Title: 12/13 PSI - Gravitational Physics Review Lecture 3

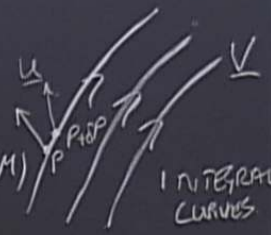
Date: Jan 30, 2013 09:00 AM

URL: http://pirsa.org/13010095

Abstract:

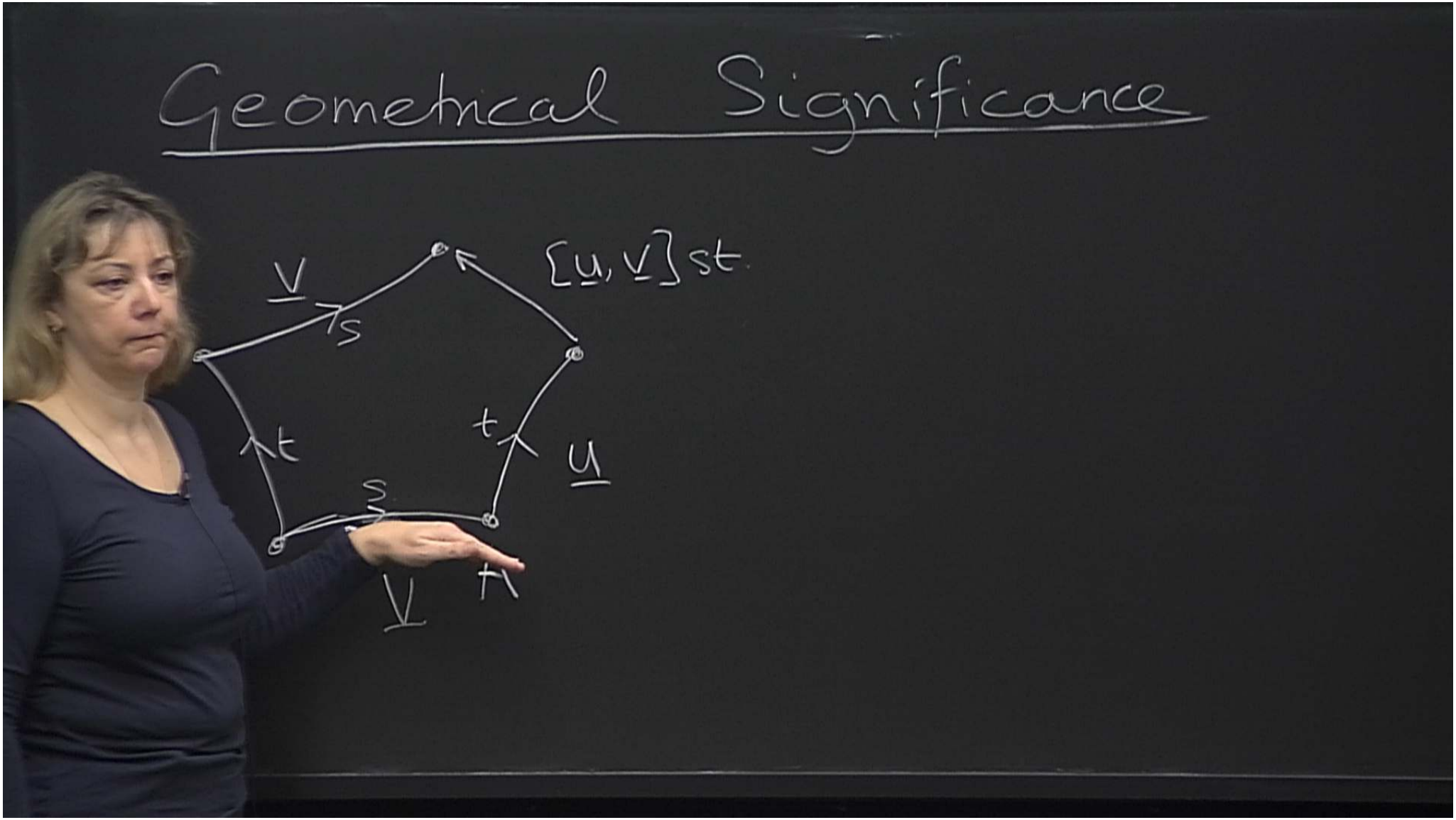
Lie Derivative

$[\underline{u}, \underline{v}]f = (\underline{u}\underline{v} - \underline{v}\underline{u})f$



$\forall f \in C^\infty(M)$

$$\mathcal{L}_V Q^{a_1 \cdots a_n}_{b_1 \cdots b_m} = V^c \partial_c Q^{a_1 \cdots a_n}_{b_1 \cdots b_m} - \sum_{i=1}^{n} Q^{a_1 \cdots \overset{i}{c} \cdots a_n}_{b_1 \cdots b_m} \partial_c V^{a_i} + \sum_{i=1}^{m} Q^{a_1 \cdots a_n}_{b_1 \cdots c \cdots b_m} \partial_{b_i} V^c$$


Geometrical Significance
V
s
[u, v] st
t
t
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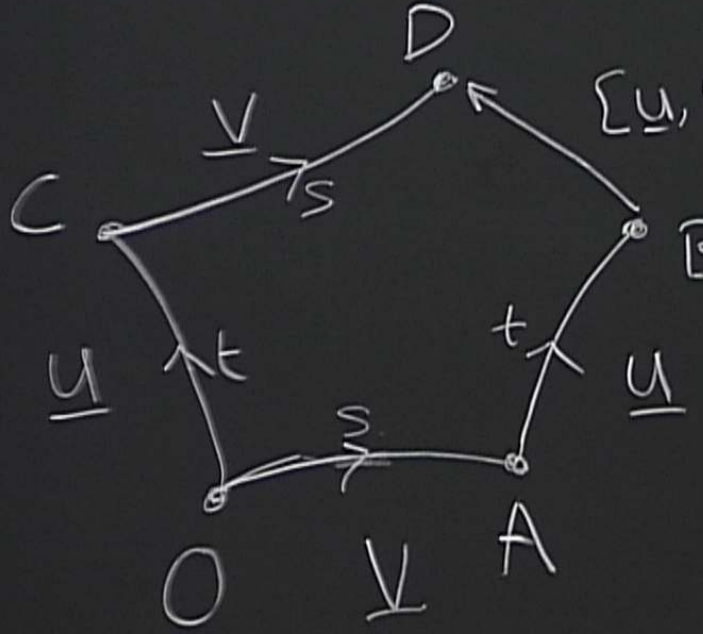
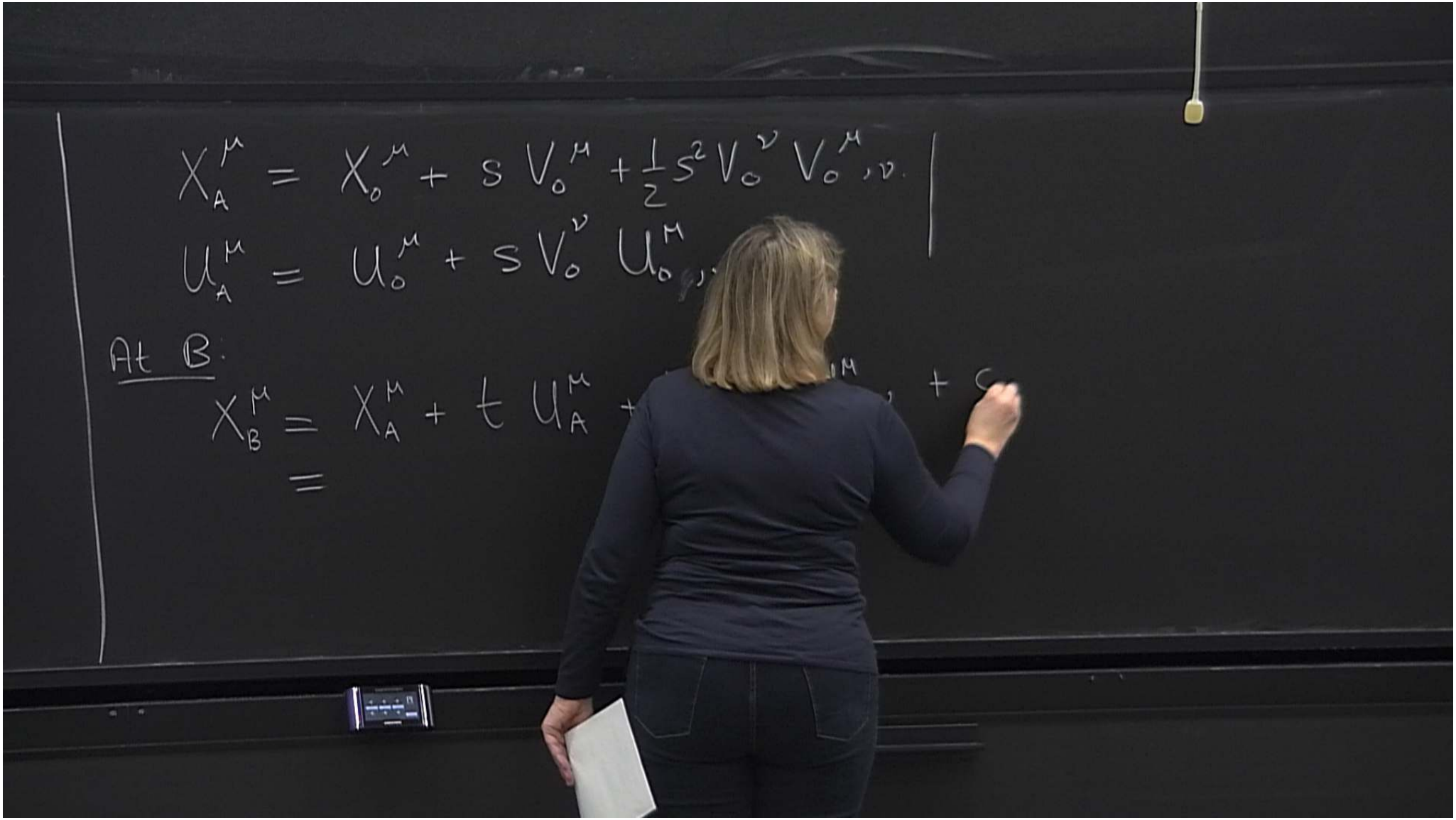
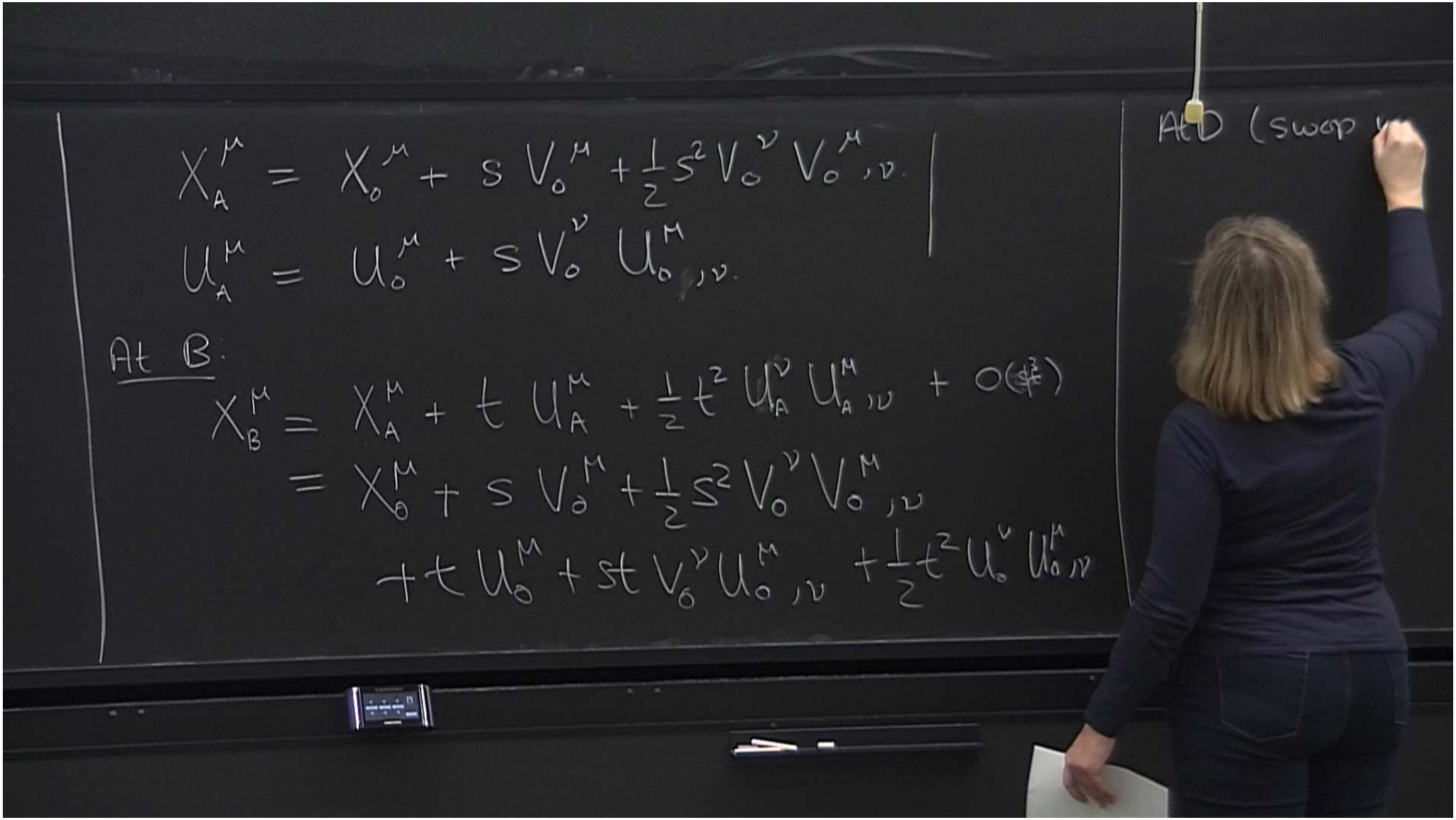
Geometrical Significance

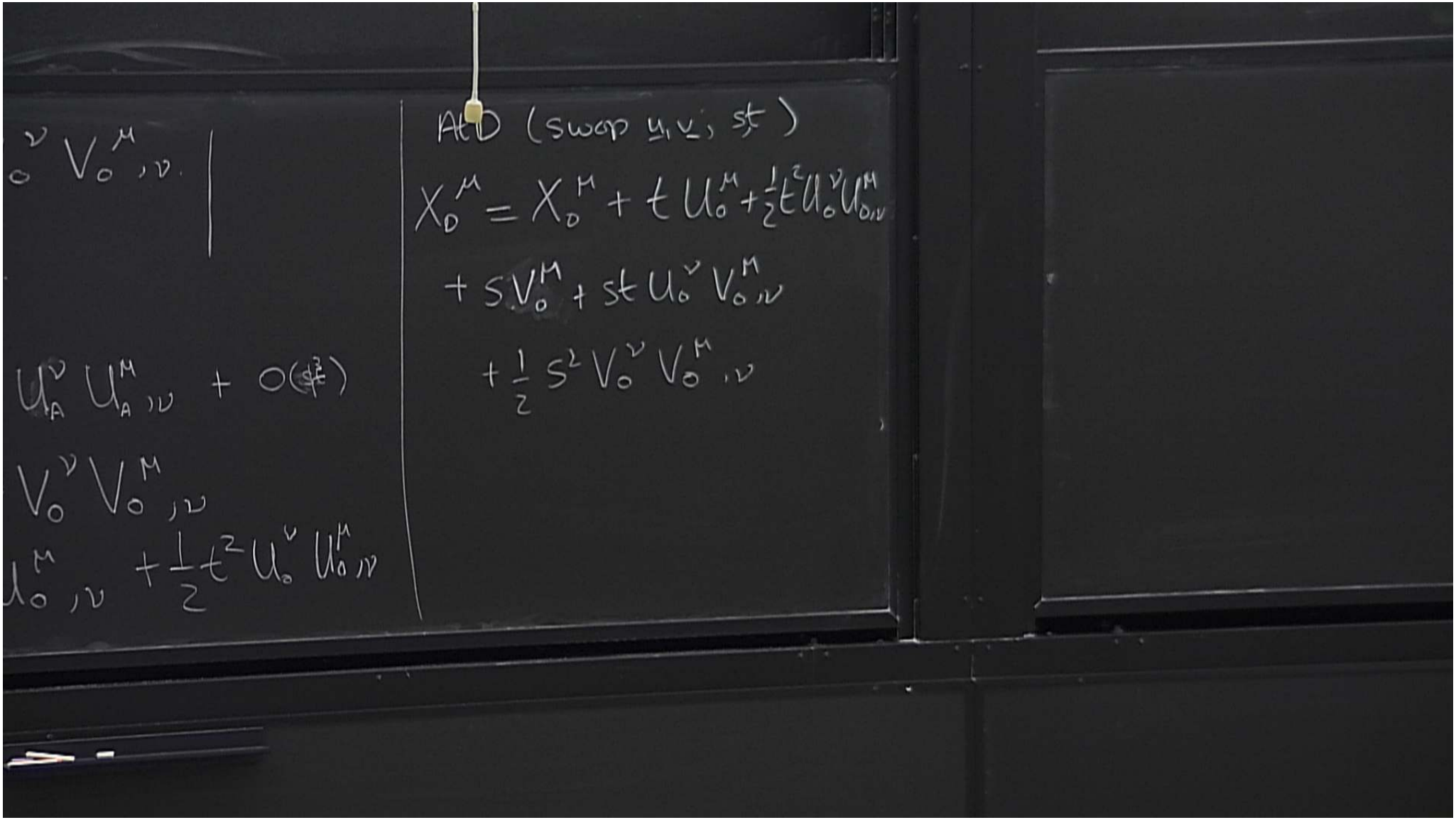


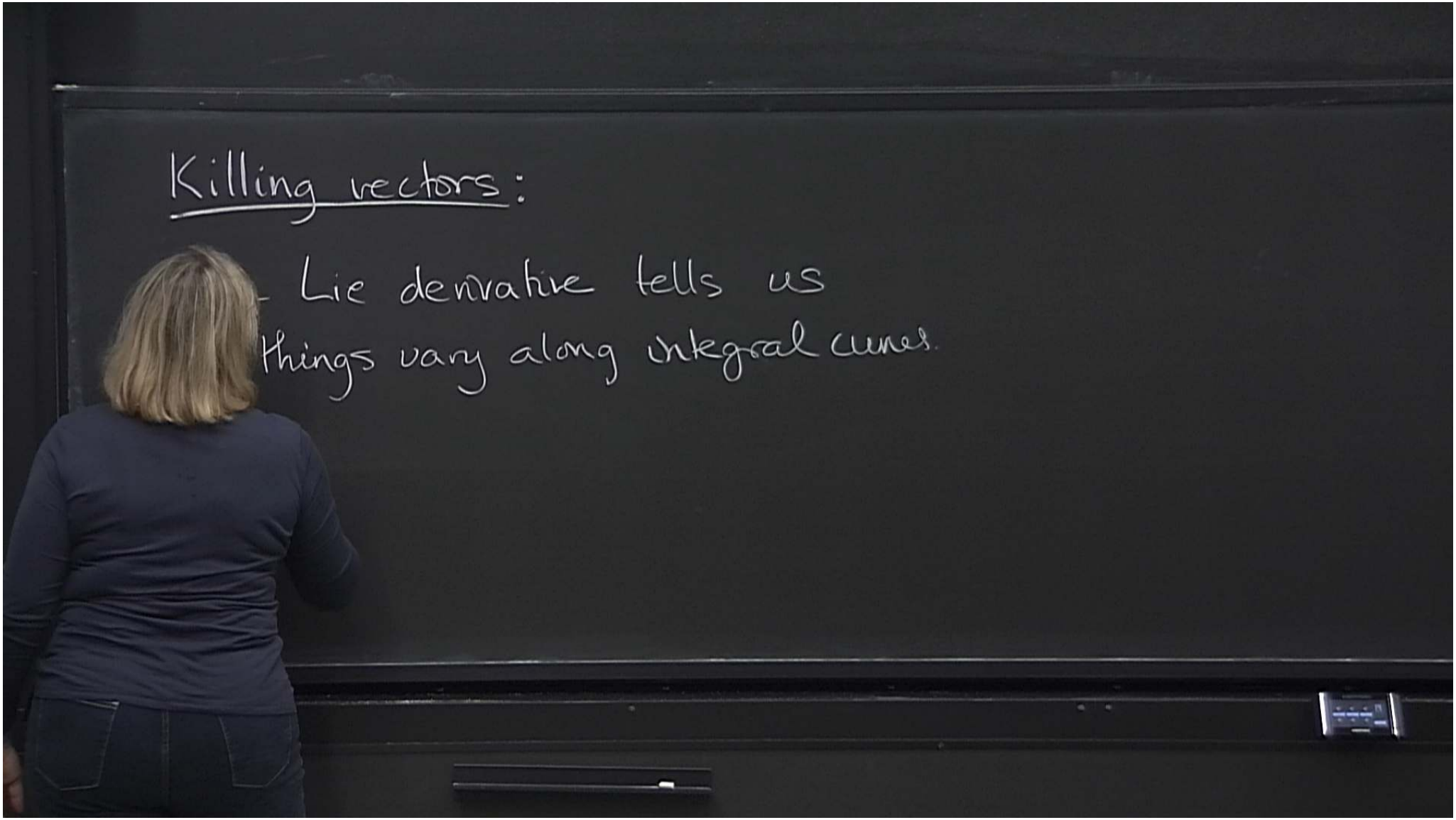
$[u, v]_{st}$

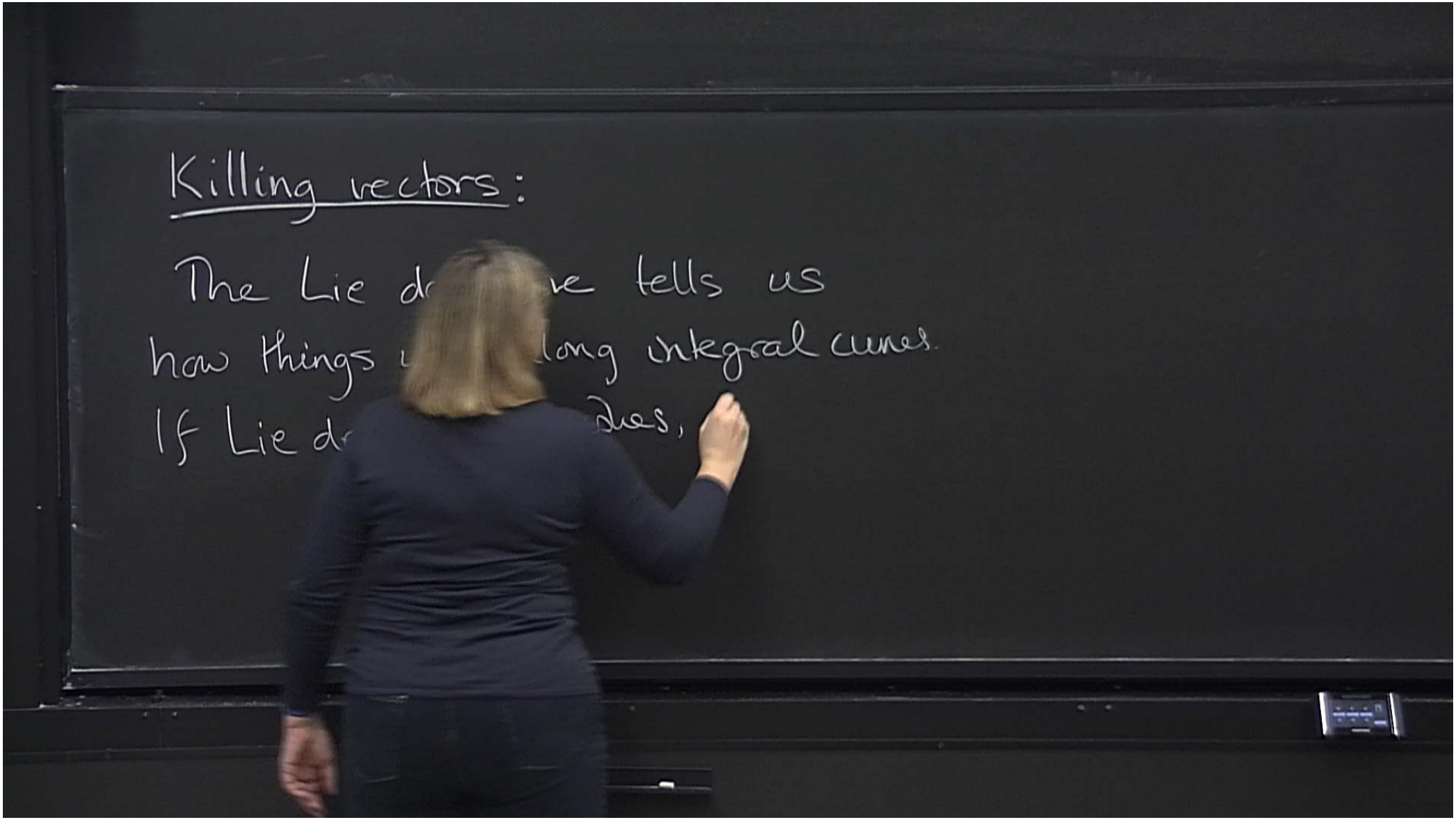
Lie bracket indicates how 'non-commutative' a u-v curve is


