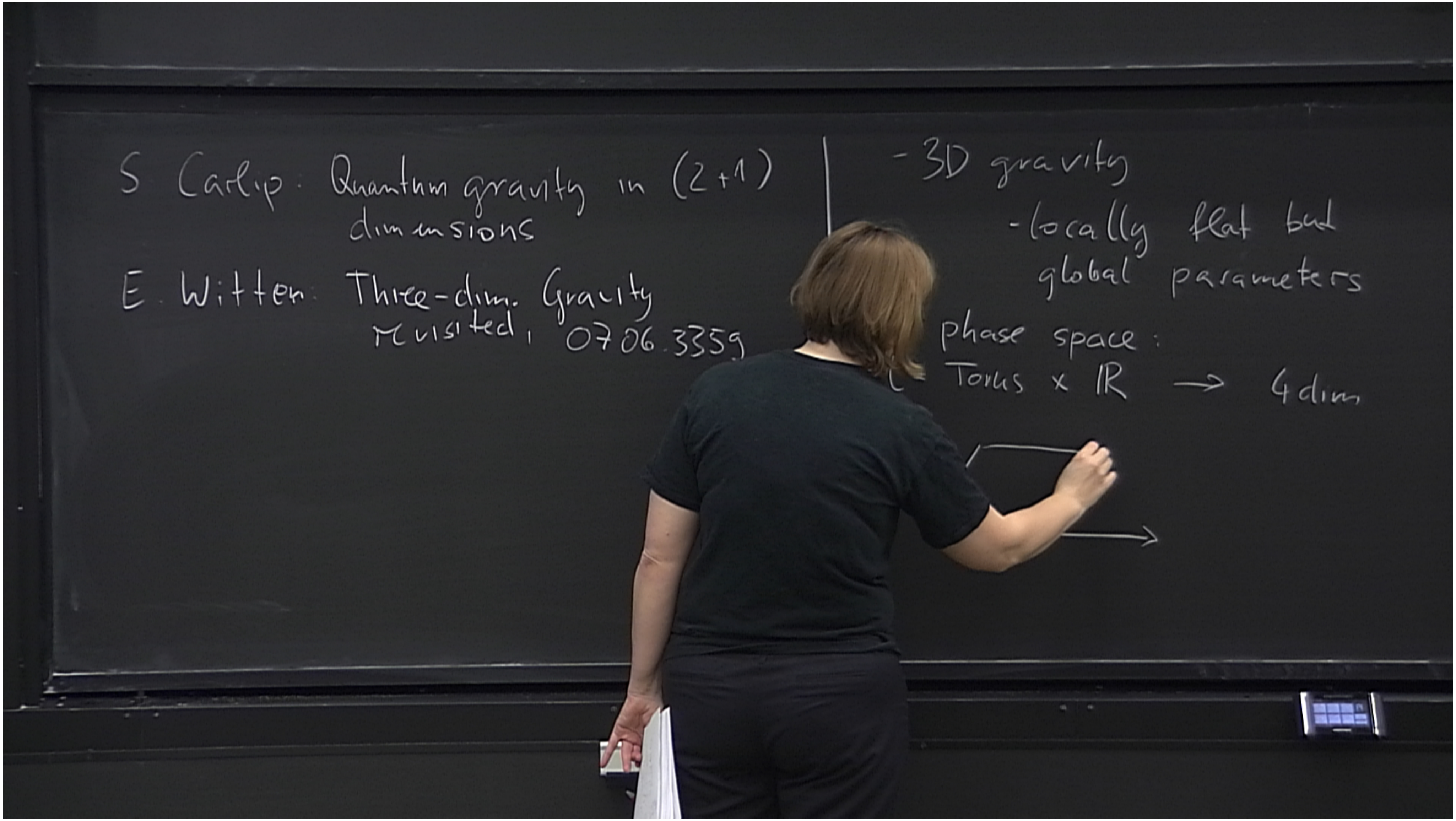
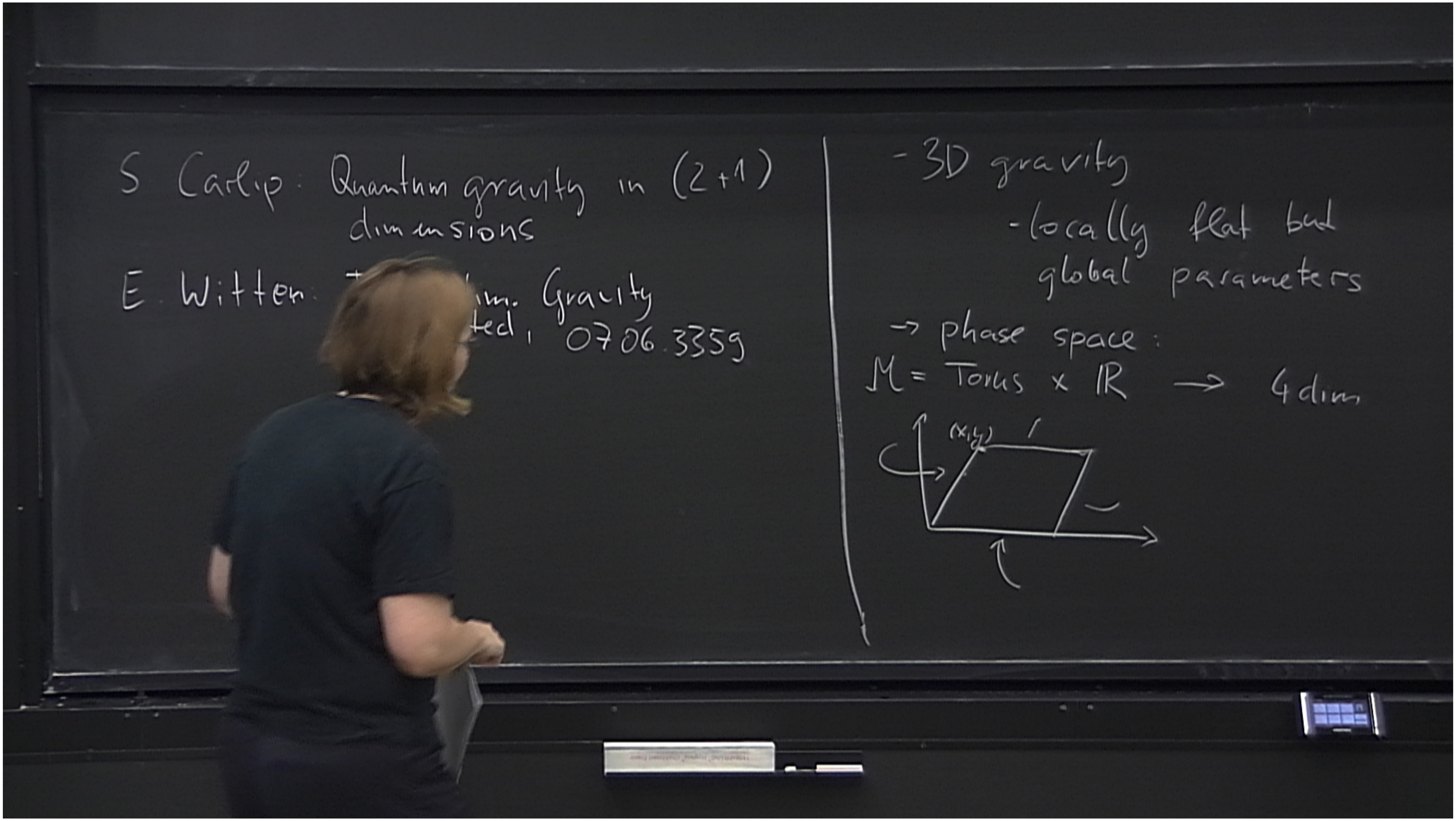
Title: 12/13 PSI - Quantum Gravity Review Lecture 2

Date: Jan 29, 2013 10:15 AM

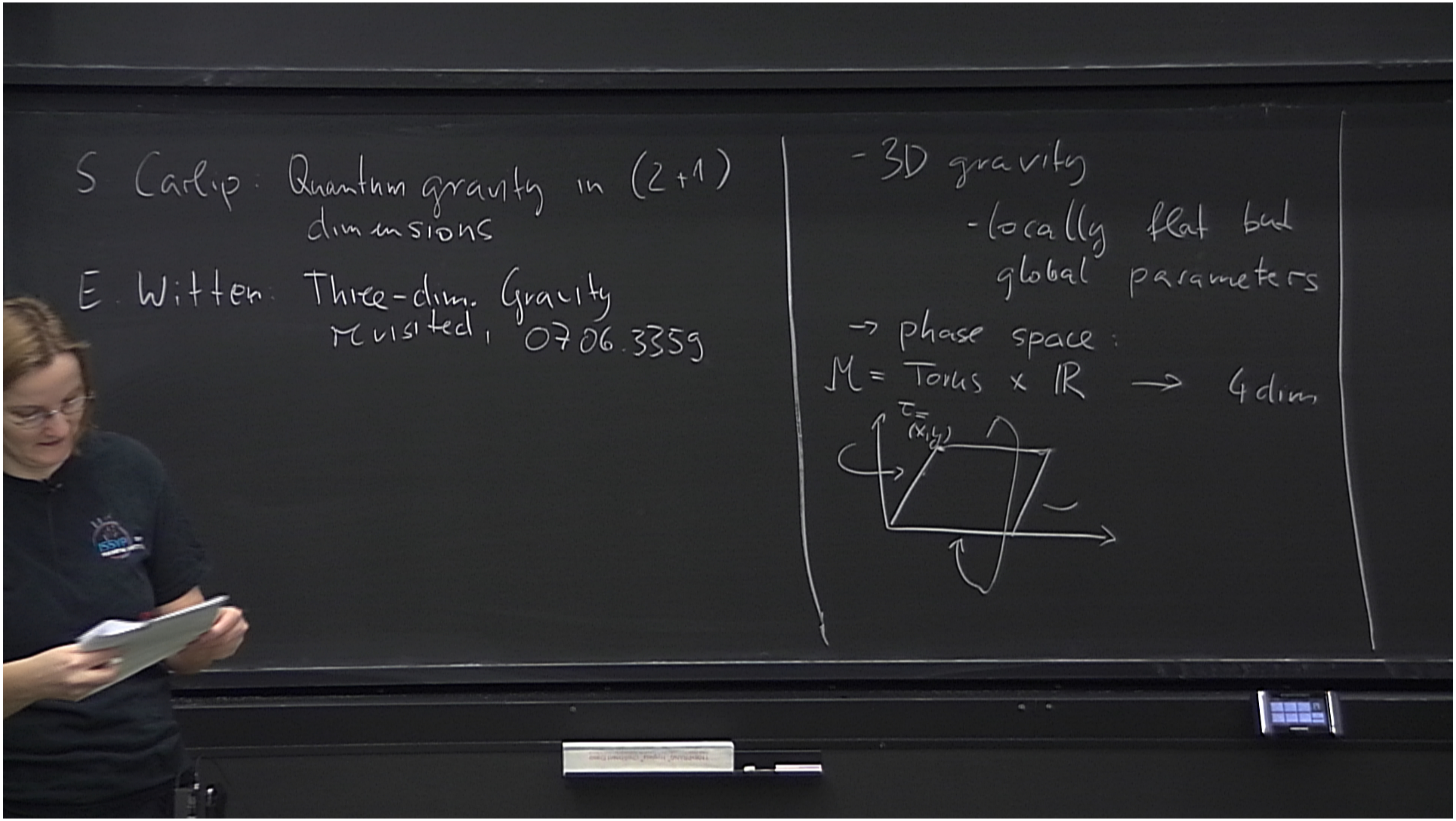
URL: http://pirsa.org/13010090

Abstract:


S. Carlip: Quantum gravity in (2+1) dimensions
E. Witten: Three-dim. Gravity revisited, 0706.3359
- 3D gravity
- locally flat but global parameters
phase space:
Torus x IR → 4 dim



S. Carlip: Quantum gravity in (2+1) dimensions
E. Witten:
dim. Gravity
0706.3359
- 3D gravity
- locally flat but global parameters
→ phase space:
M = Torus × IR → 4 dim
(x,y)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- 3D gravity
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→ phase space:
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τ = (x,y)

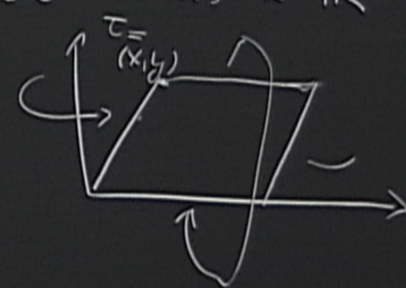
S. Carlip: Quantum gravity in (2+1) dimensions

E. Witten: Three-dim. Gravity revisited, 0706.3359

- 3D gravity
  - locally flat but global parameters

→ phase space:

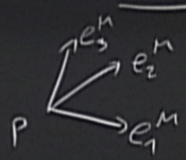
$M = \text{Torus} \times \mathbb{R} \rightarrow 4 \text{ dim}$



flat but
parameters

$\rightarrow$ 4 dim

Triads / Orthonormal Frames / n-beins



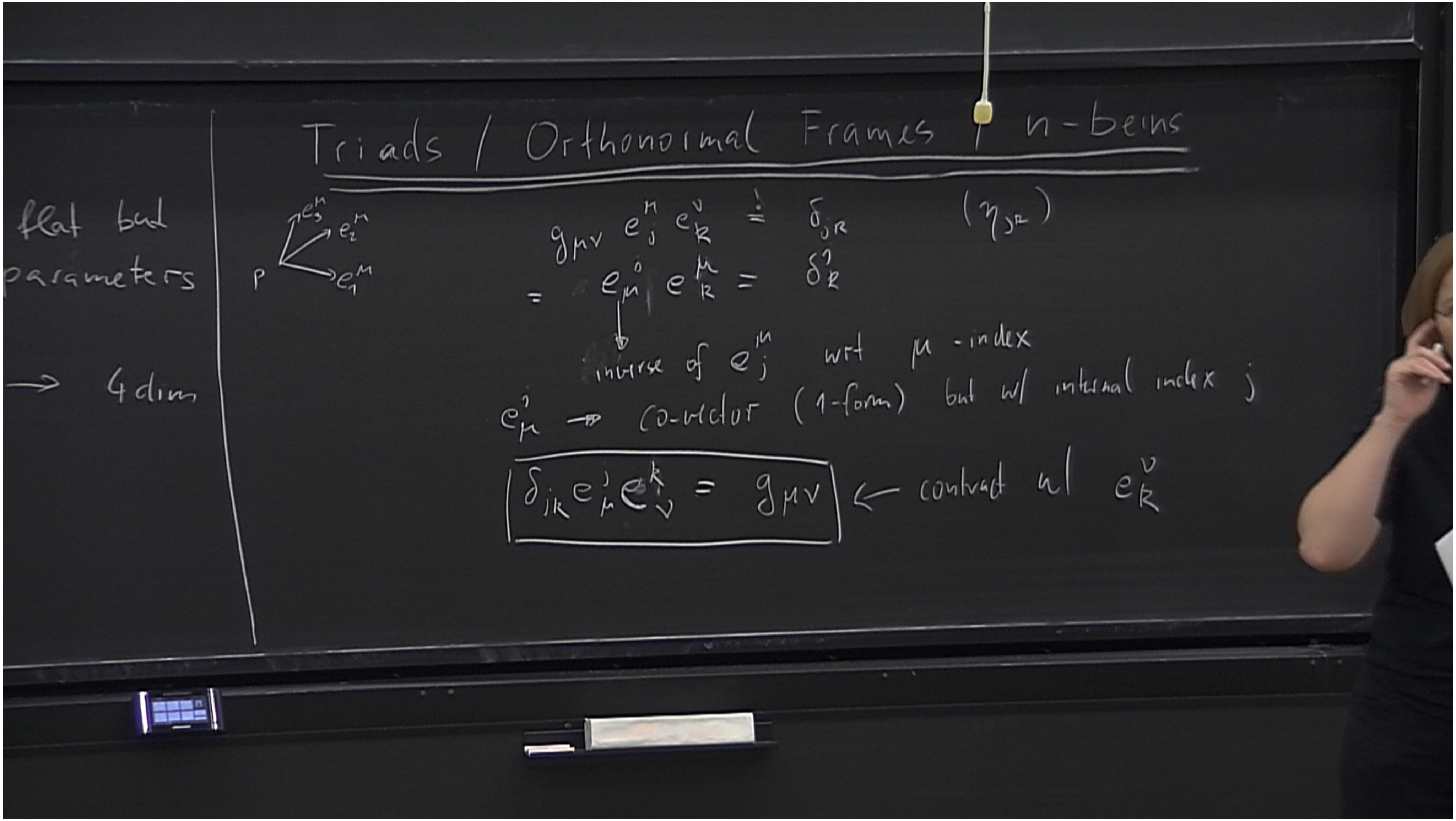
$$g_{\mu\nu}\, e^\mu_j\, e^\nu_k \overset{!}{=} \delta_{jk} \qquad (\eta_{jk})$$

