

Title: 12/13 PSI - Quantum Gravity Review Lecture 4

Date: Jan 31, 2013 10:15 AM

URL: <http://www.pirsa.org/13010092>

Abstract:

Symmetries of 3D gravity

Internal Rotations

Translation symmetry

Diffeos = "Rot + Trans"
+ EOM

Symmetries of 3D gravity

- Internal Rotations
- Translation symmetry
- Diffos = "Rot + Trans"
+ EOM

Symmetries of 3D gravity

- Internal Rotations
- Translation symmetry
- Diffeos = "Rot + Trans" + EOM

Recap: canonical analysis.

$$S = \int L(q, \dot{q}) dt = \int (p \dot{q} - H(q, p)) dt$$

Recap: canonical analysis.

$$S = \int L(q, \dot{q}) dt = \int (p \dot{q} - \underbrace{H(q, p)}_{\text{Hamiltonian}}) dt ; \quad p^{(i)} = \frac{\delta S}{\delta \dot{q}^{(i)}} = \frac{\partial L}{\partial \dot{q}} \quad (+)$$

Legendre transform

Recap: canonical analysis:

$$S = \int L(q, \dot{q}) dt = \int (\underbrace{p \dot{q}}_{\text{Hamiltonian}} - H(q, p)) dt ; \quad p^{(i)} = \frac{\delta S}{\delta \dot{q}^{(i)}} = \frac{\partial L}{\partial \dot{q}} \quad (1)$$

Legendre transform

First order action:

$$\pi_q = \frac{\delta S}{\delta \dot{q}} = p$$

Recap: canonical analysis.

$$S = \int L(q, \dot{q}) dt = \int (\underbrace{p\dot{q}}_{\text{Hamiltonian}} - H(q, p)) dt ; \quad p^{(i)} = \frac{\delta S}{\delta \dot{q}^{(i)}} = \frac{\partial L}{\partial \dot{q}} \quad (1)$$

Legendre transform

$$\{q, p\} = 1$$

First order action:

$$\pi_q = \frac{\delta S}{\delta \dot{q}} = p$$

$$\pi_p = \frac{\delta S}{\delta \dot{p}}$$

Recap: canonical analysis.

$$S = \int L(q, \dot{q}) dt = \int (\underbrace{p \dot{q}}_{\text{Hamiltonian}} - H(q, p)) dt ; \quad p^{(1)} = \frac{\delta S}{\delta \dot{q}} = \frac{\partial L}{\partial \dot{q}} \quad (+)$$

Legendre transform

$$\{q, p\} = 1$$

First order action:

$$\pi_q = \frac{\delta S}{\delta \dot{q}} = p$$

$$\pi_p = \frac{\delta S}{\delta \dot{p}} = -q$$

$$S = \int (-\dot{p} q - H) dt$$

Recap: canonical analysis.

$$S = \int L(q, \dot{q}) dt = \int (\underbrace{p \dot{q}}_{\text{Hamiltonian}} - H(q, p)) dt ; \quad p^{(i)} = \frac{\delta S}{\delta \dot{q}^{(i)}} = \frac{\partial L}{\partial \dot{q}} \quad (+)$$

Legendre transform

$$\{q, p\} = 1$$

First order action:

$$\pi_q = \frac{\delta S}{\delta \dot{q}} = p$$

$$\pi_p = \frac{\delta S}{\delta \dot{p}} = -q$$

$$1 = \{q, \pi_q\} = \{q, p\}$$

$$S = \int (-\dot{p} q - H) dt$$

$$\{p, \pi_p\} = 1 = \{p, -q\}$$

Recap: canonical analysis.

$$S = \int L(q, \dot{q}) dt = \int (\underbrace{p \dot{q}}_{\text{Hamiltonian}} - H(q, p)) dt ; \quad p^{(1)} = \frac{\delta S}{\delta \dot{q}} = \frac{\partial L}{\partial \dot{q}} \quad (1)$$

Legendre transform

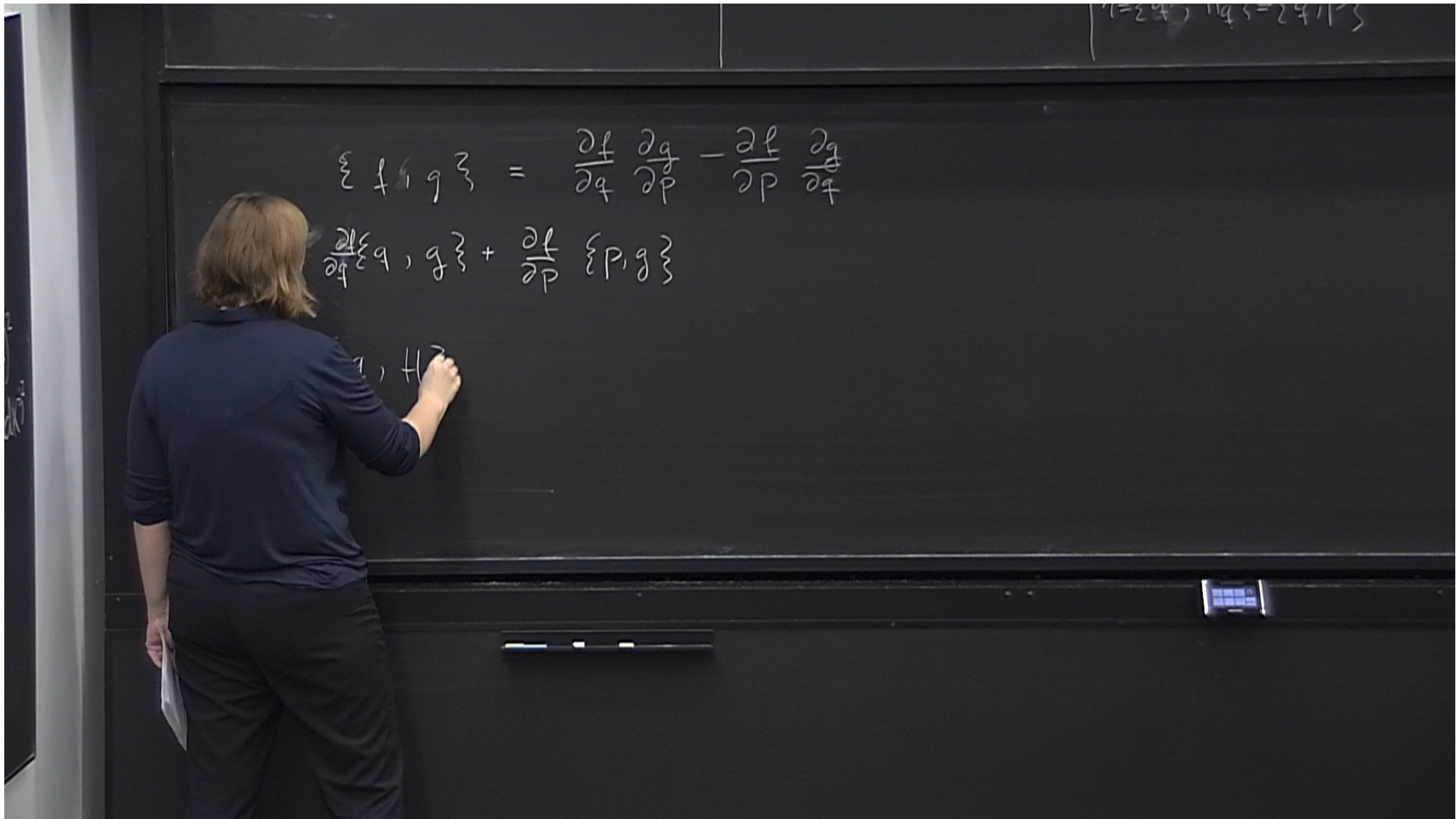
$$\{q, p\} = 1$$

F action:

$$\pi_p = \frac{\delta S}{\delta \dot{p}} = -q$$

$$S = \int (-\dot{p} q - H) dt$$

$$\{p, \pi_p\} = 1 = \{p, -q\}$$



$$\{f, g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}$$

$$= \frac{\partial f}{\partial q} \{q, g\} + \frac{\partial f}{\partial p} \{p, g\}$$

$$\dot{q} = \{q, H\}$$

$$p = \{p, H\}$$

time ev

$$\{f, g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}$$

$$= \frac{\partial f}{\partial q} \{q, g\} + \frac{\partial f}{\partial p} \{p, g\}$$

$$\dot{q} = \{q, H\} = \frac{\partial H}{\partial p} \quad \text{evolution}$$

$$p = \{p, H\} = -\frac{\partial H}{\partial q}$$

$$\{f, g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}$$

$$= \frac{\partial f}{\partial q} \{q, g\} + \frac{\partial f}{\partial p}$$

$$\dot{q} = \{q, H\} = ?$$

time evolution
"flow"

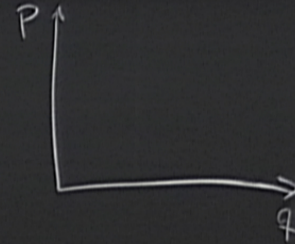
$$p = \{p, H\} = -\frac{\partial H}{\partial q}$$