$X_A^\mu = X_0^\mu + s V_0^\mu + \frac{1}{2} s^2 V_0^\nu V_{0,\nu}^\mu$
$U_A^\mu = U_0^\mu + s V_0^\nu U_{0,\nu}^\mu$
At B:
$X_B^\mu = X_A^\mu + t U_A^\mu$
$=$


$X_A^\mu = X_0^\mu + s V_0^\mu + \frac{1}{2} s^2 V_0^\nu V_{0,\nu}^\mu$
$U_A^\mu = U_0^\mu + s V_0^\nu U_{0,\nu}^\mu$
At B:
$X_B^\mu = X_A^\mu + t U_A^\mu + \frac{1}{2} t^2 U_A^\nu U_{A,\nu}^\mu + O(t^3)$
$= X_0^\mu + s V_0^\mu + \frac{1}{2} s^2 V_0^\nu V_{0,\nu}^\mu$
$+ t U_0^\mu + st V_0^\nu U_{0,\nu}^\mu + \frac{1}{2} t^2 U_0^\nu U_{0,\nu}^\mu$
At D (swap


$V_0^\nu V_{0,\nu}^\mu$
$U_A^\nu U_{A,\nu}^\mu + O(s^3)$
$V_0^\nu V_{0,\nu}^\mu$
$U_{0,\nu}^\mu + \frac{1}{2}t^2 U_0^\nu U_{0,\nu}^\mu$
AtD (swap μ,ν; s,t)
$X_0^\mu = X_0^\mu + t U_0^\mu + \frac{1}{2}t^2 U_0^\nu U_{0,\nu}^\mu$
$+ s V_0^\mu + st U_0^\nu V_{0,\nu}^\mu$
$+ \frac{1}{2} s^2 V_0^\nu V_{0,\nu}^\mu$


Killing vectors:
- Lie derivative tells us
things vary along integral curves.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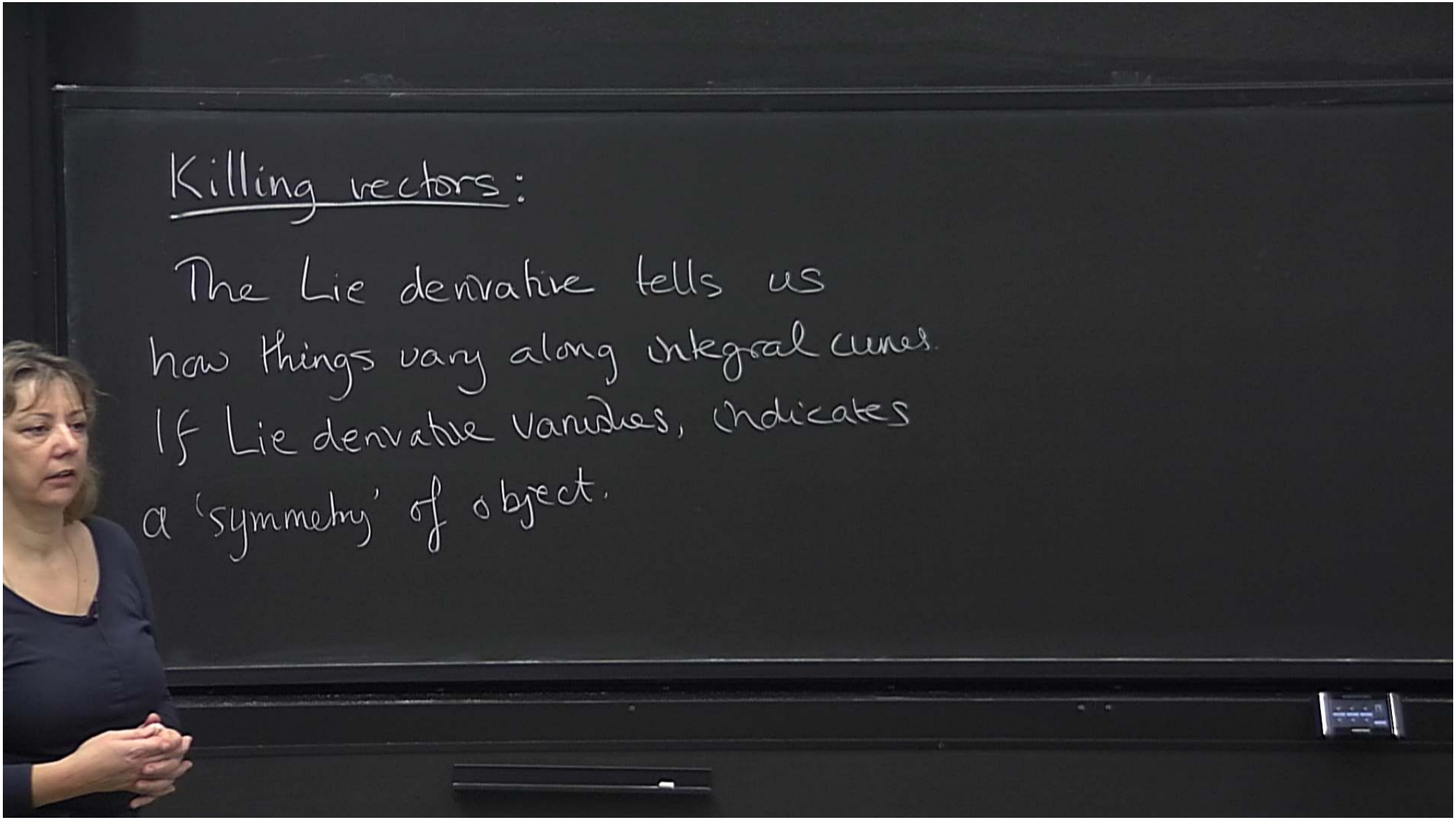
If Lie d ues,


Killing vectors:
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If Lie derivative vanishes, indicates
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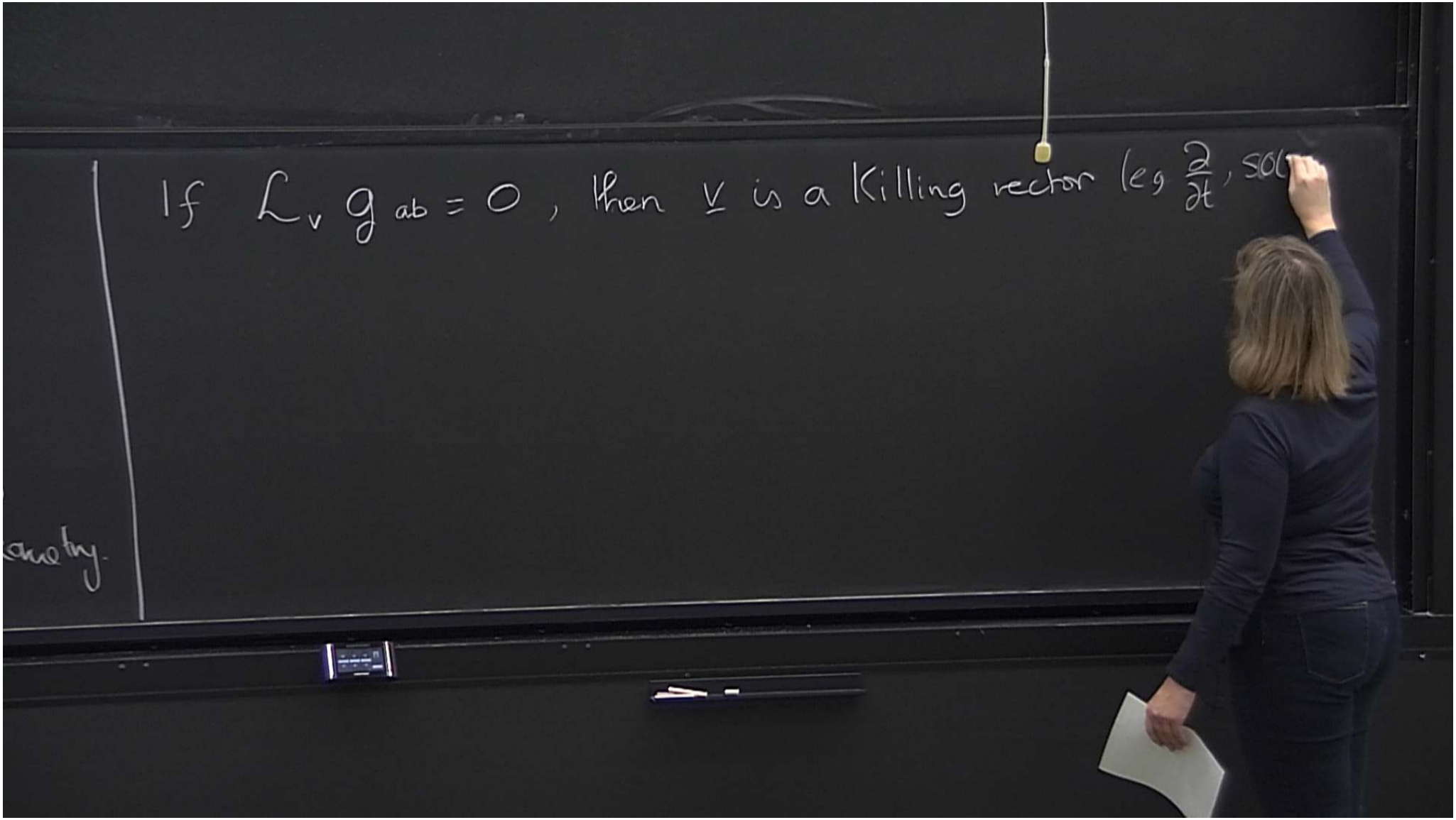
Killing vectors:

The Lie derivative tells us how things vary along integral curves. If Lie derivative vanishes, indicates a 'symmetry' of object. For the metric, this gives a symmetry of our spacetime geometry.


If $\mathcal{L}_v g_{ab} = 0$, then $\underline{v}$ is a Killing vector (eg $\frac{\partial}{\partial t}$, SO(
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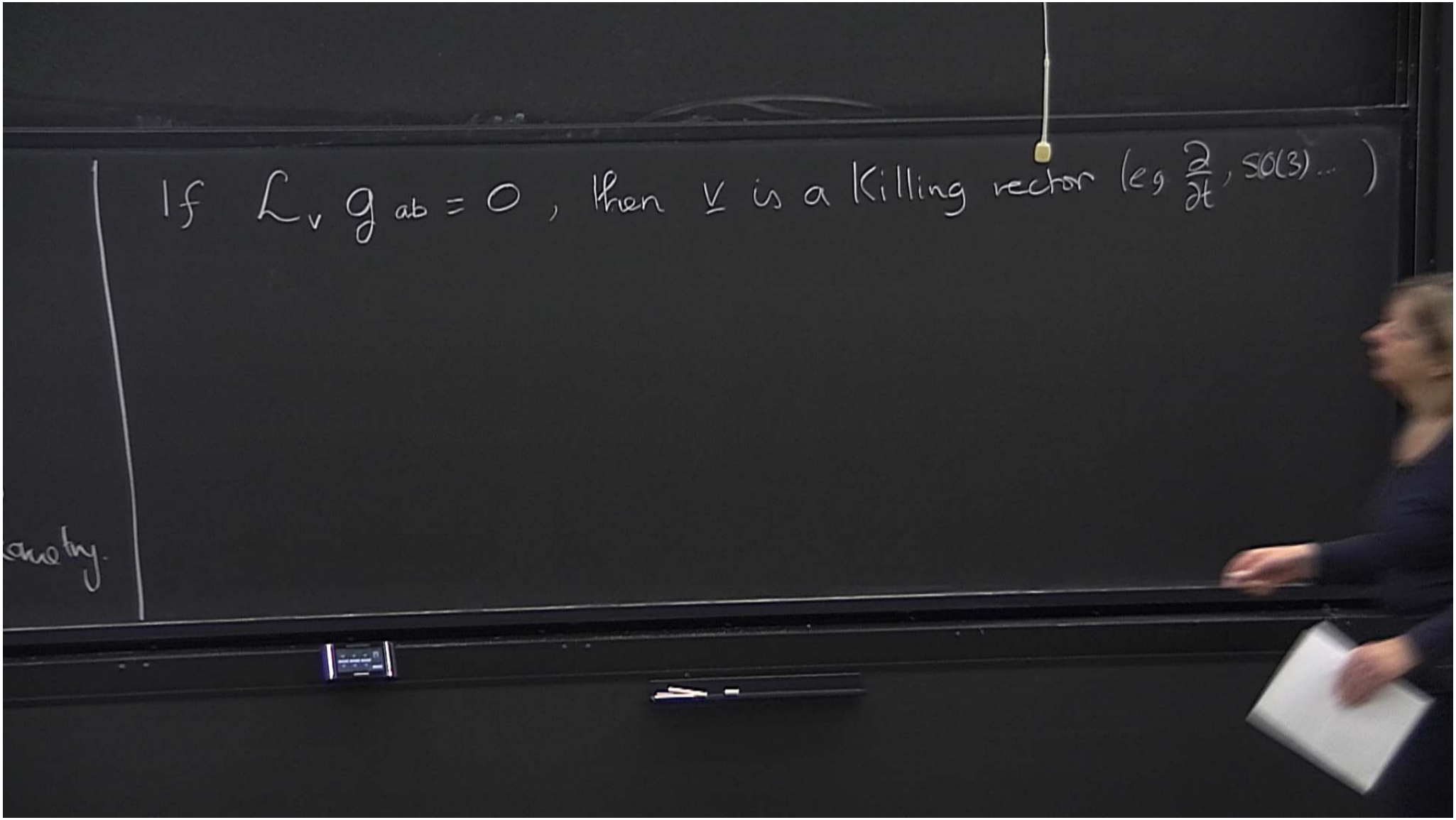
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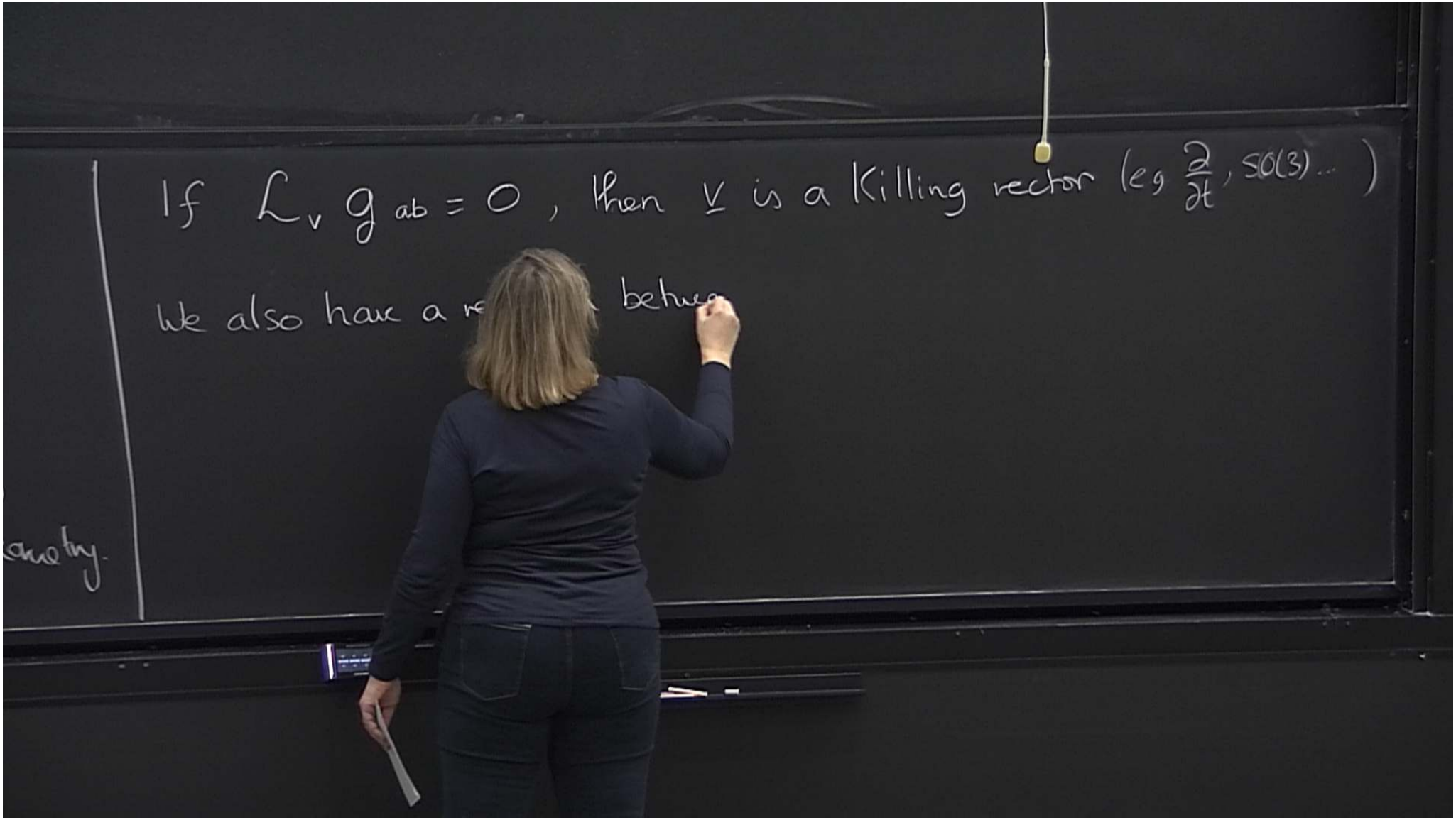
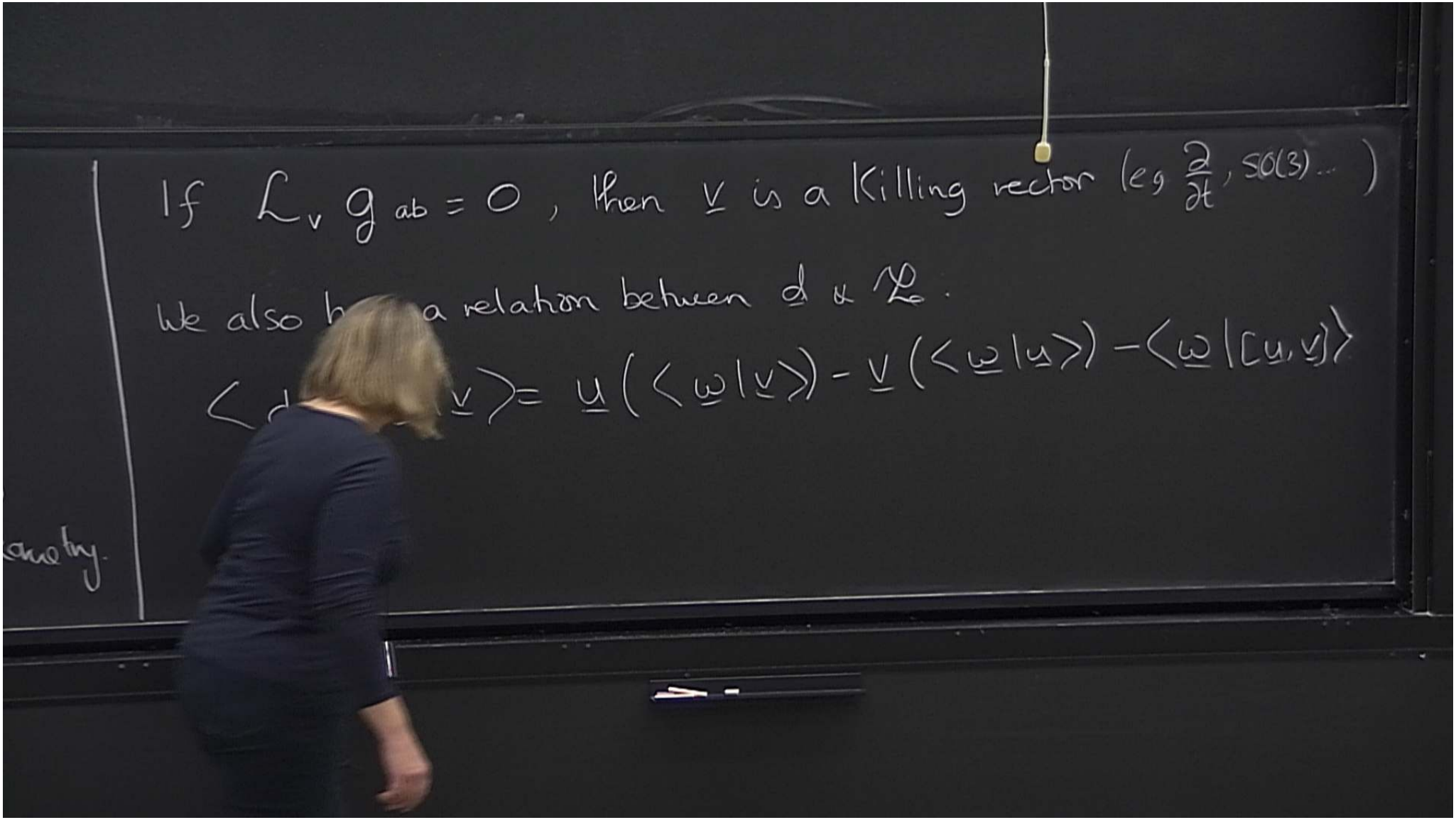
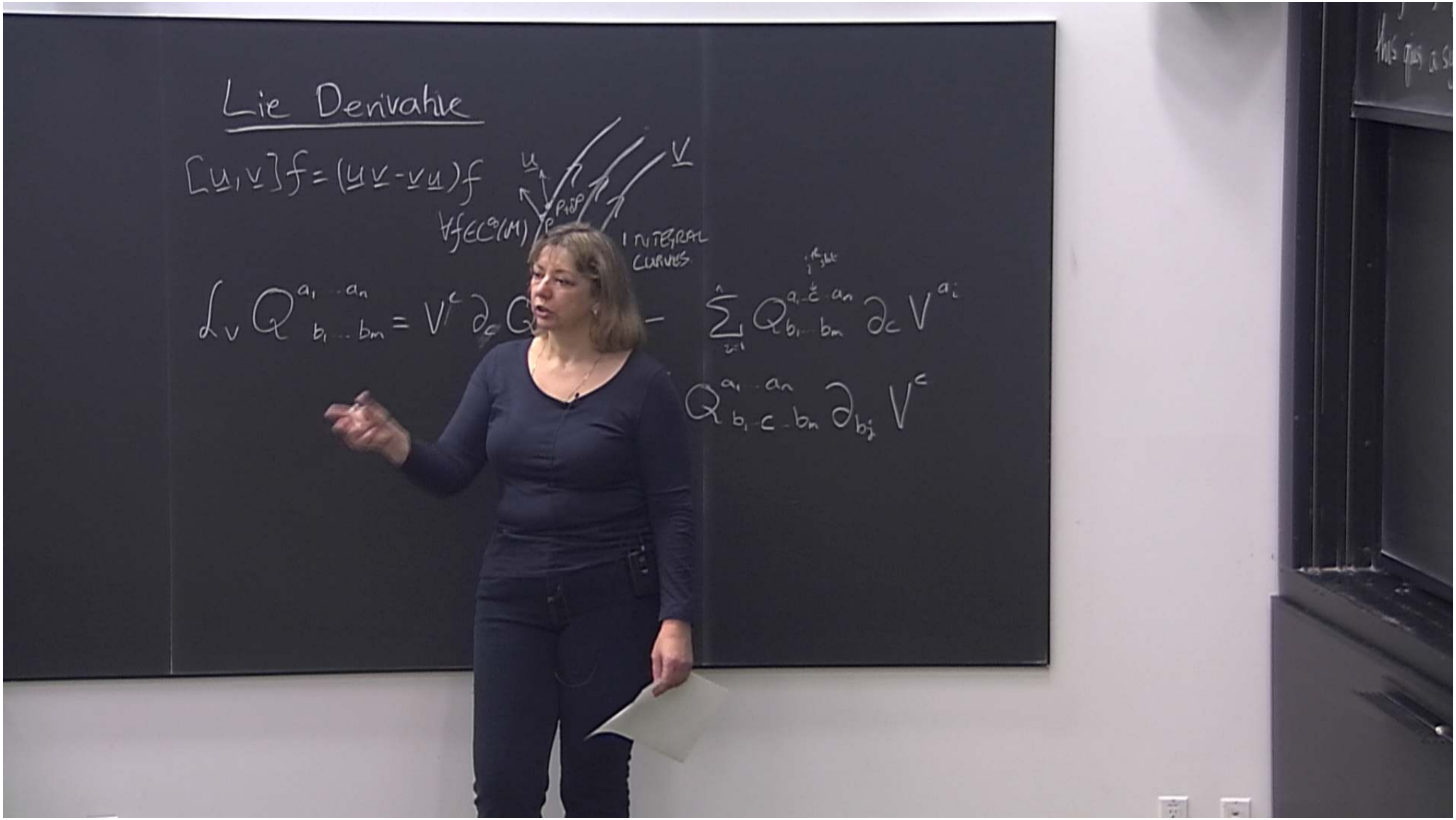
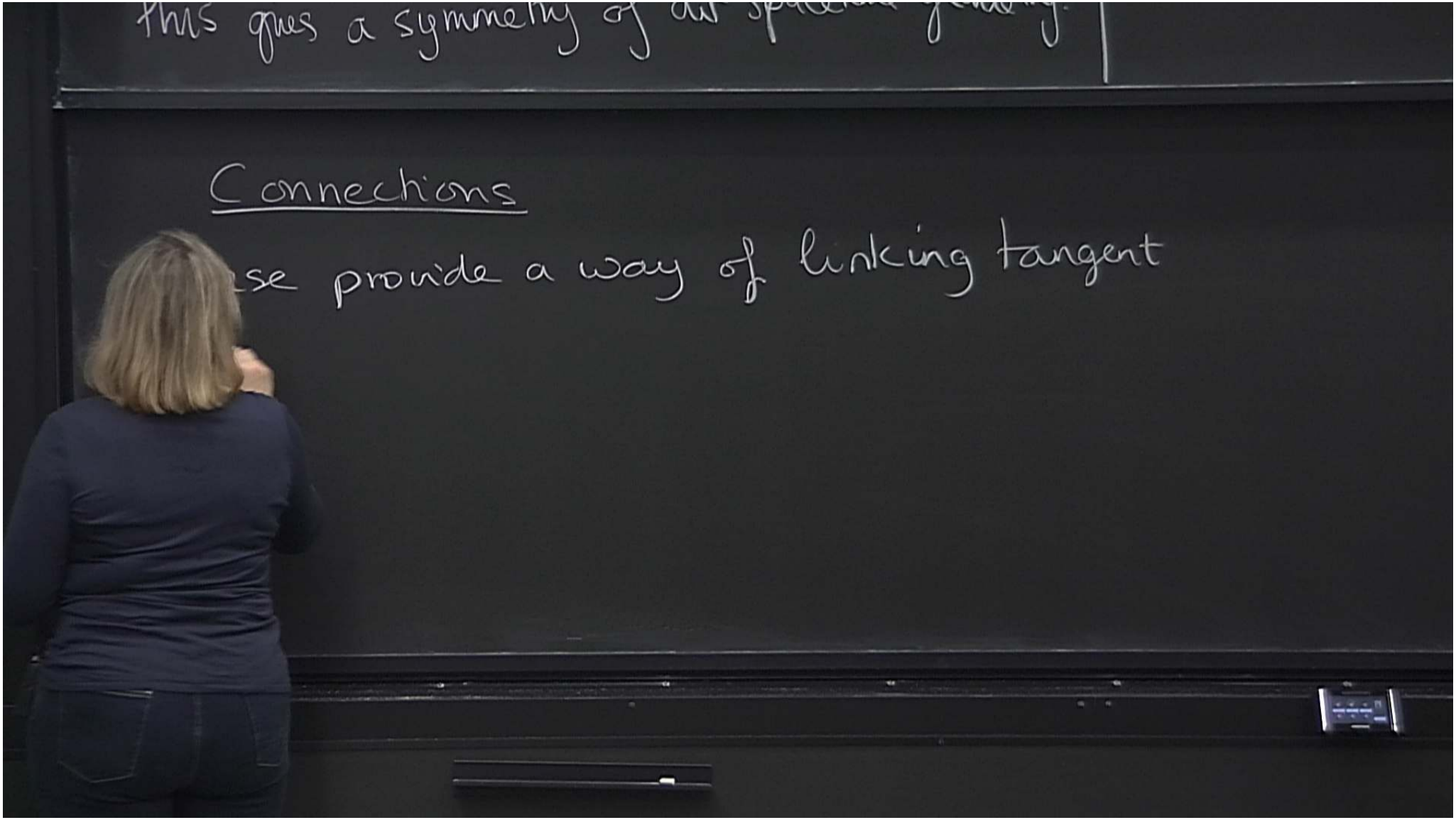
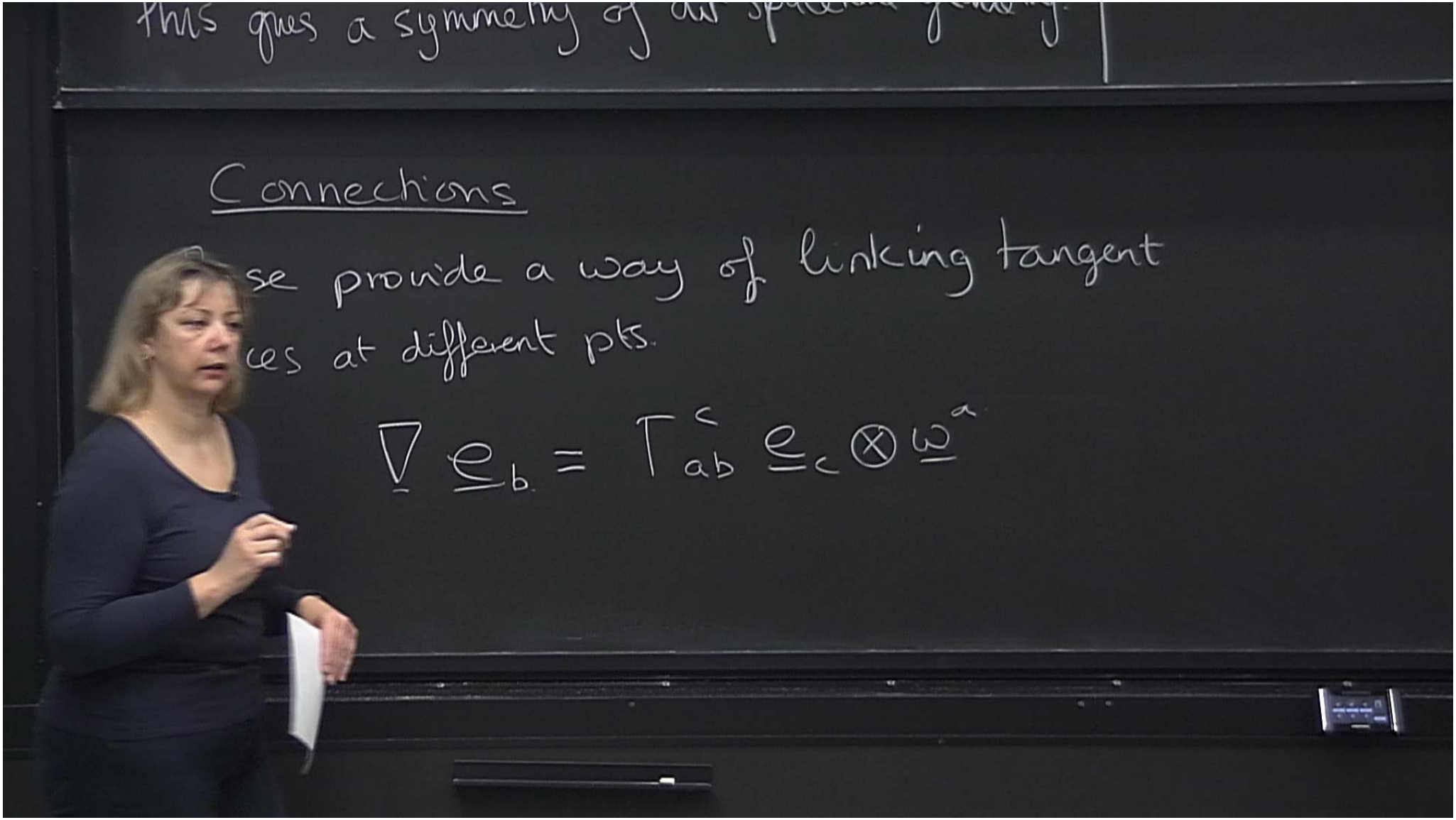

