamma}{2} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \frac{1-\gamma}{2} \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y + h \hat{\sigma}_i^z$

$\gamma, h \quad \gamma=1 \rightarrow H_I$

Can be diagonalized
by the same procedure

$\langle \hat{\sigma}_i^z \hat{\sigma}_{i+n}^z \rangle = \sum T^{1/4} G_I\left(\frac{\Lambda}{T}\right) \exp\left(-\frac{T}{c} n F_I\left(\frac{\Lambda}{T}\right)\right)$
large n

$F_I(s) = |s| \theta(-s) + \frac{1}{\pi} \int_0^\infty dy \ln \coth \frac{\sqrt{y^2 + s^2}}{2}$
 $G_I(s) = \int \frac{dy}{y} \left(\frac{dF_I}{dy}\right)^2$
Sachdev 10

$$\hat{\sigma}_i^y \hat{\sigma}_{i+1}^y + h \hat{\sigma}_i^z$$

be diagonalized
the same procedure

$$F_I(s) = |s| \theta(-s) + \frac{1}{\pi} \int_0^\infty dy \ln \coth \sqrt{\frac{y^2 + s^2}{2}}$$

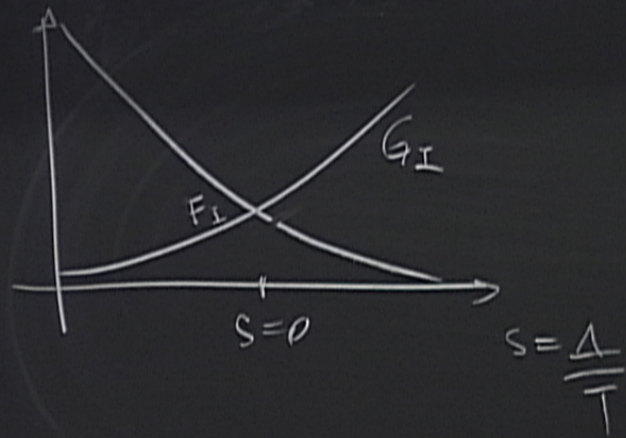
Sachdev ^{Ch.} 10

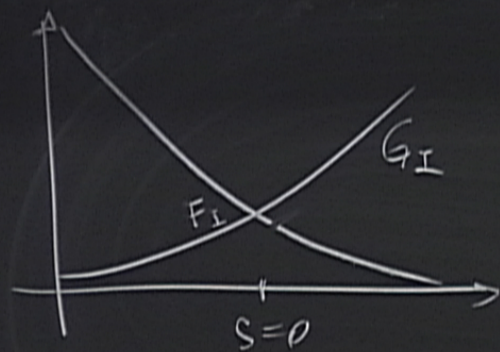
$$s = \frac{A}{T}$$

$$G_I(s) = \int \frac{dy}{y} \left(\frac{\partial F_I}{\partial y} \right)^2 + \dots$$

$T > 0$ $F_I(s)$ is analytic
 $\forall s$

$\left(\begin{matrix} T=0 \\ s \rightarrow 0 \end{matrix} \right)$ non analytic
point

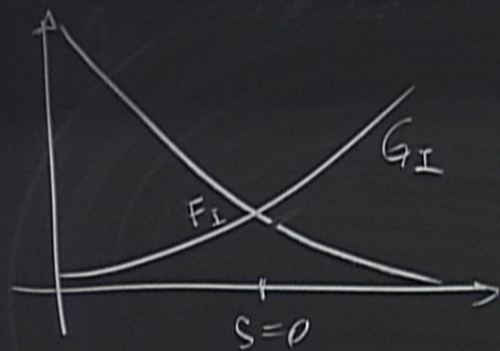




$$\langle \sigma_i^z \sigma_{i+n}^z \rangle =$$

Identity: $1 - 2c_i^+ c_i = (c_i^+ + c_i)(c_i^+ - c_i)$

$$\sigma_i^z = \prod_{j < i} (1 - 2c_j^+ c_j) (c_i + c_i^+)$$



Identity: $1 - 2c_i^+ c_i^- = (c_i^+ + c_i)(c_i^+ - c_i)$

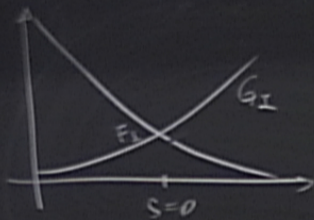
$$\frac{12}{\sigma_i} = \prod_{j < i} (1 - 2c_j^+ c_j^-) (c_i + c_i^+)$$

$$\langle \sigma_i^{12} \sigma_{i+n}^{12} \rangle = \left\langle (c_i^+ + c_i) \left[\prod_{j=1}^{i+n-1} (c_j^+ + c_j) \right] \right\rangle$$

$$\prod_{j=1}^{i+n-1} (C_j^+ + C_j)(C_j^+ - C_j) (C_{i+n}^+ + C_{i+n})$$

$$C_i^+)$$

Ann. Phys 16, 406
(196



$$\langle \sigma_i^z \sigma_{i+n}^z \rangle = \left\langle (c_i^\dagger c_i) \left[\prod_{j=i+1}^{i+n-1} (c_j^\dagger + c_j)(c_j^\dagger - c_j) \right] (c_{i+n}^\dagger + c_{i+n}) \right\rangle$$

