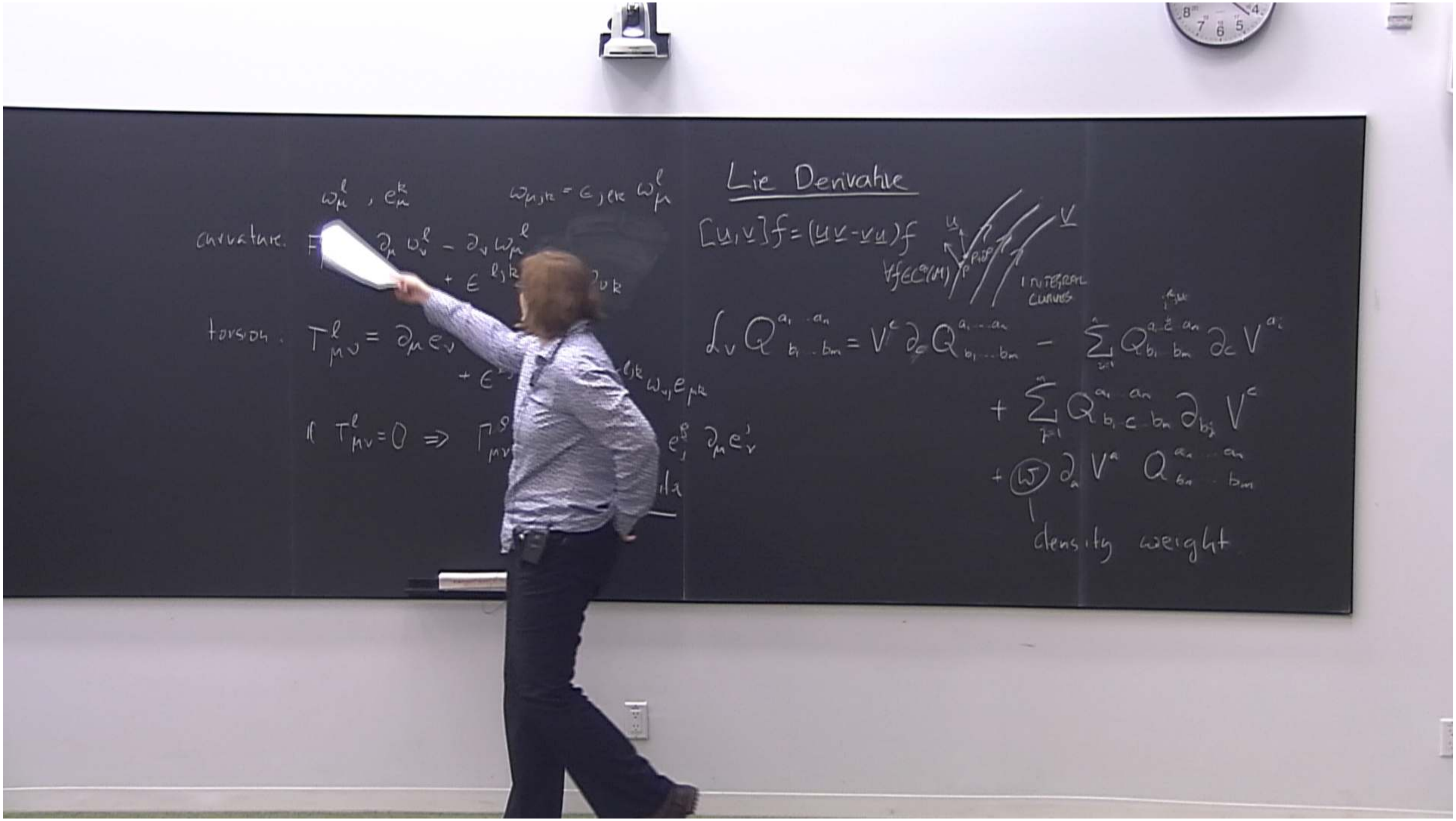
Title: 12/13 PSI - Quantum Gravity Review Lecture 3

Date: Jan 30, 2013 10:15 AM

URL: http://pirsa.org/13010091

Abstract:


Lie Derivative
curvature.
torsion.
INTEGRAL CURVES
density weight

$\omega^{\ell}_{\mu}\ ,\ e^{k}_{\mu}$ $\qquad \omega_{\mu j k} = \epsilon_{j\ell k}\,\omega^{\ell}_{\mu}$

curvature: $F^{\ell}_{\mu\nu} = \partial_\mu \omega^{\ell}_{\nu} - \partial_\nu \omega^{\ell}_{\mu} + \epsilon^{\ell j k}\,\omega_{\mu j}\,\omega_{\nu k}$

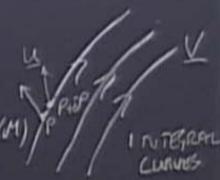
torsion: $T^{\ell}_{\mu\nu} = \partial_\mu e^{\ell}_{\nu} - \partial_\nu e^{\ell}_{\mu} + \epsilon^{\ell j k}\omega_{\mu j} e_{\nu k} - \epsilon^{\ell j k}\omega_{\nu j} e_{\mu k}$

If $T^{\ell}_{\mu\nu} = 0 \Rightarrow \Gamma^{\rho}_{\mu\nu} = e^{k}_{\nu}\,\omega^{j}_{\mu\nu}\,e^{\rho}_{j} + e^{\rho}_{j}\,\partial_\mu e^{j}_{\nu}$

is Levi-Civita

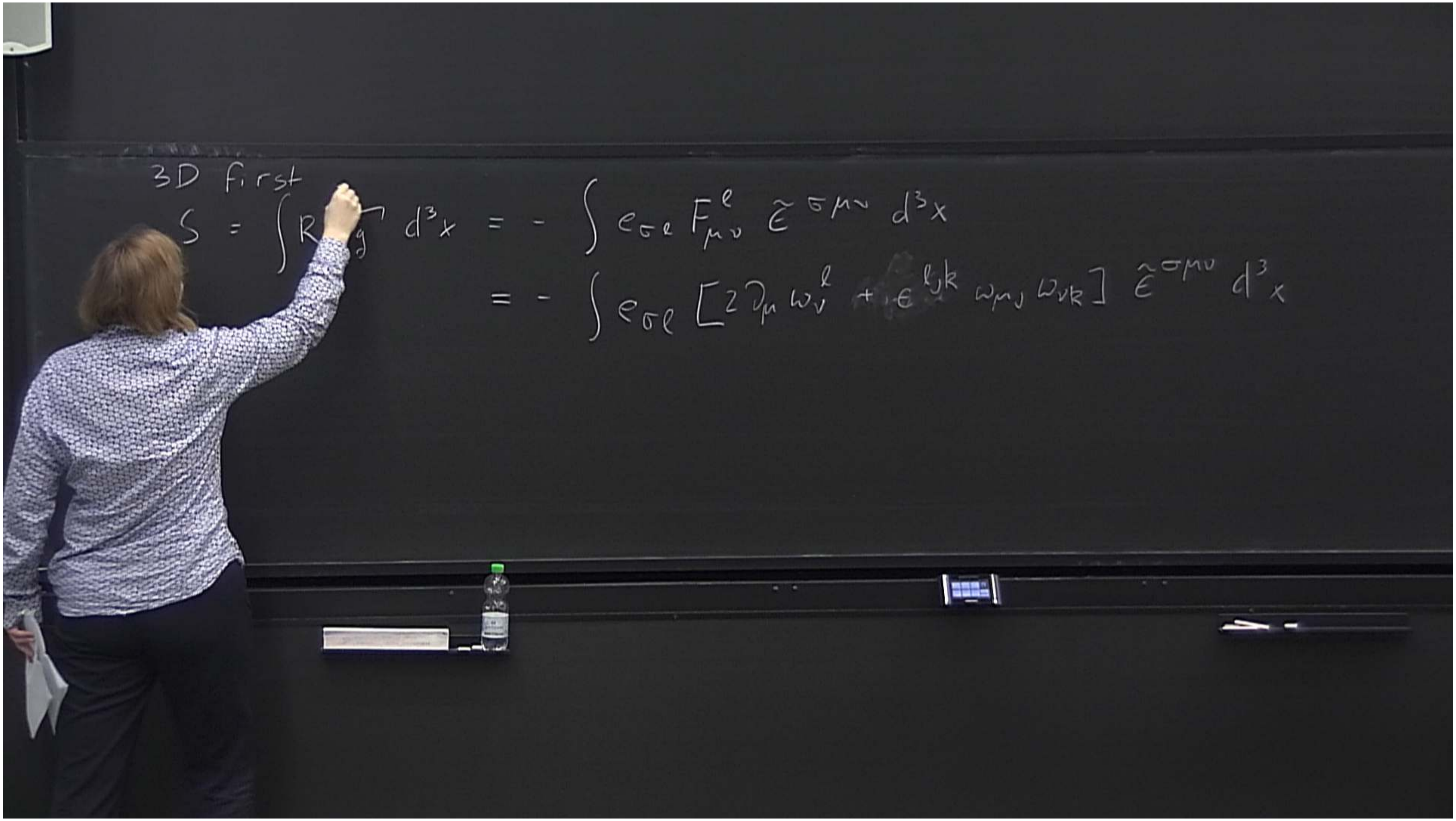
Lie Derivative

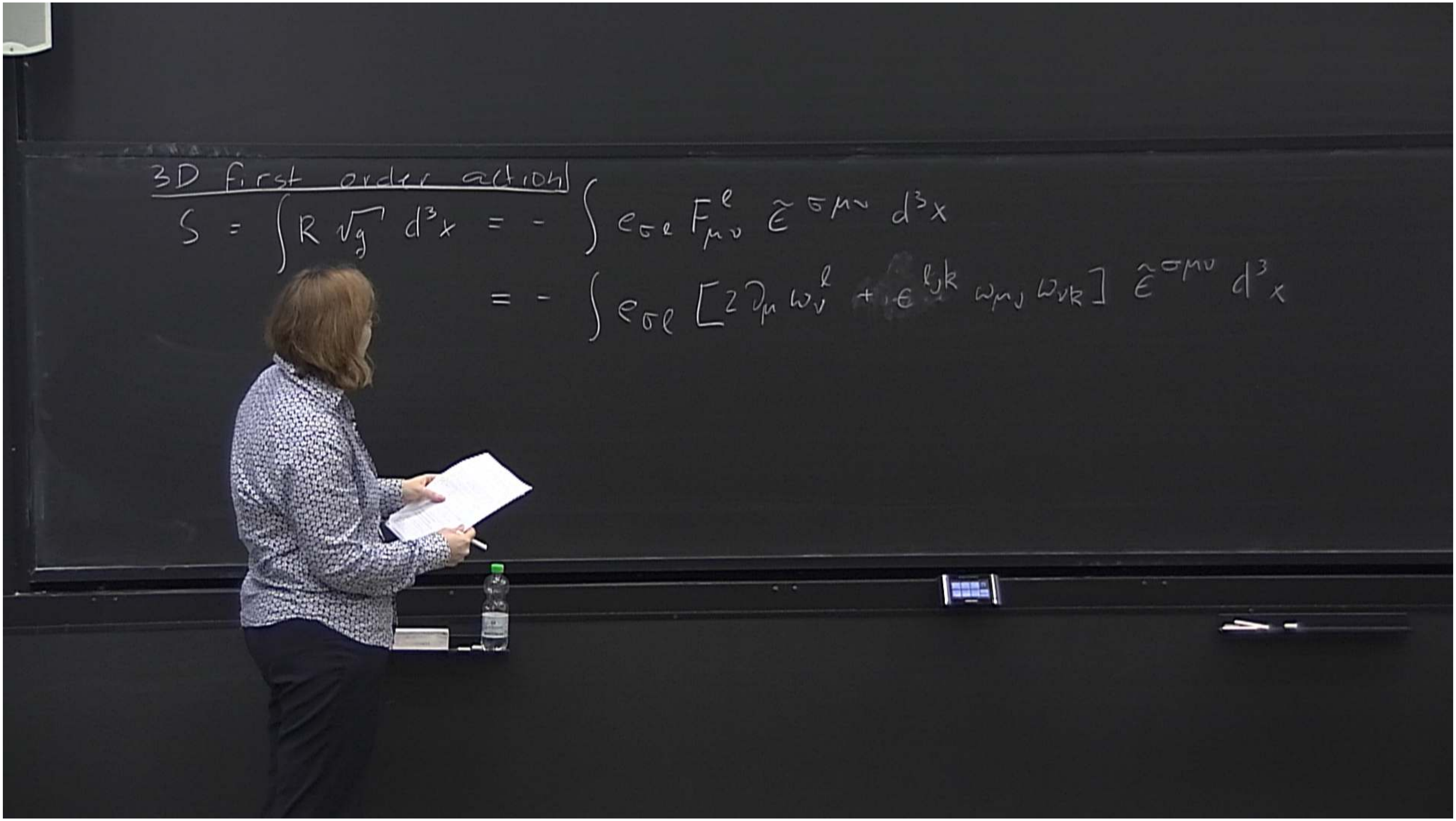
$[\underline{u},\underline{v}]f = (\underline{u}\,\underline{v} - \underline{v}\,\underline{u})f$ $\quad \forall f \in C^\infty(M)$



$$\mathcal{L}_V Q^{a_1 \ldots a_n}_{b_1 \ldots b_m} = V^c \partial_c Q^{a_1 \ldots a_n}_{b_1 \ldots b_m} - \sum_{i=1}^{n} Q^{a_1 \ldots c \ldots a_n}_{b_1 \ldots b_m}\,\partial_c V^{a_i} + \sum_{j=1}^{m} Q^{a_1 \ldots a_n}_{b_1 \ldots c \ldots b_m}\,\partial_{b_j} V^c + \omega\,\partial_a V^a\, Q^{a_1 \ldots a_n}_{b_1 \ldots b_m}$$

ω — density weight


3D first
$S = \int R\sqrt{g}\, d^3x = -\int e_{\sigma\ell}\, F^{\ell}_{\mu\nu}\, \tilde{\epsilon}^{\sigma\mu\nu}\, d^3x$
$= -\int e_{\sigma\ell}\, [2\partial_\mu \omega_\nu^{\ell} + \epsilon^{\ell jk}\, \omega_{\mu j}\, \omega_{\nu k}]\, \tilde{\epsilon}^{\sigma\mu\nu}\, d^3x$


3D first order action
$S = \int R \sqrt{g}\, d^3x = - \int e_{\sigma\ell} F^{\ell}_{\mu\nu} \tilde{\epsilon}^{\sigma\mu\nu} d^3x$
$= - \int e_{\sigma\ell} [2 \partial_\mu \omega_\nu^\ell + \epsilon^{\ell}{}_{jk} \omega_{\mu j} \omega_{\nu k}] \tilde{\epsilon}^{\sigma\mu\nu} d^3x$

3D first order action

$$S = \int R\sqrt{g}\, d^3x = -\int e_{\sigma\ell} F^{\ell}_{\mu\nu}\, \tilde{\epsilon}^{\sigma\mu\nu}\, d^3x$$

$$= -\int e_{\sigma\ell} \left[2\partial_\mu \omega^{\ell}_{\nu} + \epsilon^{\ell jk} \omega_{\mu j} \omega_{\nu k} \right] \tilde{\epsilon}^{\sigma\mu\nu}\, d^3x$$