$$= e^j_\mu\, e^\nu_k = \delta^j_k$$

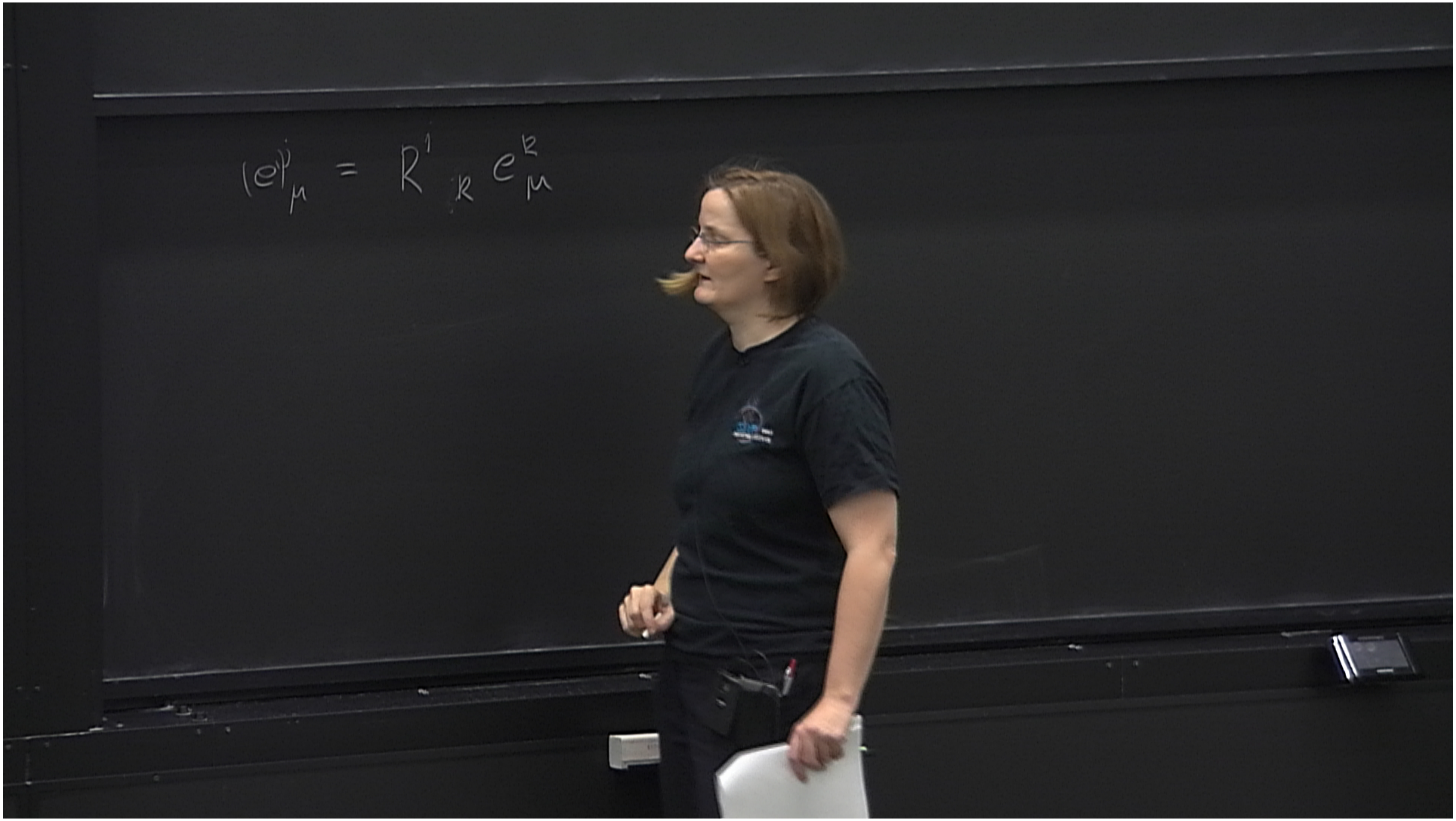
inverse of e^μ_j wrt μ-index

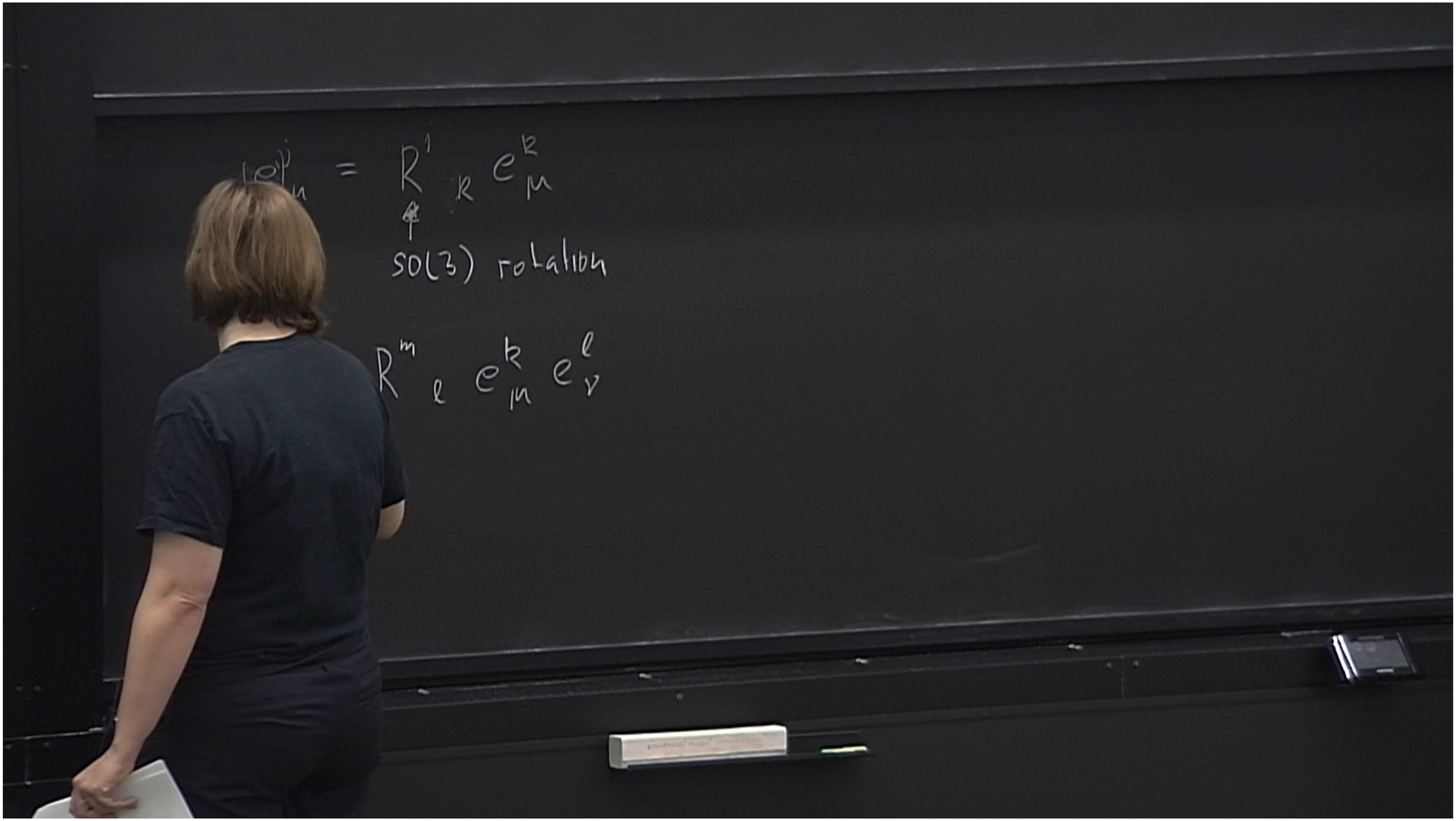
$e^j_\mu \rightarrow$ co-vector (1-form) but w/ internal index j

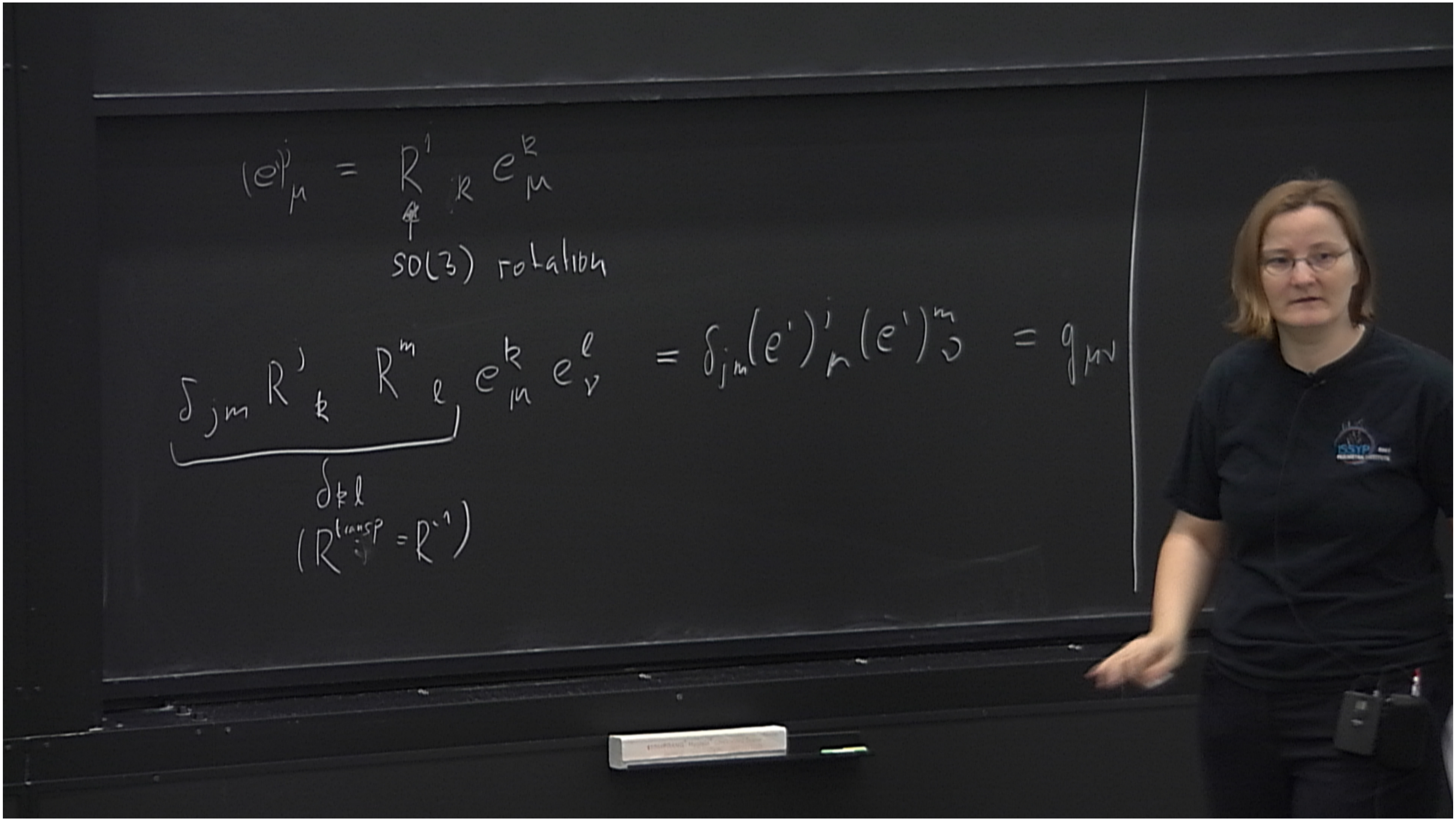
$$\boxed{\delta_{jk}\, e^j_\mu\, e^k_\nu = g_{\mu\nu}} \leftarrow \text{contract w/ } e^\nu_k$$