If $\mathcal{L}_v g_{ab} = 0$, then $\underline{v}$ is a Killing vector (eg $\frac{\partial}{\partial t}$, SO(3)...)
We also have a
bet
metry.


If $\mathcal{L}_v g_{ab} = 0$, then $\underline{v}$ is a Killing vector (eg $\frac{\partial}{\partial t}$, SO(3) ...)
We also h a relation between d & $\mathcal{L}_v$.
$\langle d \quad \underline{v} \rangle = \underline{u}(\langle \omega | \underline{v} \rangle) - \underline{v}(\langle \omega | \underline{u} \rangle) - \langle \omega | [\underline{u}, \underline{v}] \rangle$
ometry.


Lie Derivative
$[\underline{u},\underline{v}]f=(\underline{u}\underline{v}-\underline{v}\underline{u})f$
$\forall f\in C^\infty(M)$
$\underline{u}$
$\underline{v}$
p
INTEGRAL CURVES
$\mathcal{L}_v Q^{a_1\dots a_n}_{b_1\dots b_m} = V^c \partial_c Q$
$-\sum_{i=1} Q^{a_1\dots c\dots a_n}_{b_1\dots b_m} \partial_c V^{a_i}$
$Q^{a_1\dots a_n}_{b_1\dots c\dots b_m} \partial_{b_j} V^c$


this gives a symmetry of
Connections
se provide a way of linking tangent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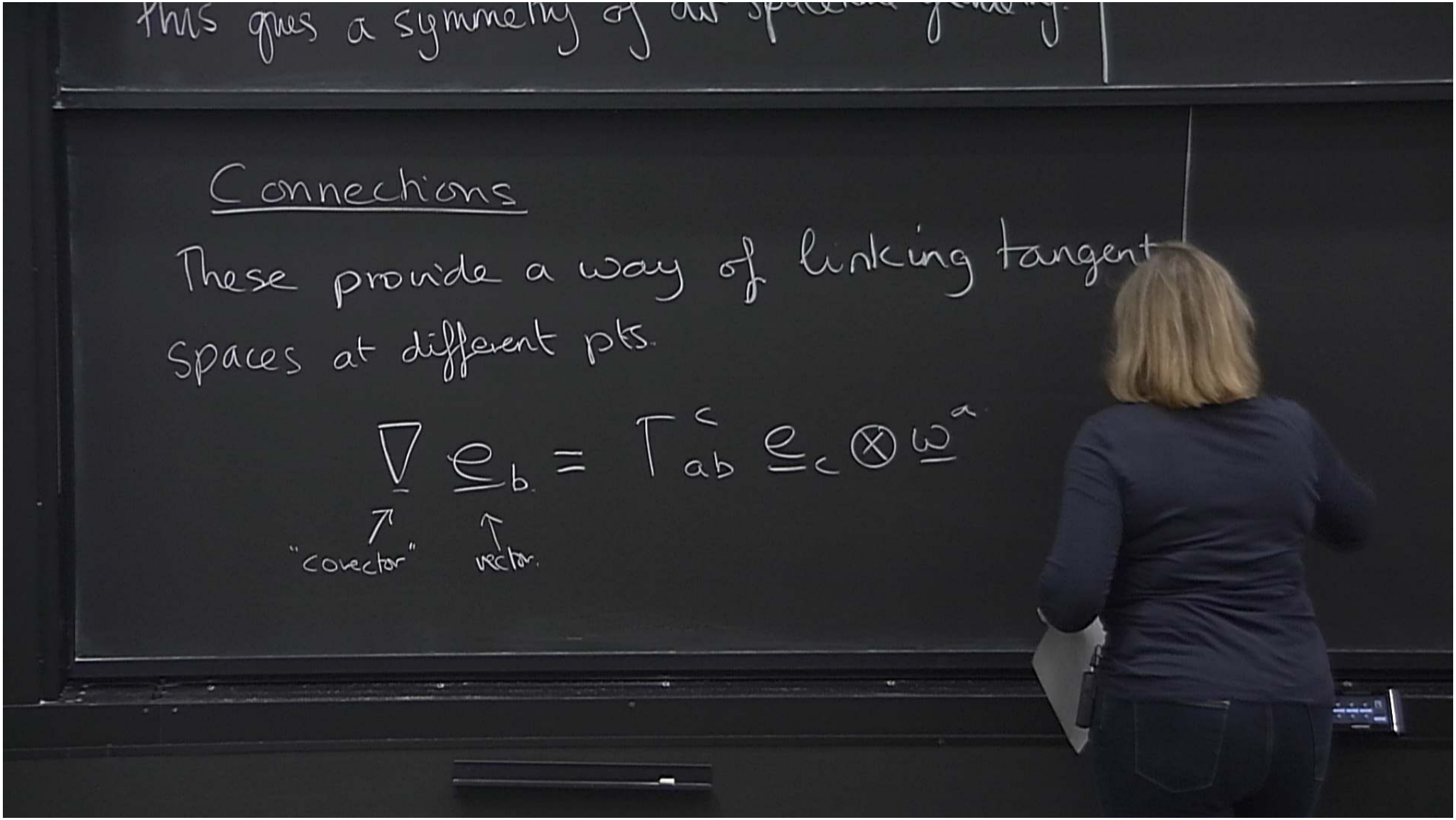
ces at different pts.
$\nabla \underline{e}_b = \Gamma^c_{ab}\, \underline{e}_c \otimes \underline{\omega}^a$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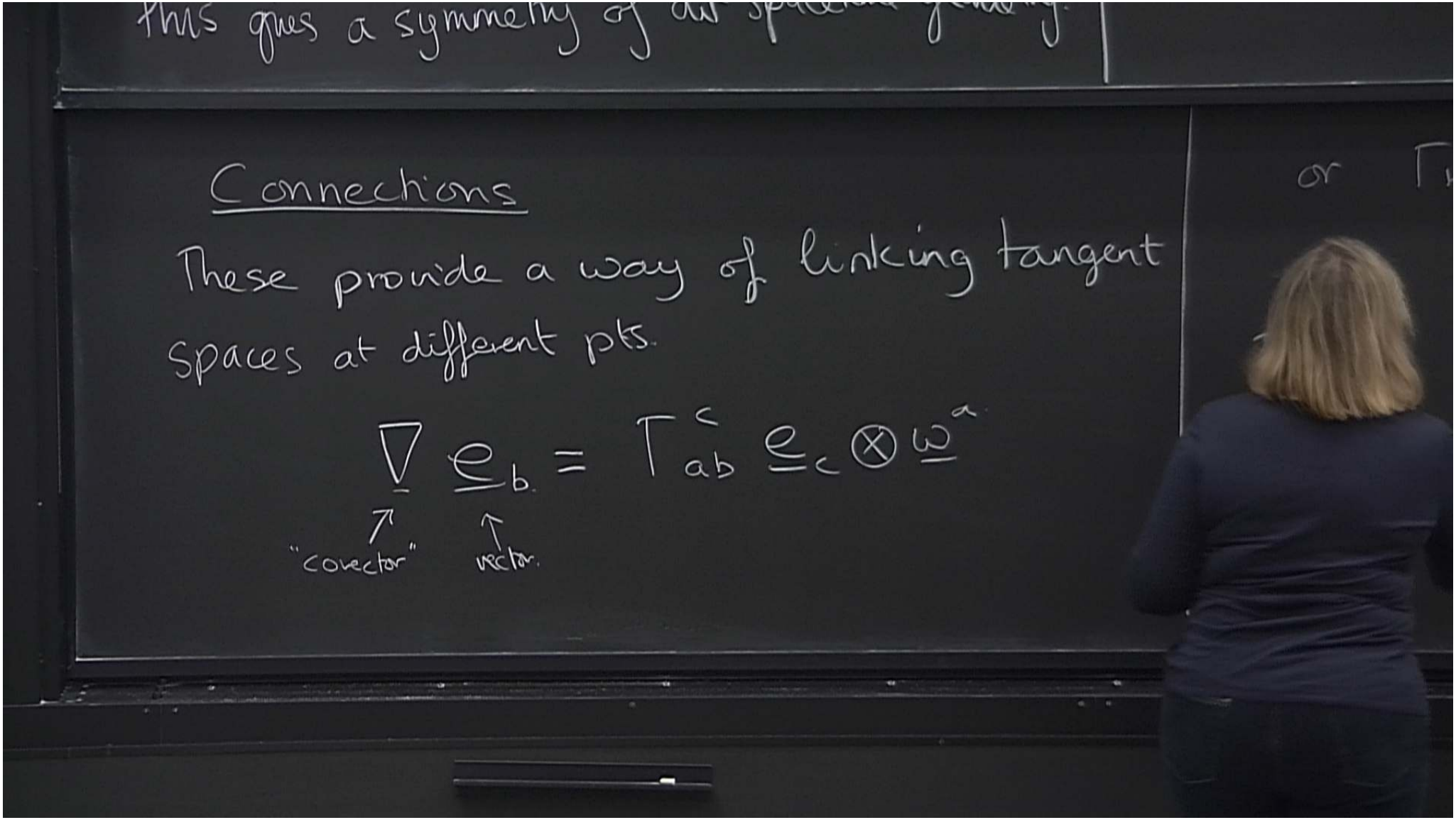
"covector"
vector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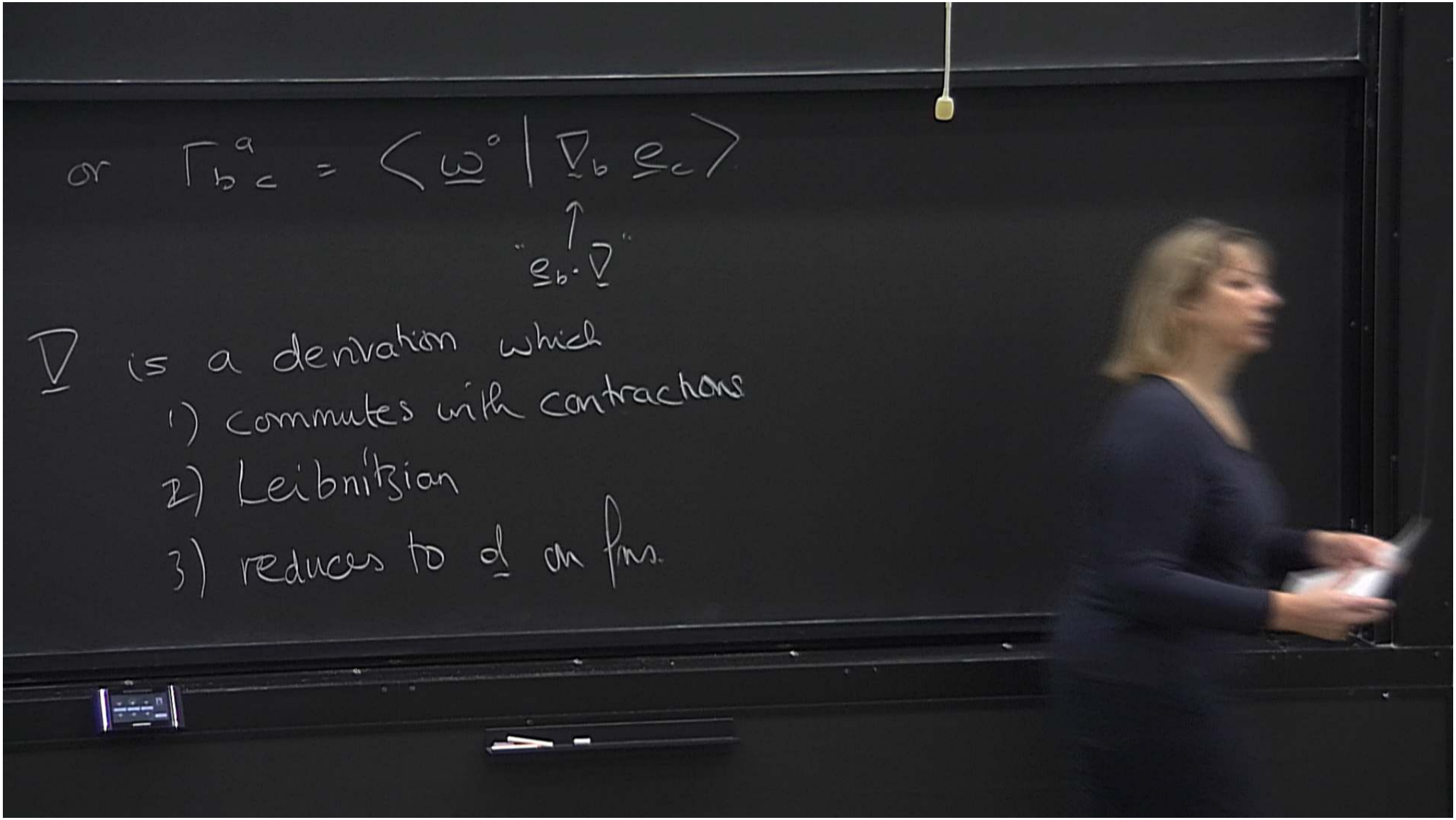
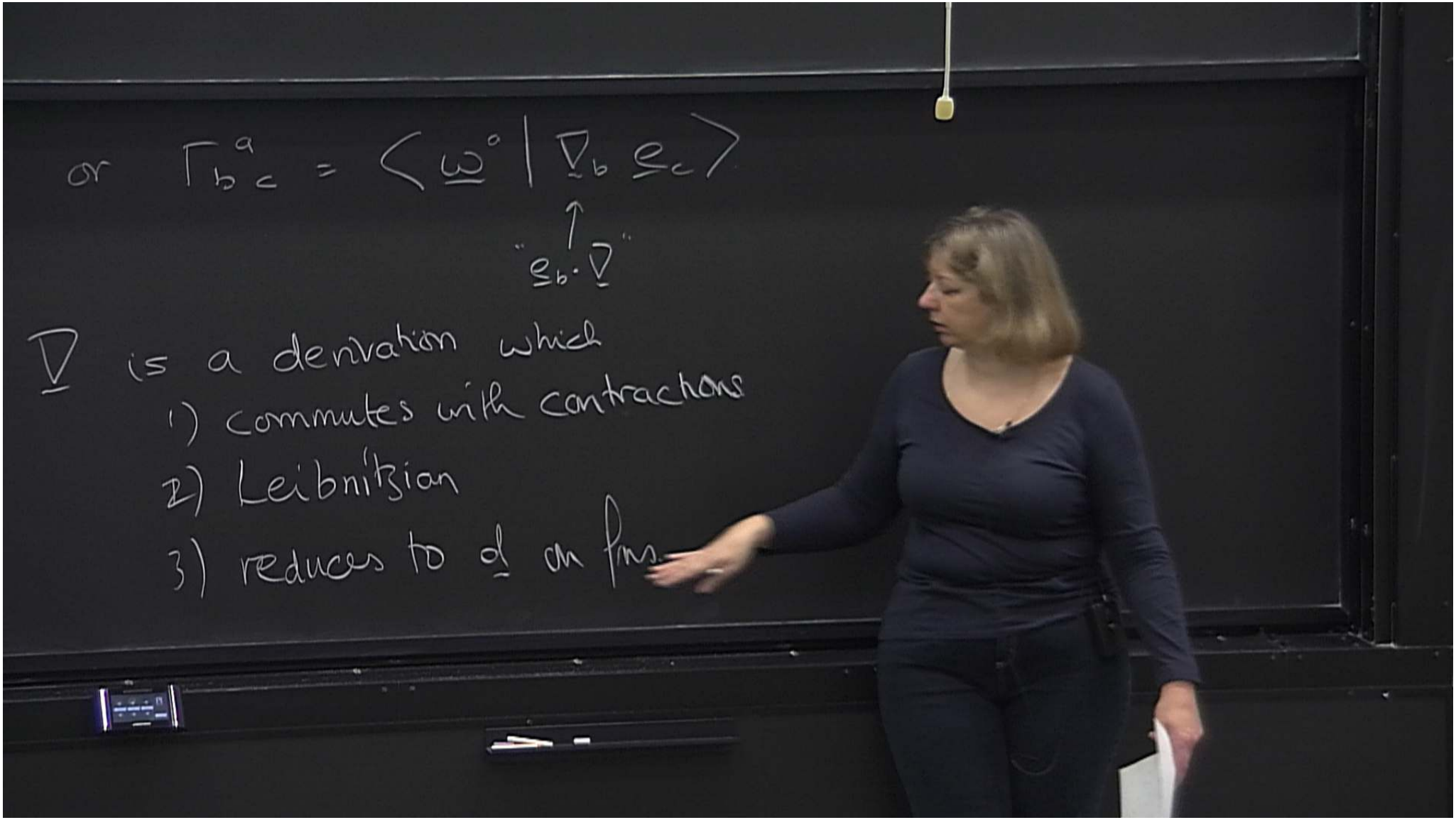
or