2 order formalism. $\omega(e)$

1 order — " — : e, ω independent

Equatio

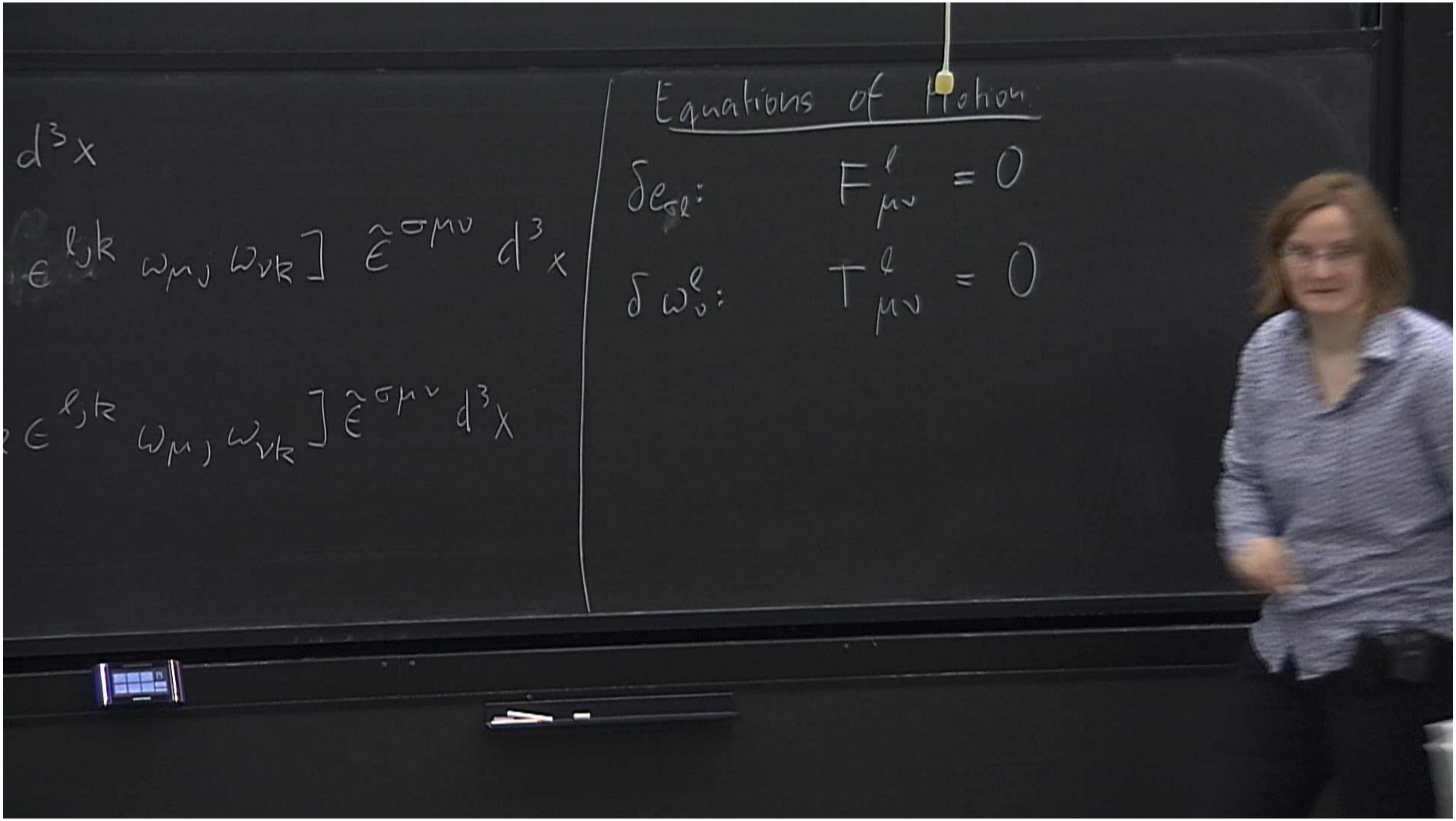
d^3x

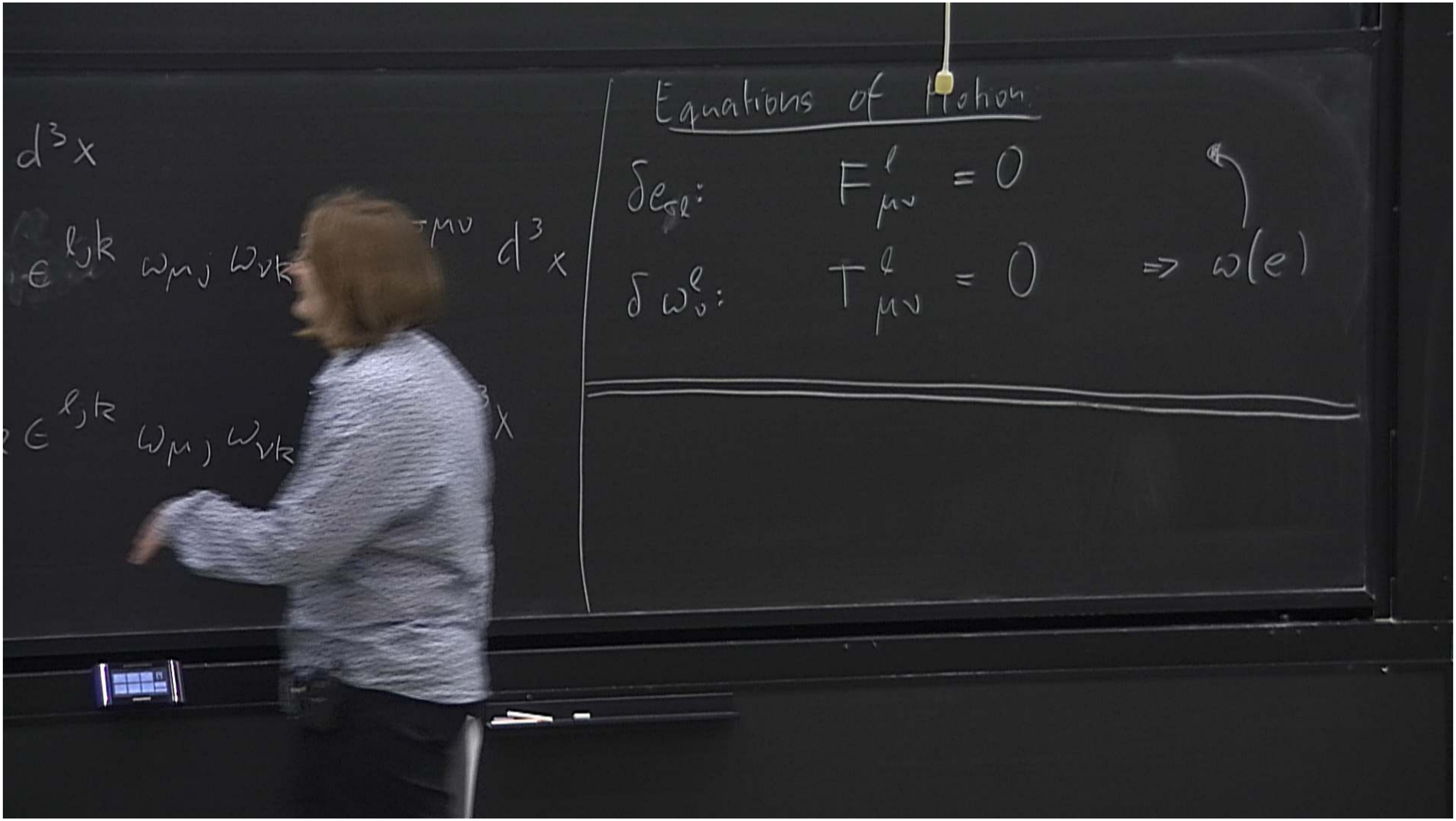
$\epsilon^{ljk}\, \omega_{\mu j}\, \omega_{\nu k}\,]\ \tilde{\epsilon}^{\sigma\mu\nu}\, d^3x$

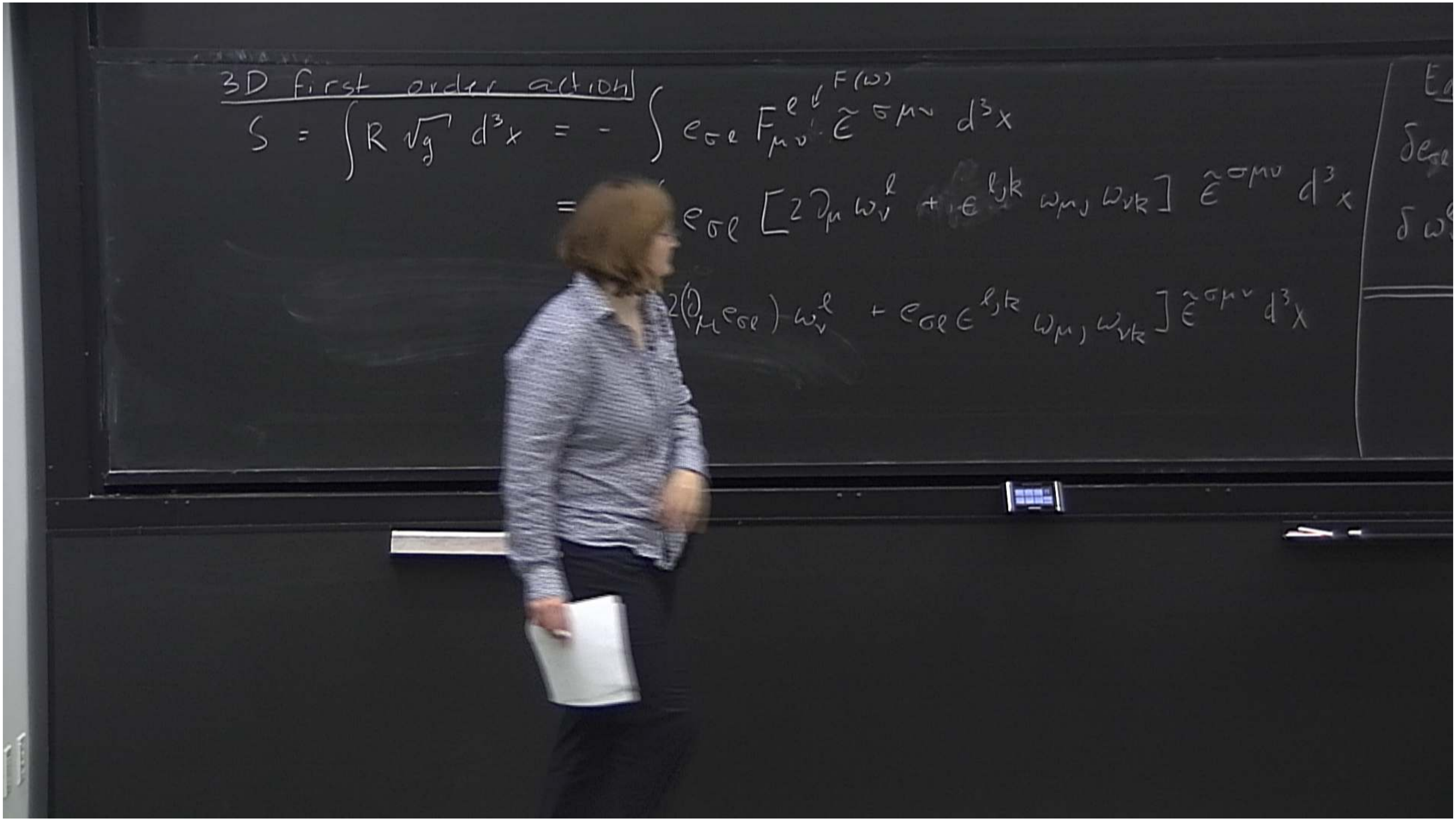
<u>Equations of Motion:</u>

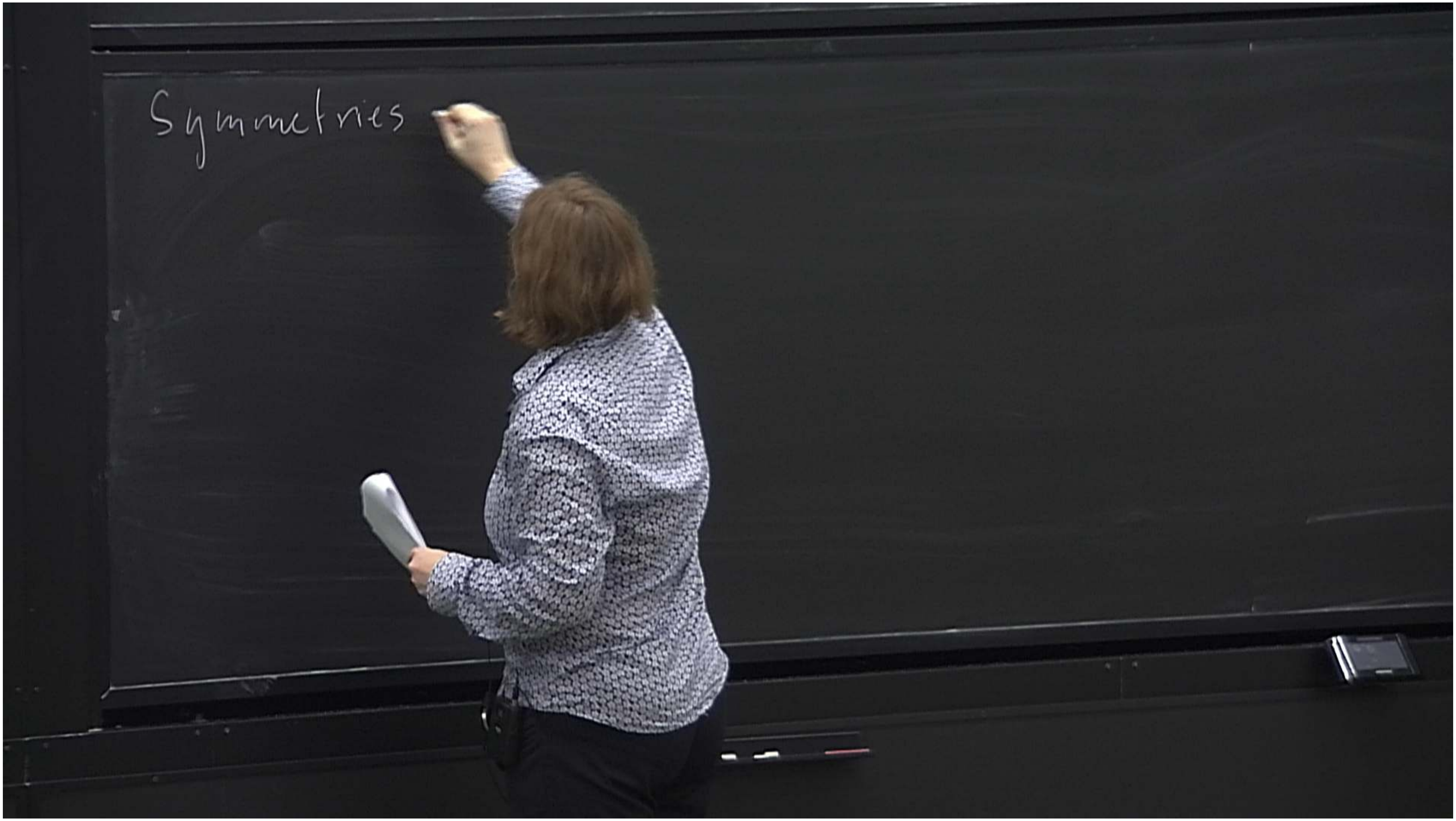
$\delta e_{\sigma l}$: $\quad F^{l}_{\mu\nu} = 0$