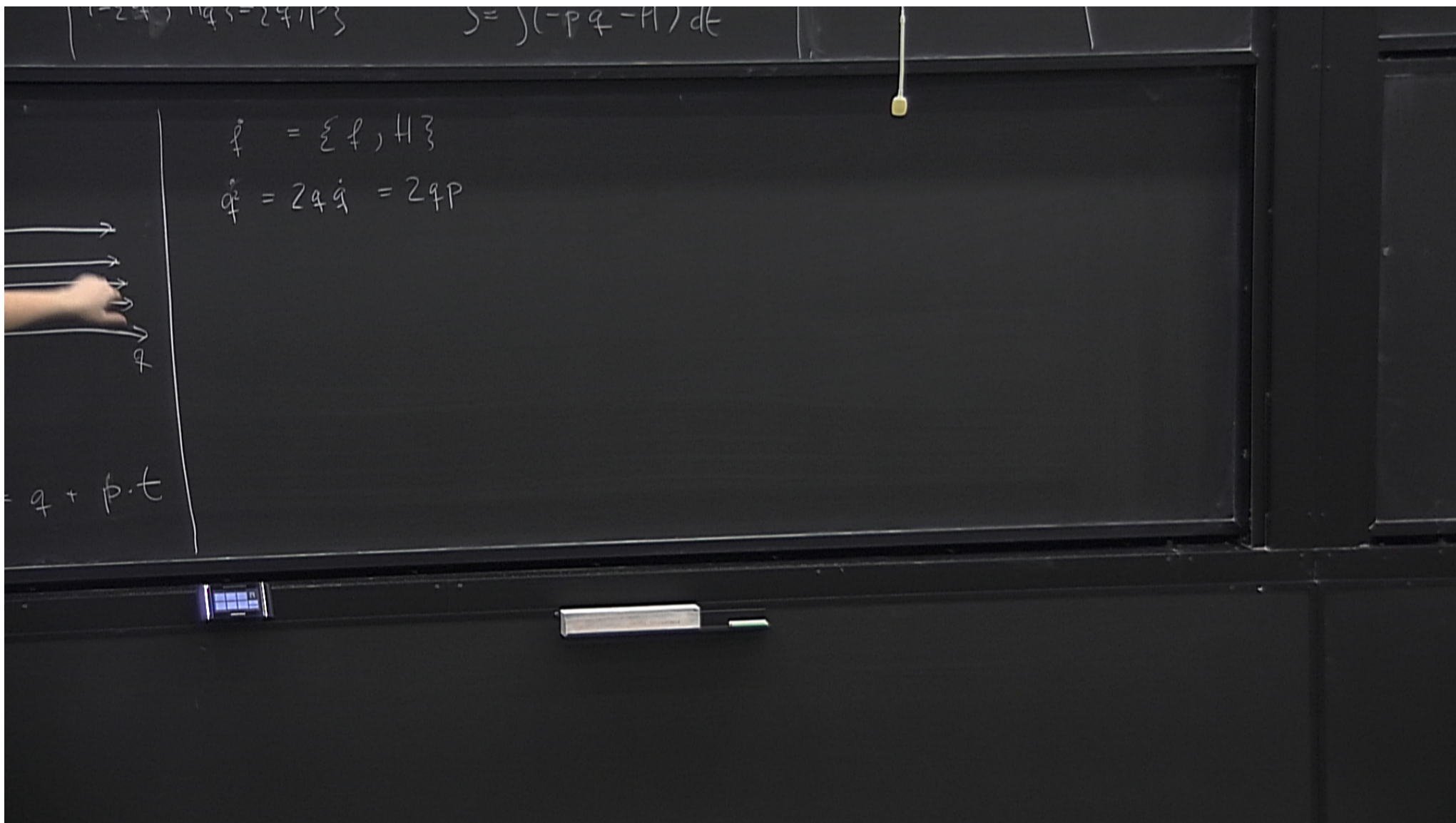
Example $H = \frac{p^2}{2}$

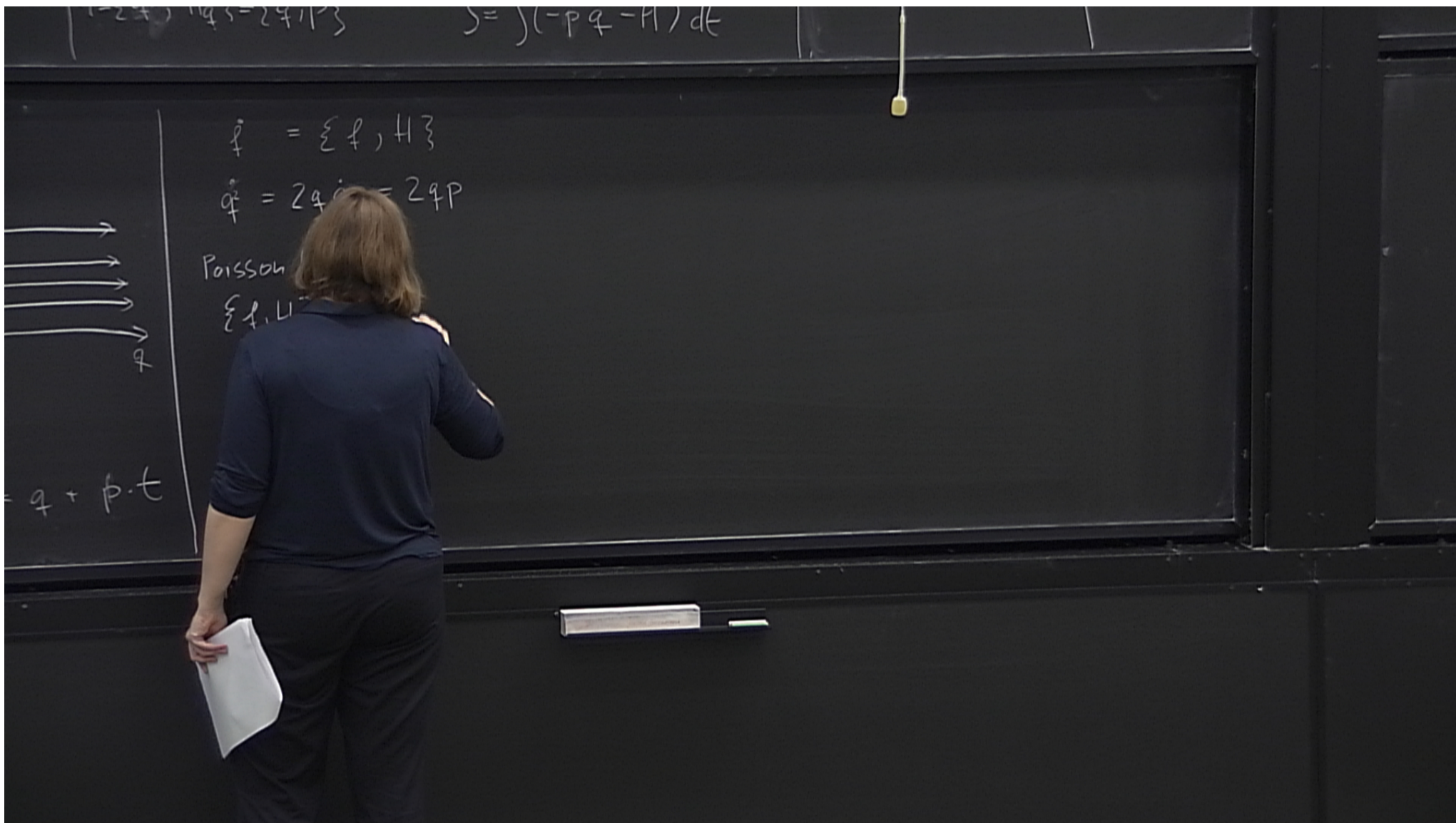
$$\dot{q}(t) = p(t)$$

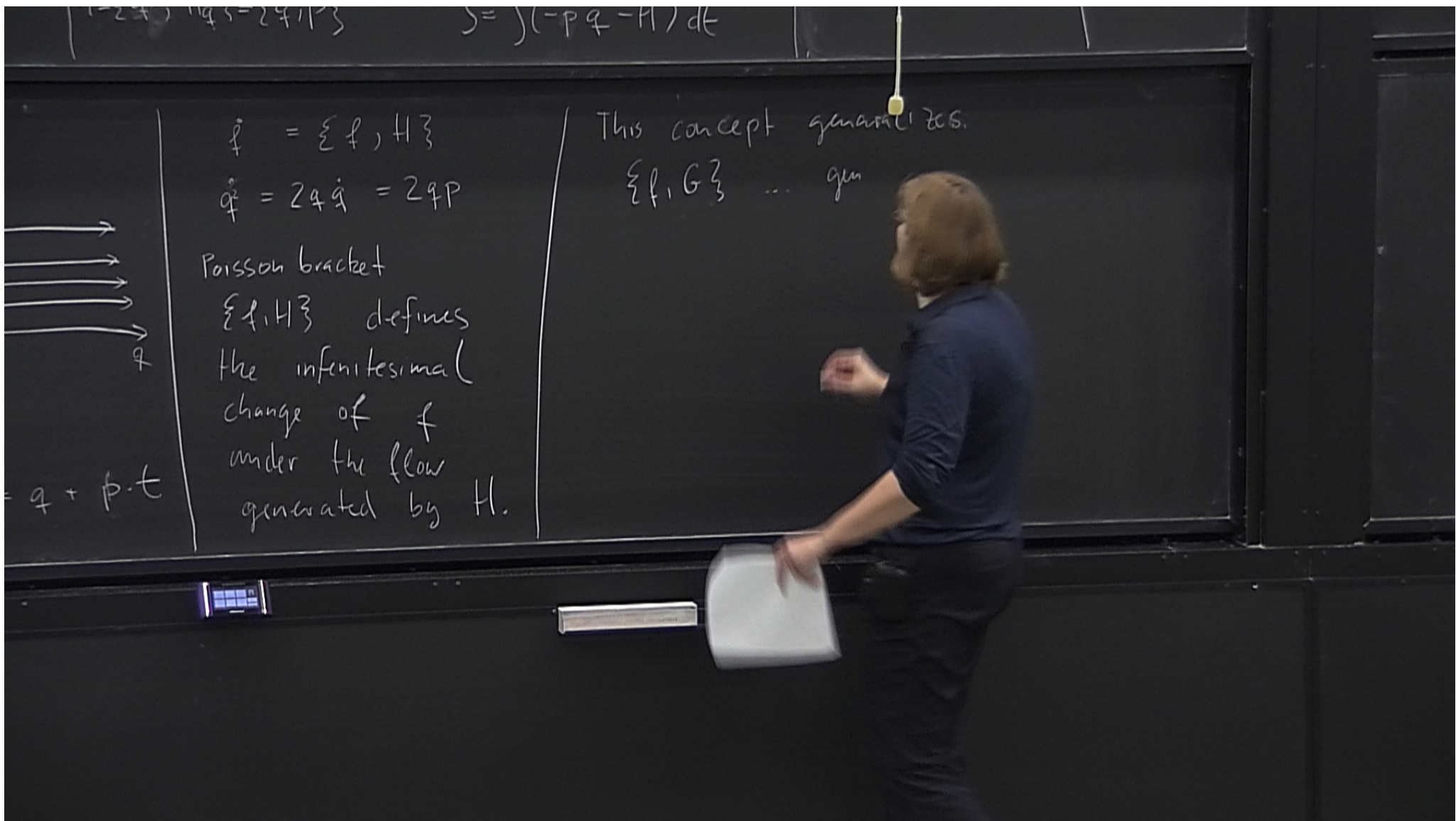
$$\dot{p}(t) = 0$$

$$\Rightarrow q(t) = q + p \cdot t$$









$$\dot{f} = \{f, H\}$$

$$\dot{q} = 2q\dot{q} = 2qP$$

Poisson bracket

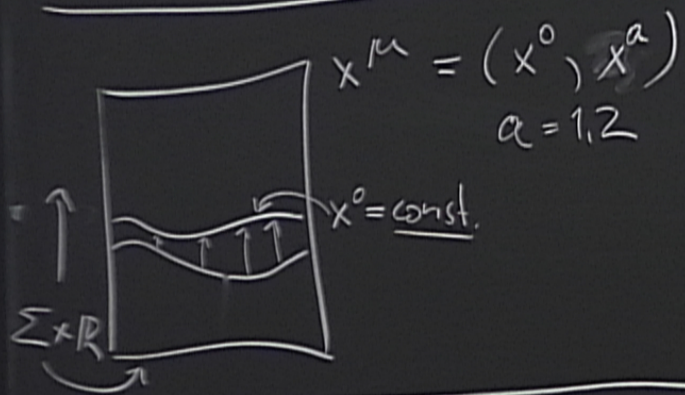
$\{f, H\}$ defines
the infinitesimal
change of f
under the flow
generated by H .

This concept generalizes.

$\{f, G\}$... generated by G .



$3D = (2+1)D$ gravity



$$\omega \rightarrow A$$

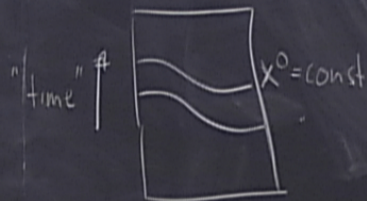
$$S_{\text{old}} \rightarrow S_{\text{new}} = \left(-\frac{1}{2}\right) S_{\text{old}}$$



curvature: $F_{\mu\nu}^{\ell} = \partial_{\mu} A_{\nu}^{\ell} - \partial_{\nu} A_{\mu}^{\ell} + \epsilon^{\ell ik} A_{\mu i} A_{\nu k}$

torsion: $T_{\mu\nu}^{\ell} = \partial_{\mu} e_{\nu}^{\ell} - \partial_{\nu} e_{\mu}^{\ell} + \epsilon^{\ell ik} A_{\mu i} e_{\nu k} - \epsilon^{\ell ik} A_{\nu i} e_{\mu k}$

(2+1) split: $\mu = (0, a)$
 $(a = 1, 2)$



Applying Cartan

① Gravitational wave

$$ds^2 = dt^2 - dr^2 + H(t-r, x^i, x^i) (dt - dr)^2 - dx^{i2}$$

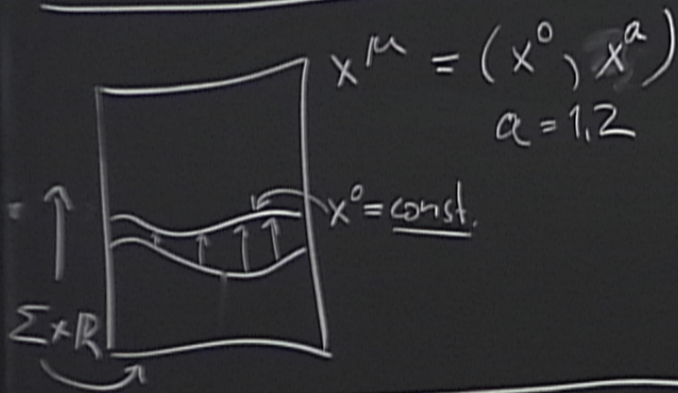
• orthon basis

$$\underline{\omega}^t = dt + \frac{1}{2} H d(t-r)$$

$$\underline{\omega}^r = dr + \frac{1}{2} H d(t-r)$$

$$\underline{\omega}^i = dx^i$$

3D = (2 + 1) D gravity