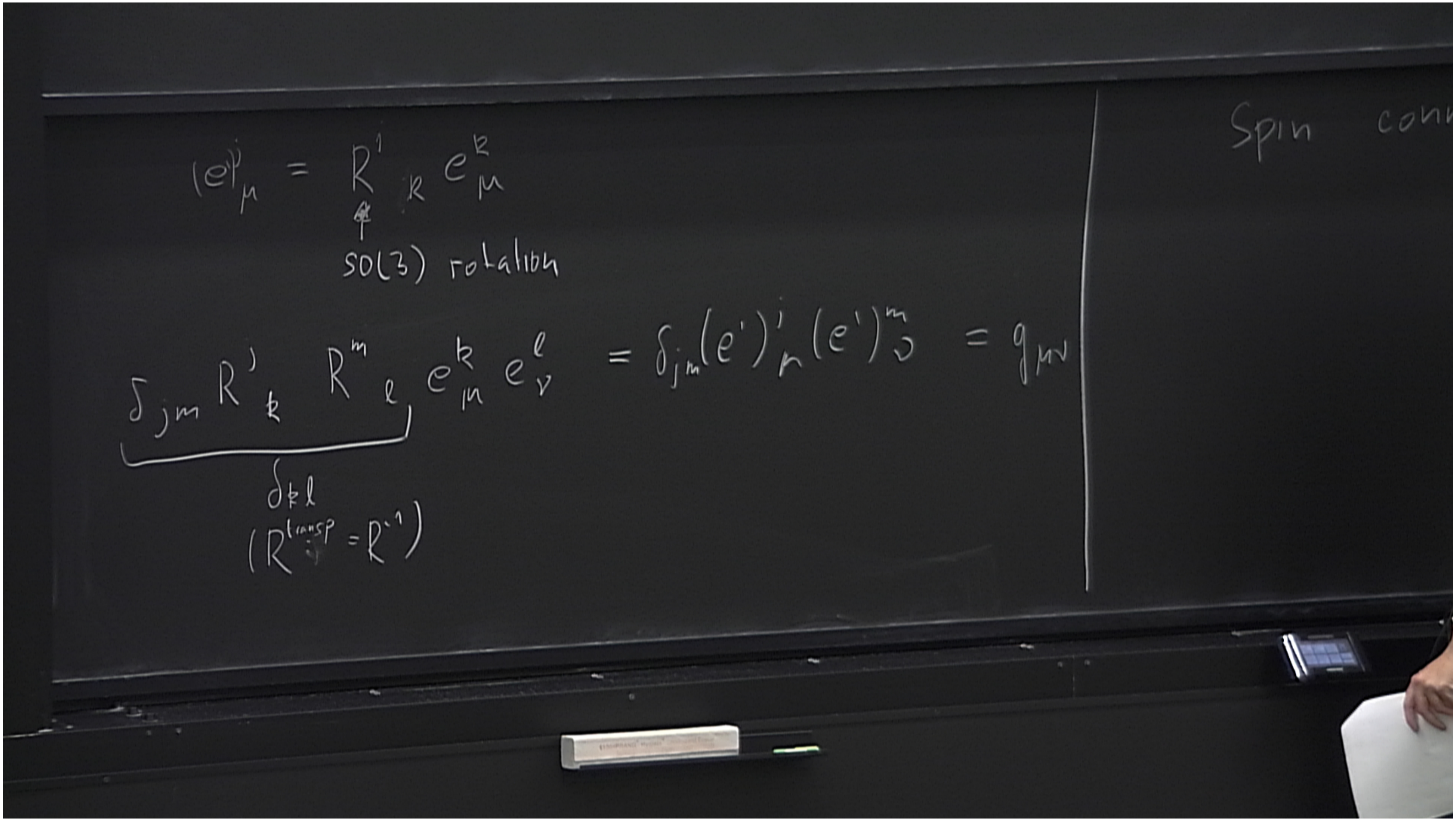


Triads / Orthonormal Frames / n-beins
flat but parameters
→ 4 dim
p
e_1^μ e_2^μ e_3^μ
$g_{\mu\nu}\, e_j^\mu e_k^\nu \stackrel{!}{=} \delta_{jk}$ (η_{jk})
$= e_\mu^j\, e_k^\mu = \delta_k^j$
inverse of e_j^μ wrt μ-index
e_μ^j → co-vector (1-form) but w/ internal index j
$\boxed{\delta_{jk}\, e_\mu^j e_\nu^k = g_{\mu\nu}}$ ← contract w/ e_k^ν

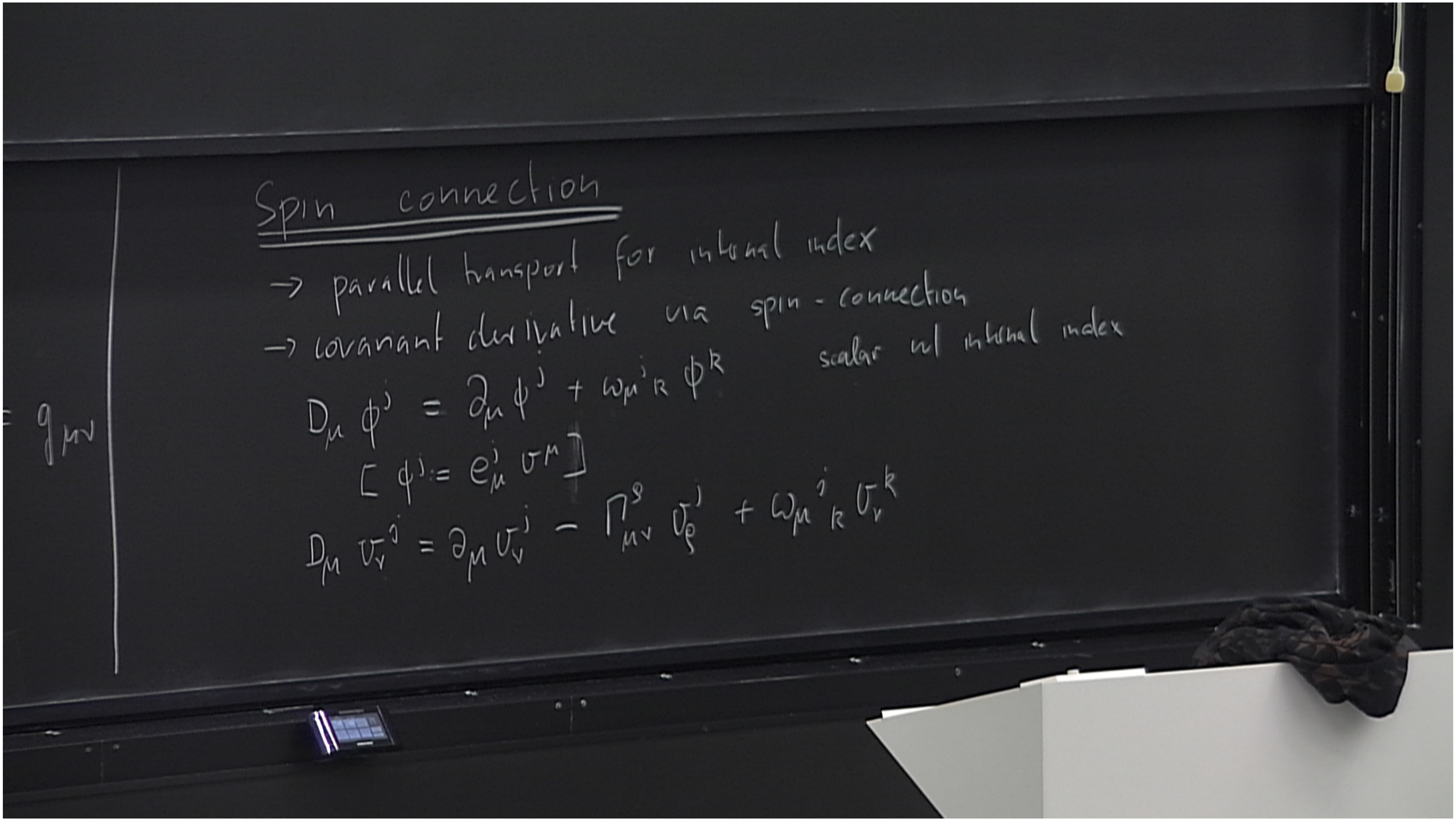

$(e')^i{}_\mu = R^i{}_k \, e^k{}_\mu$

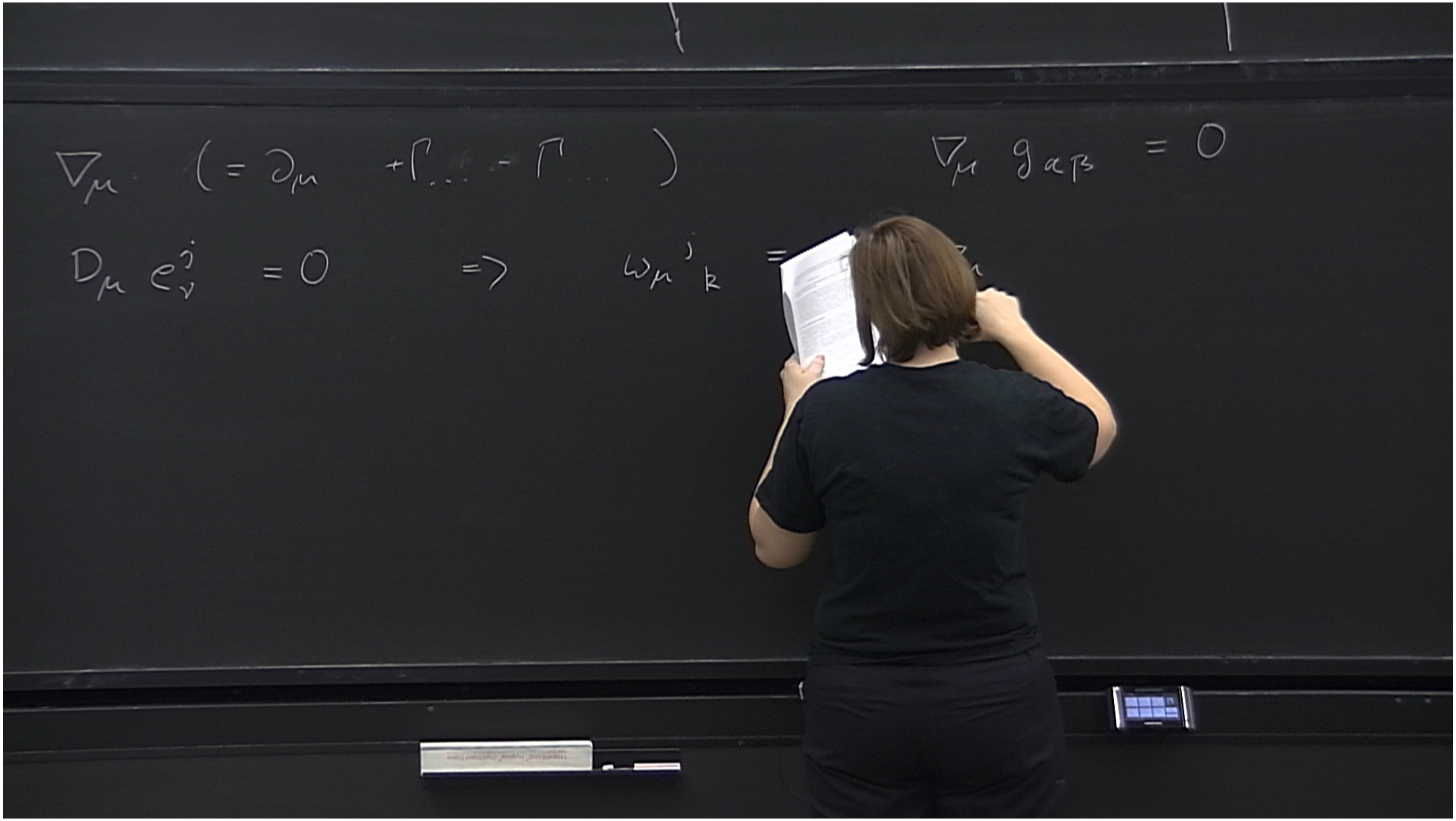

SO(3) rotation


$(e')^j{}_\mu = R^j{}_k \, e^k_\mu$
SO(3) rotation
$\underbrace{\delta_{jm} R^j{}_k \, R^m{}_l}_{\delta_{kl}} \, e^k_\mu \, e^l_\nu = \delta_{jm} (e')^j{}_\mu (e')^m{}_\nu = g_{\mu\nu}$
$(R^{\text{transp}} = R^{-1})$