$\delta \omega^{l}_{\nu}$:


Equations of Motion
$\delta e_{\sigma l}$: $F^{l}_{\mu\nu} = 0$
$\delta \omega^{l}_{\nu}$: $T^{l}_{\mu\nu} = 0$
d^3x
$\epsilon^{ljk} \omega_{\mu j} \omega_{\nu k}] \tilde{\epsilon}^{\sigma\mu\nu} d^3x$
$\epsilon^{ljk} \omega_{\mu j} \omega_{\nu k}] \tilde{\epsilon}^{\sigma\mu\nu} d^3x$


Equations of Motion
δe: $F^{l}_{\mu\nu} = 0$
$\delta \omega^{l}_{\nu}$: $T^{l}_{\mu\nu} = 0$ $\Rightarrow \omega(e)$
d^3x

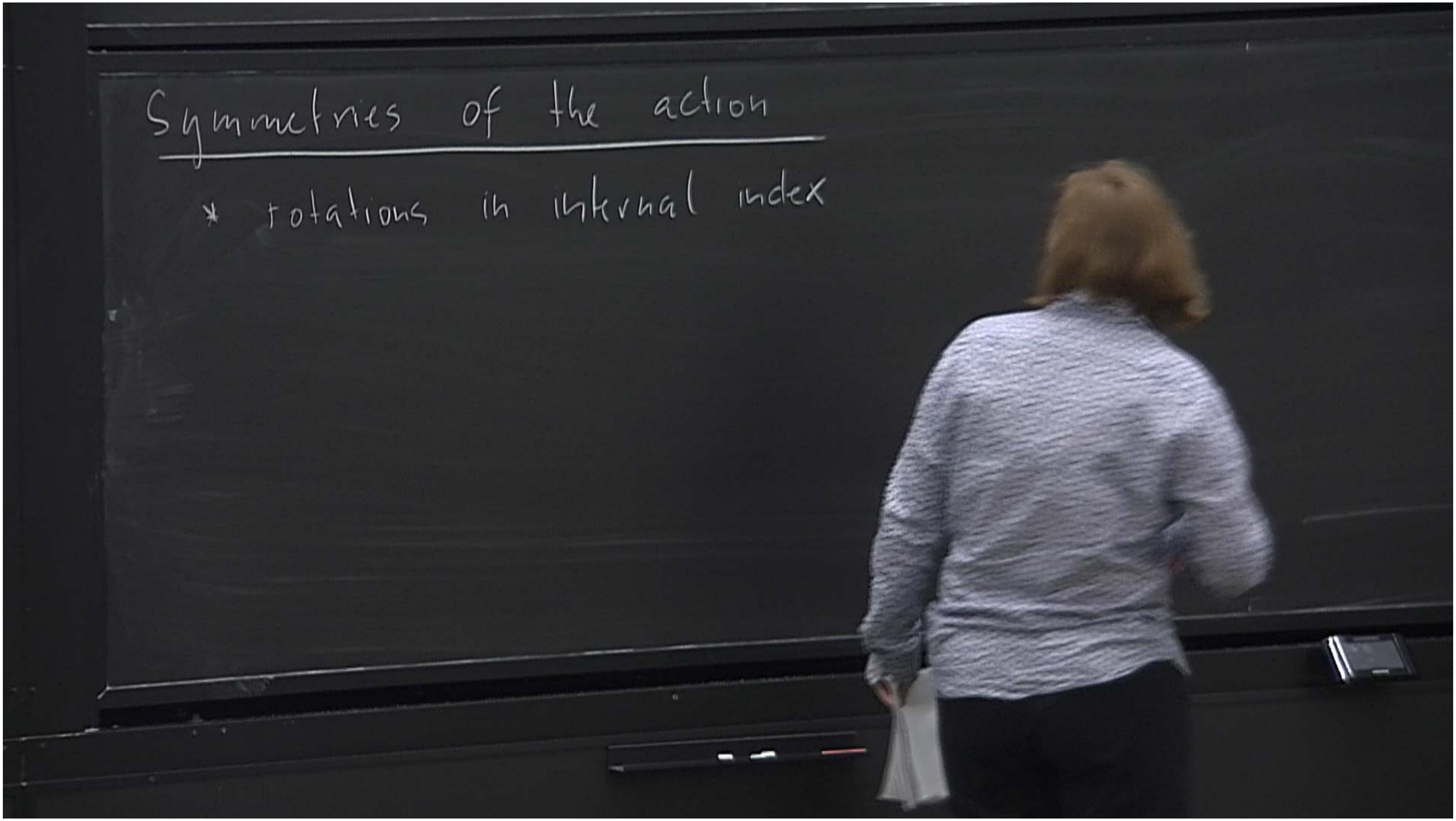

3D first order action
$S = \int R \sqrt{g}\, d^3x = -\int e_{\sigma\ell} F^{\ell}_{\mu\nu}(\omega)\, \tilde{\epsilon}^{\sigma\mu\nu}\, d^3x$
$= \quad e_{\sigma\ell} \left[2\partial_\mu \omega_\nu^\ell + \epsilon^{\ell jk} \omega_{\mu j} \omega_{\nu k} \right] \tilde{\epsilon}^{\sigma\mu\nu}\, d^3x$
$2(\partial_\mu e_{\sigma\ell})\, \omega_\nu^\ell + e_{\sigma\ell} \epsilon^{\ell jk} \omega_{\mu j} \omega_{\nu k}] \tilde{\epsilon}^{\sigma\mu\nu}\, d^3x$