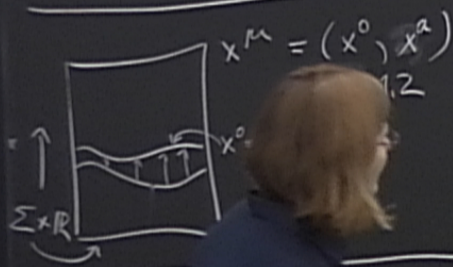


$$\omega \rightarrow A$$

$$S_{\text{old}} \rightarrow S_{\text{new}} = \left(-\frac{1}{2}\right) S_{\text{old}}$$

$$S = \frac{1}{2} \int e_{\sigma\alpha} F_{\mu\nu}^e \tilde{\epsilon}^{\sigma\mu\nu} d^3x$$

3D = (2+1)D gravity



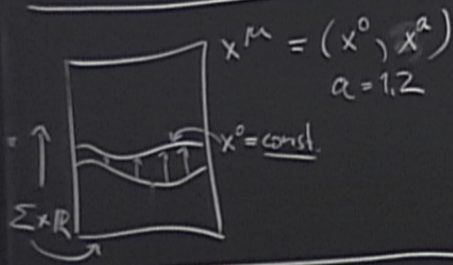
$\omega \rightarrow /$

$S_{\text{old}} \rightarrow S_{\text{new}}$

$$S = \frac{1}{2} \int e_{\sigma\ell} F_{\mu\nu}^{\ell} \tilde{E}^{\sigma\mu\nu} d^3x \quad \mu \rightarrow (0, a)$$

$$= \frac{1}{2} \int [e_{0\ell} F_{ab}^{\ell} \tilde{E}^{0ab} + 2e_{a\ell} F_{0b}^{\ell} \tilde{E}^{a0b}] d^2x dx^0$$

3D = (2+1)D gravity



$\omega \rightarrow A$

$S_{\text{old}} \rightarrow S_{\text{new}} = \left(\frac{1}{2}\right) \int \dots$

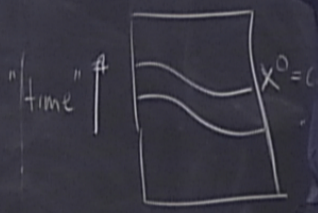
$$S = \frac{1}{2} \int e_{\alpha\ell} F_{\mu\nu}^{\ell} \tilde{\epsilon}^{\alpha\mu\nu} d^3x \quad \mu \rightarrow (0, a)$$

$$= \frac{1}{2} \int [e_{0\ell} F_{ab}^{\ell} \tilde{\epsilon}^{0ab} F_{0b}^{\ell} \tilde{\epsilon}^{a0b}] d^2x dx^0$$

$$= \frac{1}{2} \int [e_{0\ell} F_{ab}^{\ell} \tilde{\epsilon}^{0a}]$$

curvature: $F_{\mu\nu}^{\ell} = \partial_{\mu} A_{\nu}^{\ell} - \partial_{\nu} A_{\mu}^{\ell} + \epsilon^{\ell ik} A_{\mu i} A_{\nu k}$
 torsion: $T_{\mu\nu}^{\ell} = \partial_{\mu} e_{\nu}^{\ell} - \partial_{\nu} e_{\mu}^{\ell} + \epsilon^{\ell ik} A_{\mu i} e_{\nu k}$