or $\Gamma^a_{bc} = \langle \omega^a | \nabla_b e_c \rangle$
"$e_b \cdot \nabla$"
∇ is a derivation which
1) commutes with contractions
2) Leibnitzian
3) reduces to d on fns.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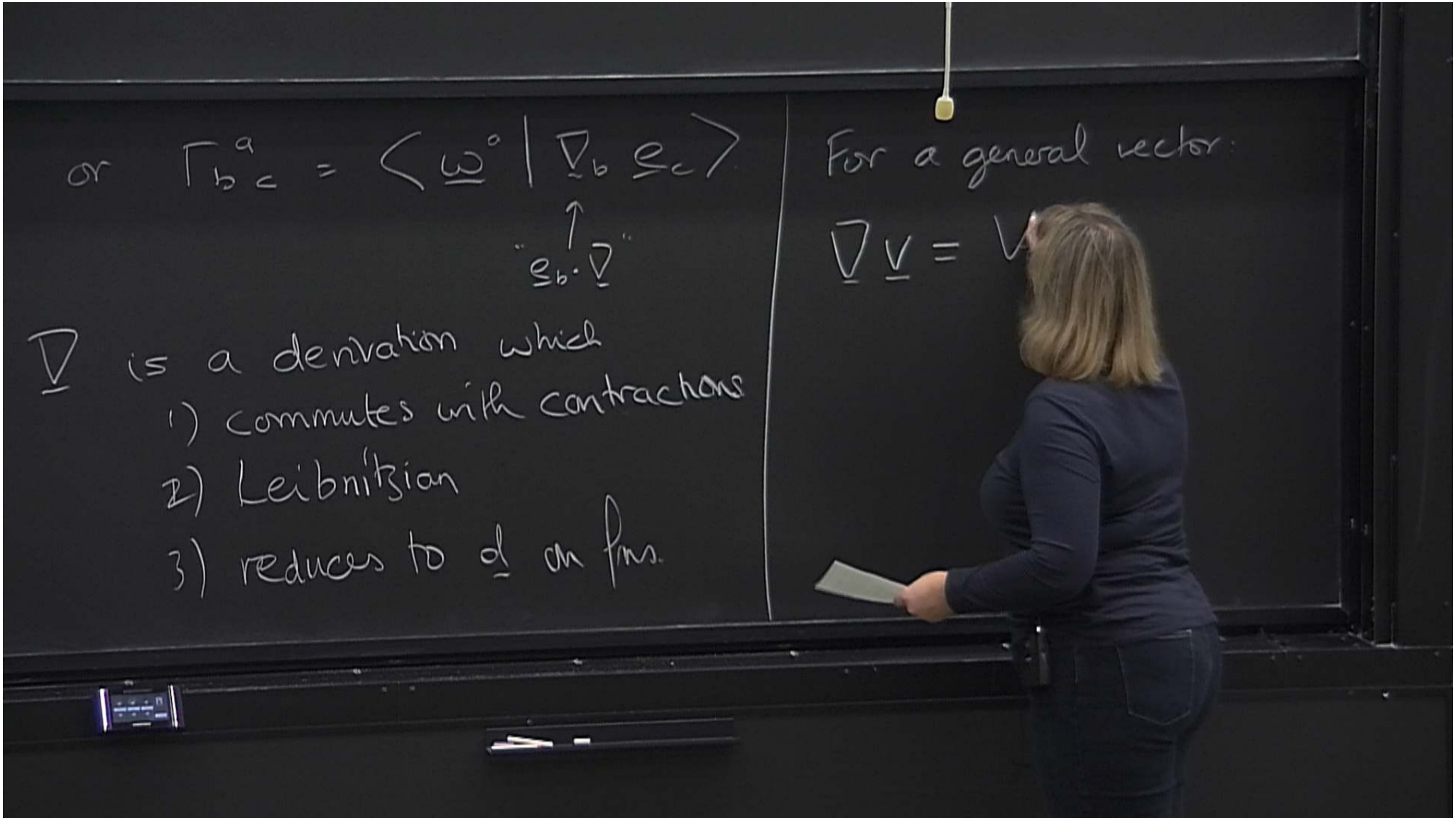
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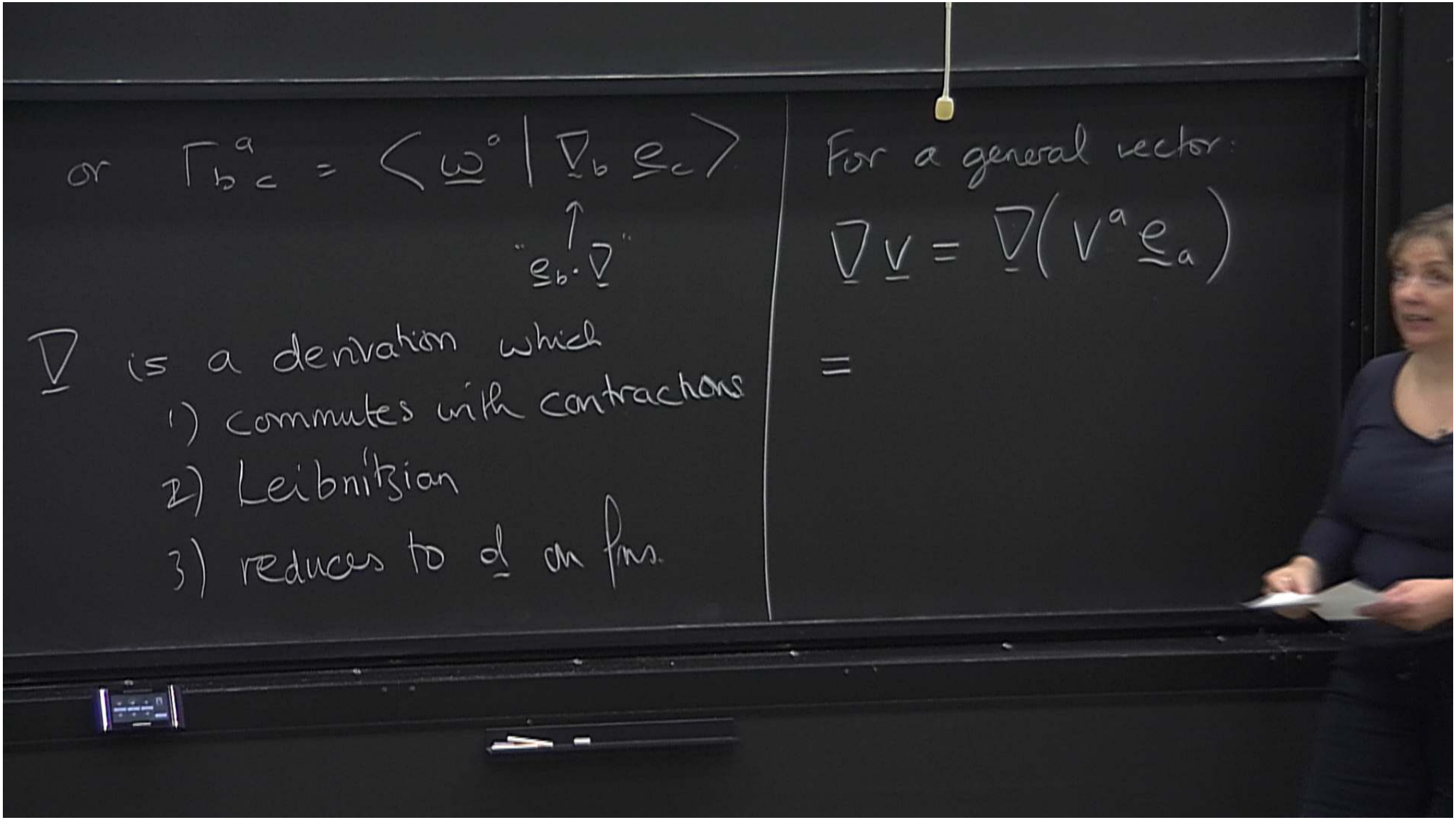
3) reduces to d on fns.
For a general vector:
$\nabla V =$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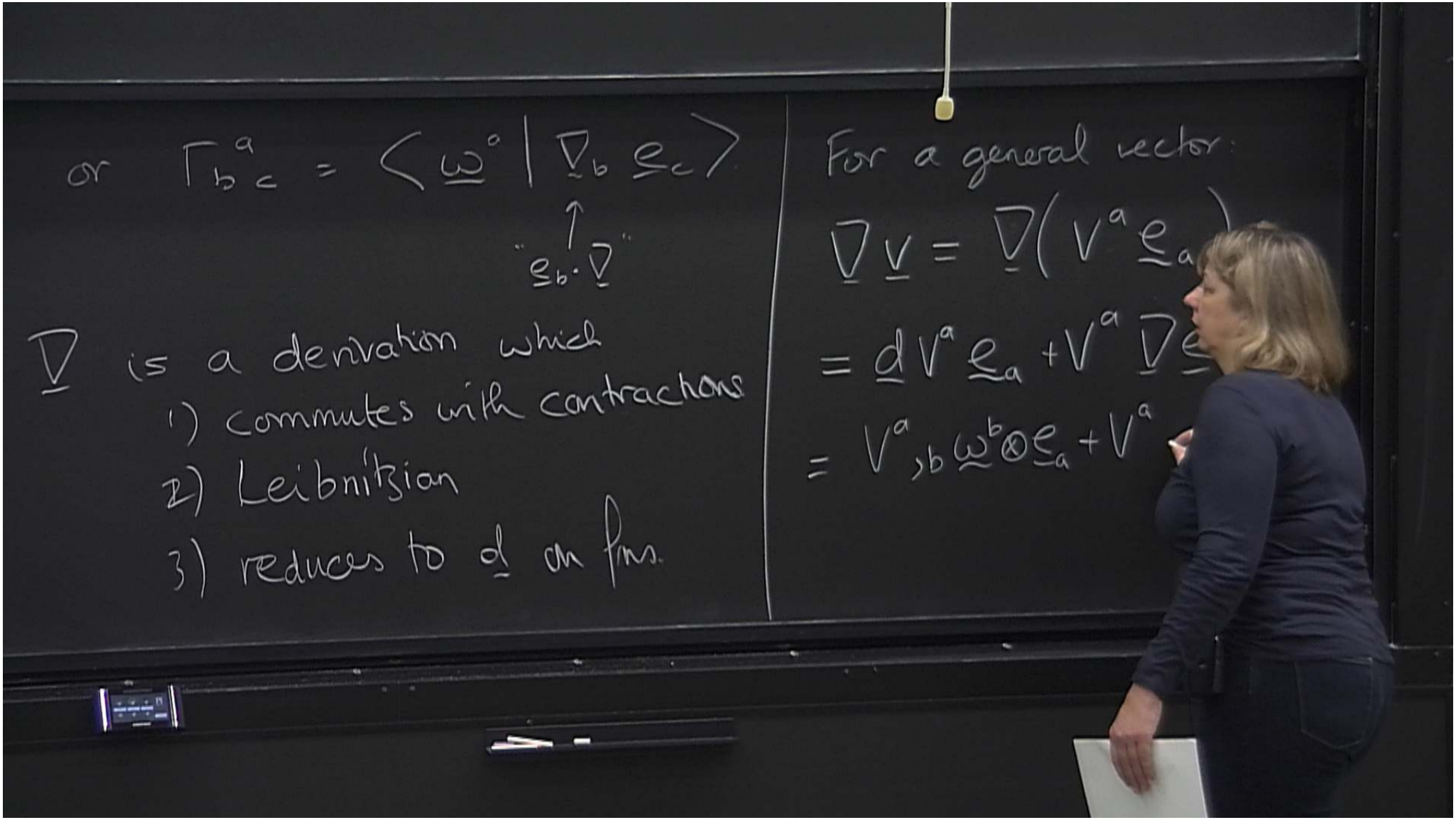
For a general vector:
$\nabla V = \nabla(V^a e_a)$
=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1) commutes with contractions
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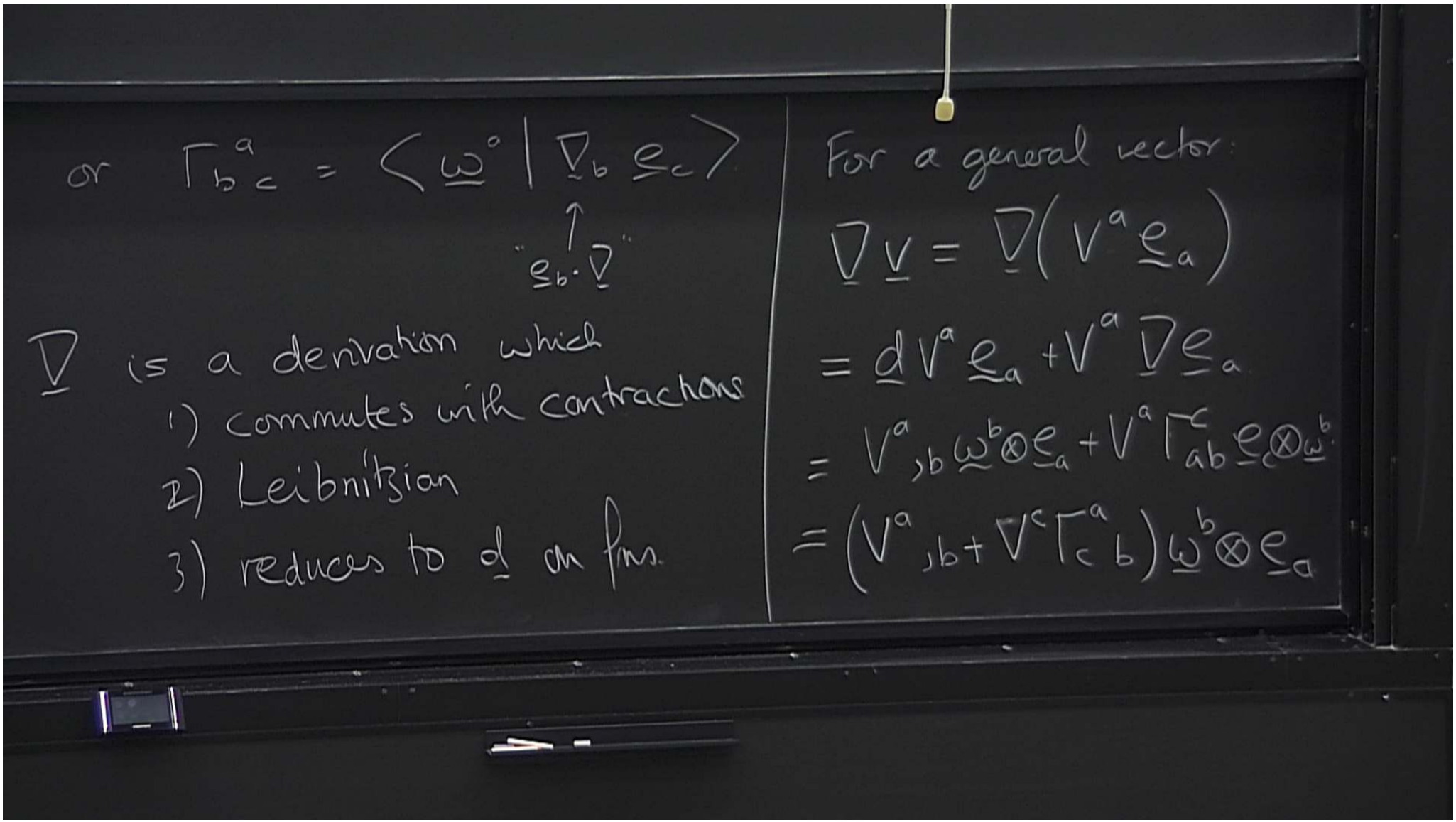
3) reduces to d on fns.
For a general vector:
$\nabla v = \nabla(v^a e_a)$
$= d v^a e_a + v^a \nabla e$
$= v^a{}_{,b}\, \omega^b \otimes e_a + v^a$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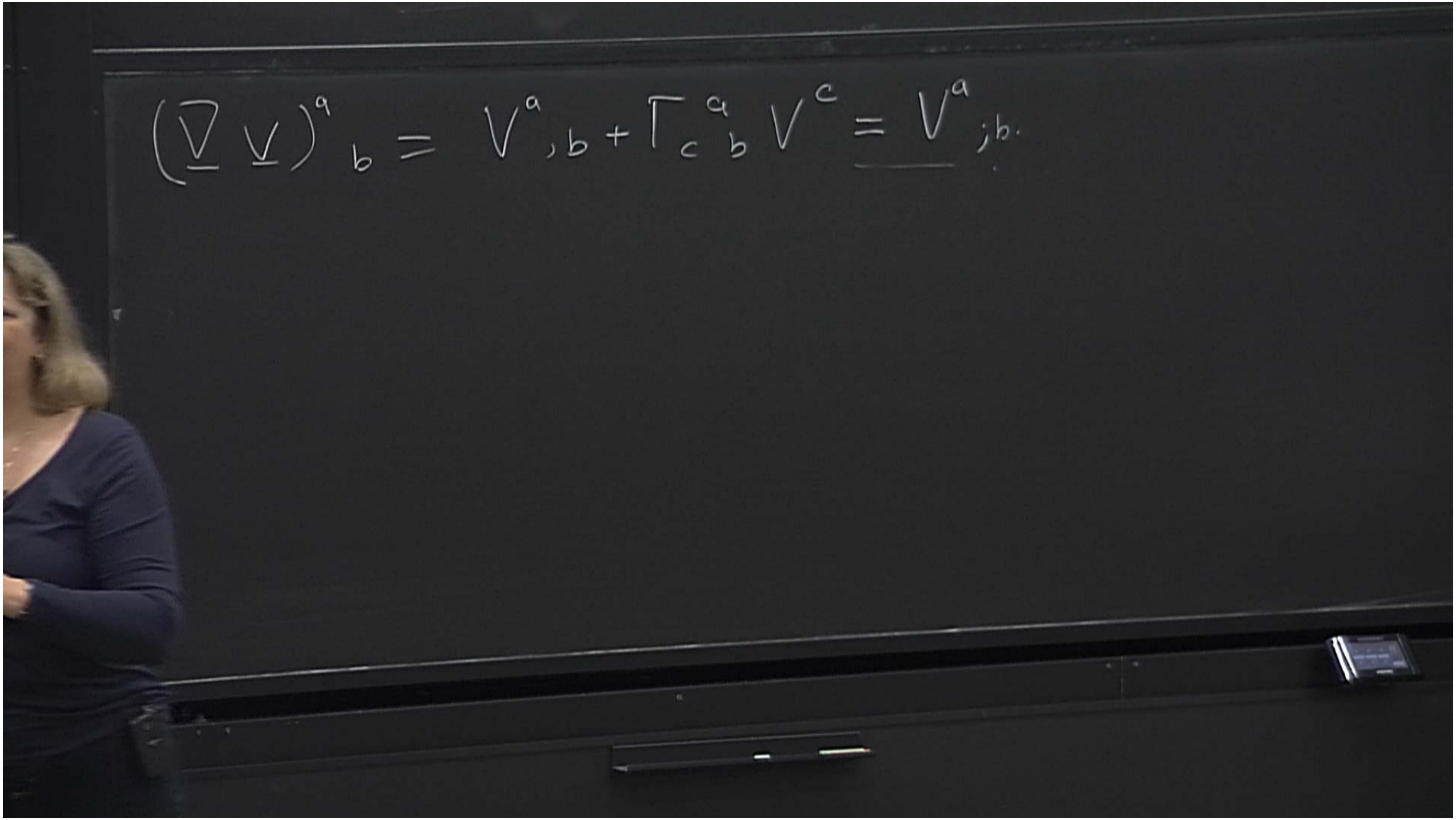
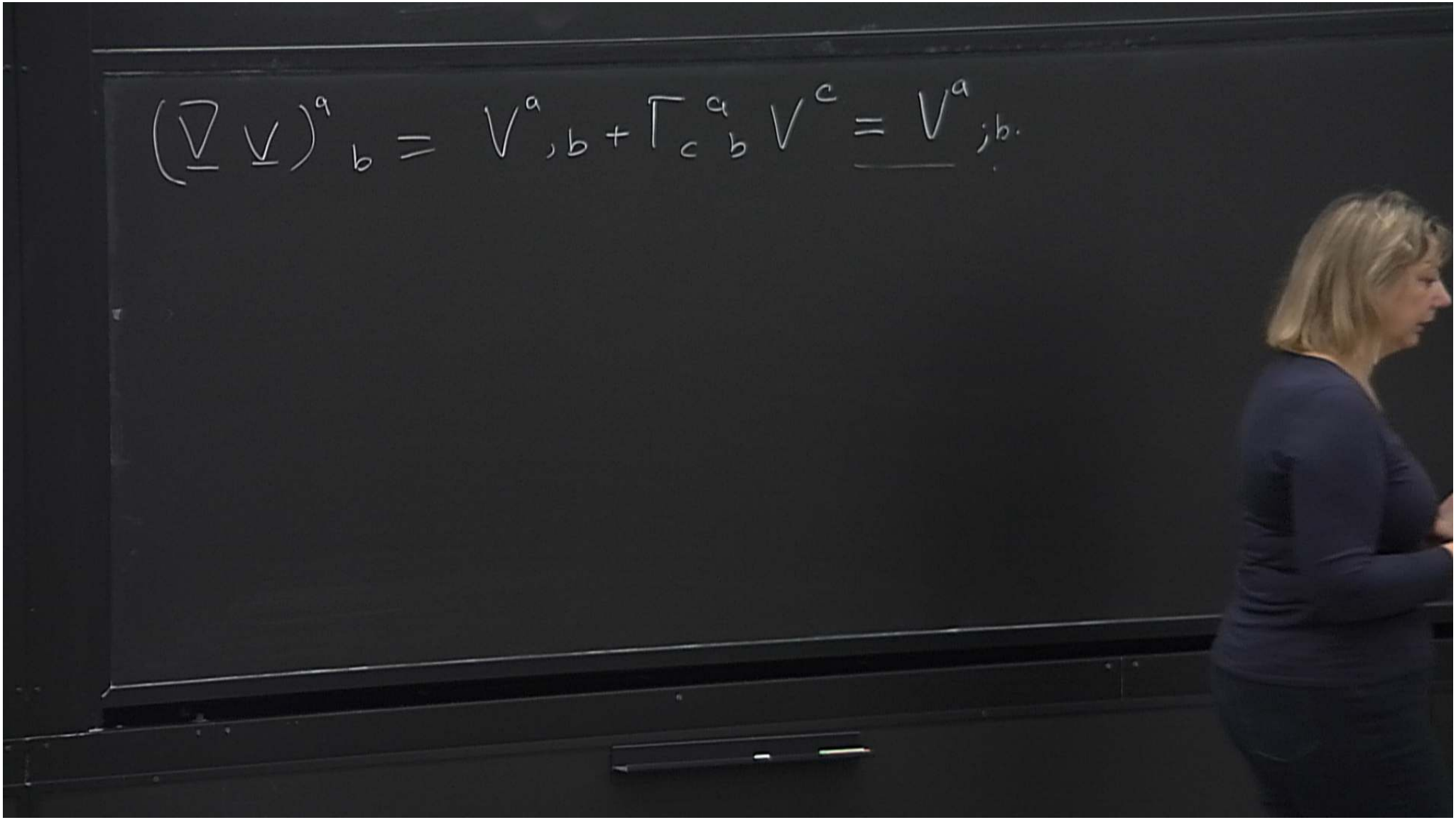
For a general vector:
$\nabla v = \nabla(v^a e_a)$
$= d v^a e_a + v^a \nabla e_a$
$= v^a{}_{,b}\, \omega^b \otimes e_a + v^a \Gamma^c_{ab} e_c \otimes \omega^b$
$= (v^a{}_{,b} + v^c \Gamma^a_{cb}) \omega^b \otimes e_a$