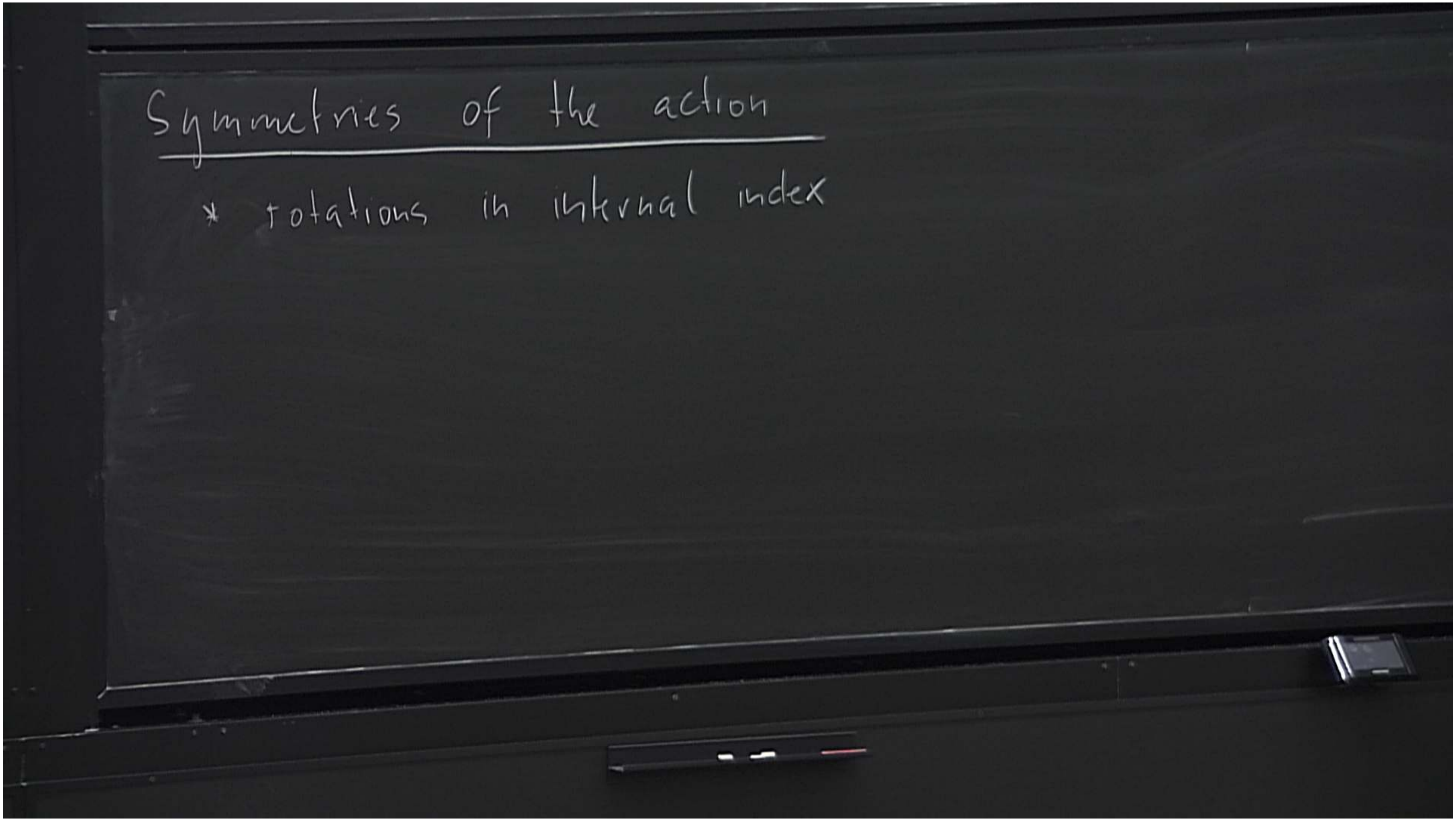

Symmetries

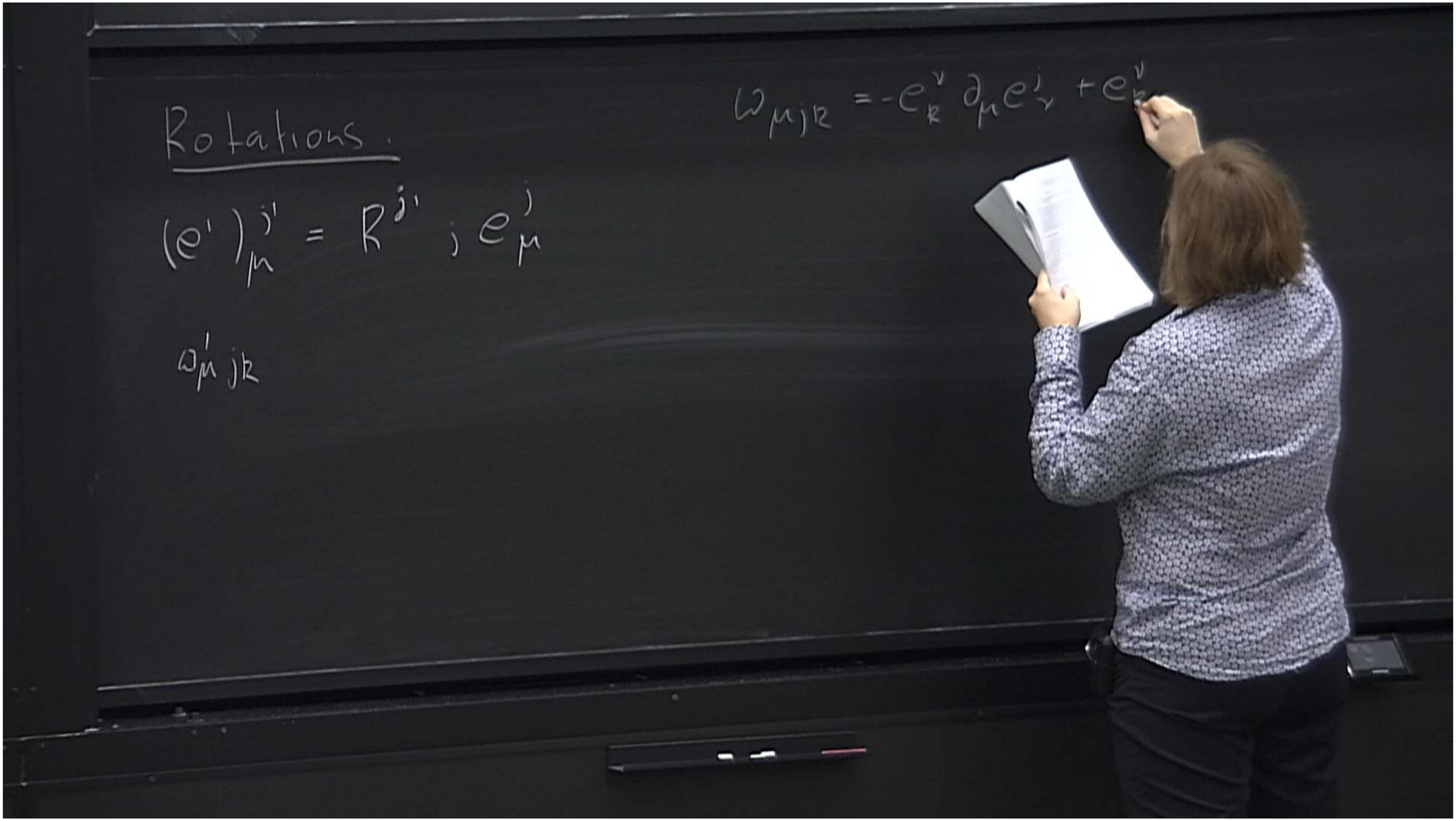
Symmetries of the action

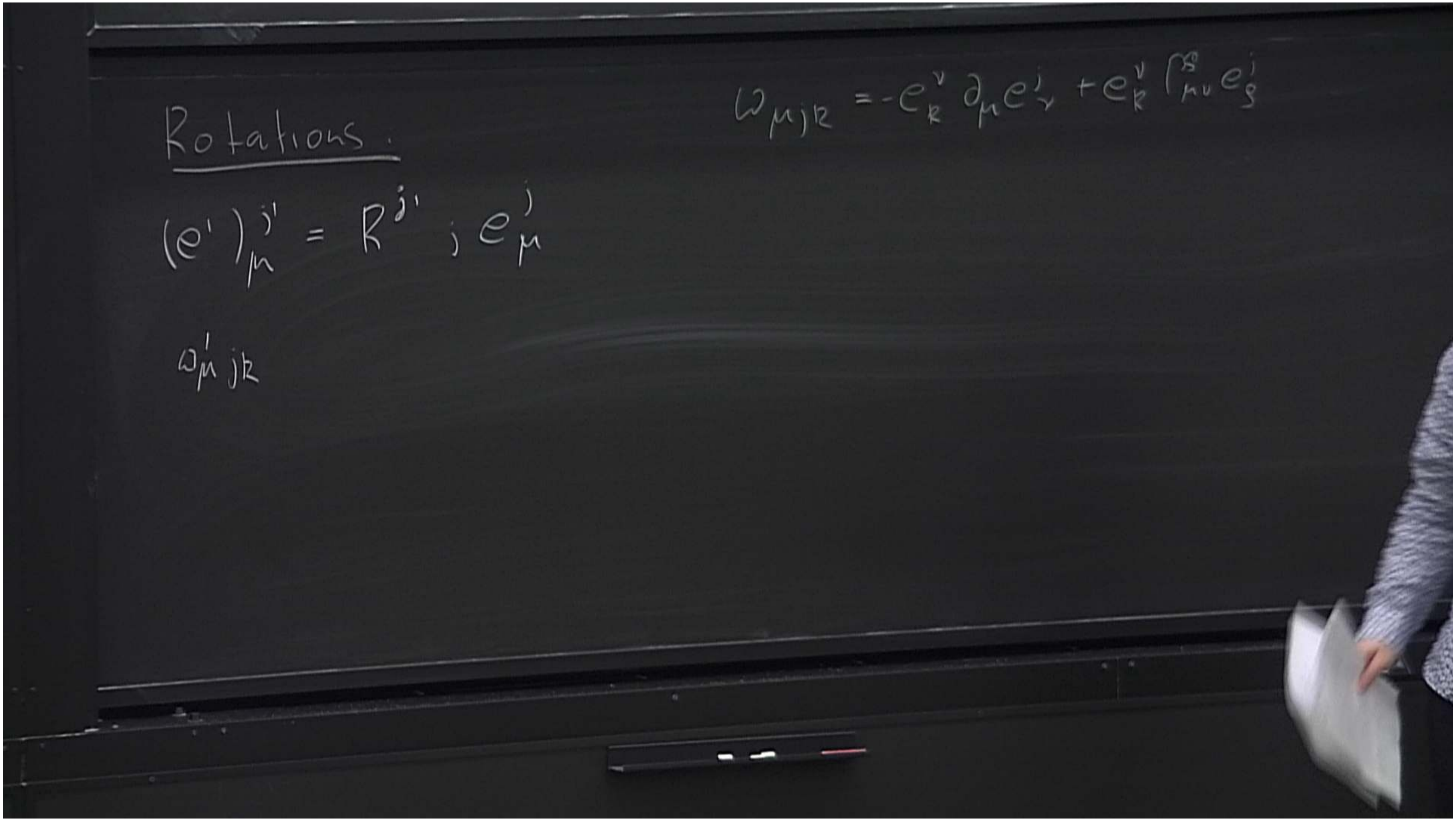
Symmetries of the action

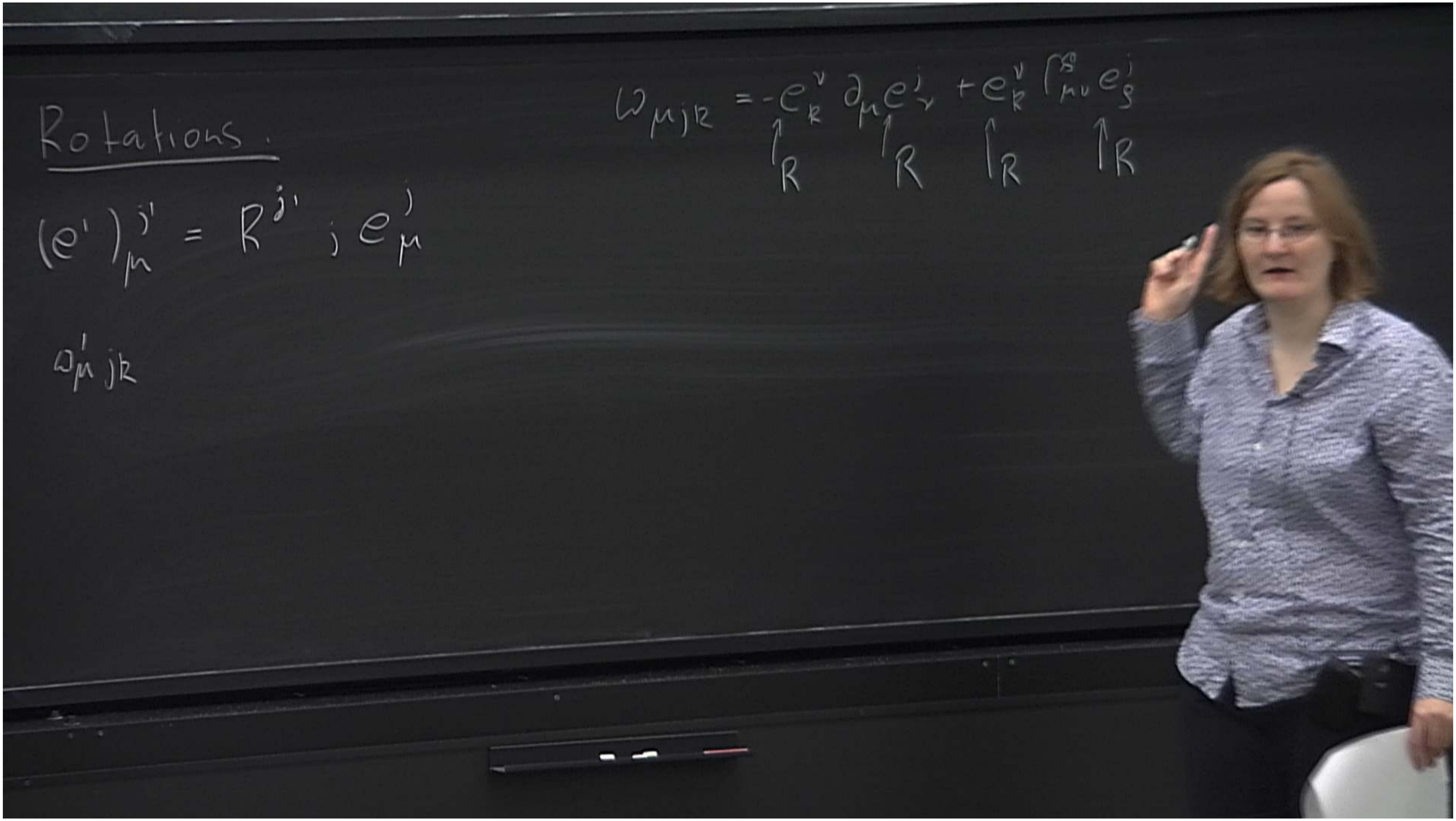
* rotations in internal index

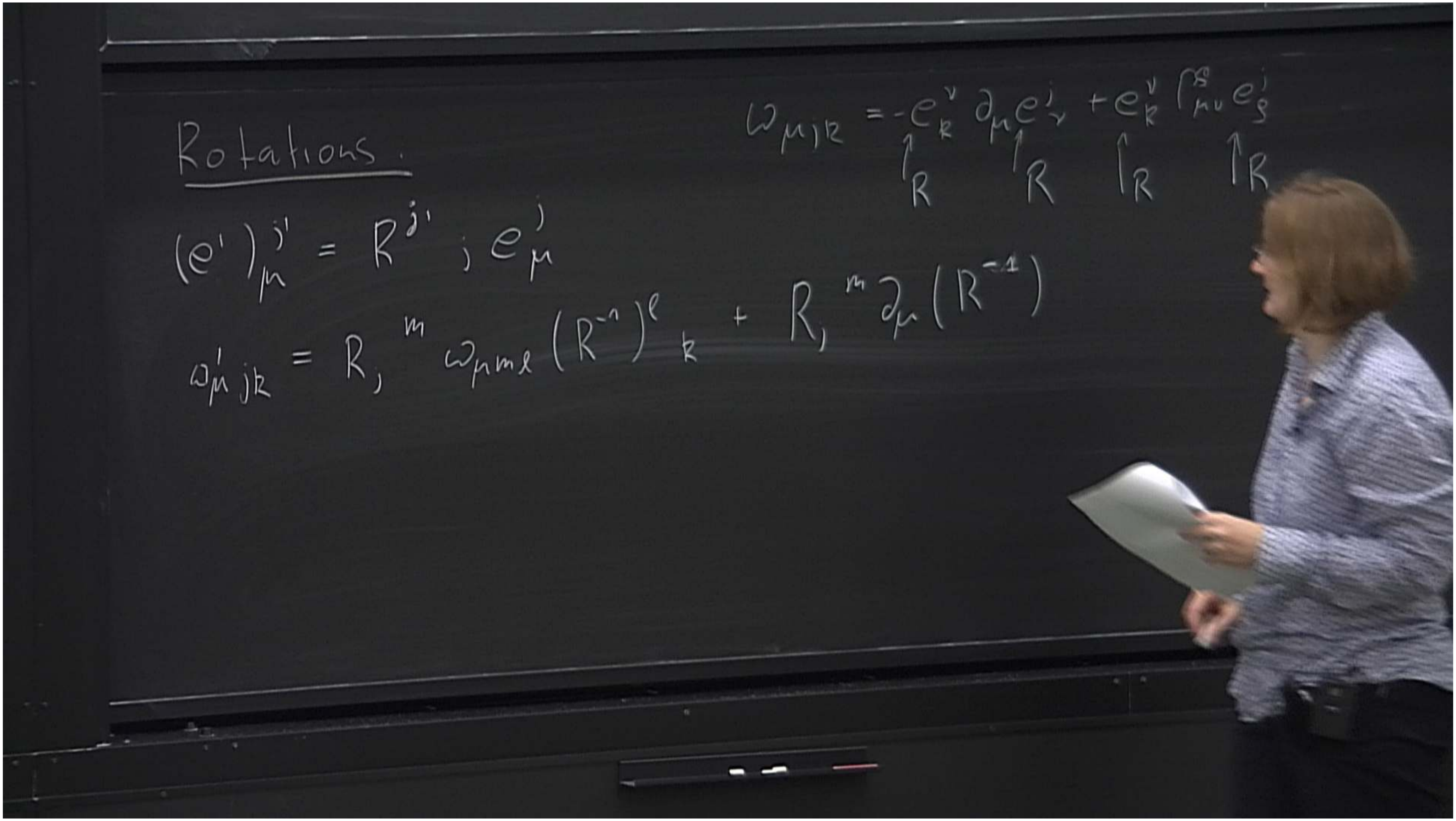

Symmetries of the action
* rotations in internal index

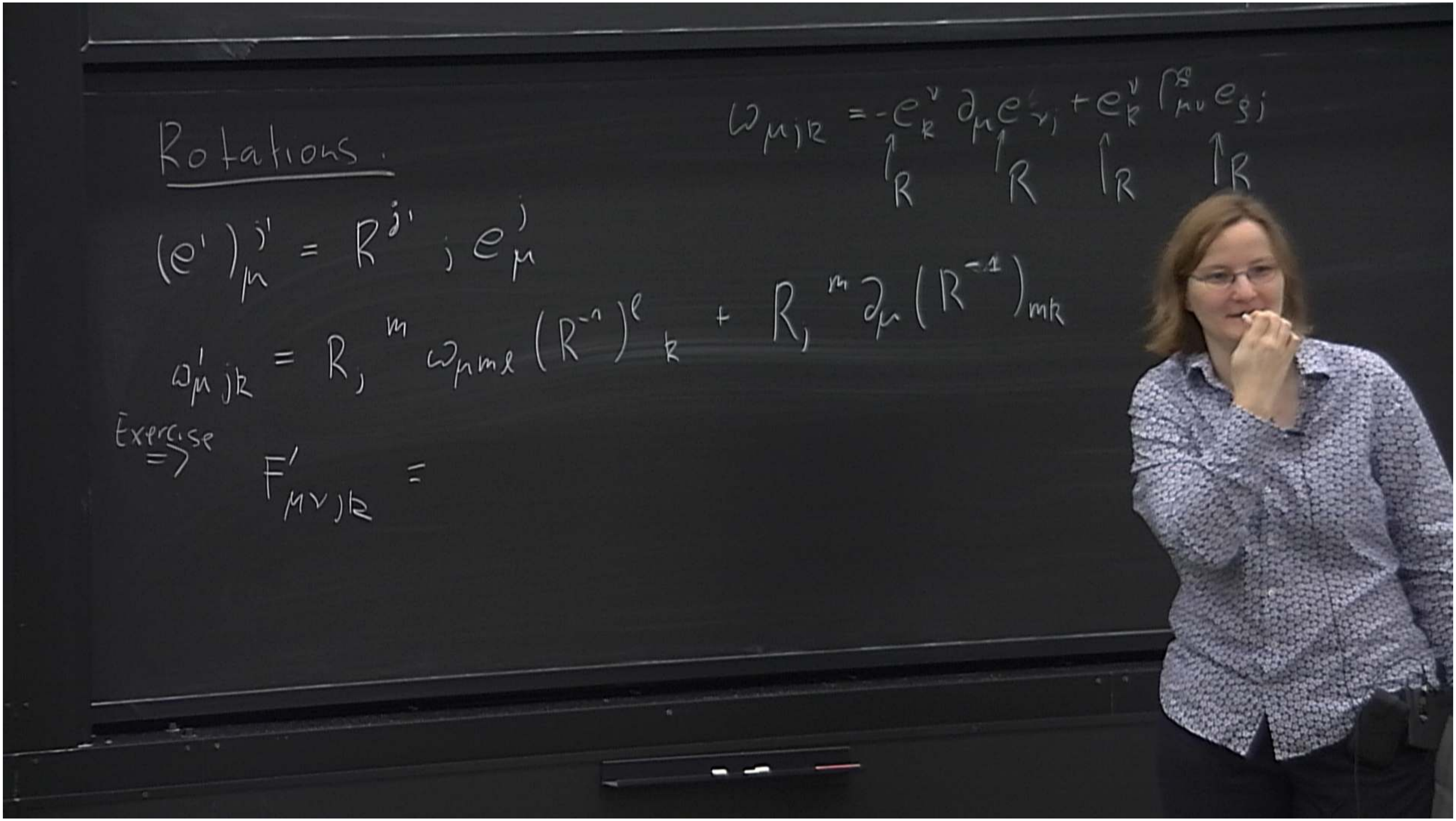

Symmetries of the action
* rotations in internal index

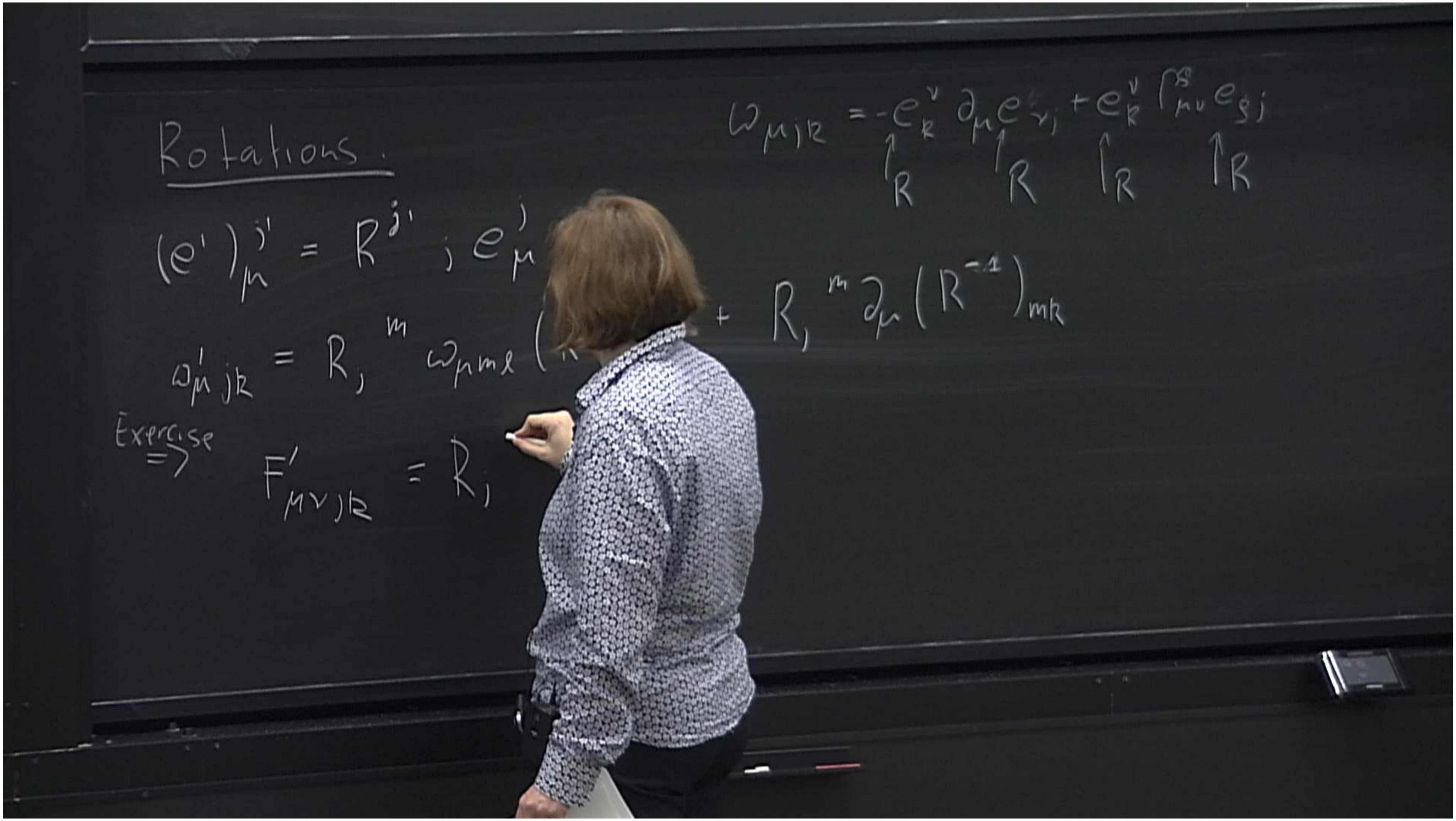

Rotations.
$(e')_\mu^{j'} = R^{j'}{}_j \, e_\mu^j$
$\omega'_\mu{}_{jk}$
$\omega_{\mu jk} = -e_k^\nu \partial_\mu e^j_\nu + e_k^\nu$


Rotations.
$(e')^{j'}_{\mu} = R^{j'}{}_{j}\, e^{j}_{\mu}$
$\omega'_{\mu\, jk}$
$\omega_{\mu jk} = -e^{\nu}_{k}\, \partial_{\mu} e^{j}_{\nu} + e^{\nu}_{k}\, \Gamma^{\rho}_{\mu\nu}\, e^{j}_{\rho}$


Rotations.
$\omega_{\mu jk} = -e_k^{\nu}\,\partial_\mu e^j_\nu + e_k^{\nu}\,\Gamma^{\rho}_{\mu\nu}\,e^j_\rho$
R R R R
$(e')_\mu^{j'} = R^{j'}{}_j\, e^j_\mu$
$\omega'_{\mu jk}$


Rotations.
$\omega_{\mu jk} = -e_k^{\ \nu}\, \partial_\mu e^j_{\ \nu} + e_k^{\ \nu}\, \Gamma^\rho_{\mu\nu}\, e^j_{\ \rho}$
$(e')^{j'}_{\ \mu} = R^{j'}_{\ \ j}\, e^j_{\ \mu}$