$(e')^j_\mu = R^j{}_k \, e^k_\mu$
SO(3) rotation
$\underbrace{\delta_{jm} R^j{}_k \, R^m{}_l}_{\delta_{kl}} e^k_\mu \, e^l_\nu = \delta_{jm} (e')^j_\mu (e')^m_\nu = g_{\mu\nu}$
$(R^{transp} = R^{-1})$
Spin conn


$= g_{\mu\nu}$
Spin connection
→ parallel transport for internal index
→ covariant derivative via spin - connection
scalar w/ internal index
$D_\mu \phi^j = \partial_\mu \phi^j + \omega_\mu{}^j{}_k \phi^k$
$[\phi^j := e^j_\mu v^\mu]$
$D_\mu v_\nu^j = \partial_\mu v_\nu^j - \Gamma^\rho_{\mu\nu} v_\rho^j + \omega_\mu{}^j{}_k v_\nu^k$


$\nabla_\mu \, (= \partial_\mu + \Gamma_{\cdots} - \Gamma_{\cdots})$
$\nabla_\mu g_{\alpha\beta} = 0$
$D_\mu e^j_\nu = 0 \Rightarrow \omega_\mu{}^j{}_k =$

$$\nabla_\mu \cdot \quad (= \partial_\mu + \Gamma_{\dots} - \Gamma_{\dots}) \qquad \nabla_\mu g_{\alpha\beta} = 0$$

$$D_\mu e^j_\nu = 0 \quad \Rightarrow \quad \boxed{\omega_\mu{}^j{}_k = -e^\nu_k \nabla_\mu e^j_\nu} \qquad e_{\nu j} \dots k$$

$$\omega_{\mu j k} = -e^\nu_k \nabla_\mu e_{\nu j}$$

$$+\Gamma_{\ldots} - \Gamma_{\ldots} \;)$$

$$\nabla_\mu g_{\alpha\beta} = 0$$

$$\Rightarrow \quad \boxed{\omega_\mu{}^j{}_k = -e^\nu_k \nabla_\mu e^j_\nu} \qquad e_{\nu j} = \delta_{jk} e^k_\nu$$

$$\omega_{\mu jk} = -e^\nu_k \nabla_\mu e_{\nu j}$$

$$\ldots_{\alpha\beta} = 0$$

$$e_{\nu j} = \delta_{jk}\, e_{\nu}^{\;k}$$

$$\ldots - \omega_{\mu k j}$$

$$\boxed{\omega_{\mu j k} = -\omega_{\mu k j}} \quad \text{implements} \quad \nabla_{\rho}\, g_{\mu\sigma} = 0$$

$$\alpha\beta \;= 0$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = 0$$

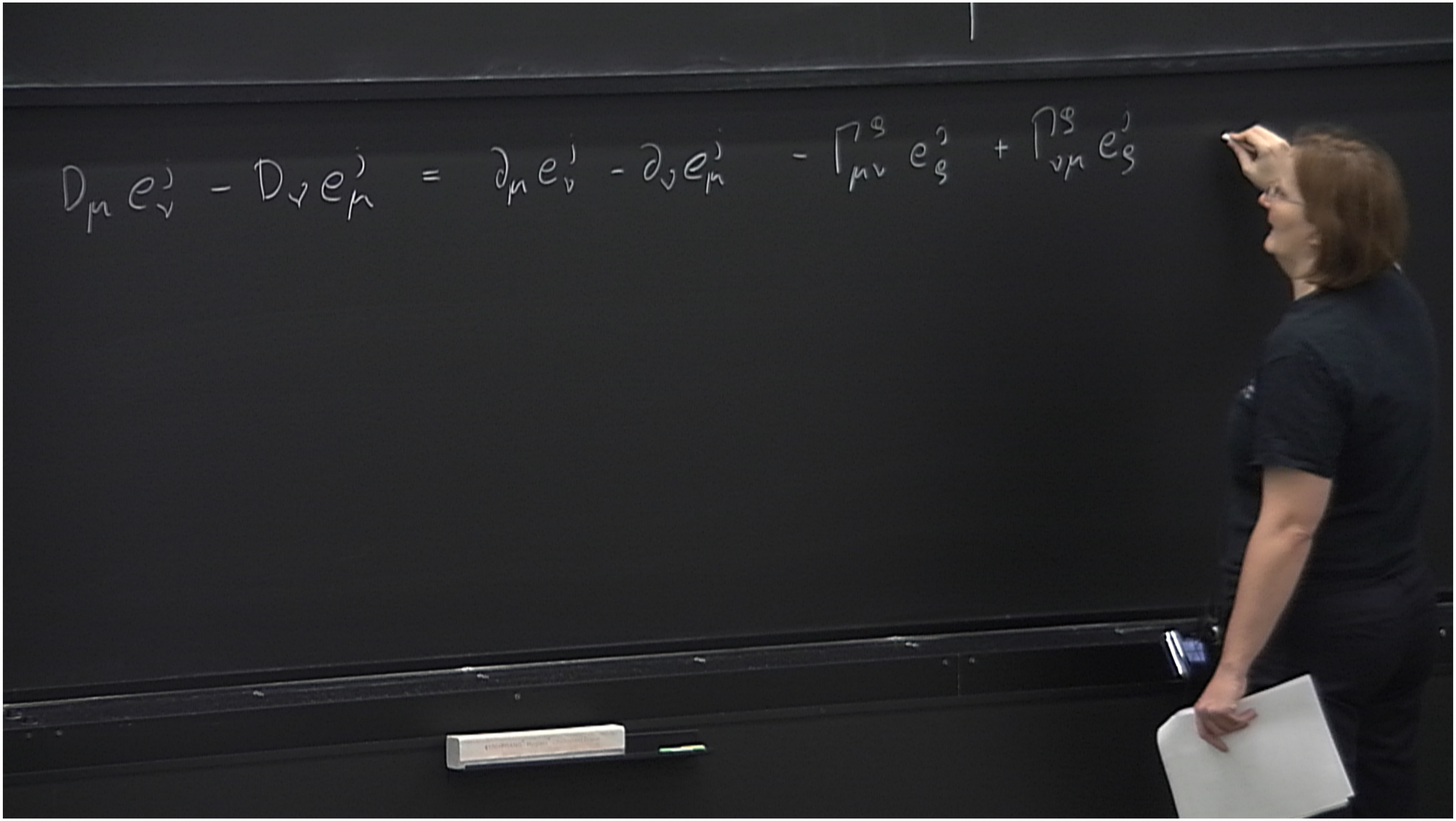
Torsion free, $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$