$(\underline{\nabla}\,\underline{V})^a{}_b = V^a{}_{,b} + \Gamma^a_{cb} V^c = V^a{}_{;b}$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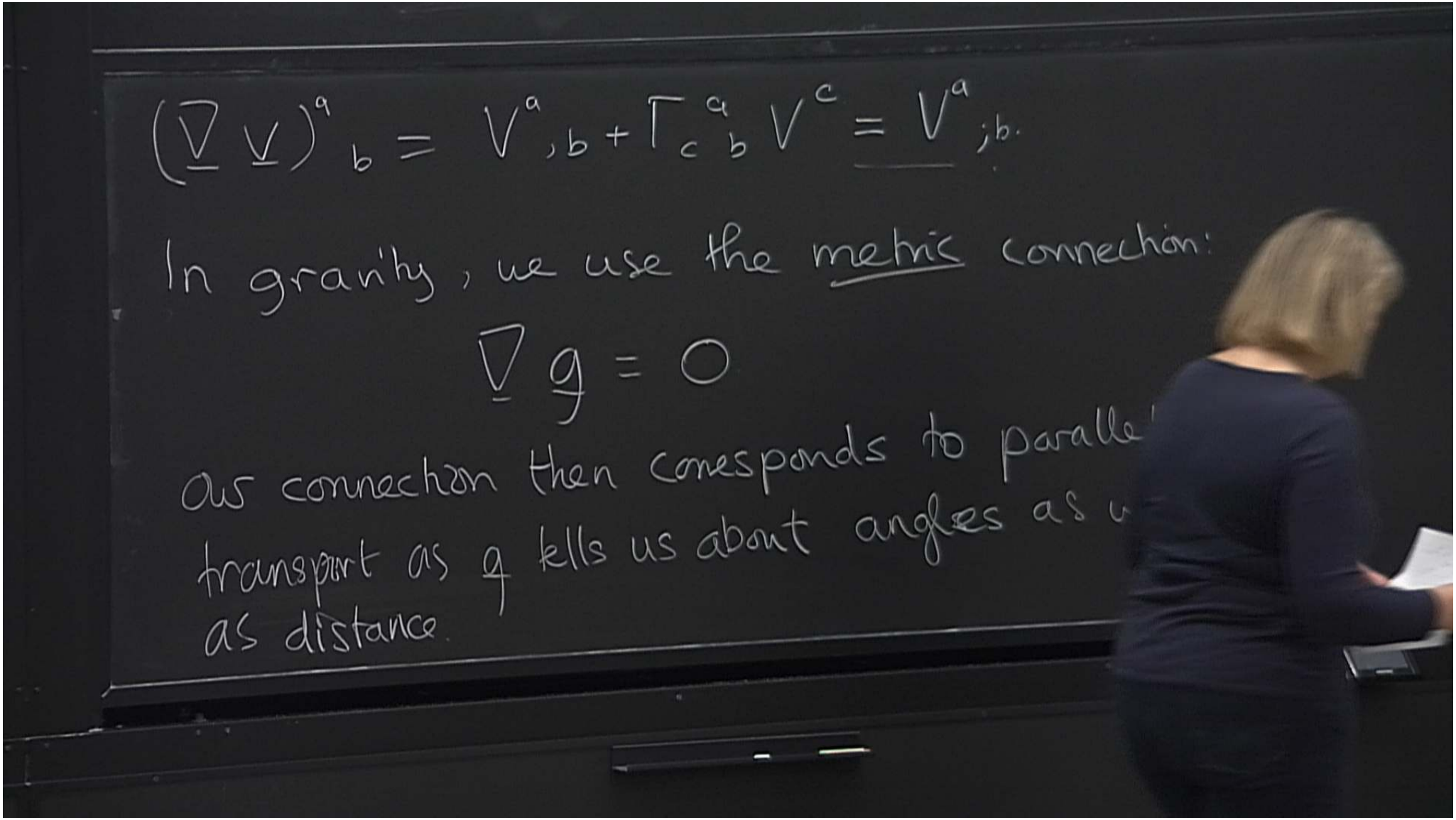
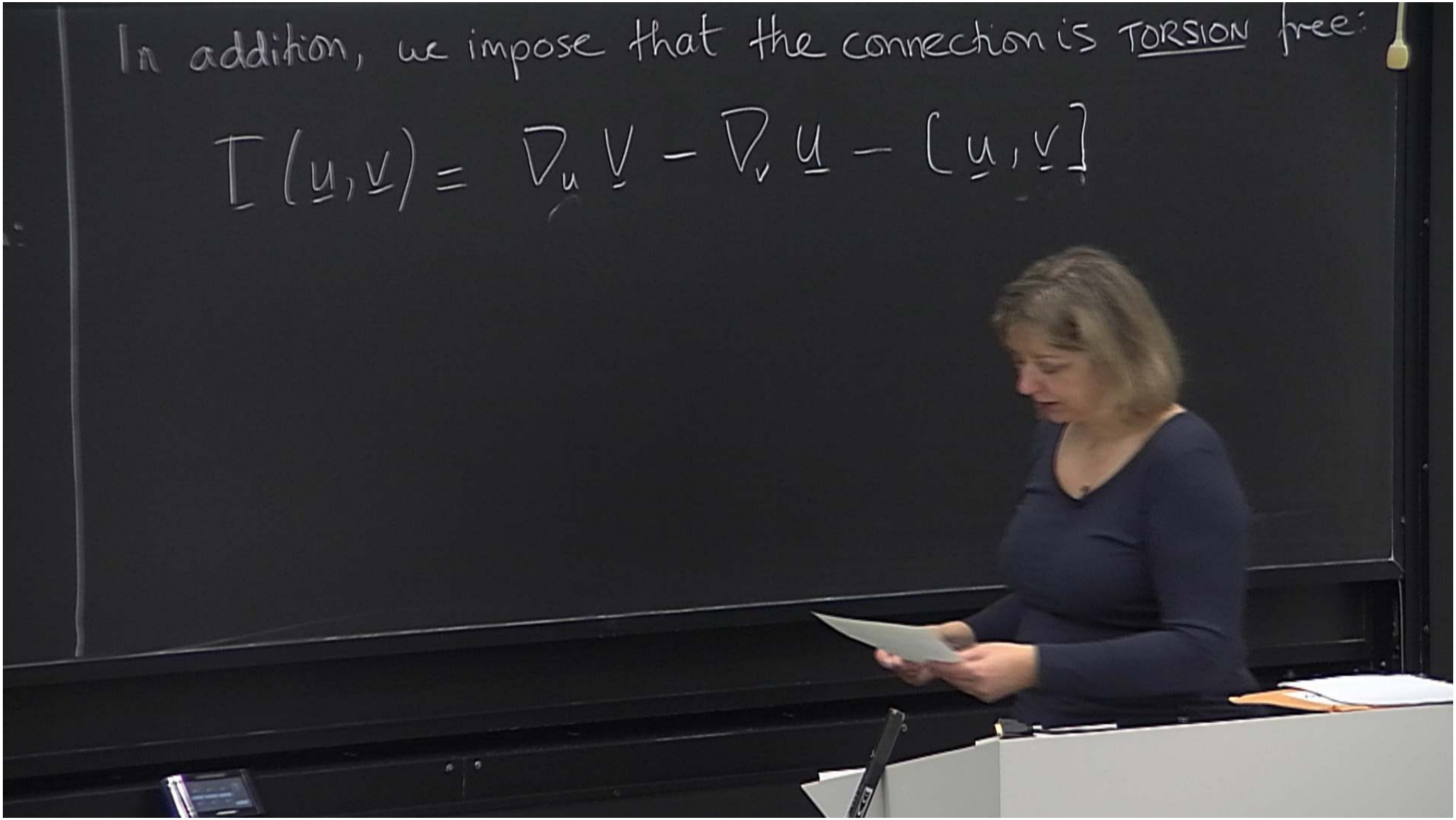
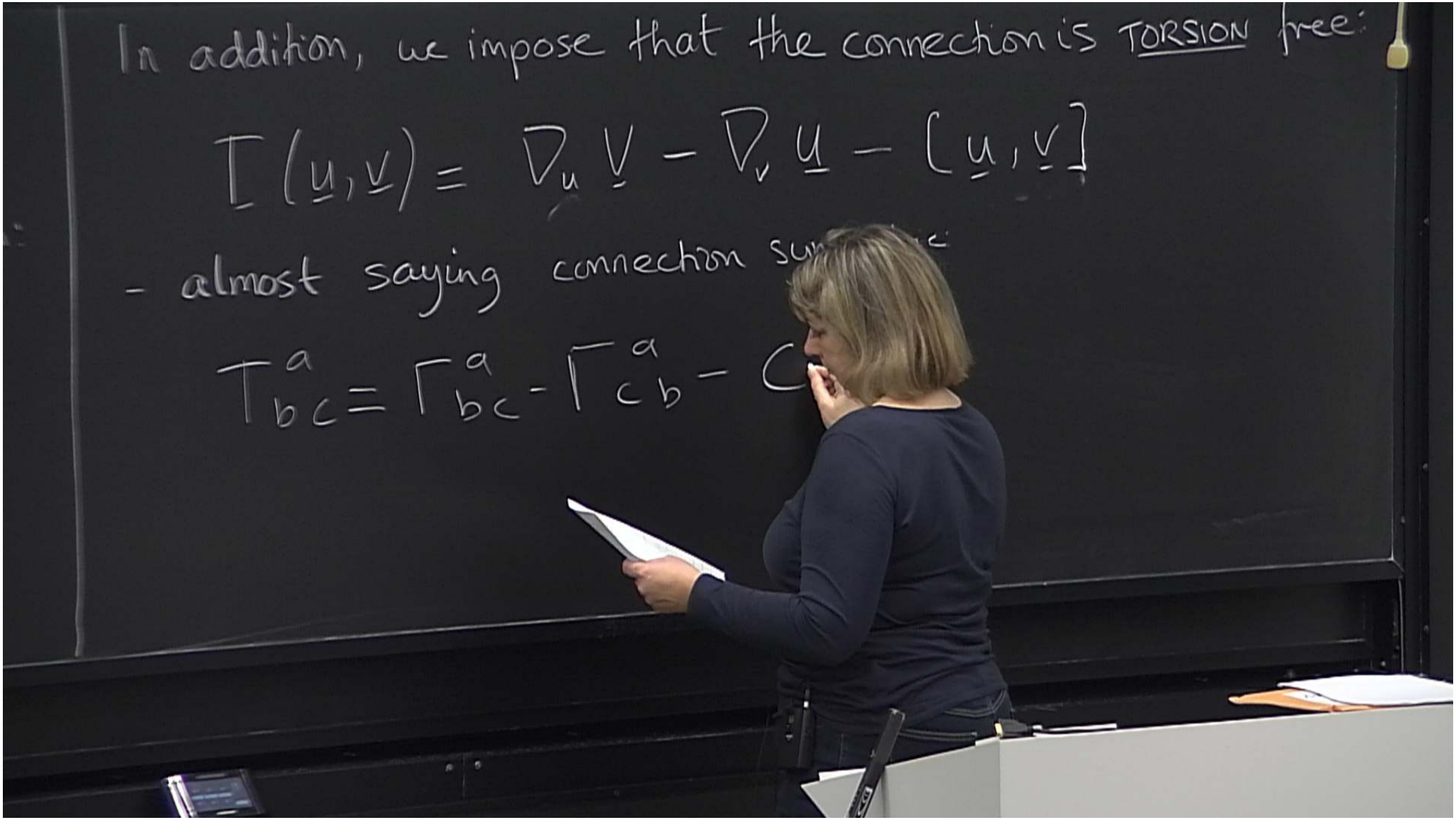
$$(\underline{\nabla}\,\underline{V})^a{}_b = V^a{}_{,b} + \Gamma^a_{c\,b} V^c = \underline{V^a{}_{;b}}$$

In gravity, we use the metric connection:

$$\underline{\nabla}\,\underline{g} = 0$$


$(\underline{\nabla}\, \underline{V})^a{}_b = V^a{}_{,b} + \Gamma^a_{cb} V^c = \underline{V^a{}_{;b}}$
In gravity, we use the metric connection:
$\underline{\nabla}\, g = 0$
our connection then corresponds to parall
transport as g tells us about angles as
as distance


In addition, we impose that the connection is TORSION free:
$T(\underline{u},\underline{v}) = \nabla_{\underline{u}}\underline{v} - \nabla_{\underline{v}}\underline{u} - [\underline{u},\underline{v}]$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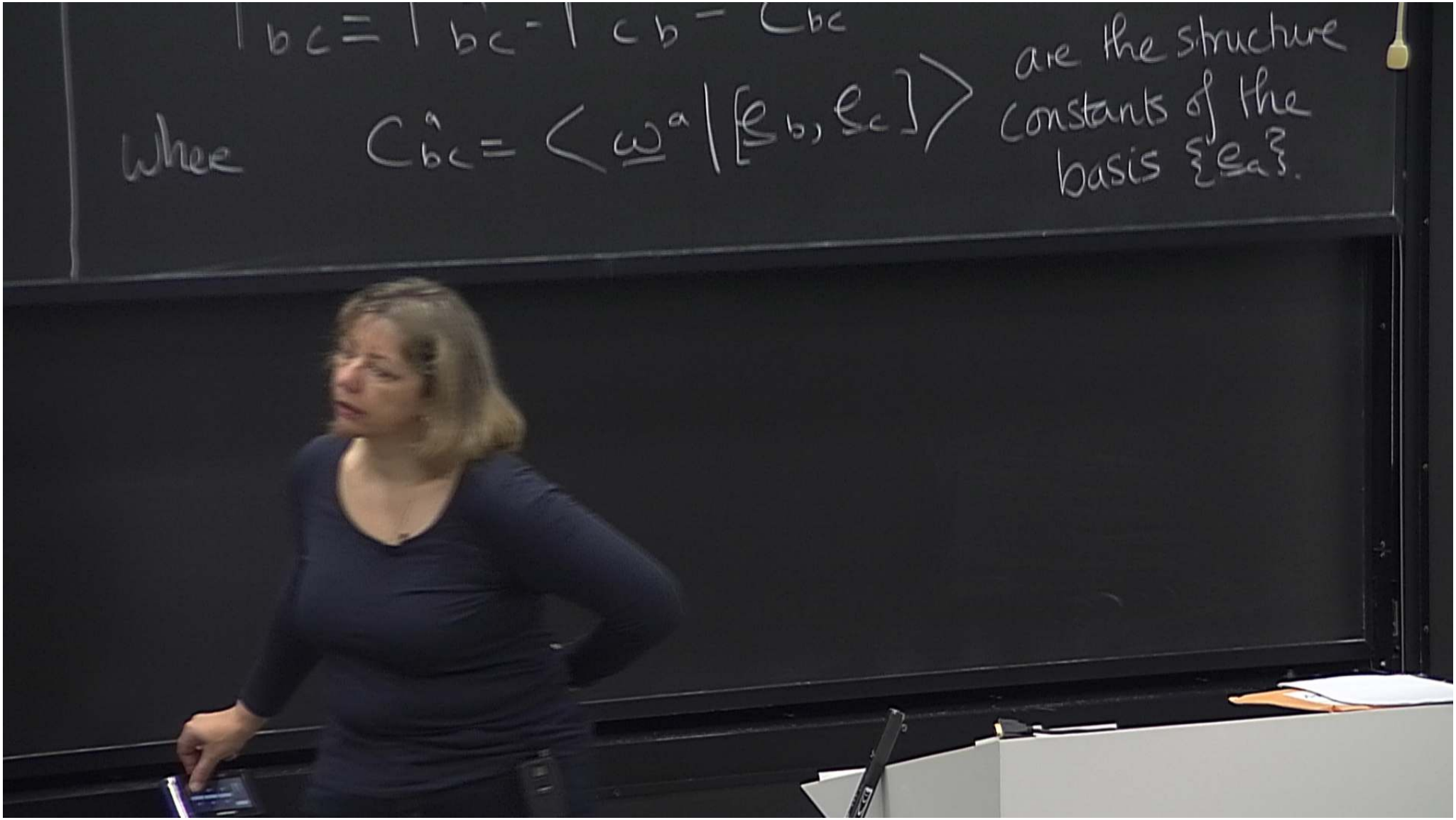
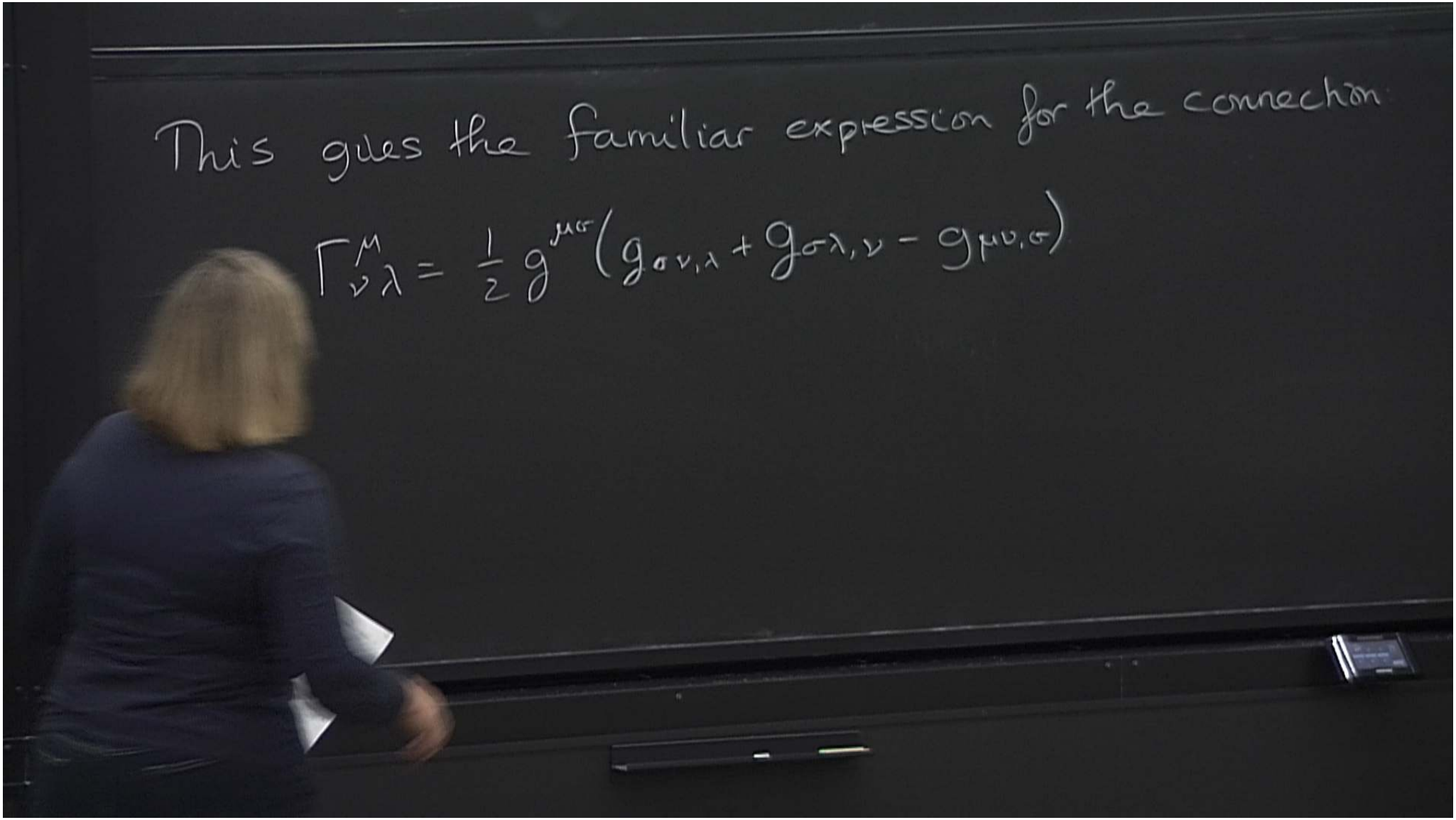
$T(\underline{u},\underline{v}) = \nabla_{\underline{u}}\underline{v} - \nabla_{\underline{v}}\underline{u} - [\underline{u},\underline{v}]$
- almost saying connection
$T^{a}_{bc} = \Gamma^{a}_{bc} - \Gamma^{a}_{cb} - C$


$C_{bc} = -C_{cb} - C_{bc}$
Where $C^a_{bc} = \langle \omega^a | [e_b, e_c] \rangle$ are the structure constants of the basis $\{e_a\}$.