(2+1)



Applying Cartan

① Gravitational wave

$$ds^2 = dt^2 - dr^2 + H(t-r, x^i, x^i) (dt - dr)^2 - dx^{i^2}$$

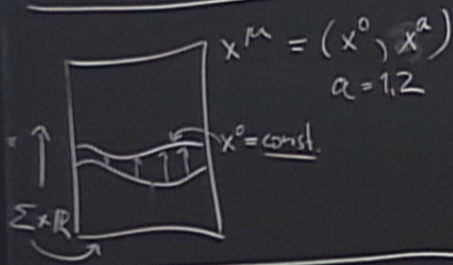
• gln basis

$$\underline{\omega}^t = dt + \frac{1}{2} H d(t-r)$$

$$\underline{\omega}^r = dr + \frac{1}{2} H d(t-r)$$

$$\underline{\omega}^i = dx^i$$

3D = (2+1)D gravity



$$\omega \rightarrow A$$

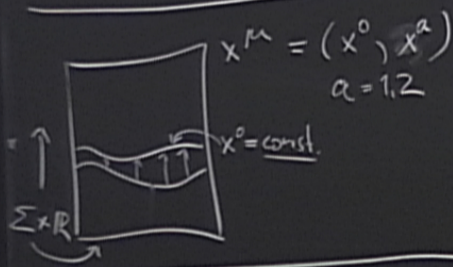
$$S_{\text{old}} \rightarrow S_{\text{new}} = \left(\frac{1}{2}\right) \int d^3x$$

$$S = \frac{1}{2} \int e_{\sigma\ell} F_{\mu\nu}^{\ell} \tilde{\epsilon}^{\sigma\mu\nu} d^3x \quad \mu \rightarrow (0, a)$$

$$= \frac{1}{2} \int [e_{0\ell} F_{ab}^{\ell} \tilde{\epsilon}^{0ab} + 2e_{a\ell} F_{0b}^{\ell} \tilde{\epsilon}^{a0b}] d^2x dx^0$$

$$= \frac{1}{2} \int [e_{0\ell} F_{ab}^{\ell} \tilde{\epsilon}^{0ab} + 2e_{a\ell} (\partial_0 A_b^{\ell} - \partial_b A_0^{\ell} + \epsilon^{\ell k} A_b^i A_b^k) \tilde{\epsilon}^{a0b}] d^2x dx^0$$

3D = (2+1)D gravity



$$\omega \rightarrow A$$

$$S_{\text{old}} \rightarrow S_{\text{new}} = \left(\frac{1}{2}\right) \int d^3x$$

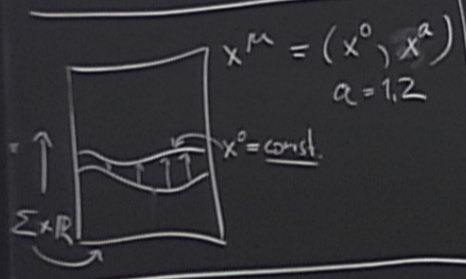
$$S = \frac{1}{2} \int e_{\alpha\ell} F_{\mu\nu}^{\ell} \tilde{\epsilon}^{\alpha\mu\nu} d^3x \quad \mu \rightarrow (0, a)$$

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$$= \frac{1}{2} \int [2e_{a\ell} \partial_0 A_b^{\ell} \tilde{\epsilon}^{a0b} + e_{0\ell} F_{ab}^{\ell} \tilde{\epsilon}^{0ab} + 2A_0^{\ell} \partial_b (e_{a\ell}) \tilde{\epsilon}^{a0b}] d^2x dx^0$$

3D = (2+1)D gravity



$\omega \rightarrow A$
 $S_{\text{old}} \rightarrow S_{\text{new}} = \left(\frac{1}{2}\right) \int d^3x$

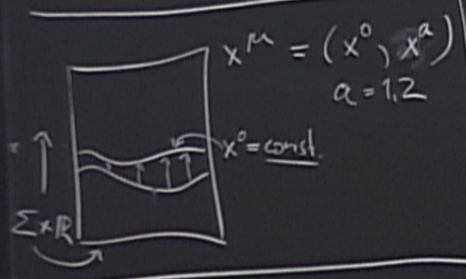
$$S = \frac{1}{2} \int e_{\alpha\ell} F_{\mu\nu}^{\ell} \tilde{\epsilon}^{\alpha\mu\nu} d^3x \quad \mu \rightarrow (0, a)$$

$$= \frac{1}{2} \int [e_{0\ell} F_{ab}^{\ell} \tilde{\epsilon}^{0ab} + 2e_{a\ell} F_{0b}^{\ell} \tilde{\epsilon}^{a0b}] d^2x dx^0$$

$$= \frac{1}{2} \int [e_{0\ell} F_{ab}^{\ell} \tilde{\epsilon}^{0ab} + 2e_{a\ell} (\partial_0 A_b^{\ell} - \partial_b A_0^{\ell} + \epsilon^{\ell jk} A_j^i A_k^b) \tilde{\epsilon}^{a0b}] d^2x dx^0$$

$$= \frac{1}{2} \int [2e_{a\ell} \partial_0 A_b^{\ell} \tilde{\epsilon}^{a0b} + e_{0\ell} F_{ab}^{\ell} \tilde{\epsilon}^{0ab} + 2A_0^{\ell} [\partial_0(e_{a\ell}) + \frac{1}{2} \epsilon^{\ell jk} A_j^i A_k^b] \tilde{\epsilon}^{a0b}] d^2x dx^0$$