$\omega'_{\mu jk} = R_j^{\ m}\, \omega_{\mu m l}\, (R^{-1})^l_{\ k} + R_j^{\ m}\, \partial_\mu (R^{-1})$


Rotations.
$\omega_{\mu jk} = -e_k^{\nu} \partial_\mu e_{\nu j} + e_k^{\nu} \Gamma^{\rho}_{\mu\nu} e_{\rho j}$
R R R R
$(e')_\mu^{j'} = R^{j'}{}_{j} e_\mu^{j}$
$\omega'_{\mu jk} = R_j{}^m \omega_{\mu m l} (R^{-1})^l{}_k + R_j{}^m \partial_\mu (R^{-1})_{mk}$
Exercise $\Rightarrow$
$F'_{\mu\nu jk} =$


Rotations.
$(e')_\mu^{j'} = R^{j'}{}_j \, e_\mu^j$
$\omega_{\mu jk} = -e_k^\nu \partial_\mu e_{\nu j} + e_k^\nu \Gamma^\rho_{\mu\nu} e_{\rho j}$
R R R R
$\omega'_{\mu jk} = R_j{}^m \omega_{\mu m l} ($
$+ R_j{}^m \partial_\mu (R^{-1})_{mk}$
Exercise
$\Rightarrow$
$F'_{\mu\nu jk} = R_j$

Rotations.

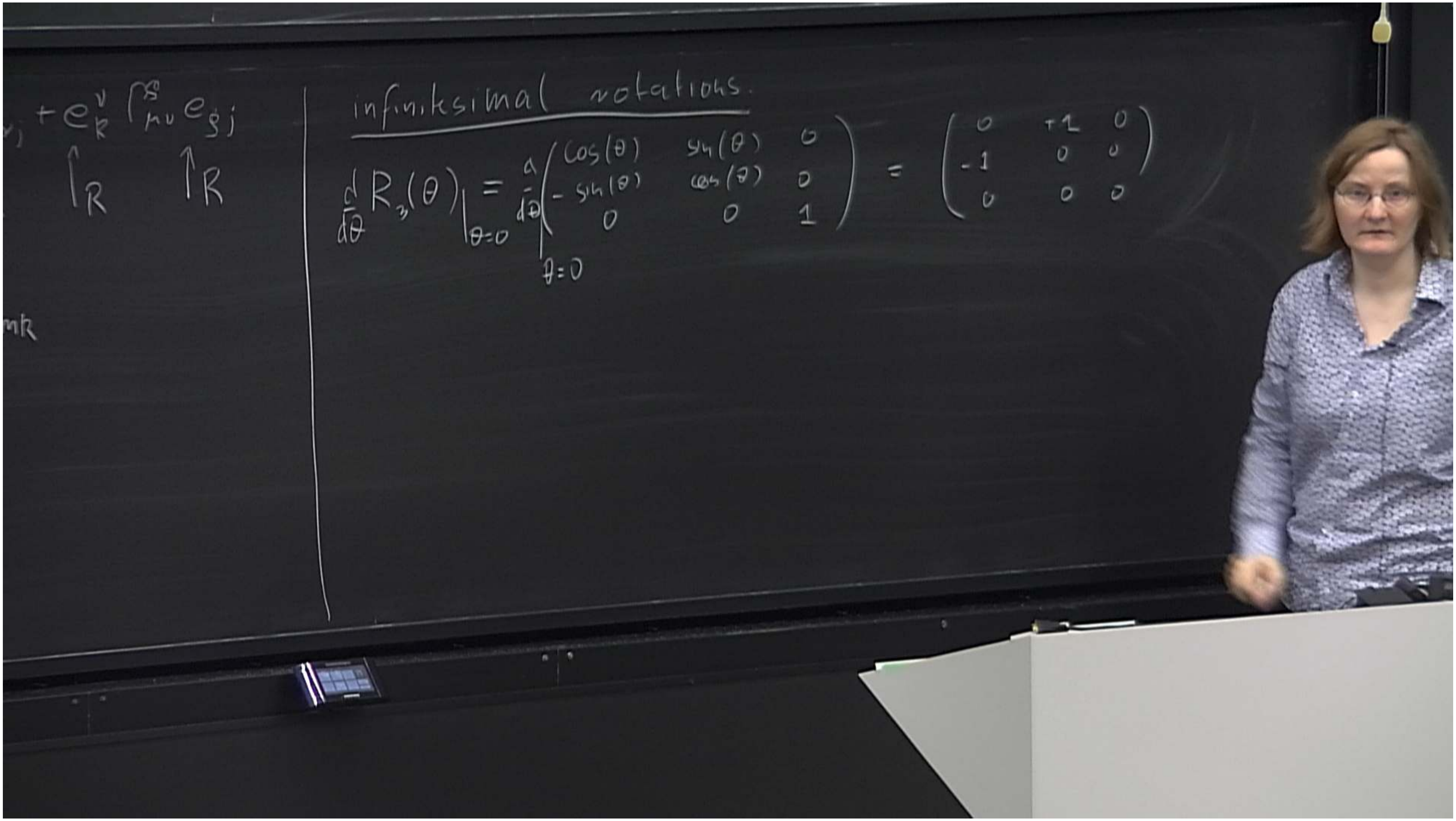
$$\omega_{\mu jk} = -\underset{\uparrow R}{e_k^{\ \nu}}\, \partial_\mu \underset{\uparrow R}{e_{\nu j}} + \underset{\uparrow R}{e_k^{\ \nu}}\, \Gamma^\rho_{\mu\nu} \underset{\uparrow R}{e_{\rho j}}$$

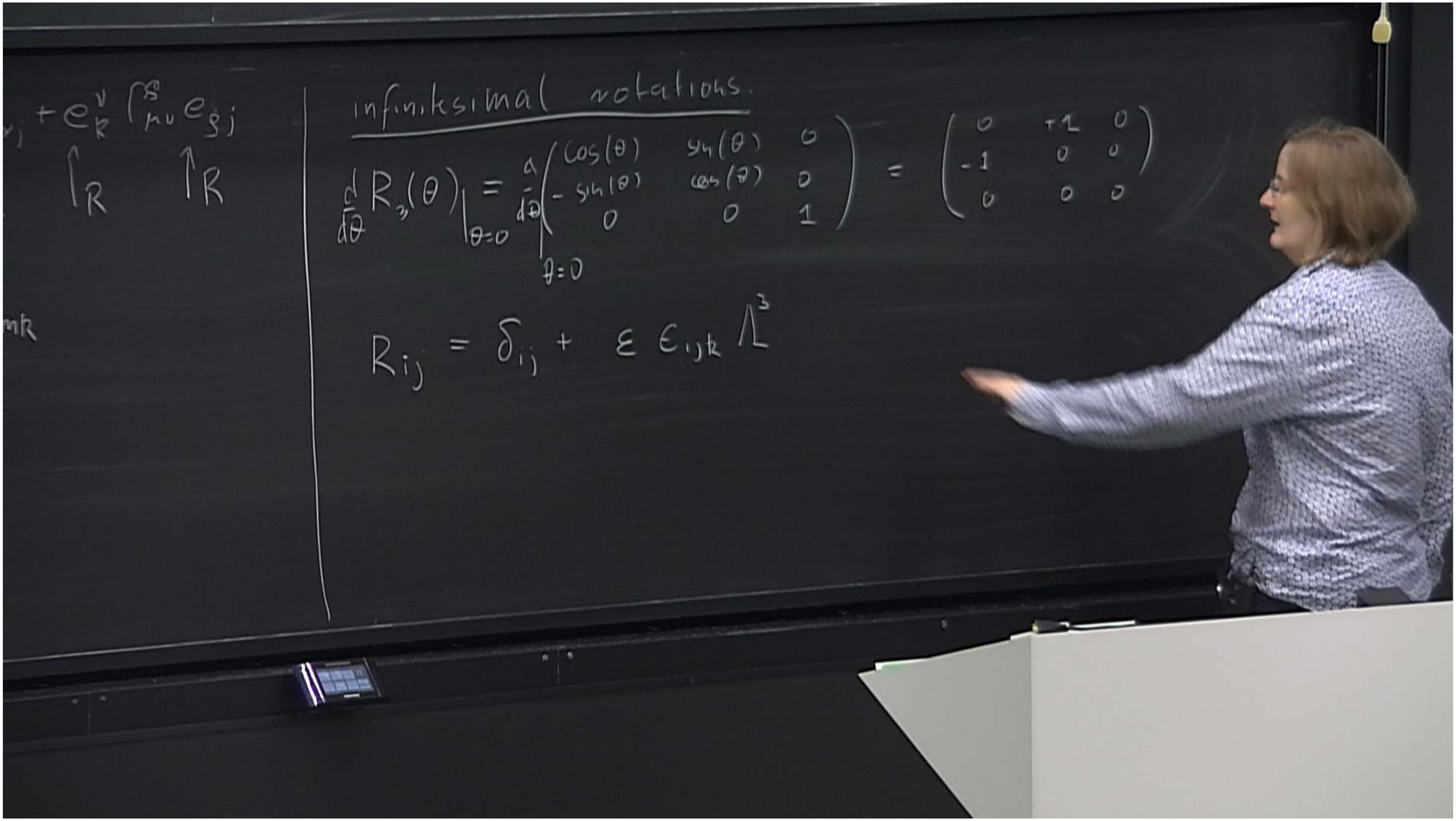
$$(e')_\mu^{\ j'} = R^{j'}{}_j\, e^j_\mu$$

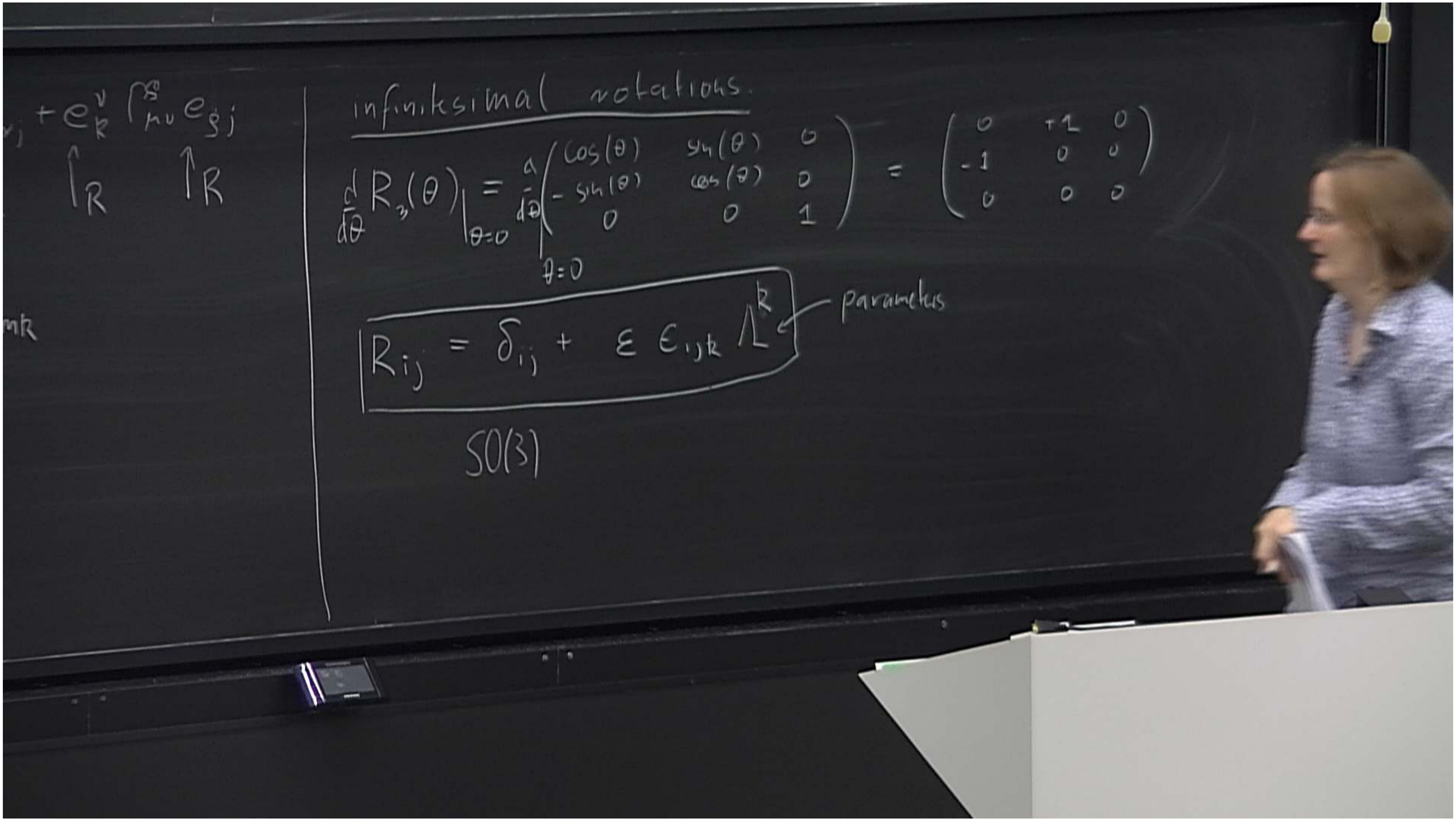
$$\omega'_{\mu jk} = R_j{}^m\, \omega_{\mu m l} (R^{-1})^l{}_k + R_j{}^m\, \partial_\mu (R^{-1})_{mk}$$