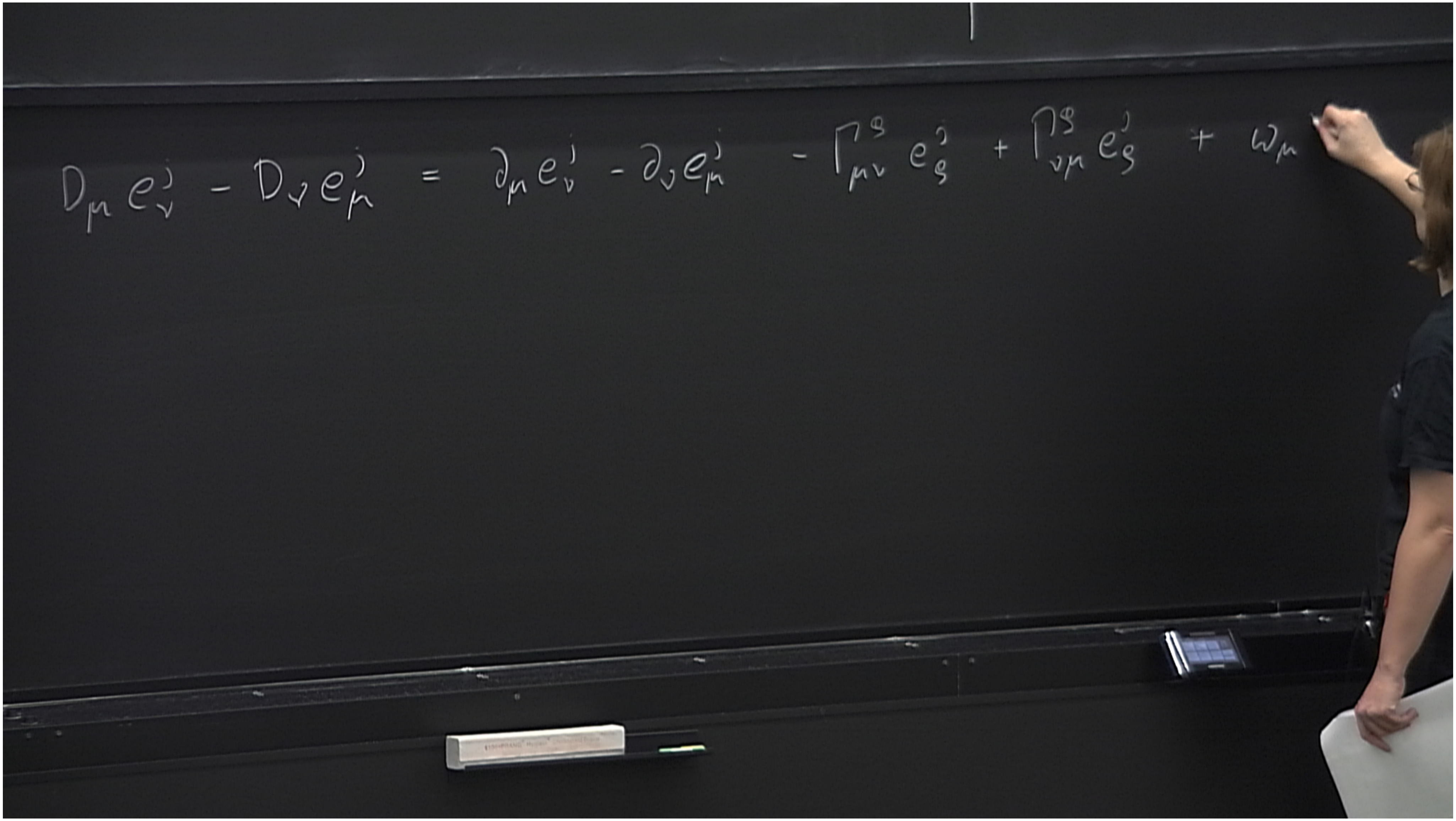
$$e_{\nu j} = \delta_{jk}\, e^k_\nu$$

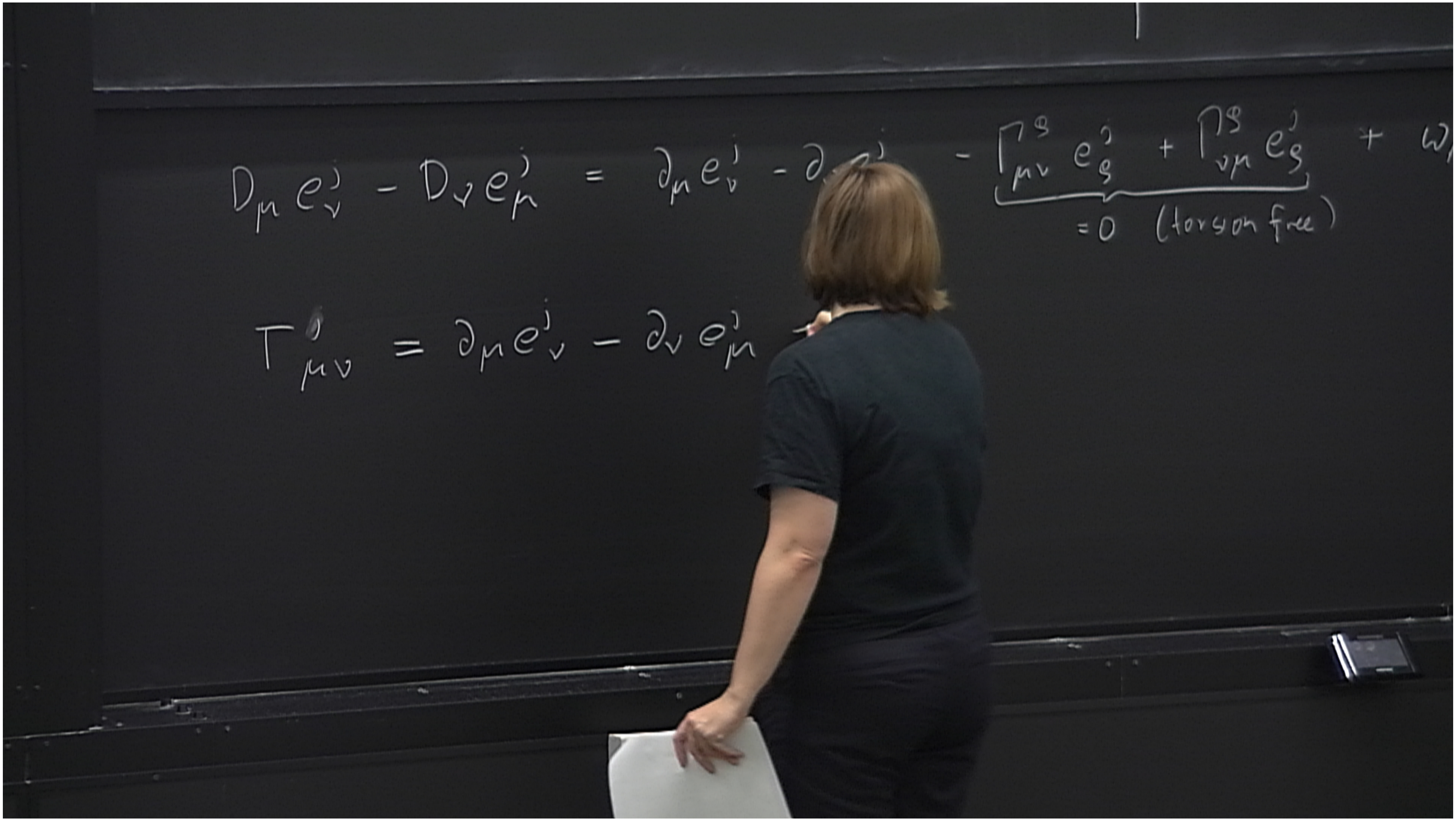
$$- \omega_{\mu k j}$$

$$\boxed{\omega_{\mu jk} = -\omega_{\mu kj}}$$

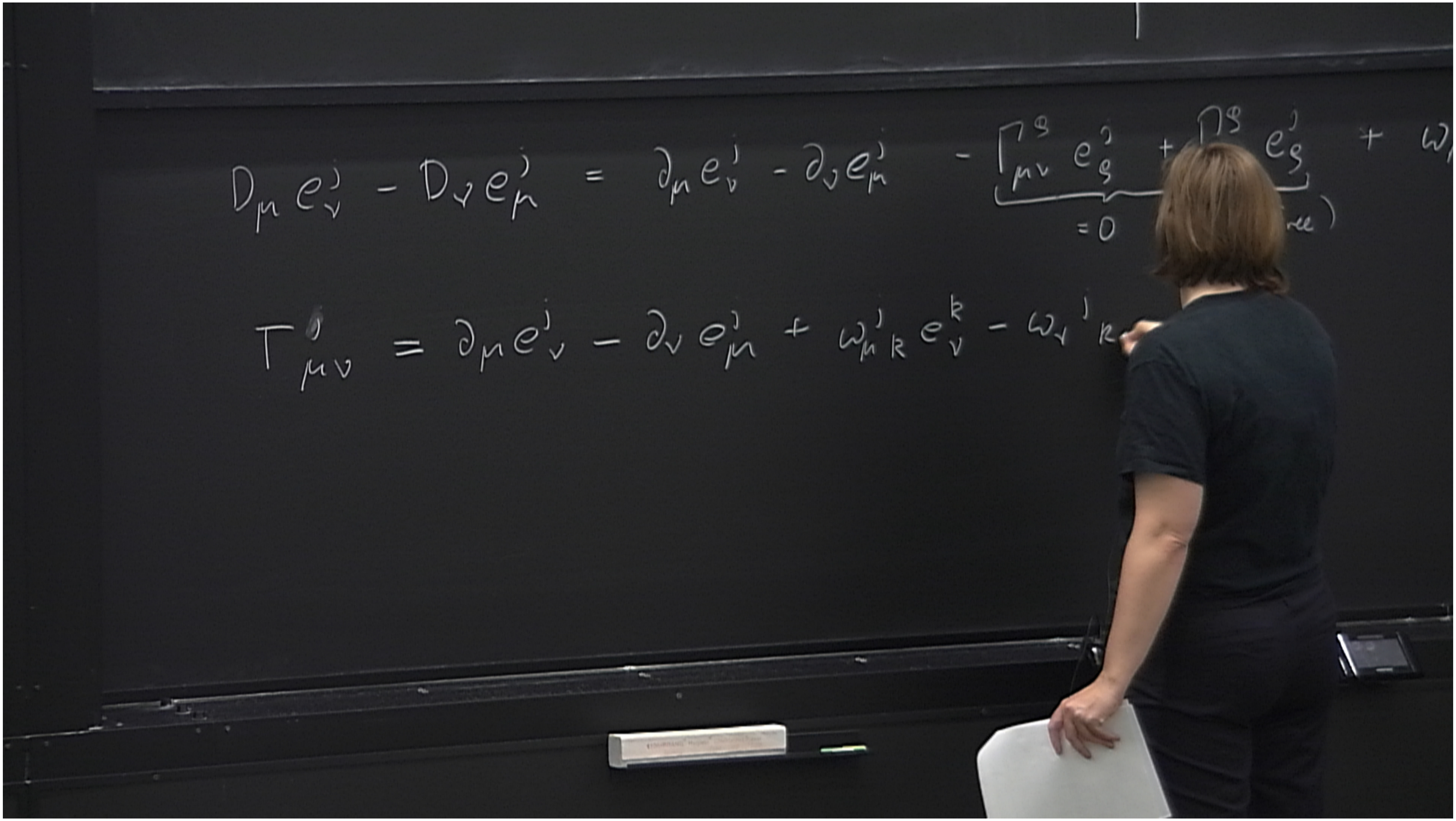
implements $\nabla_\rho g_{\mu\sigma} = 0$