3D = (2+1)D gravity



$$\omega \rightarrow A$$

$$S_{\text{red}} \rightarrow S_{\text{new}} = \left(\frac{1}{2}\right) \int \dots$$

$$S = \frac{1}{2} \int e_{\sigma\ell} F_{\mu\nu}^{\ell} \tilde{E}^{\sigma\mu\nu} d^3x \quad \mu \rightarrow (0, a)$$

$$= \frac{1}{2} \int [e_{0\ell} F_{ab}^{\ell} \tilde{E}^{0ab} + 2e_{a\ell} F_{\ell b}^{\ell} \tilde{E}^{a0b}] d^2x dx^0$$

$$= \frac{1}{2} \int [e_{0\ell} F_{ab}^{\ell} \tilde{E}^{0ab} + 2e_{a\ell} (\partial_0 A_b^{\ell} - \epsilon^{\ell jk} A_j^i A_b^k) \tilde{E}^{a0b}] d^2x dx^0$$

$$= \frac{1}{2} \int [2e_{a\ell} \partial_0 A_b^{\ell} \tilde{E}^{a0b} + e_{0\ell} F_{ab}^{\ell} \tilde{E}^{0ab} + \frac{1}{2} e_{a\ell} \epsilon^{\ell jk} A_j^i A_b^k \tilde{E}^{a0b}] d^2x dx^0$$

$$S = \int e_b^i \partial_0 A_{aj} (\tilde{E}^{ab}) + e_0 \frac{1}{2} F_{ab}^e \tilde{E}^{ab} + A_0^e ($$

$$S = \int \left[e_b^i \partial_0 A_{aj} (+\tilde{E}^{ab}) + e_0 \left(\frac{1}{2} F_{ab}^e \tilde{E}^{ab} + A_0^e (\partial_a e_{be} + \epsilon_{lmn} A_a^m e_b^n) \tilde{E}^{ab} \right) \right] d^2x dx^0$$

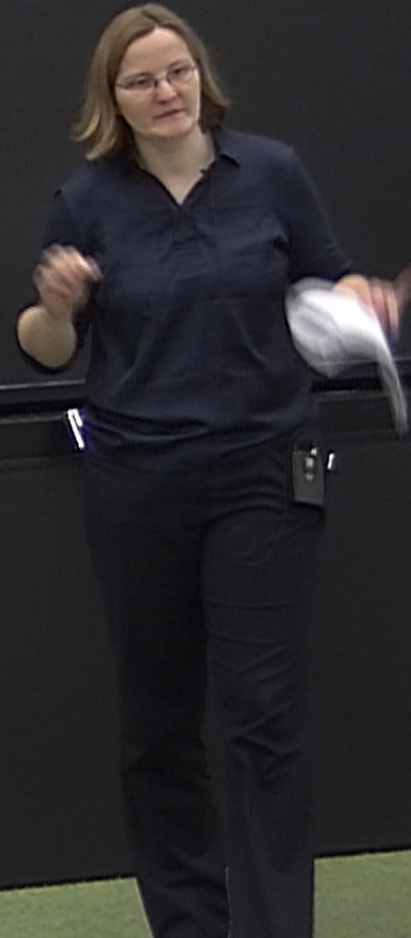
$$S = \int [e_b^i \partial_0 A_{aj} (\tilde{E}^{ab}) + e_0 \frac{1}{2} F_{ab}^e \tilde{E}^{ab} + A_0^e (\partial_a e_{be} + \epsilon_{lmn} A_a^m e_b^n) \tilde{E}^{ab}] d^2x dx^0$$

$$(\pi_{A_a^i} =) E$$

$$S = \int [e_b^i \partial_0 A_{aj} (\tilde{E}^{ab}) + e_0 \frac{1}{2} F_{ab}^e \tilde{E}^{ab} + A_0^e (\partial_a e_{be} + \epsilon_{lmn} A_a^m e_b^n) \tilde{E}^{ab}] d^2x dx^0$$

$$(\pi_{A_a^i}) = E_j^a = \frac{\delta S}{\delta (A_a^i)} = e_b^i \tilde{E}^{ab}$$

vector



$$S = \int [e_b^i \partial_0 A_{aj} (+\tilde{E}^{ab}) + e_0] \frac{1}{2} F_{ab}^e \tilde{E}^{ab} + A_0^e (\partial_a e_{be} + \epsilon_{lmn} A_a^m e_b^n) \tilde{E}^{ab}] d^2x dx^0$$

$$(\Pi_{A_a^i}) = E_j^a = \frac{\delta S}{\delta (A_a^i)} = e_b^i \tilde{E}^{ab}$$

vector density
weight +1

$$\{A_a^i(x), E_R^b(y)\} = \delta_R^i \delta_a^b \delta(x, y)$$

$$S = \int \left[e_b^i \partial_0 A_{aj} (\tilde{E}^{ab}) + e_0 \left(\frac{1}{2} F_{ab}^e \tilde{E}^{ab} + A_0^e (\partial_a e_{be} + \epsilon_{lmn} A_a^m e_b^n) \tilde{E}^{ab} \right) \right] d^2x dx^0$$

$$(\Pi_{A_a^i}) = E_j^a = \frac{\delta S}{\delta (A_a^i)} = e_b^i \tilde{E}^{ab}$$

vector density
weight +1

No time derivatives for e_0, A_0^e

$$\{A_a^i(x), E_R^b(y)\} = \delta_K^i \delta_a^b \delta(x, y)$$

$$S = \int \left[e_b^i \partial_0 A_{aj} (+\tilde{E}^{ab}) + e_0 \left(\frac{1}{2} \frac{F_{ab}^e \tilde{E}^{ab}}{F^i} + A_0^e (\partial_a e_{be} + \epsilon_{lmn} A_a^m e_b^n) \tilde{E}^{ab} \right) \right] d^2x dx^0$$

time derivatives for e_{0j}, A_0^e

$$(\Pi_{A_a^i}) = E_j^a = \frac{\delta S}{\delta (A_a^i)} = e_b^i \tilde{E}^{ab}$$

$$S = \int e_{0j} F^j$$

$$\frac{\delta S}{\delta e_{0j}} =$$

$$\{A_a^j(x), E_R^b(y)\} = \delta_K^j \delta_a^b \delta(x-y)$$

$$\int_0^1 \left(\partial_a e_{be} + \epsilon_{lmn} A_a^m e_b^n \right) \tilde{e}^{ab} \Big] d^2x dx^0$$

variables for e_{0j} , A_0^e
 Lagrange multipliers
 F^j

(equation of motion)

(but no time derivatives)

Constraint:

- EDM w/out time derivatives (of phase space variables)
- Only phase space points p , that satisfy the constraints, leads to a solution.

$$S_{old} \rightarrow S_{new} = \left(-\frac{1}{2}\right) S_{old}$$

$$-\epsilon^{ab} = -\epsilon^{ab}$$

$$S = \int \left[e_b^a \partial_0 A_{aj} (+\epsilon^{ab}) + e_0^i \bar{F}_{ab} \frac{e}{F^i} + A_0 (\partial_a e_b^a) \right]$$

$$= E_j^a = \frac{\delta S}{\delta (\partial_0 A_a^j)} = e_b^j \epsilon^{ab}$$

vector density
weight +1

$$\{E_R^b(y)\} = \delta_k^j \delta_a^b \delta(x, y)$$

No time derivatives for e_0^i
Lagrange multipliers

$$S = \int \dots e_0^i F^i$$

$$\frac{\delta S}{\delta e_0^i} = F^i \stackrel{!}{=} 0 \quad (\text{equation of motion})$$