Exercise $\Rightarrow$

$$F'_{\mu\nu jk} = R_j{}^m\, F_{\mu\nu m l} (R^{-1})^l{}_k$$


infinitesimal rotations.
$\frac{d}{d\theta} R_3(\theta)\Big|_{\theta=0} = \frac{d}{d\theta}\begin{pmatrix}\cos(\theta) & \sin(\theta) & 0\\ -\sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{pmatrix}\Bigg|_{\theta=0} = \begin{pmatrix}0 & +1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$


infinitesimal rotations.
$R_{ij} = \delta_{ij} + \varepsilon\, \epsilon_{ijk} \Lambda^3$


infinitesimal rotations
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$\boxed{R_{ij} = \delta_{ij} + \varepsilon\, \epsilon_{ijk} \Lambda^k}$
parameters
SO(3)

$$\ldots_j + e^{\nu}_{k} \Gamma^{\rho}_{\mu\nu} e_{\rho j}$$

$\uparrow R \qquad \uparrow R$

mk

infinitesimal rotations.

$$\frac{d}{d\theta} R_3(\theta)\Big|_{\theta=0} = \frac{d}{d\theta}\begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}\Bigg|_{\theta=0} = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

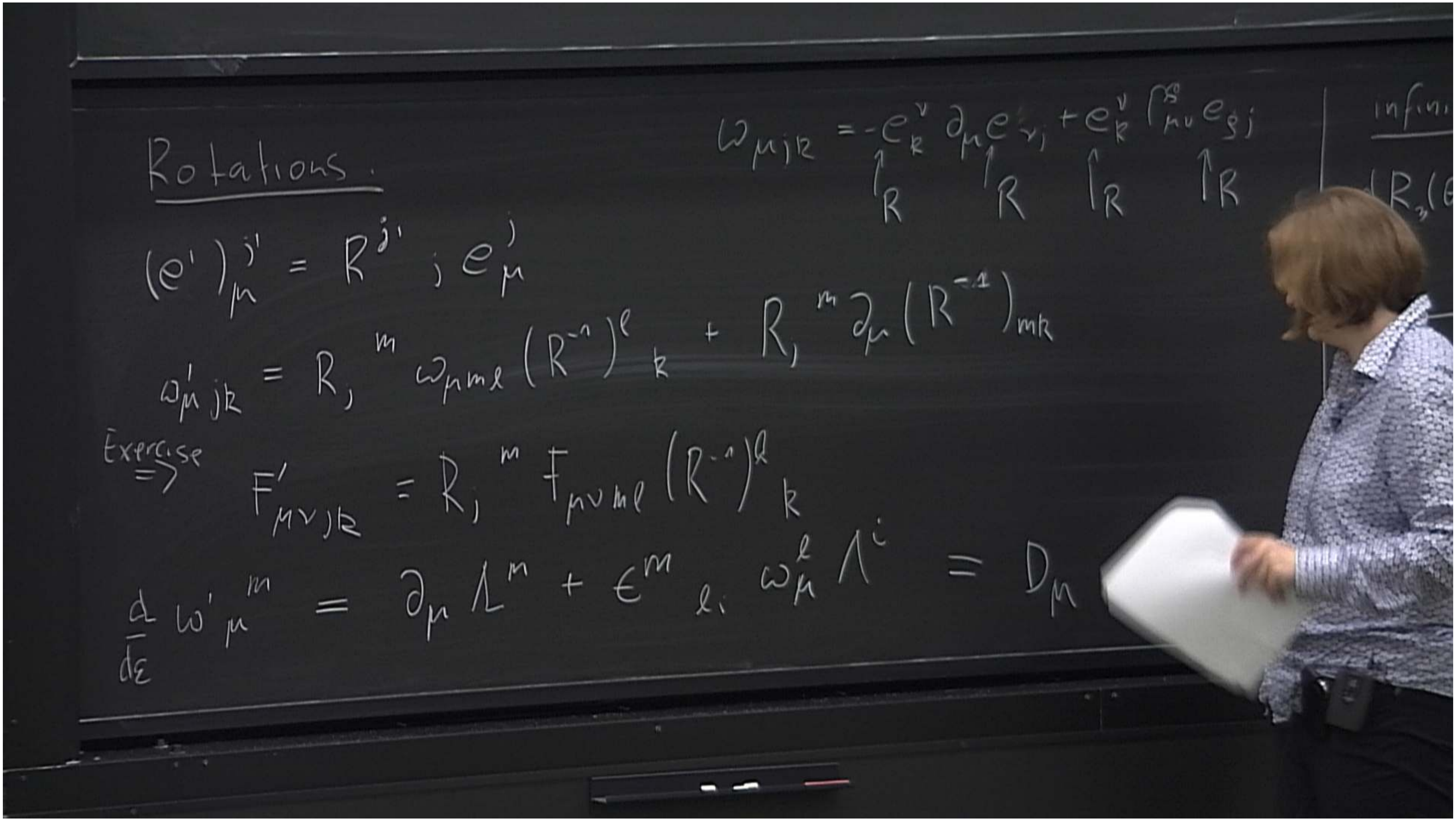
$$\boxed{R_{ij} = \delta_{ij} + \varepsilon\, \epsilon_{ijk} \Lambda^{k}}$$

parameters

SO(3)

basis of Lie algebra $so(3) = su(2)$

$$(T_k)_{ij} = \epsilon_{ijk}$$


Rotations.
$\omega_{\mu jk} = -e^\nu_k \partial_\mu e_{\nu j} + e^\nu_k \Gamma^\rho_{\mu\nu} e_{\rho j}$
R R R R
$(e')_\mu^{j'} = R^{j'}{}_j e^j_\mu$
$\omega'_{\mu jk} = R_j{}^m \omega_{\mu m l} (R^{-1})^l{}_k + R_j{}^m \partial_\mu (R^{-1})_{mk}$
Exercise $\Rightarrow$
$F'_{\mu\nu jk} = R_j{}^m F_{\mu\nu m l} (R^{-1})^l{}_k$
$\frac{d}{d\epsilon} \omega'_\mu{}^m = \partial_\mu \Lambda^m + \epsilon^m{}_{li} \omega^l_\mu \Lambda^i = D_\mu$

Rotations.

$$\omega_{\mu jk} = -e_k^{\nu}\, \partial_\mu e_{\nu j} + e_k^{\nu}\, \Gamma^{\rho}_{\mu\nu}\, e_{\rho j}$$

R R R R

$$(e')_\mu^{j'} = R^{j'}{}_{j}\, e_\mu^{j}$$

$$\omega'_{\mu jk} = R_j{}^{m}\, \omega_{\mu m l}\, (R^{-1})^{l}{}_{k} + R_j{}^{m}\, \partial_\mu (R^{-1})_{mk}$$

Exercise $\Rightarrow$

$$F'_{\mu\nu jk} = R_j{}^{m}\, F_{\mu\nu m l}\, (R^{-1})^{l}{}_{k}$$

$$\frac{d}{d\epsilon}\, \omega'_\mu{}^{m} = \partial_\mu \Lambda^m + \epsilon^{m}{}_{li}\, \omega_\mu^{l}\, \Lambda^{i} = D_\mu \Lambda_m = \delta^R$$

infini

$\frac{d}{d\theta} R_3(\theta$

$\boxed{R_{ij}}$

Rotations.

$$\omega_{\mu jk} = -e^\nu_k \, \partial_\mu e_{\nu j} + e^\nu_k \, \Gamma^\rho_{\mu\nu} \, e_{\rho j}$$

$$\uparrow R \quad \uparrow R \quad \uparrow R \quad \uparrow R$$

$$(e')^{j'}_\mu = R^{j'}{}_j \, e^j_\mu$$

$$\omega'_{\mu jk} = R_j{}^m \, \omega_{\mu m l} (R^{-1})^l{}_k + R_j{}^m \, \partial_\mu (R^{-1})_{mk}$$

Exercise $\Rightarrow$

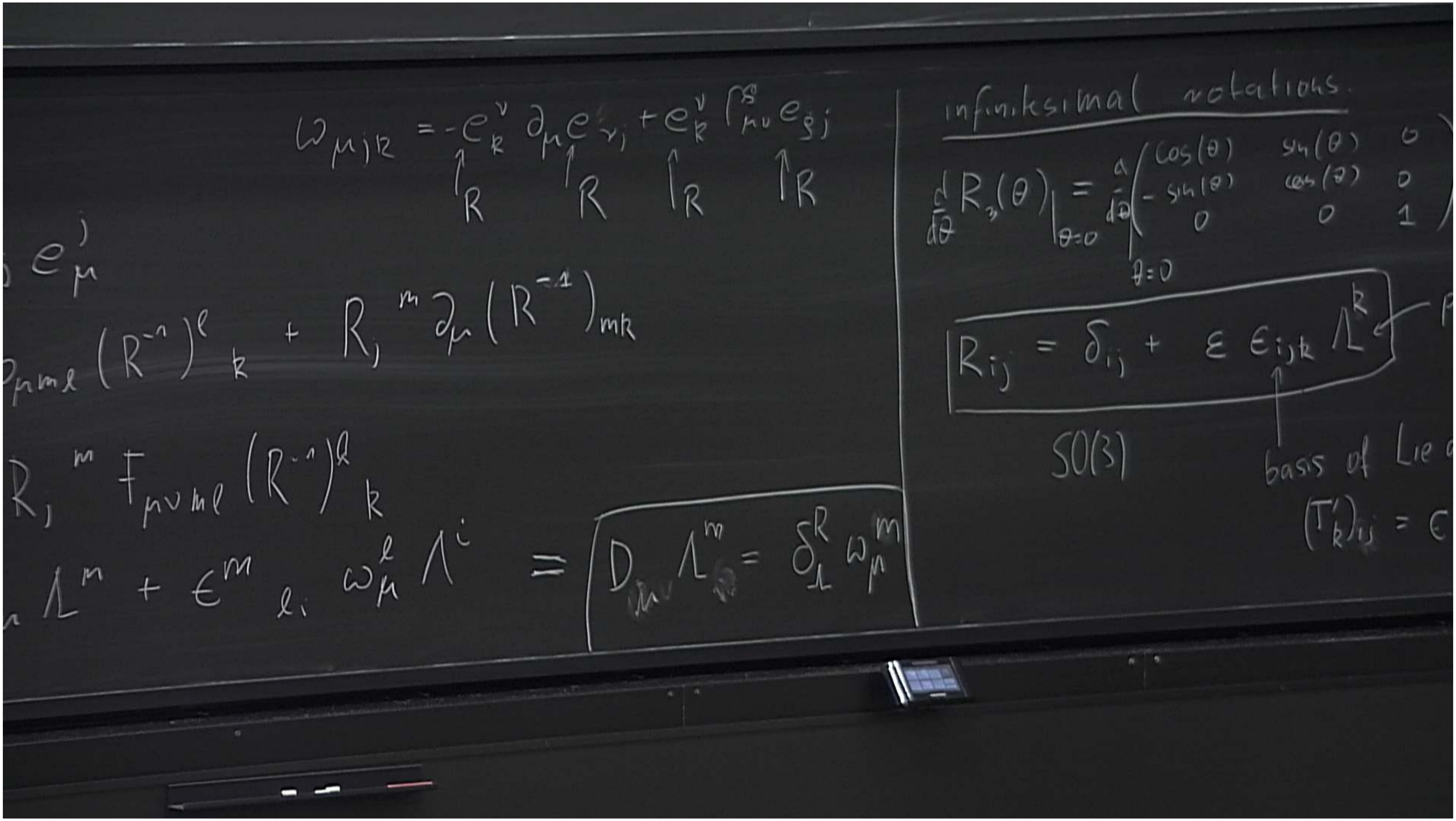
$$F'_{\mu\nu jk} = R_j{}^m \, F_{\mu\nu m l} (R^{-1})^l{}_k$$

$$\frac{d}{d\varepsilon} \omega'_\mu{}^m = \partial_\mu \Lambda^m + \epsilon^m{}_{li} \, \omega^l_\mu \Lambda^i = D_\mu \Lambda_m = \delta^R_\Lambda \omega_\mu$$

infini...

$\frac{d}{d\theta} R_z($...

R_{ij} ...


infinitesimal rotations.
ω_{μjk} = -e_k^ν ∂_μ e_{νj} + e_k^ν Γ^ρ_{μν} e_{ρj}
R R R R
e_μ^j
R_j^m ∂_μ (R^{-1})_{mk}
R_j^m F_{μνml} (R^{-1})^l_k
Λ^m + ε^m_{li} ω_μ^l Λ^i
d/dθ R_3(θ)|_{θ=0} = d/dθ (cos(θ) sin(θ) 0 ; -sin(θ) cos(θ) 0 ; 0 0 1)|_{θ=0}
R_{ij} = δ_{ij} + ε ε_{ijk} Λ^k
SO(3)
basis of Lie

Rotations.