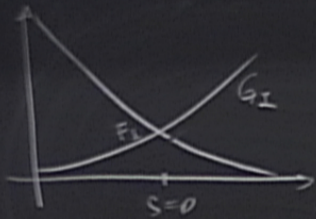
Identity: $1 - 2c_i^\dagger c_i = (c_i^\dagger + c_i)(c_i^\dagger - c_i)$

$$\frac{\sigma_i^z}{2} = \prod_{j=i} (1 - 2c_j^\dagger c_j) (c_i + c_i^\dagger)$$

$$= \langle B_i A_{i+1} B_{i+1} \dots A_{i+n-1} B_{i+n-1} A_{i+n} \rangle$$

$A_i \equiv c_i^\dagger + c_i$
 $B_i \equiv c_i^\dagger - c_i$

Ann. Phys. 16, 466
(1961)



$$\langle \sigma_i^z \sigma_{i+n}^z \rangle = \left\langle (c_i^\dagger - c_i) \left[\prod_{j=i+1}^{i+n-1} (c_j^\dagger + c_j)(c_j^\dagger - c_j) \right] (c_{i+n}^\dagger + c_{i+n}) \right\rangle$$

Identity: $1 - 2c_i^\dagger c_i = (c_i^\dagger + c_i)(c_i^\dagger - c_i)$

$$\frac{\sigma_i^z}{2} = \prod_{j=i} (1 - 2c_j^\dagger c_j) (c_i + c_i^\dagger)$$

$$= \langle B_i A_{i+1} B_{i+1} \dots A_{i+n-1} B_{i+n-1} A_{i+n} \rangle$$

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Ann. Phys. 16, 466
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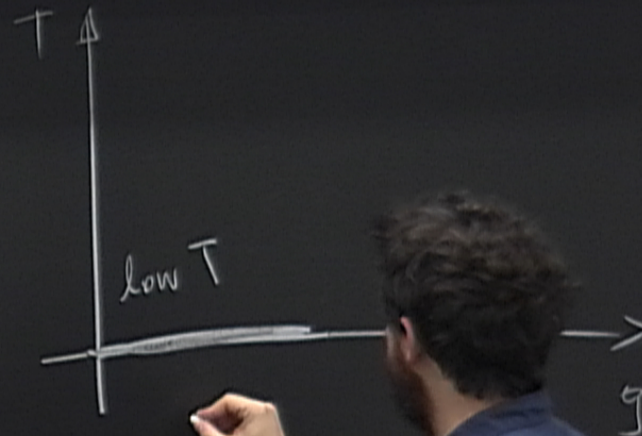
→ Fetter & ...

$$\xi^{-1} = \frac{T}{c} F_I \left(\frac{\Delta}{T} \right)$$

$$\xi = \begin{cases} c \sqrt{\frac{\pi}{2\Delta T}} e^{\Delta/T} & \Delta \gg T \\ \frac{4c}{\pi T} & |\Delta| < T \\ -T & \end{cases}$$

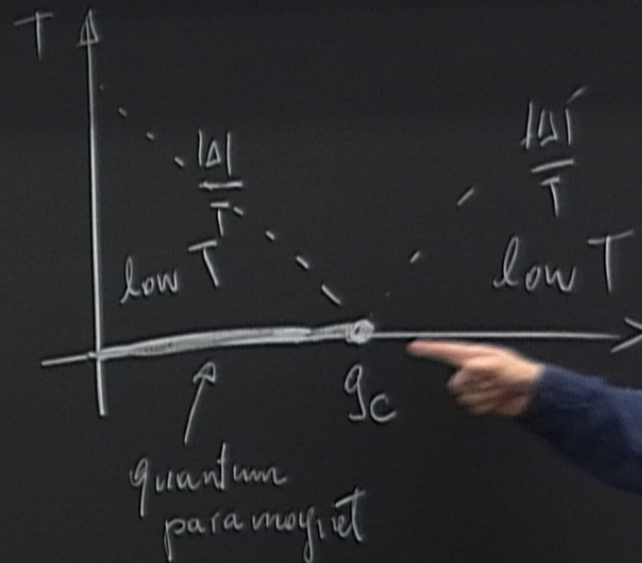
$$\xi^{-1} = \frac{T}{c} F_I \left(\frac{\Delta}{T} \right)$$

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$$\chi^{-1} = \frac{T}{c} F_I \left(\frac{\Delta}{T} \right)$$

$$\chi = \begin{cases} c \sqrt{\frac{\pi}{2\Delta T}} e^{\Delta/T} & \Delta \gg T \\ \frac{4c}{\pi T} & |\Delta| < T \\ \frac{c}{|\Delta|} & \Delta \ll -T \end{cases}$$



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