This gives the familiar expression for the connection

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} - g_{\mu\nu,\sigma} \right)$$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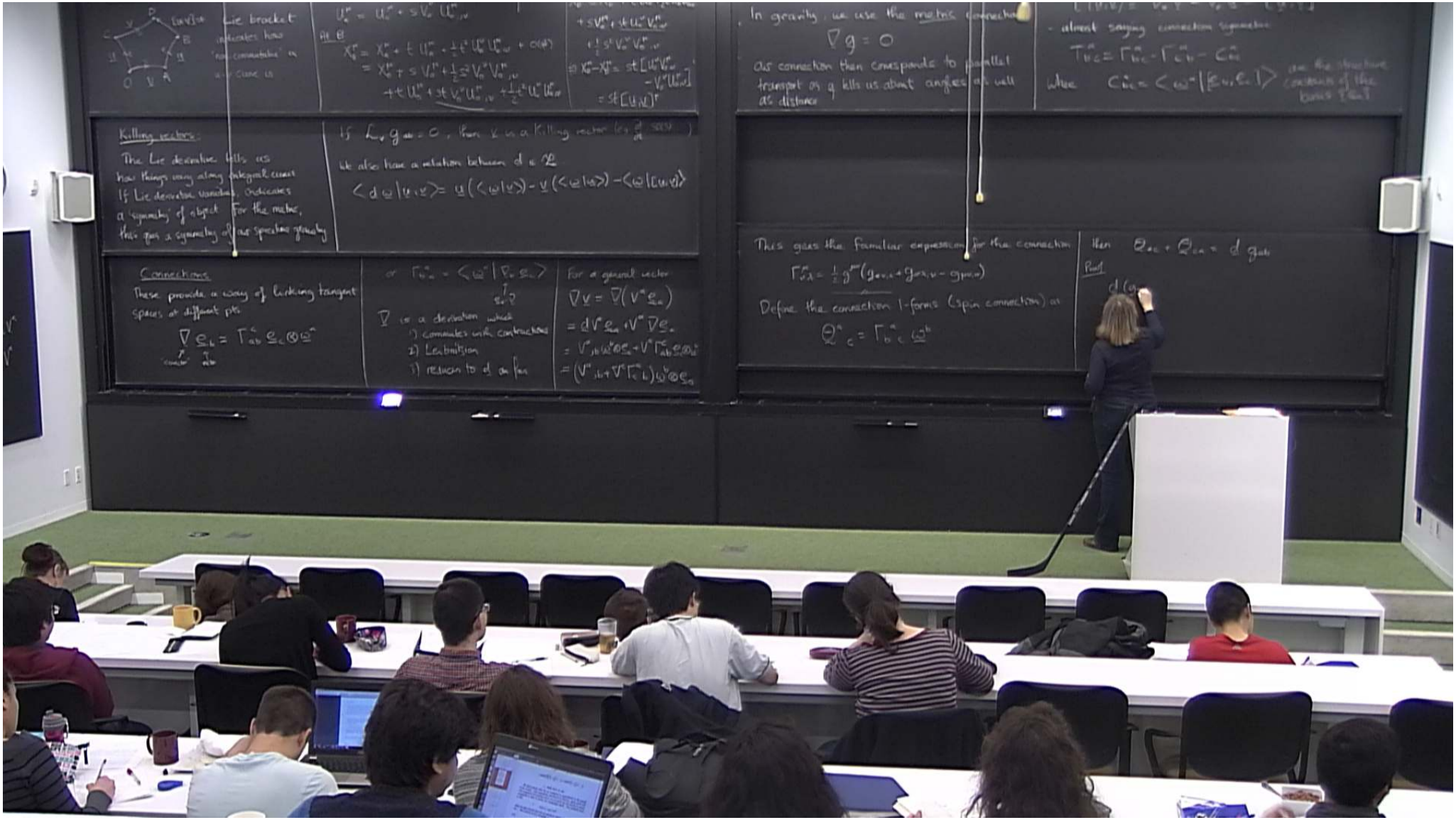
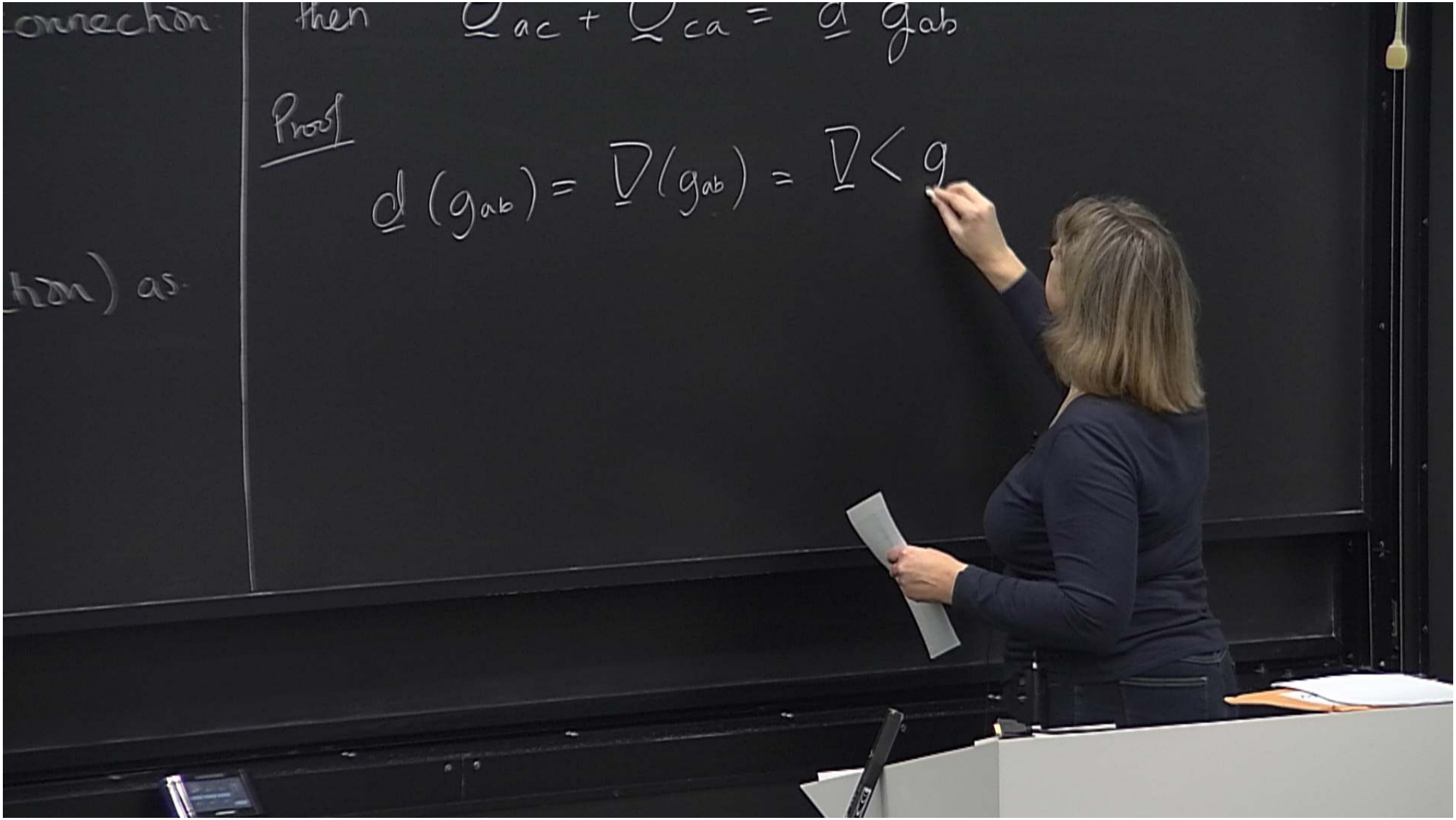
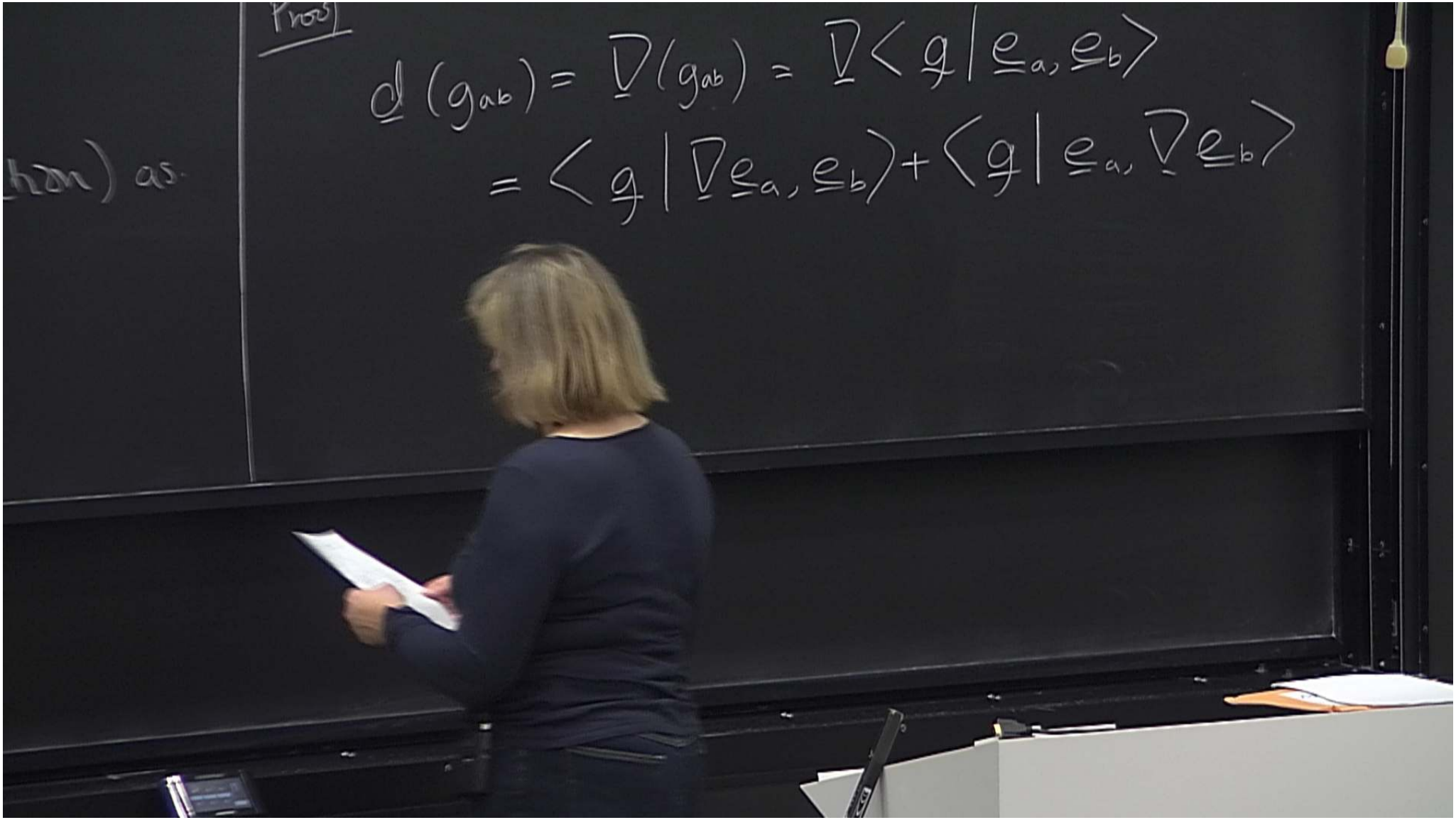
$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}\left(g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} - g_{\mu\nu,\sigma}\right)$$

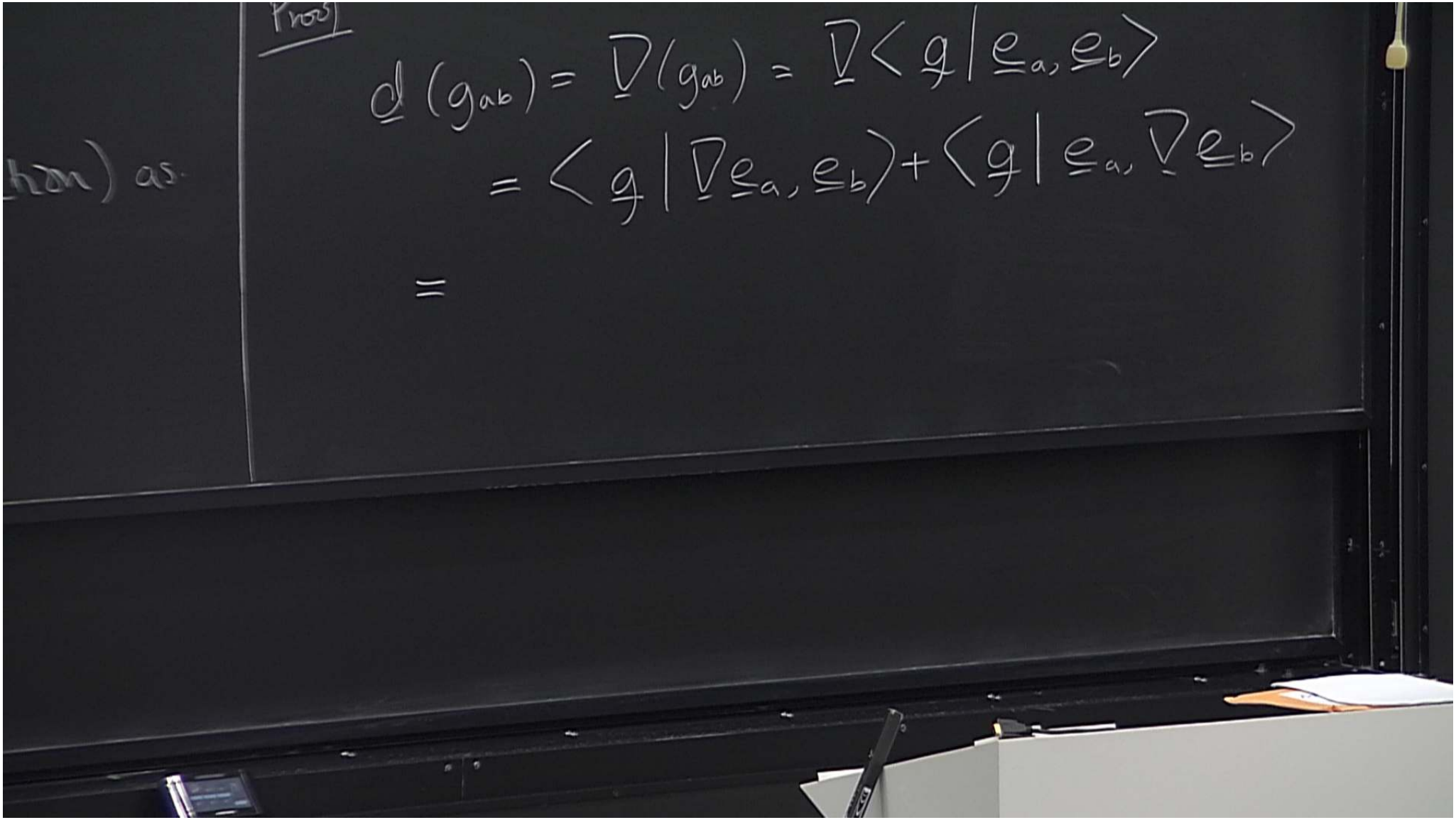
Define the connection 1-forms (spin connection) as

$$\underset{\sim}{\theta}{}^{a}{}_{c} = \Gamma^{a}_{bc}\,\underline{\omega}^{b}$$


Killing vectors:
Connections:
In gravity, we use the metric connection
$\nabla g = 0$
Define the connection 1-forms (spin connection) as
$\Theta^a{}_c = \Gamma^a_{bc}\,\omega^b$


onnection
then
Proof
(hon) as


Proof
$\nabla_c(g_{ab}) = \nabla(g_{ab}) = \nabla\langle g | e_a, e_b\rangle$
$= \langle g | \nabla e_a, e_b\rangle + \langle g | e_a, \nabla e_b\rangle$


Proof
$\nabla(g_{ab}) = \nabla(g_{ab}) = \nabla\langle g | e_a, e_b\rangle$
$= \langle g | \nabla e_a, e_b\rangle + \langle g | e_a, \nabla e_b\rangle$
$=$

this gives a symmetry of the spacetime geometry.

Connections

These provide a way of linking tangent spaces at different pts.

$$\underline{\nabla}\,\underline{e}_b = \Gamma^{c}_{ab}\,\underline{e}_c \otimes \underline{\omega}^{a}$$

$\underline{\nabla}$: "covector" — $\underline{e}_b$: vector. — Γ^{c}_{ab}: $\underline{\theta}^{c}{}_{b} \otimes \underline{e}_c$

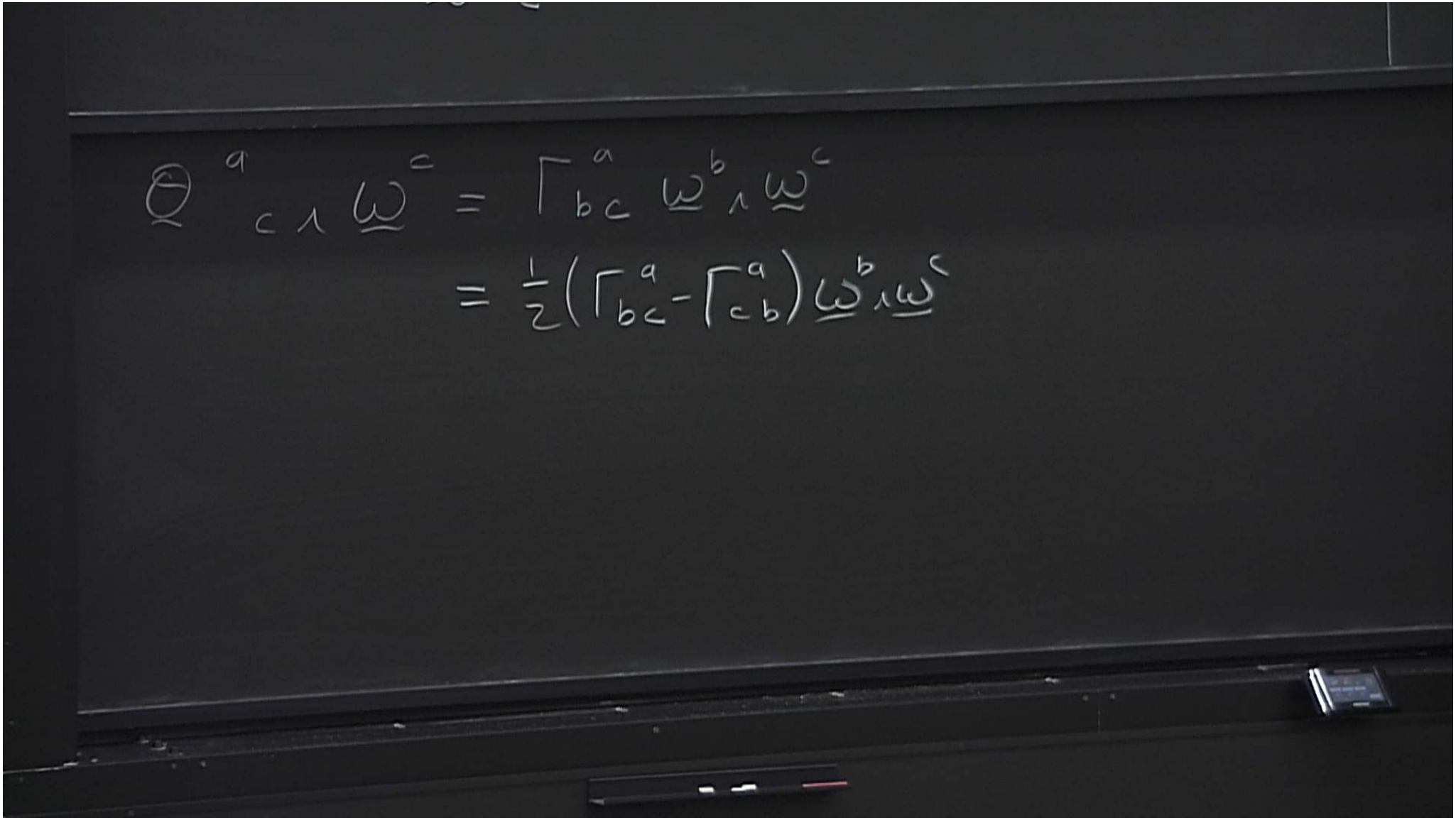
or Γ

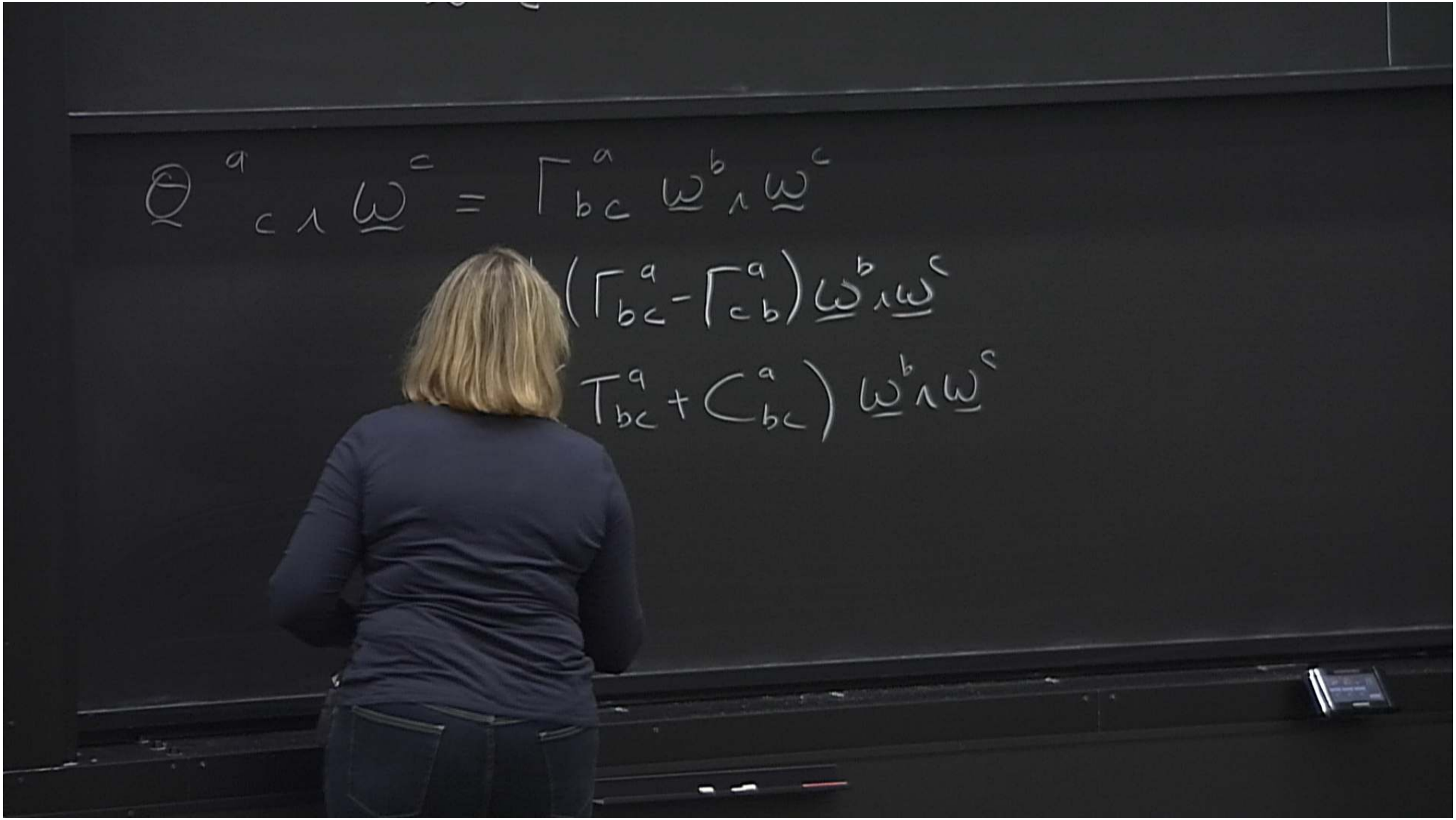
$\underline{\nabla}$ is a

1)

2)

3)


$\Theta^a{}_{c\wedge}\omega^c = \Gamma^a_{bc}\,\omega^b\wedge\omega^c$
$= \frac{1}{2}\left(\Gamma^a_{bc} - \Gamma^a_{cb}\right)\omega^b\wedge\omega^c$


$\Theta^{a}{}_{c} \wedge \omega^{c} = \Gamma^{a}_{bc}\, \omega^{b} \wedge \omega^{c}$
$(\Gamma^{a}_{bc} - \Gamma^{a}_{cb})\, \omega^{b} \wedge \omega^{c}$
$T^{a}_{bc} + C^{a}_{bc})\, \omega^{b} \wedge \omega^{c}$

$$\Theta^a{}_{c\wedge}\,\underline{\omega}^c = \Gamma^a_{bc}\,\underline{\omega}^b\wedge\underline{\omega}^c$$

$$= \frac{1}{2}\left(\Gamma^a_{bc} - \Gamma^a_{cb}\right)\underline{\omega}^b\wedge\underline{\omega}^c$$

$$= \frac{1}{2}\left(T^a_{bc} + C^a_{bc}\right)\underline{\omega}^b\wedge\underline{\omega}^c$$

$$= \underline{T}^a$$

TORSION 2-FORM

$$\Theta^{a}{}_{c\wedge}\underline{\omega}^{c} = \Gamma^{a}_{bc}\,\underline{\omega}^{b}\wedge\underline{\omega}^{c}$$