$\omega_{\mu jk} = -e^\nu_k \partial_\mu e_{\nu j} + e^\nu_k \Gamma^\rho_{\mu\nu} e_{\rho j}$

$\uparrow R \quad \uparrow R \quad \uparrow R \quad \uparrow R$

$(e')^{j'}_\mu = R^{j'}{}_j \, e^j_\mu$

$\omega'_{\mu\, jk} = R_j{}^m \omega_{\mu m \ell} (R^{-1})^\ell{}_k + R_j{}^m \partial_\mu (R^{-1})_{mk}$

Exercise $\Rightarrow$ $F'_{\mu\nu jk} = R_j{}^m F_{\mu\nu m\ell} (R^{-1})^\ell{}_k$

$\ldots{}^m = \partial_\mu \Lambda^m + \epsilon^m{}_{\ell i} \, \omega^\ell_\mu \Lambda^i \Rightarrow \boxed{D_\mu \Lambda^m = \delta^R_\Lambda \omega^m_\mu}$

$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}$

$\frac{d}{d\theta} R$

$$\omega_{\mu jk} = -\underset{\uparrow R}{e_k^{\nu}}\, \partial_\mu \underset{\uparrow R}{e_{\nu j}} + \underset{\uparrow R}{e_k^{\nu}}\, \Gamma^{\rho}_{\mu\nu} \underset{\uparrow R}{e_{\rho j}}$$

Rotations.

$$(e')_\mu^{j'} = R^{j'}{}_j\, e^j_\mu$$

$$\omega'_{\mu jk} = R_j{}^m\, \omega_{\mu m l} (R^{-1})^l{}_k + R_j{}^m \partial_\mu (R^{-1})_{mk}$$

Exercise $\Rightarrow$

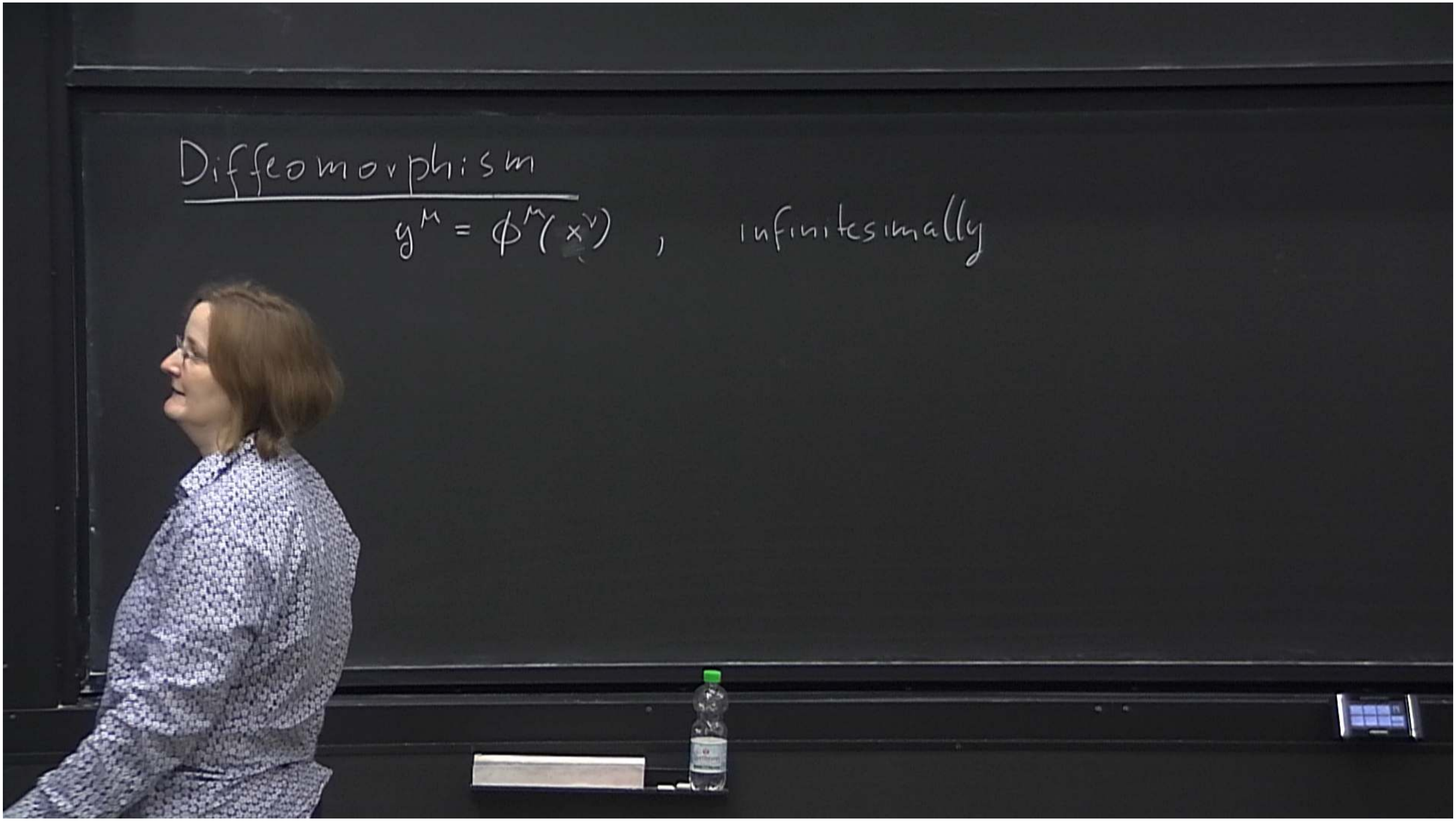
$$F'_{\mu\nu jk} = R_j{}^m\, F_{\mu\nu m l} (R^{-1})^l{}_k$$

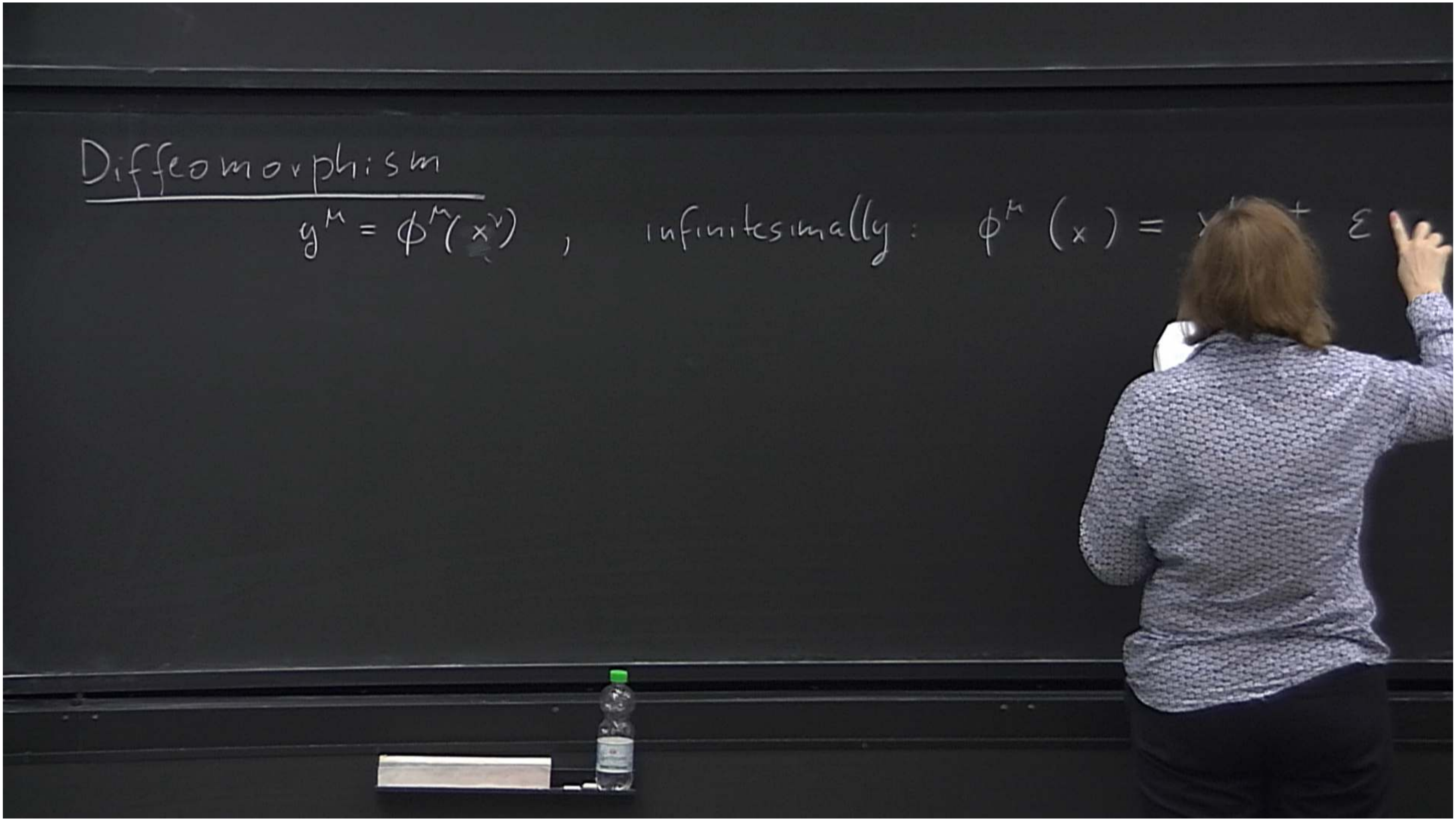
$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \omega'_\mu{}^m = \partial_\mu \Lambda^m + \epsilon^m{}_{li}\, \omega^l_\mu \Lambda^i \Rightarrow \boxed{D_\mu \Lambda^m = \delta^R_\Lambda \omega_\mu^m}$$

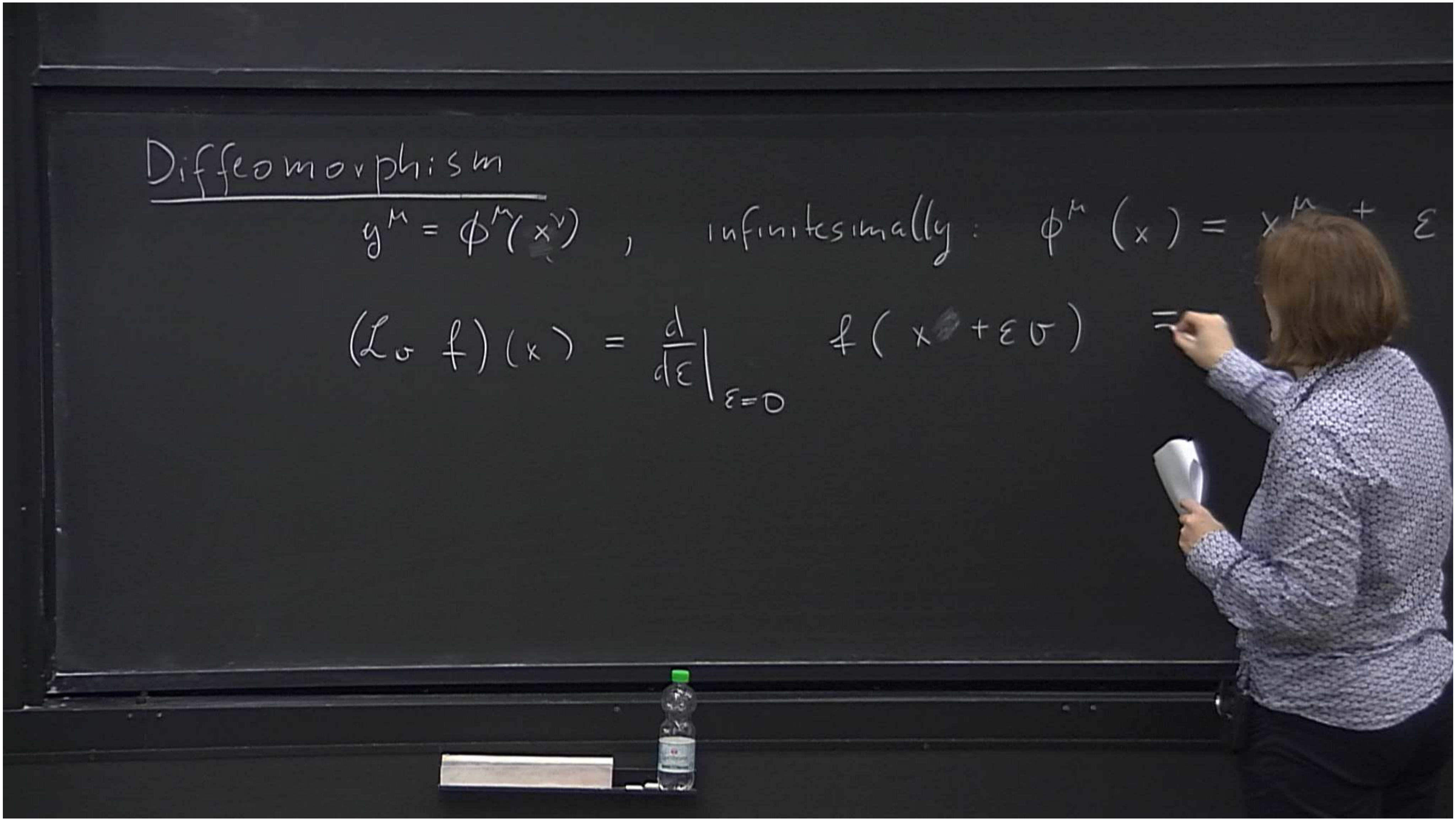
in

$\frac{d}{d\theta} R$

R


Diffeomorphism
$y^\mu = \phi^\mu(x^\nu)$, infinitesimally


Diffeomorphism
$y^\mu = \phi^\mu(x^\nu)$, infinitesimally: $\phi^\mu(x) =$


Diffeomorphism
$y^\mu = \phi^\mu(x^\nu)$, infinitesimally: $\phi^\mu(x) = x^\mu + \varepsilon$
$(\mathcal{L}_v f)(x) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(x + \varepsilon v) =$

orphism

$$y^\mu = \phi^\mu(x^\nu) \quad , \quad \text{infinitesimally: } \phi^\mu(x) = x^\mu + \varepsilon v^\mu(x)$$