(but no time deriv)

$$S_{old} \rightarrow S_{new} = \left(-\frac{1}{2}\right) S_{old}$$

$$-\tilde{e}^{\alpha ab} = -\tilde{e}^{\alpha ab}$$

$$S = \int \left[e_b^i \partial_0 A_{aj} (+\tilde{e}^{\alpha ab}) \right] + e_0 \left[\frac{F_{ab} \tilde{e}^{\alpha ab}}{F^j} \right] + A_0 \left(\partial_a e_b^a \right)$$

$$(\pi_{A_a^i}) = E_j^a = \frac{\delta S}{\delta (\partial_0 A_a^i)} = e_b^i$$

$$E_j^a = e_{bj} \tilde{e}^{\alpha ab} \quad e_{cj} = E_j^a \tilde{e}_{ac}$$

vector det.
weight +1

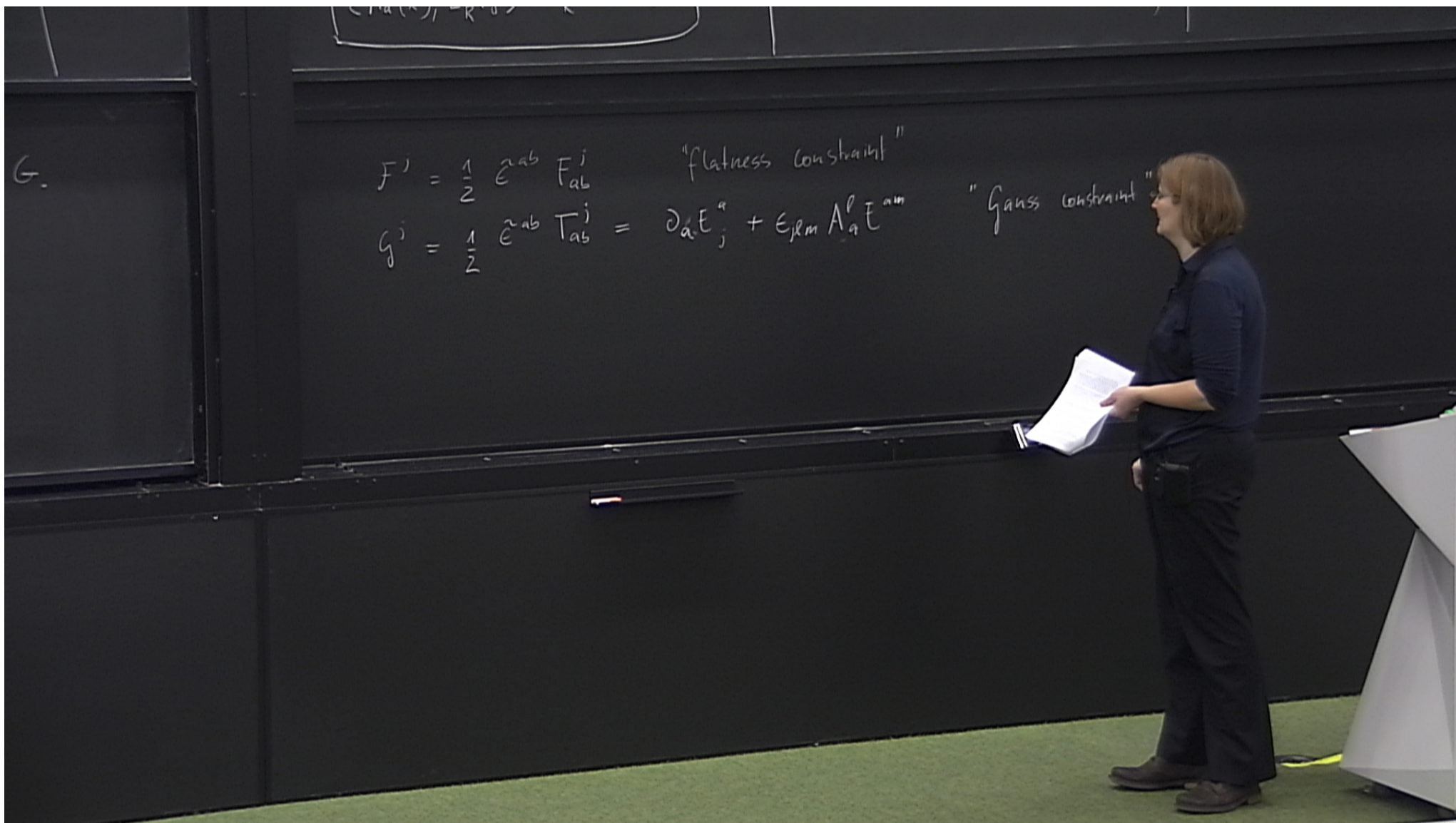
$$\{A_a^i(x), E_R^b(y)\} = \delta_K^i \delta_a^b \delta(x,y)$$

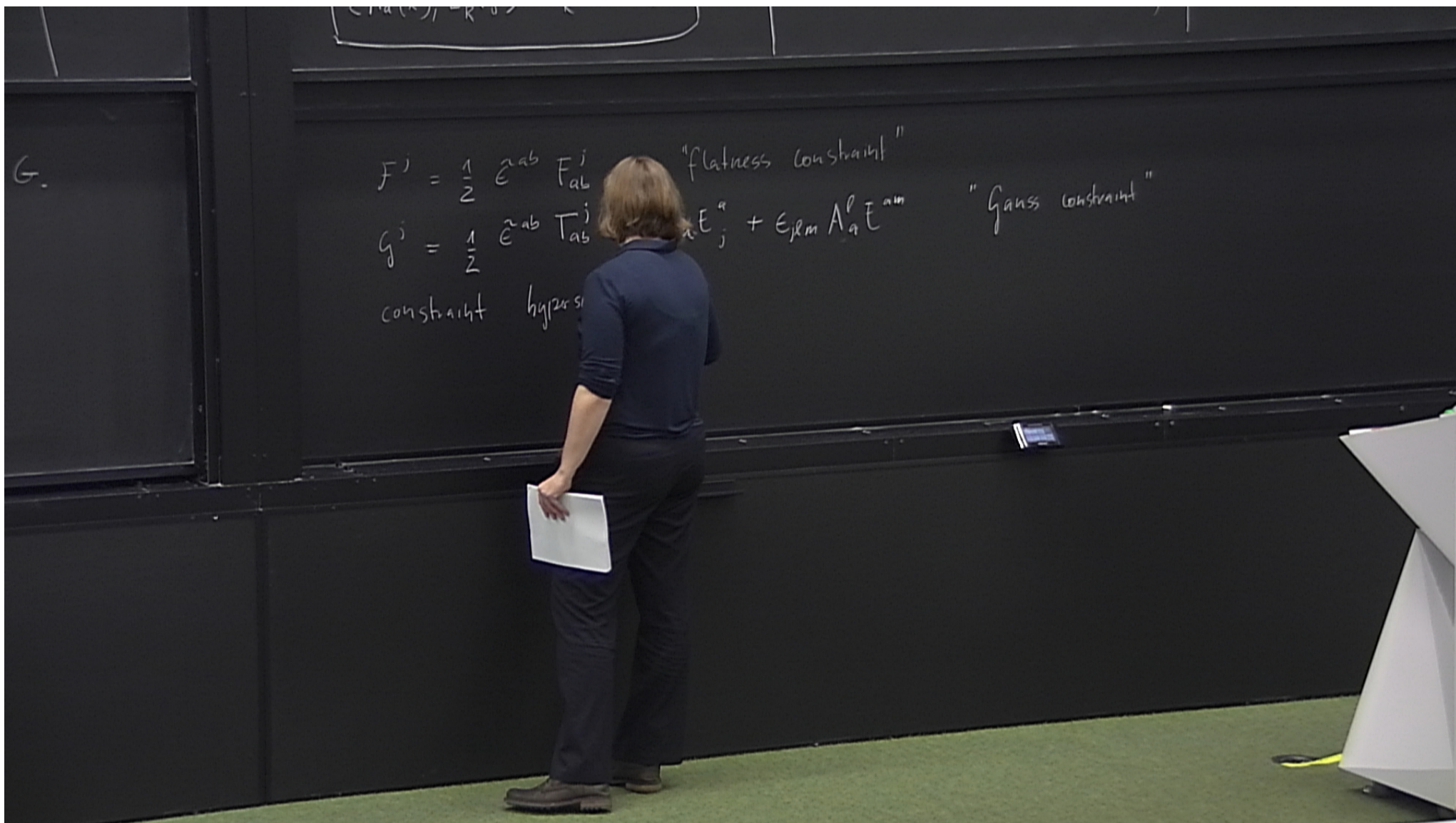
No time derivatives for e_0
Lagrange multipliers