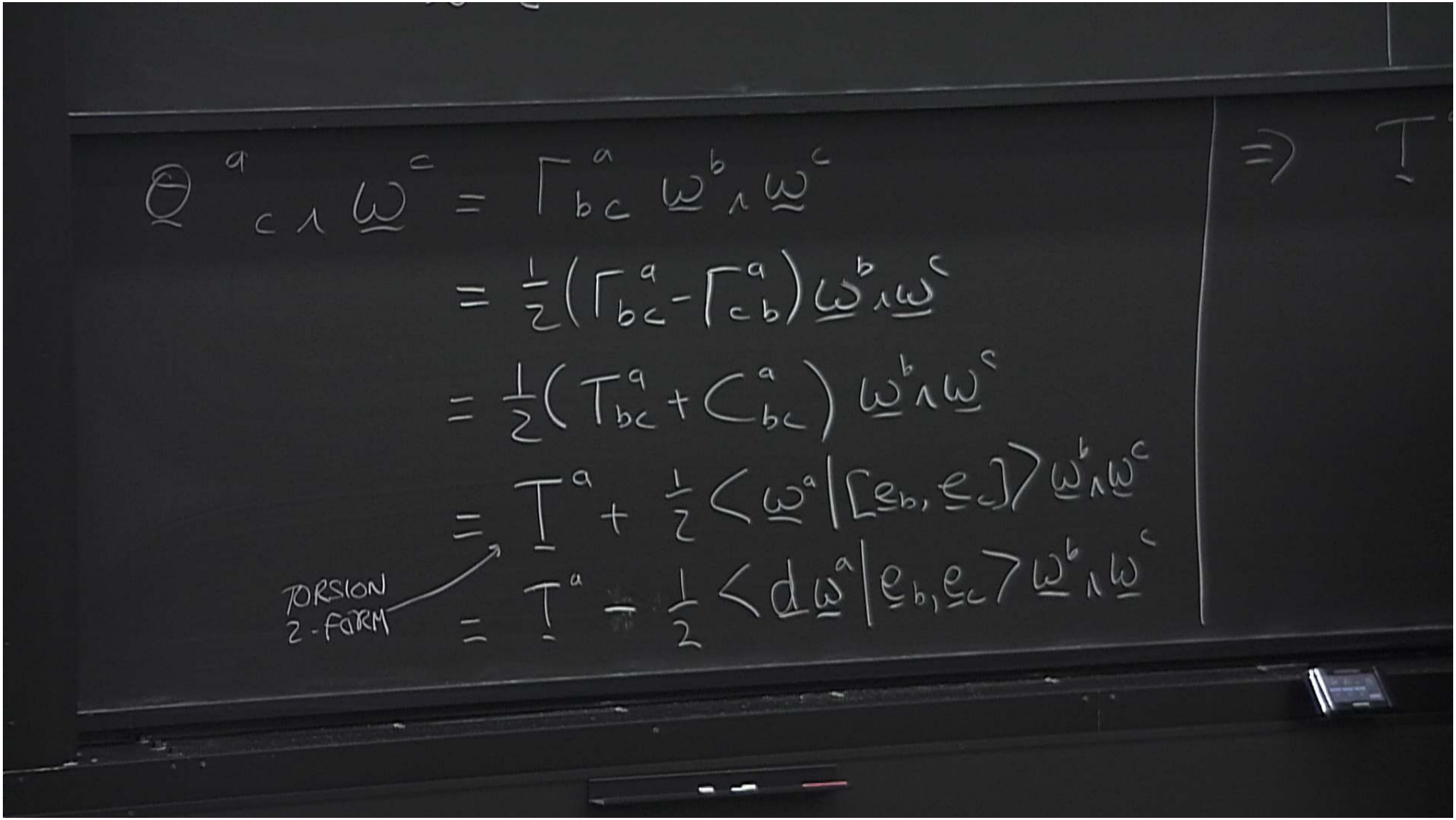
$$= \frac{1}{2}\left(\Gamma^{a}_{bc} - \Gamma^{a}_{cb}\right)\underline{\omega}^{b}\wedge\underline{\omega}^{c}$$

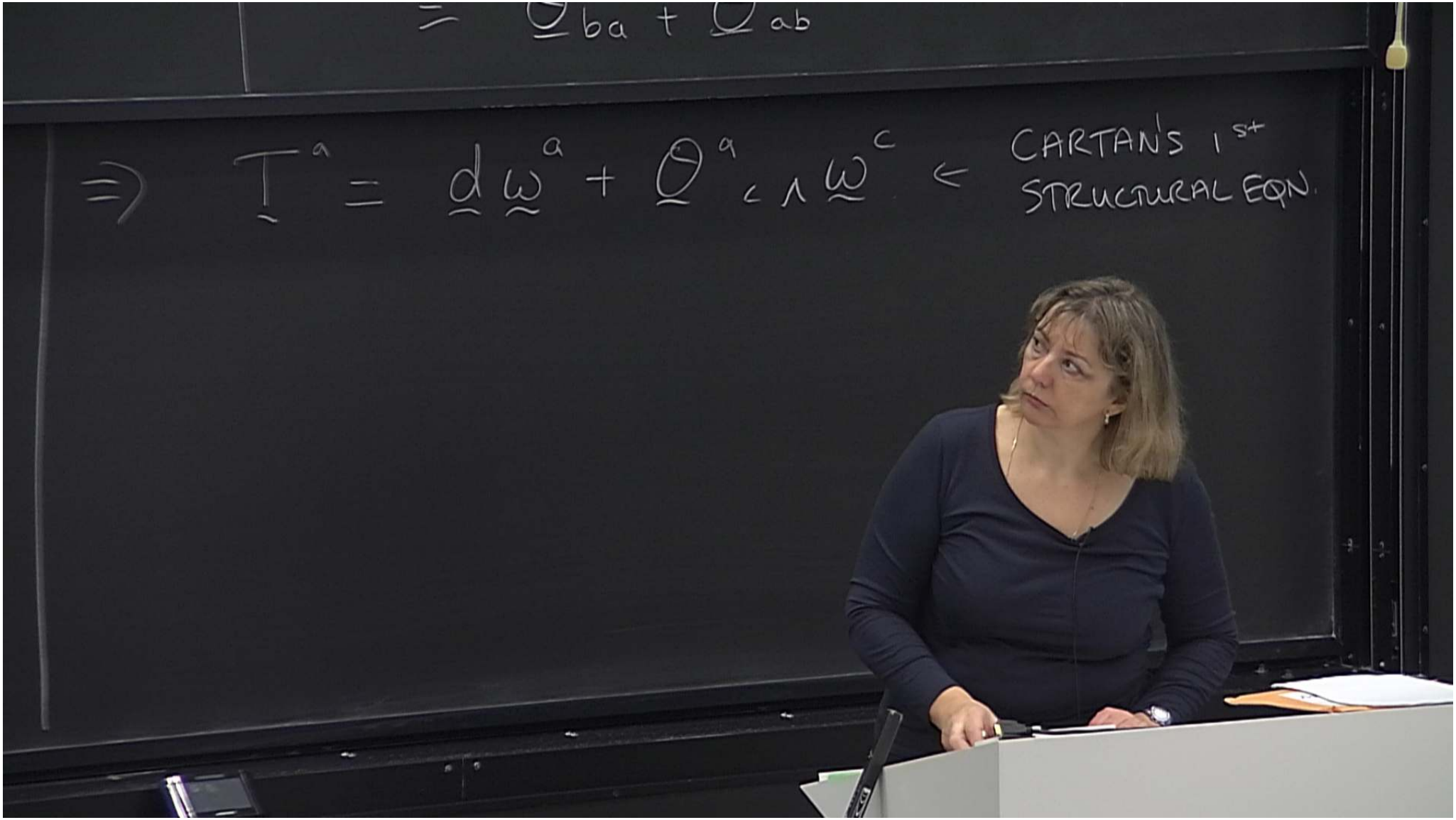
$$= \frac{1}{2}\left(T^{a}_{bc} + C^{a}_{bc}\right)\underline{\omega}^{b}\wedge\underline{\omega}^{c}$$

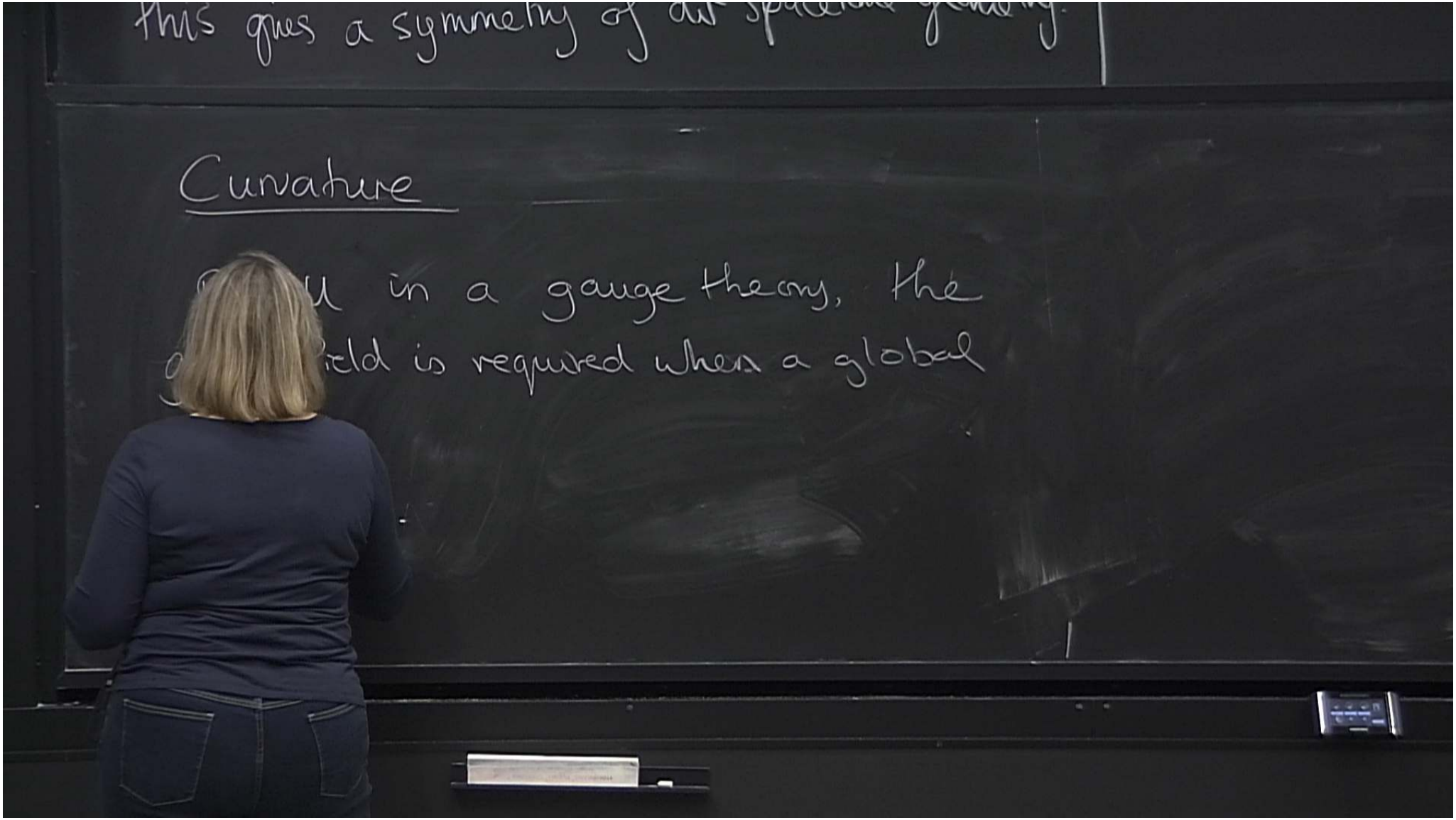
$$= \underline{T}^{a} + \frac{1}{2}\langle\underline{\omega}^{a}\,|\,[\underline{e}_{b},\underline{e}_{c}]\rangle\,\underline{\omega}^{b}\wedge\underline{\omega}^{c}$$

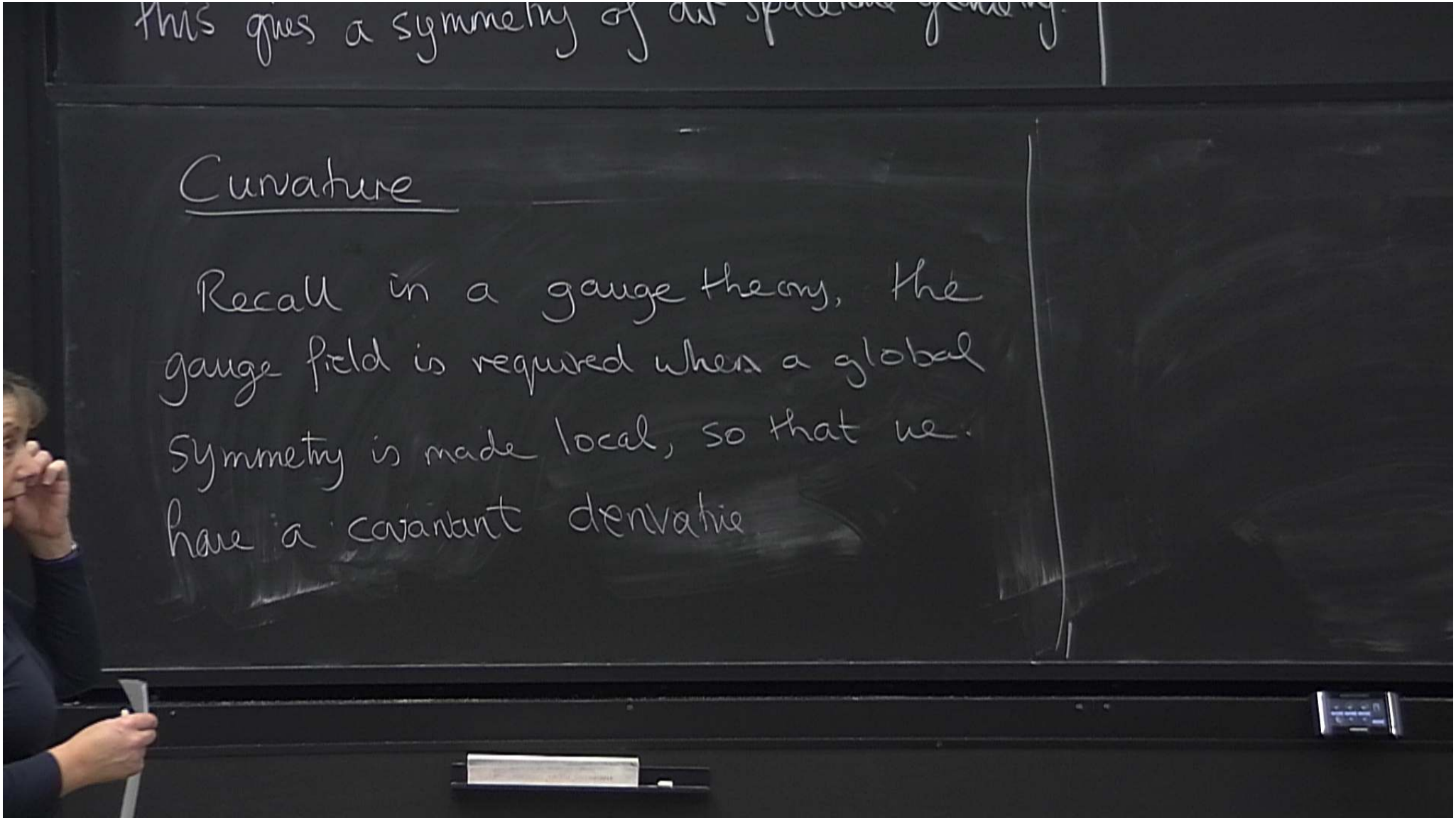
TORSION 2-FORM

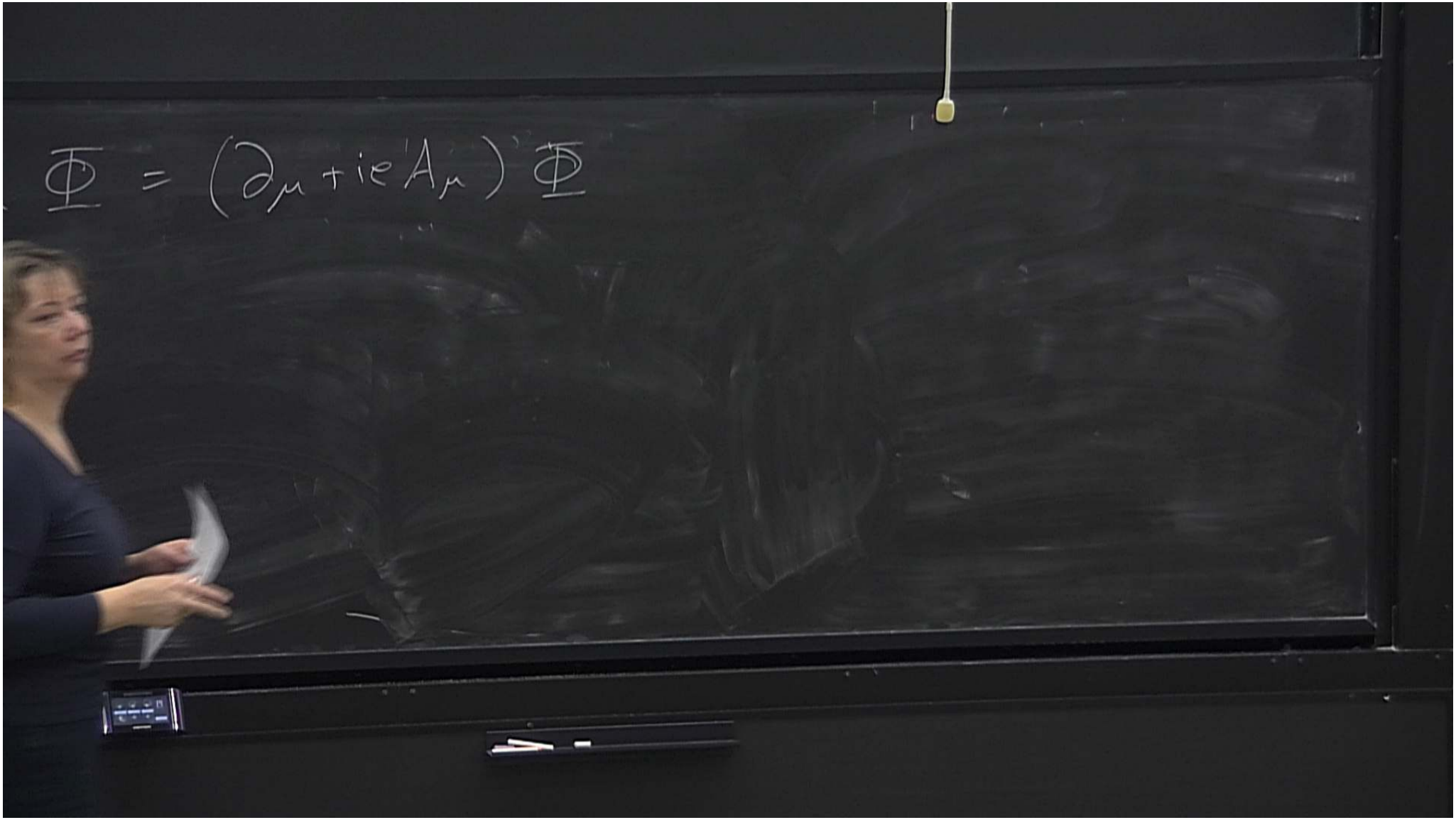
$$= \underline{T}^{a} - \frac{1}{2}\langle d\underline{\omega}^{a}\,|\,\underline{e}_{b},\underline{e}_{c}\rangle\,\underline{\omega}^{b}\wedge\underline{\omega}^{c}$$

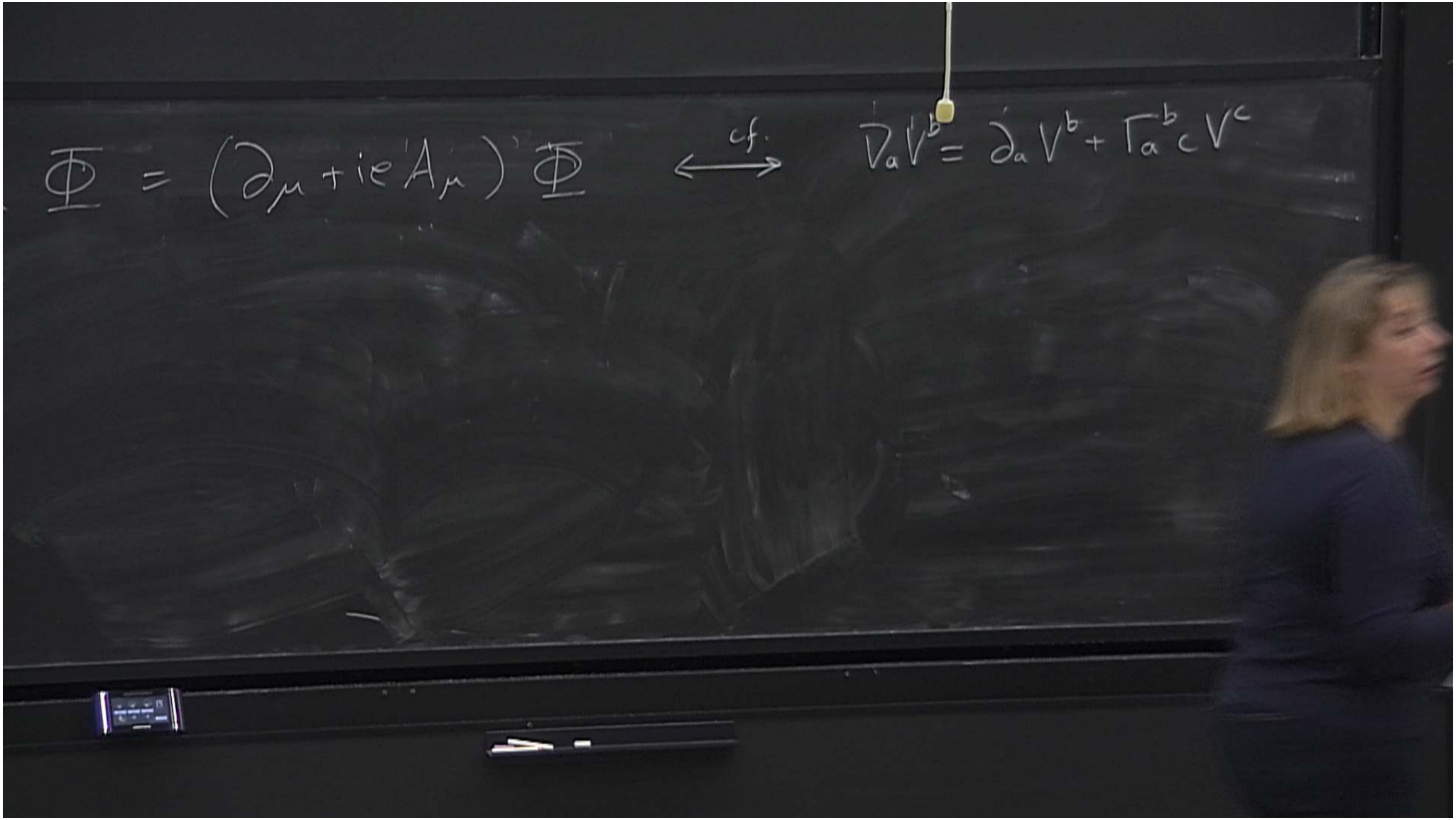

$\Theta^a{}_c \wedge \omega^c = \Gamma^a_{bc}\, \omega^b \wedge \omega^c$
$= \frac{1}{2}(\Gamma^a_{bc} - \Gamma^a_{cb})\, \omega^b \wedge \omega^c$
$= \frac{1}{2}(T^a_{bc} + C^a_{bc})\, \omega^b \wedge \omega^c$
$= T^a + \frac{1}{2} \langle \omega^a | [e_b, e_c] \rangle\, \omega^b \wedge \omega^c$
$= T^a - \frac{1}{2} \langle d\omega^a | e_b, e_c \rangle\, \omega^b \wedge \omega^c$
TORSION 2-FORM
$\Rightarrow T$


$= \Theta_{ba} + \Theta_{ab}$
$\Rightarrow T^a = d\omega^a + \Theta^a{}_c \wedge \omega^c \Leftarrow$ CARTAN'S 1st STRUCTURAL EQN.


this gives a symmetry of
Curvature
in a gauge theory, the
field is required when a global


this gives a symmetry of
Curvature
Recall in a gauge theory, the
gauge field is required when a global
symmetry is made local, so that we
have a covariant derivative