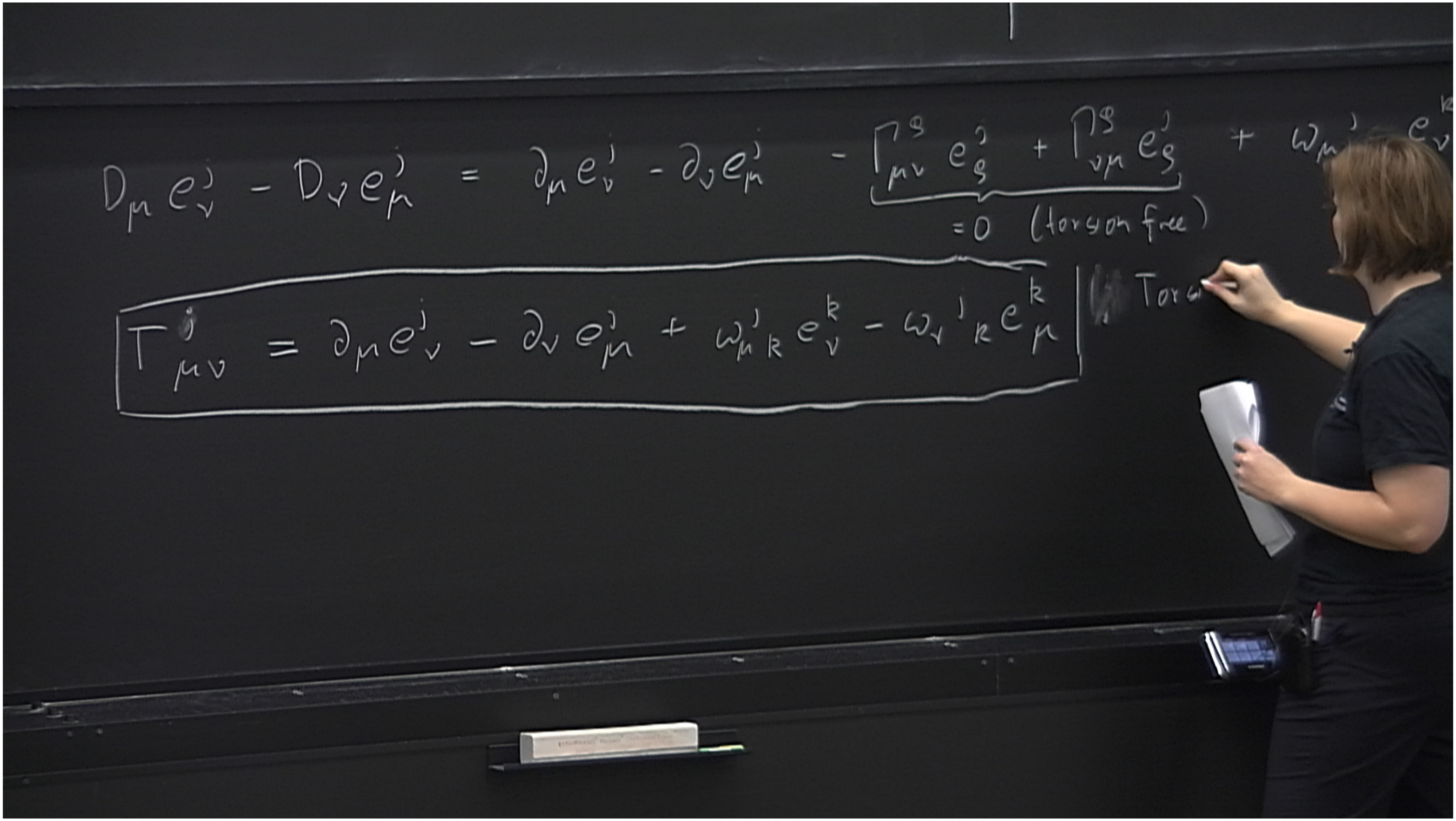

$D_\mu e^j_\nu - D_\nu e^j_\mu = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu - \Gamma^\rho_{\mu\nu} e^j_\rho + \Gamma^\rho_{\nu\mu} e^j_\rho$


$D_\mu e^j_\nu - D_\nu e^j_\mu = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu - \Gamma^\rho_{\mu\nu} e^j_\rho + \Gamma^\rho_{\nu\mu} e^j_\rho + \omega_\mu$



$D_\mu e^j_\nu - D_\nu e^j_\mu = \partial_\mu e^j_\nu - \partial$
$- \underbrace{\Gamma^\rho_{\mu\nu} e^j_\rho + \Gamma^\rho_{\nu\mu} e^j_\rho}_{=0 \text{ (torsion free)}} + \omega$
$T^j_{\mu\nu} = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu$


$D_\mu e^j_\nu - D_\nu e^j_\mu = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu - \Gamma^\rho_{\mu\nu} e^j_\rho + \Gamma^\rho e^j_\rho + \omega$
$= 0$
$T^j_{\mu\nu} = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu + \omega_\mu{}^j{}_k e^k_\nu - \omega_\nu{}^j{}_k$


$D_\mu e^j_\nu - D_\nu e^j_\mu = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu - \underbrace{\Gamma^\rho_{\mu\nu} e^j_\rho + \Gamma^\rho_{\nu\mu} e^j_\rho}_{=0 \text{ (torsion free)}} + \omega_\mu{}^j{}_k e^k_\nu$
$\boxed{T^j_{\mu\nu} = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu + \omega_\mu{}^j{}_k e^k_\nu - \omega_\nu{}^j{}_k e^k_\mu}$
Tors

$$= \partial_\mu e^j_\nu - \partial_\nu e^j_\mu \underbrace{- \Gamma^\rho_{\mu\nu} e^j_\rho + \Gamma^\rho_{\nu\mu} e^j_\rho}_{=0 \text{ (torsion free)}} + \omega_\mu{}^j{}_k e^k_\nu - \omega_\nu{}^j{}_k e^k_\mu =$$

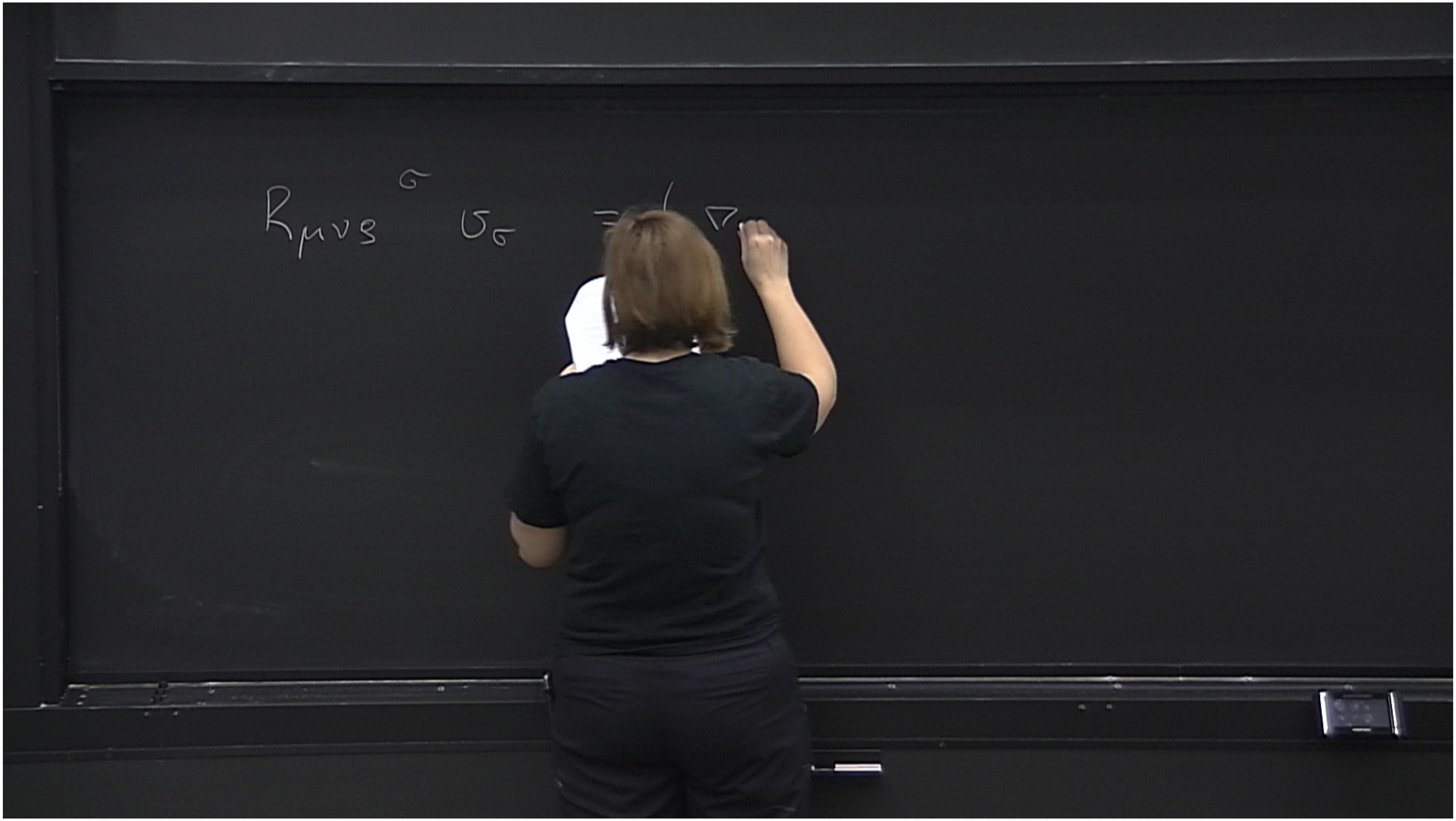
$$\boxed{- \partial_\nu e^j_\mu + \omega_\mu{}^j{}_k e^k_\nu - \omega_\nu{}^j{}_k e^k_\mu}$$