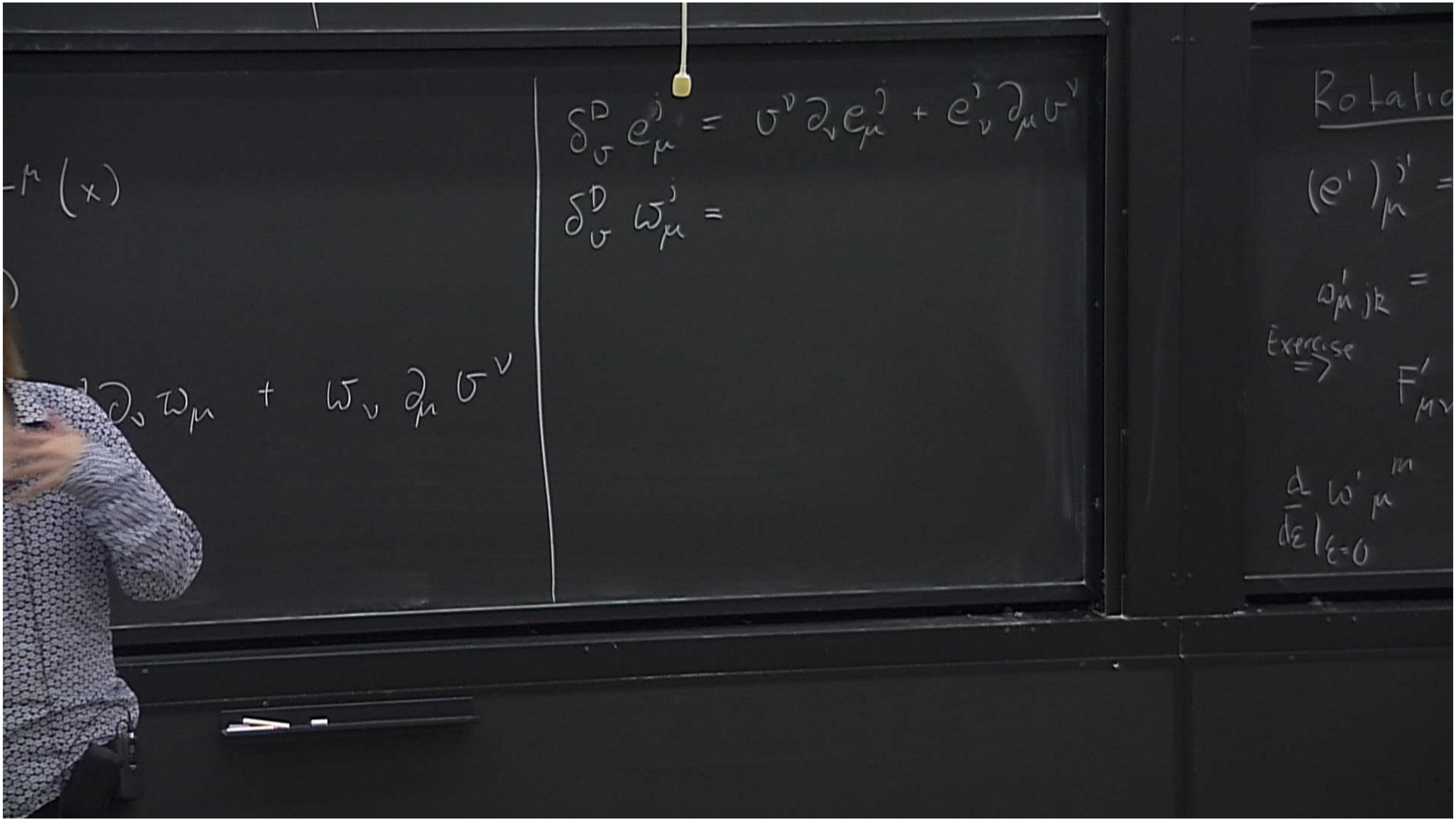
$$(\mathcal{L}_v f)(x) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(x + \varepsilon v) = v^\mu \partial_\mu f \; (x)$$

$$(\mathcal{L}_v \omega)_\mu(x) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left[\omega_\nu(x + \varepsilon v) \frac{\partial \phi^\nu}{\partial x^\mu} \right] = v^\nu \partial_\nu \omega_\mu +$$

infinitesimally: $\phi^\mu(x) = x^\mu + \varepsilon\, v^\mu(x)$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(x+\varepsilon v) = v^\mu \partial_\mu f\,(x)$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left[\omega_\nu(x+\varepsilon v)\, \frac{\partial \phi^\nu}{\partial x^\mu} \right] = v^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu v^\nu$$


$\partial_\nu \omega_\mu + \omega_\nu \partial_\mu v^\nu$
$\delta^D_v e^j_\mu = v^\nu \partial_\nu e^j_\mu + e^j_\nu \partial_\mu v^\nu$
$\delta^D_v \omega^j_\mu =$
Rotatio
$(e')^{j'}_\mu =$
$\omega'_{\mu\, jk} =$
Exercise $\Rightarrow$ $F'_{\mu\nu}$
$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \omega'_\mu{}^m$

$$\omega_{\mu jk} = -e_k^{\nu}\,\partial_\mu e_{\nu j} + e_k^{\nu}\,\Gamma^{\rho}_{\mu\nu} e_{\rho j}$$

R R R R

$R^{j}{}_{j}\, e^{j}_{\mu}$

${}^{m}\,\omega_{\mu m l}\,(R^{-1})^{l}{}_{k} + R_j{}^{m}\,\partial_\mu (R^{-1})_{mk}$

$= R_j{}^{m}\, F_{\mu\nu m l}\,(R^{-1})^{l}{}_{k}$

$$\boxed{\delta^{R}_{\Lambda} e^{j}_{\mu} = \varepsilon^{j}{}_{kl}\, e^{k}_{\mu} \Lambda^{l}}$$

$$\partial_\mu \Lambda^{m} + \epsilon^{m}{}_{li}\,\omega^{l}_{\mu} \Lambda^{i} = \boxed{D_\mu \Lambda^{m} = \delta^{R}_{\Lambda}\, \omega^{m}_{\mu}}$$

infinitesimal rotations

$$\frac{d}{d\theta} R_3(\theta)\Big|_{\theta=0} = \frac{d}{d\theta}\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \\ 0 & 0 \end{pmatrix}\Bigg|_{\theta=0}$$

$$\boxed{R_{ij} = \delta_{ij} + \varepsilon\, \epsilon_{ijk} \Lambda^{k}}$$

SO(3)

basis of

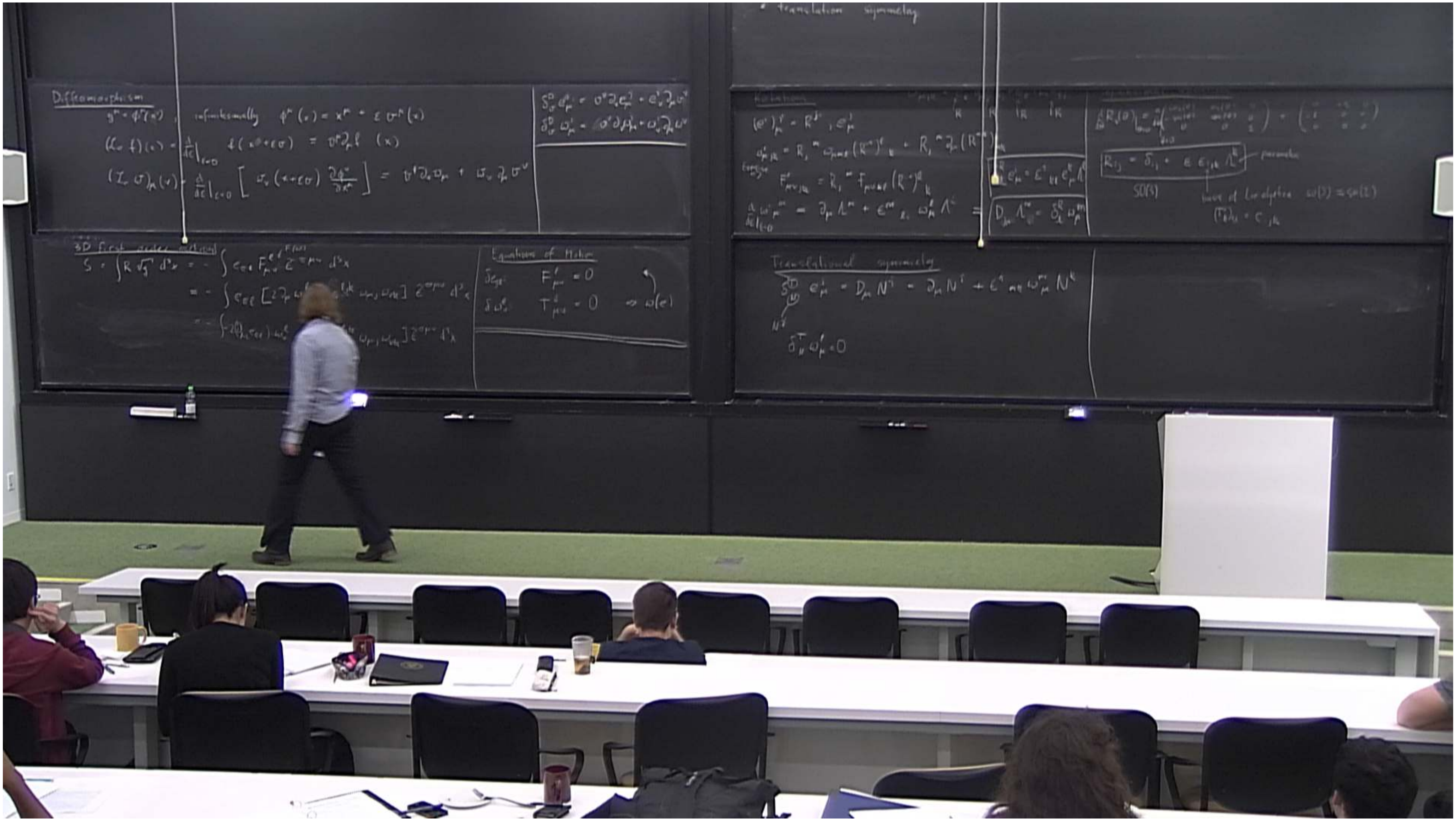
$(T^{i}_{k})_{i}$

Translational symmetry

$$\delta^{T}_{N}\, e^{i}_{\mu} = D_{\mu} N^{i} = \partial_{\mu} N^{i} + \epsilon^{i}{}_{mk}\,\omega^{m}_{\mu} N^{k}$$

N^{i}

$$\delta^{T}_{N}\, \omega^{\ell}_{\mu} = 0$$


Diffeomorphism
Equations of Motion
Translational symmetry

Translational symmetry

$$\delta^{T}_{N} e^{i}_{\mu} = D_{\mu} N^{i} = \partial_{\mu} N^{i} + \epsilon^{i}{}_{mk}\, \omega^{m}_{\mu} N^{k}$$

N^{i}

(due to Bianchi identity)

$$\delta^{T}_{N} \omega^{l}_{\mu} = 0$$

3D first order action

$$S = \int R\sqrt{g}\, d^3x = -\int e_{\sigma\ell}\, F^{\ell}_{\mu\nu}(\omega)\, \tilde{\epsilon}^{\sigma\mu\nu}\, d^3x$$

$$= -\int \overset{(e_{\sigma\ell}+\epsilon D_\sigma N_\ell)}{e_{\sigma\ell}} \left[2\partial_\mu \omega^\ell_\nu + \epsilon^{\ell jk}\, \omega_{\mu j} \right.$$

$$= -\int -2(\partial_\mu e_{\sigma\ell})\,\omega^\ell_\nu + e_{\sigma\ell}\,\epsilon^{\ell jk}$$

$$\boxed{0 = N_\ell\, D_\sigma\, F^{\ell}_{\mu\nu}\, \tilde{\epsilon}^{\sigma\mu\nu}}$$

Bianchi identity

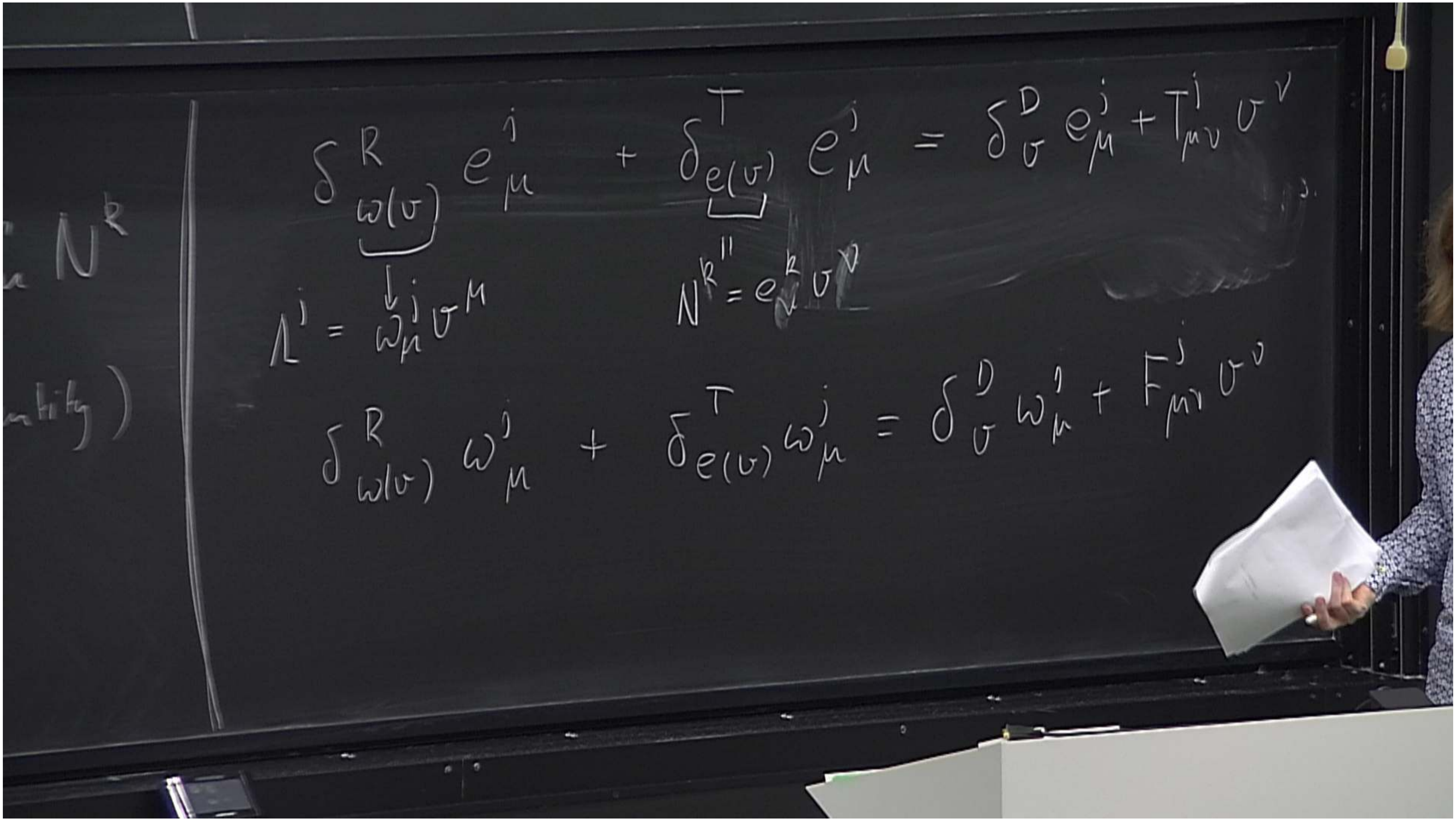
Translational symmetry

$$\delta^{T}_{N} e^{i}_{\mu} = D_\mu N^i = \partial_\mu N^i + \epsilon^{i}{}_{mk}\, \omega^{m}_{\mu} N^k$$

N^i

(due to Bianchi identity)

$$\delta^{T}_{N} \omega^{l}_{\mu} = 0$$


N^k
$\delta^R_{\omega(v)} e^j_\mu + \delta^T_{e(v)} e^j_\mu = \delta^D_v e^j_\mu + T^j_{\mu\nu} v^\nu$
$\Lambda^j = \omega^j_\mu v^\mu$
$N^k = e^k_\nu v^\nu$
$\delta^R_{\omega(v)} \omega^j_\mu + \delta^T_{e(v)} \omega^j_\mu = \delta^D_v \omega^j_\mu + F^j_{\mu\nu} v^\nu$