$\Phi = (\partial_\mu + ie'A_\mu)\Phi$

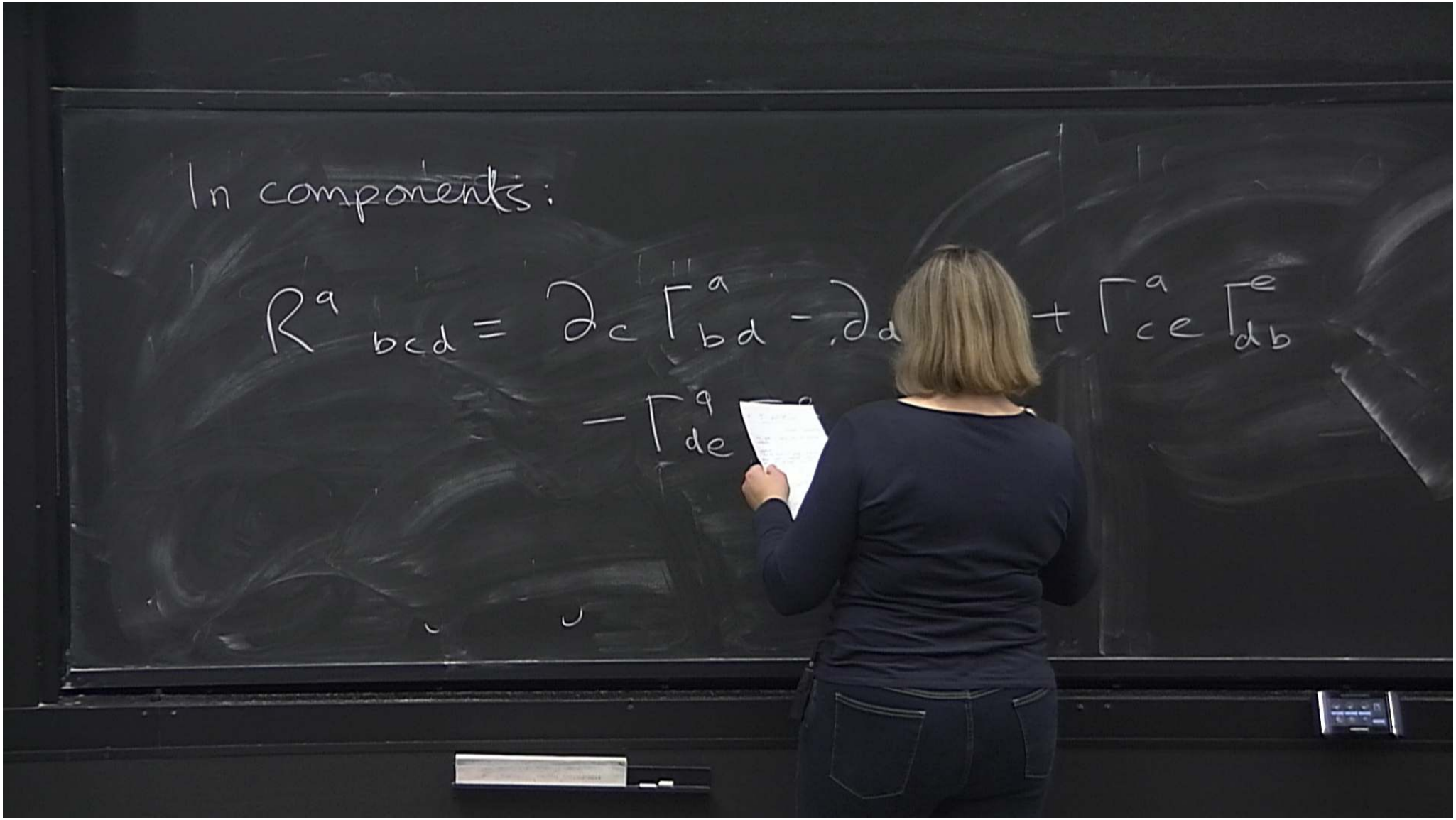

$\underline{\Phi} = (\partial_\mu + ie A_\mu)\underline{\Phi}$
cf.
$\nabla_a V^b = \partial_a V^b + \Gamma^b_{ac} V^c$

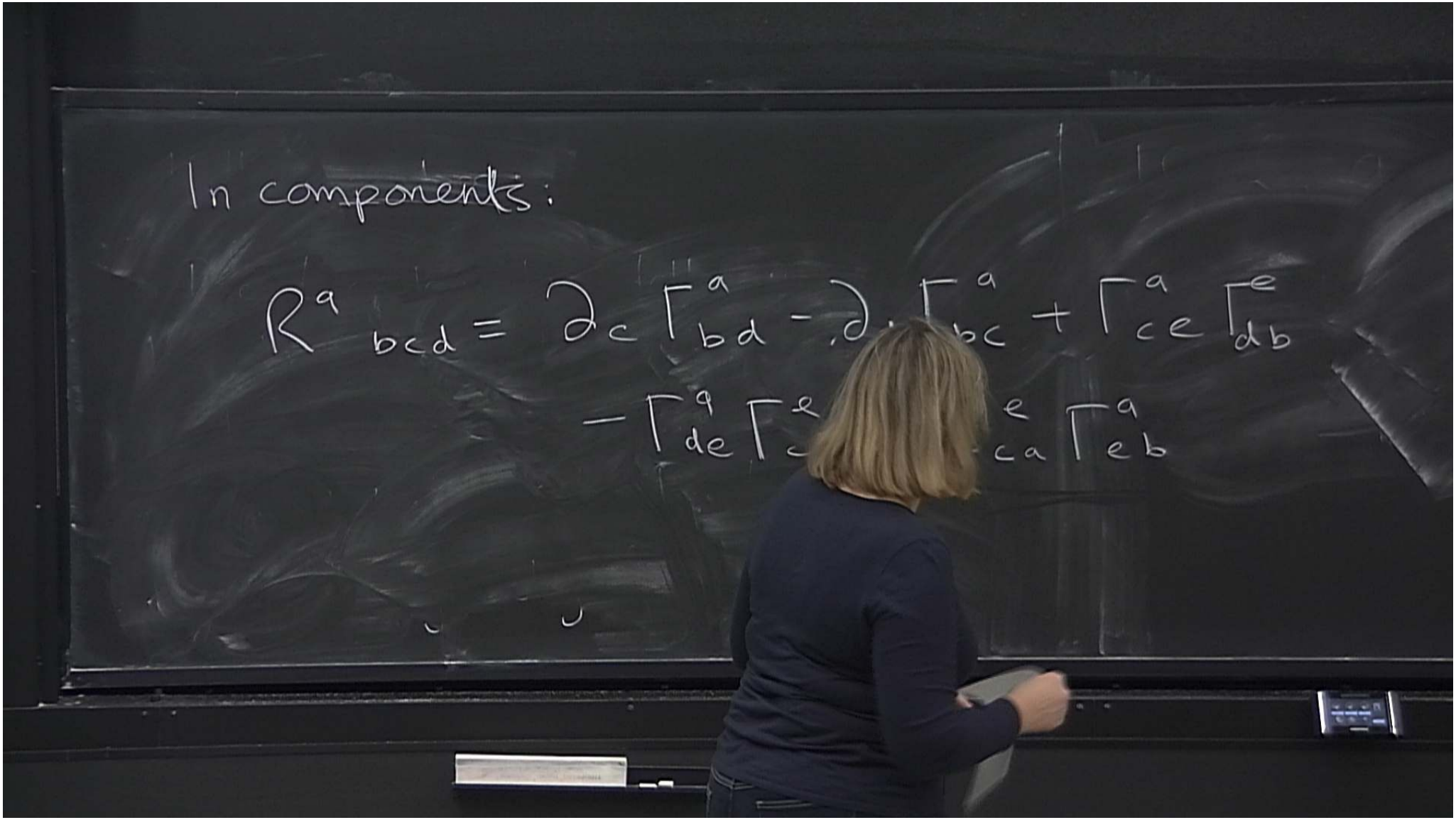
geometry.

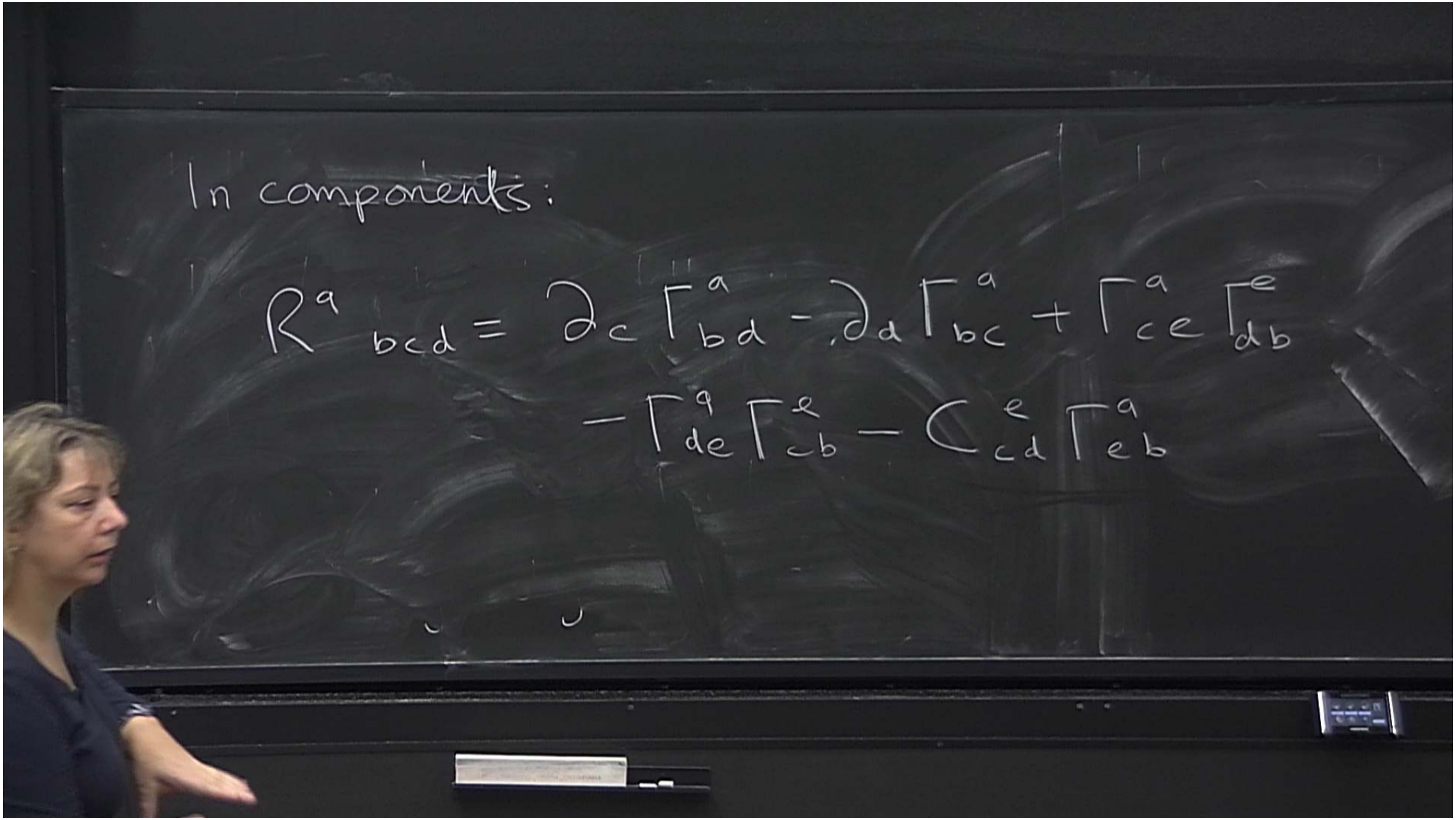
$$D_\mu \Phi = (\partial_\mu + ie A_\mu)\Phi \quad \overset{\text{cf.}}{\longleftrightarrow} \quad \nabla_a V^b = \partial_a V^b + \Gamma^b{}_{ac} V^c$$

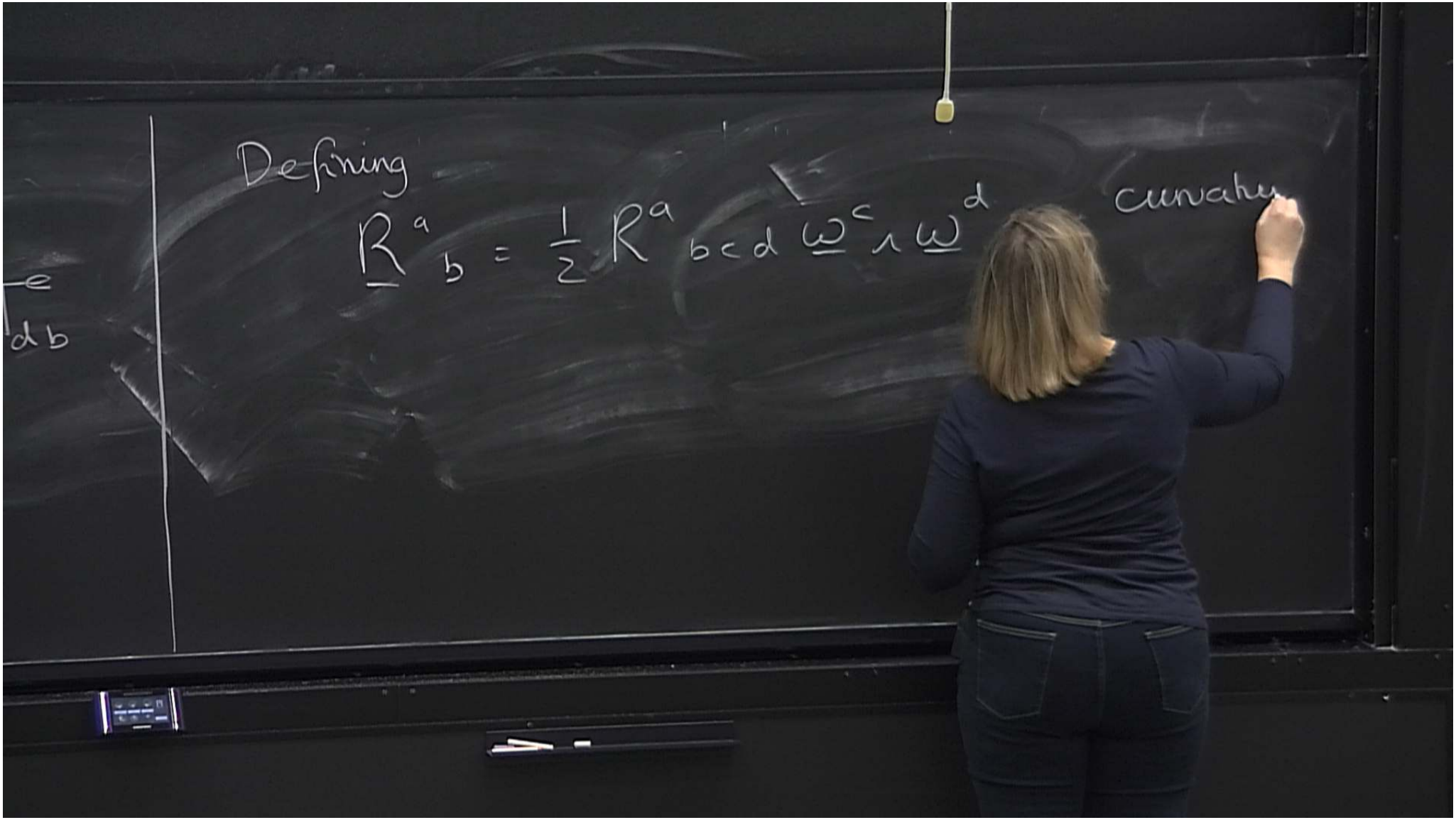
The e-m field is then defined as a commutator of derivs

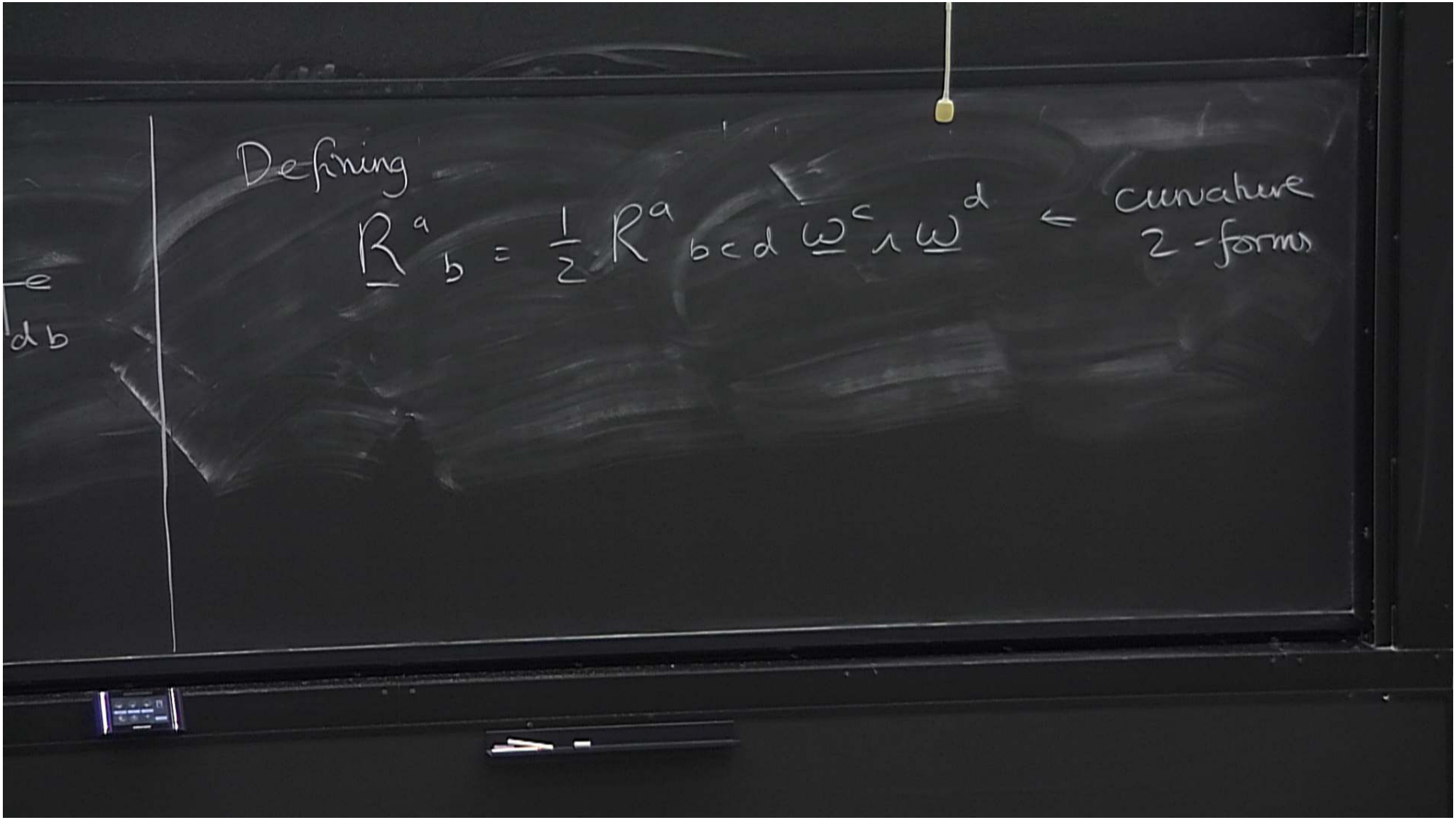
$$[D_\mu, D_\nu]\Phi = F_{\mu\nu}\Phi$$


In components:
$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d$
$+ \Gamma^a_{ce} \Gamma^e_{db}$
$- \Gamma^a_{de}$


In components:
$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{db}$
$- \Gamma^a_{de}\Gamma^e_{cb}$


In components:
$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{db}$
$- \Gamma^a_{de}\Gamma^e_{cb} - C^e_{cd}\Gamma^a_{eb}$


Defining
$\underline{R}^{a}{}_{b} = \frac{1}{2} R^{a}{}_{bcd}\, \underline{\omega}^{c} \wedge \underline{\omega}^{d}$
curvatu
e
d b

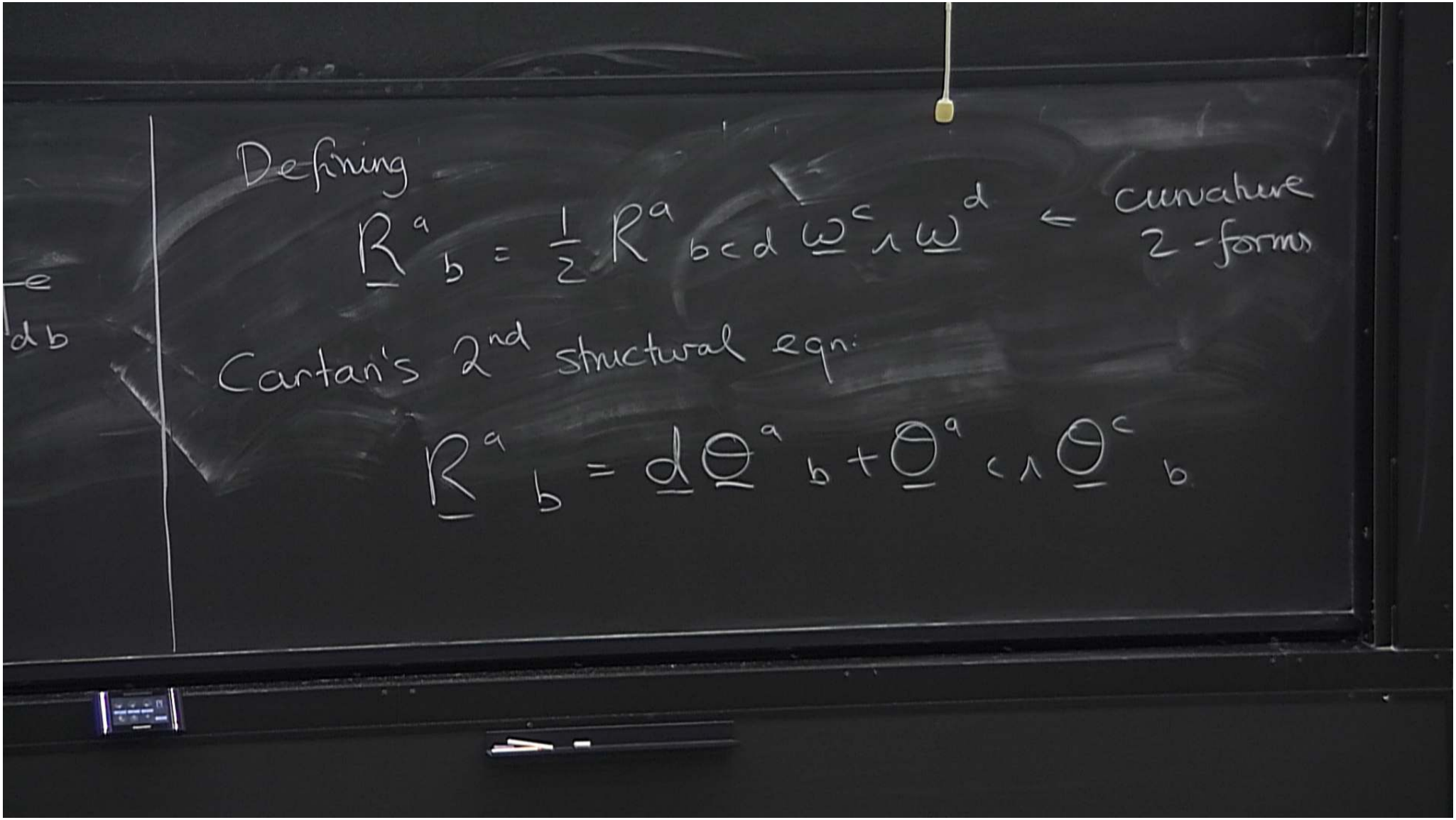

Defining
$\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$ ← curvature 2-forms
e
d b

Defining

$$\underline{R}^{a}{}_{b} = \frac{1}{2} R^{a}{}_{bcd}\, \underline{\omega}^{c} \wedge \underline{\omega}^{d} \quad \leftarrow \text{curvature 2-forms}$$

Cartan's 2nd structural eqn:

$$\underline{R}^{a}{}_{b} = \underline{d}\,\underline{\Theta}^{a}{}_{b} + \underline{\Theta}^{a}{}_{c} \wedge \underline{\Theta}^{c}{}_{b}$$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Cartan's 2nd structural eqn:
$\underline{R}^a{}_b = \underline{d}\,\underline{\Theta}^a{}_b + \underline{\Theta}^a{}_c \wedge \underline{\Theta}^c{}_b$