$$S = \int \dots e_0^j F^j$$

$$= F^j \stackrel{!}{=} 0 \quad (\text{equation of motion})$$

(but no time deriv.)





$$F^j = \frac{1}{2} \tilde{\epsilon}^{ab} F_{ab}^j \quad \text{"flatness constraint"}$$

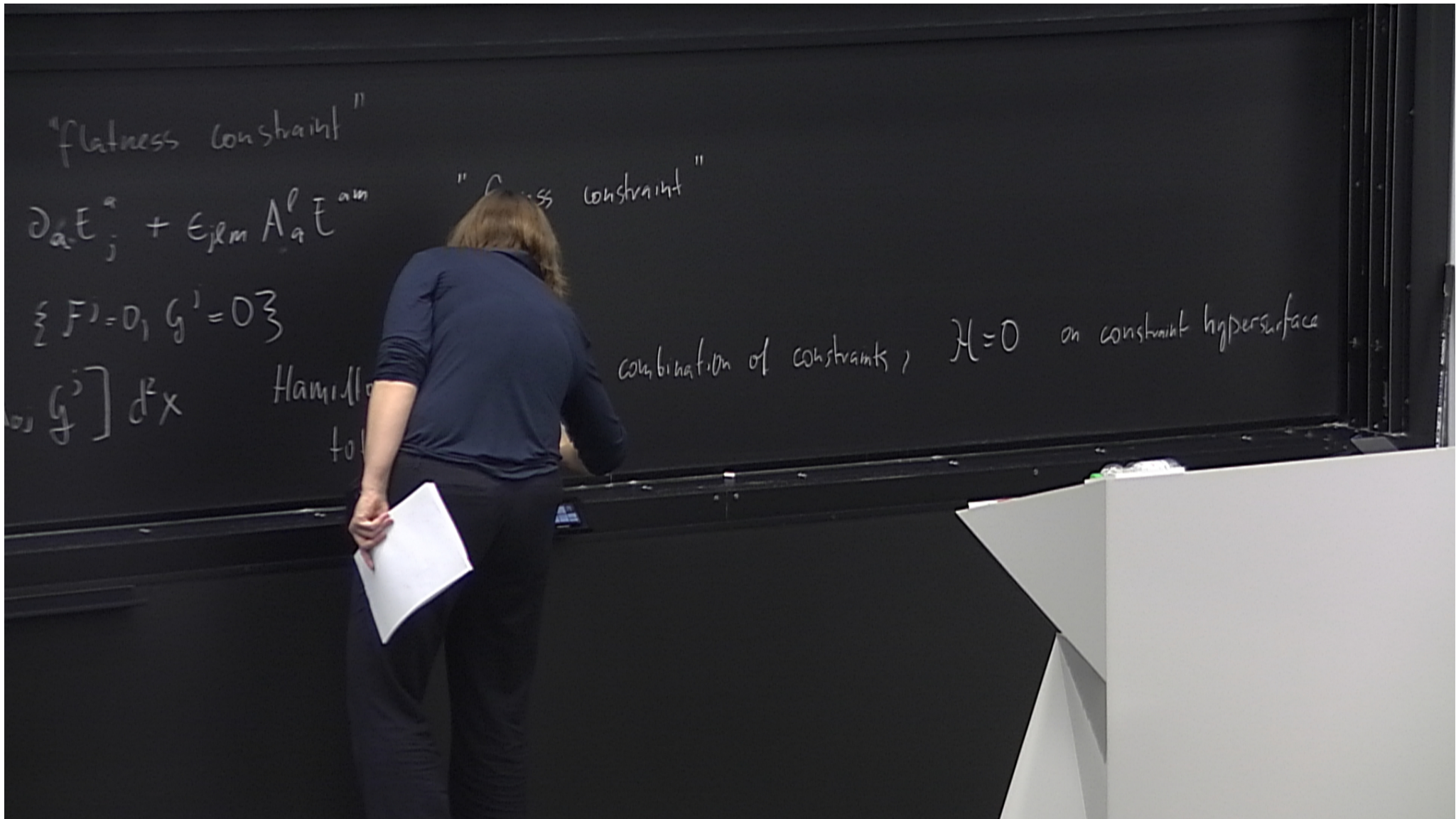
$$G^j = \frac{1}{2} \tilde{\epsilon}^{ab} T_{ab}^j = \partial_a E_j^a + \epsilon_{jkm} A_a^k E^{am} \quad \text{"Gauss constraint"}$$

constraint hypersurface $\{F^j=0, G^j=0\}$

$$\mathcal{H} = - \int [E_0, F^j + A_0, G^j] d^3x \quad \text{Hamiltonian is given as a combination of constraints}$$

"Gauss constraint"

is a combination of constraints, $\mathcal{H} = 0$ on constraint hypersurface



"flatness constraint"

$$\partial_a E^a_j + \epsilon_{jlm} A^l_a E^{am}$$

"Gauss constraint"

$$\{F^i=0, G^i=0\}$$

$$[g^j] d^2x$$

Hamilton
total

combination of constraints, $\mathcal{H}=0$ on constraint hypersurface