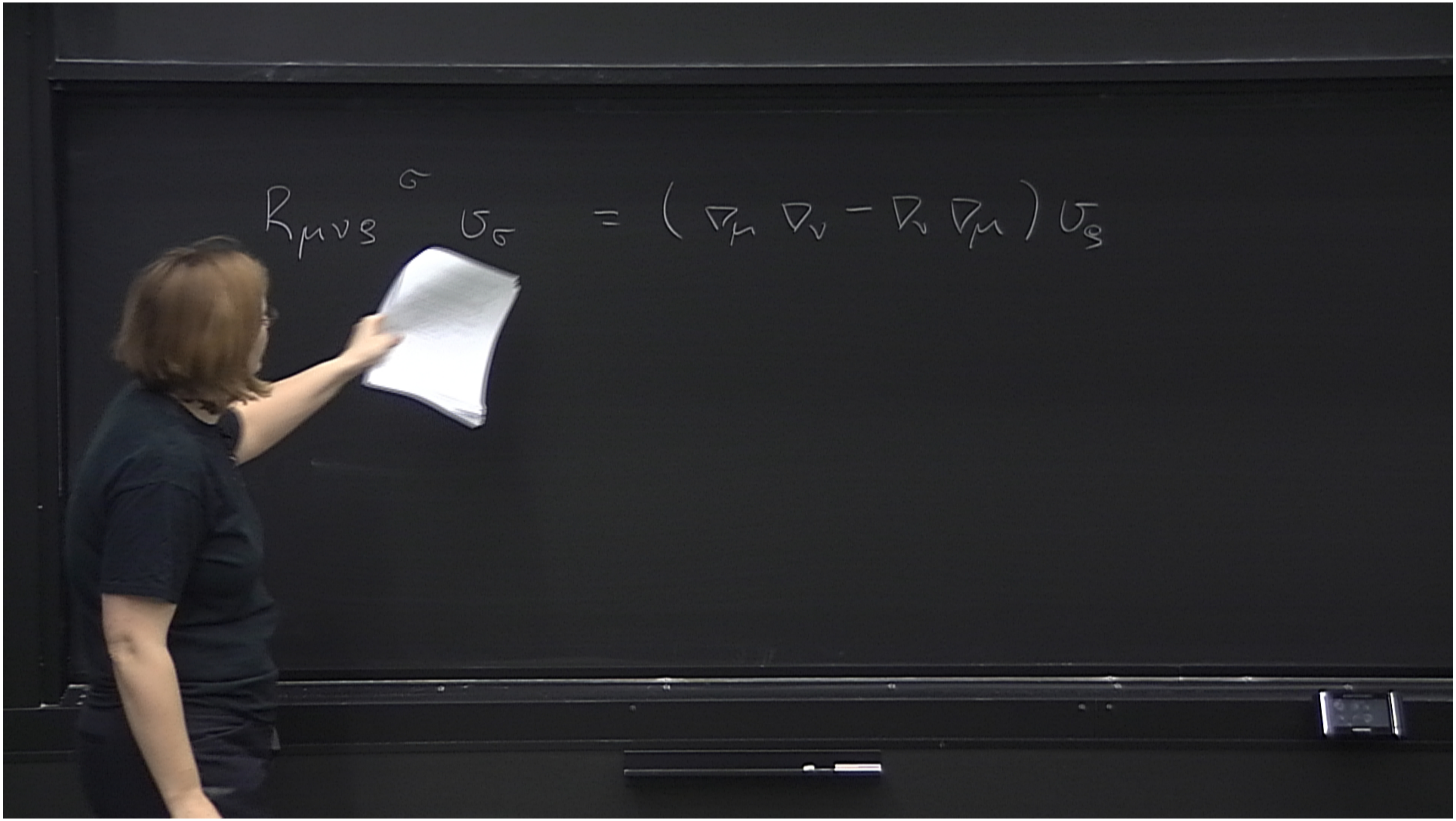
Torsion tensor

$$= \partial_\mu e_\nu^j - \partial_\nu e_\mu^j \underbrace{- \Gamma^\rho_{\mu\nu} e_\rho^j + \Gamma^\rho_{\nu\mu} e_\rho^j}_{=0 \text{ (torsion free)}} + \omega_\mu{}^j{}_k e_\nu^k - \omega_\nu{}^j{}_k e_\mu^k =$$

$$- \partial_\nu e_\mu^j + \omega_\mu{}^j{}_k e_\nu^k - \omega_\nu{}^j{}_k e_\mu^k$$

Torsion tensor


$R_{\mu\nu\rho}{}^{\sigma} \, v_\sigma = (\nabla$

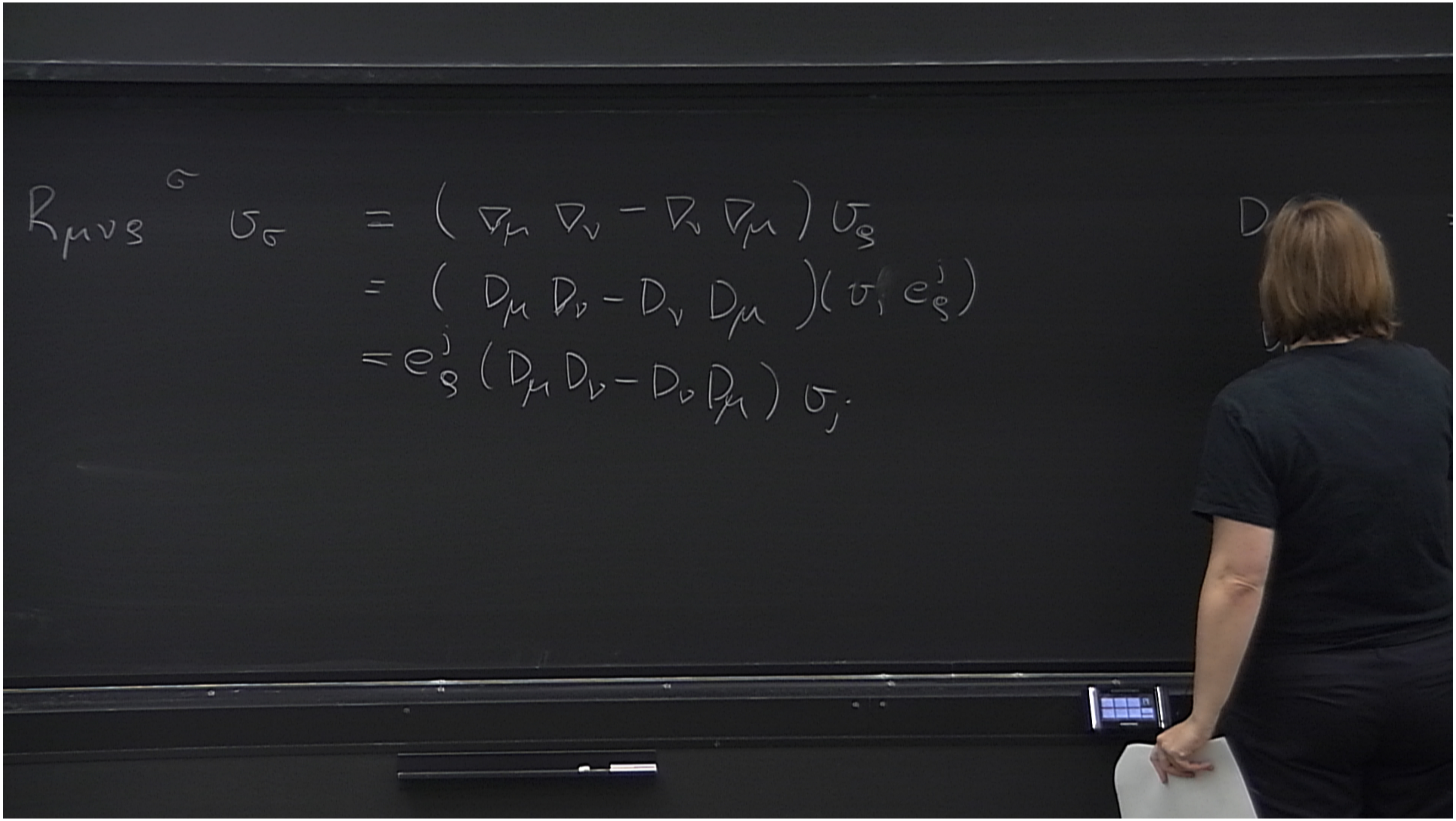

$R_{\mu\nu\rho}{}^{\sigma} v_{\sigma} = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v_\rho$


$R_{\mu\nu\rho}{}^{\sigma}\, v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, v_\rho$

$$= \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \right) v_\rho$$

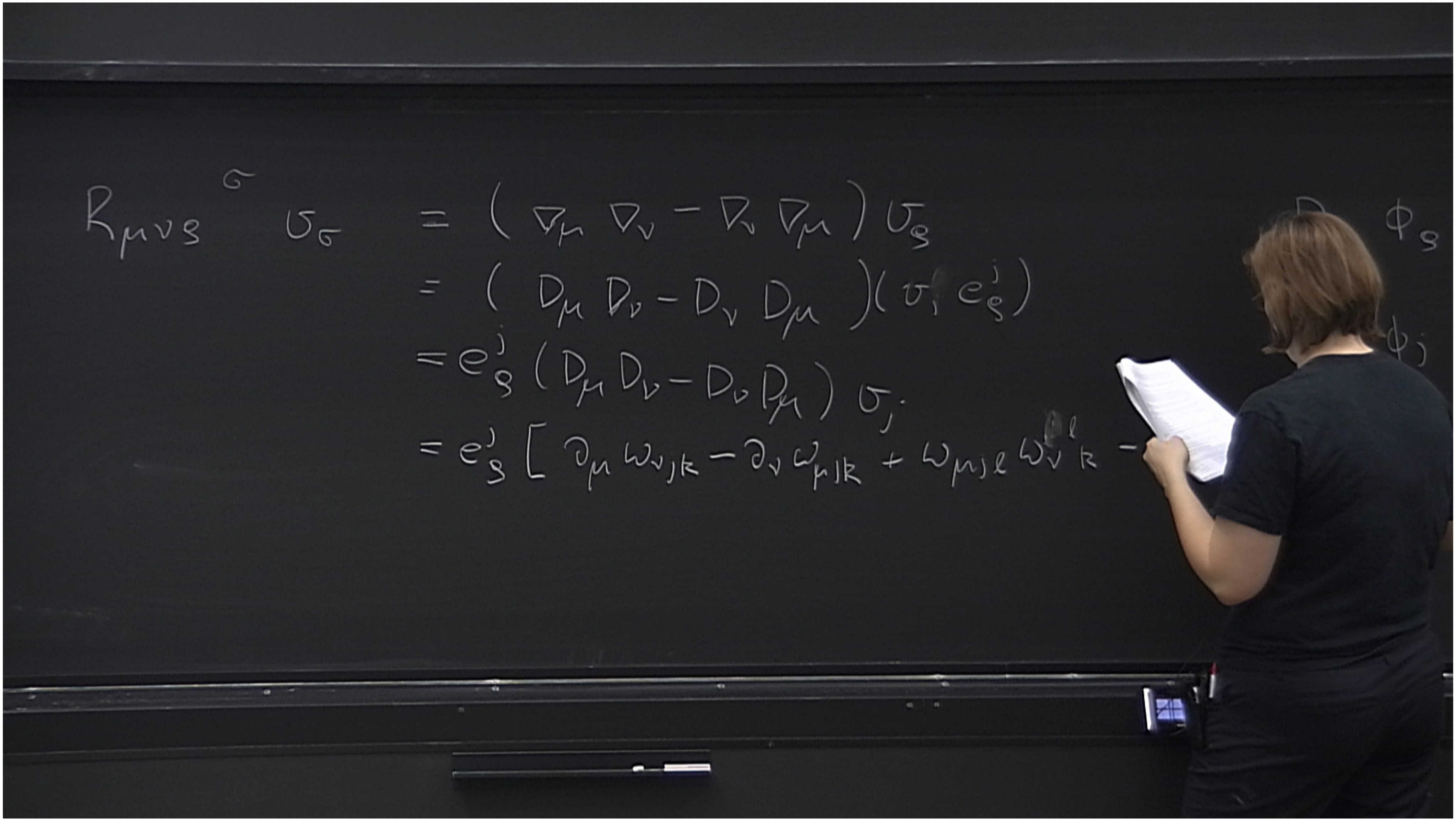
$$= \left(\qquad \right)\left(v_i \, e^i_\rho \right)$$

$$D_\mu \phi_\rho = \nabla_\mu \phi_\rho$$

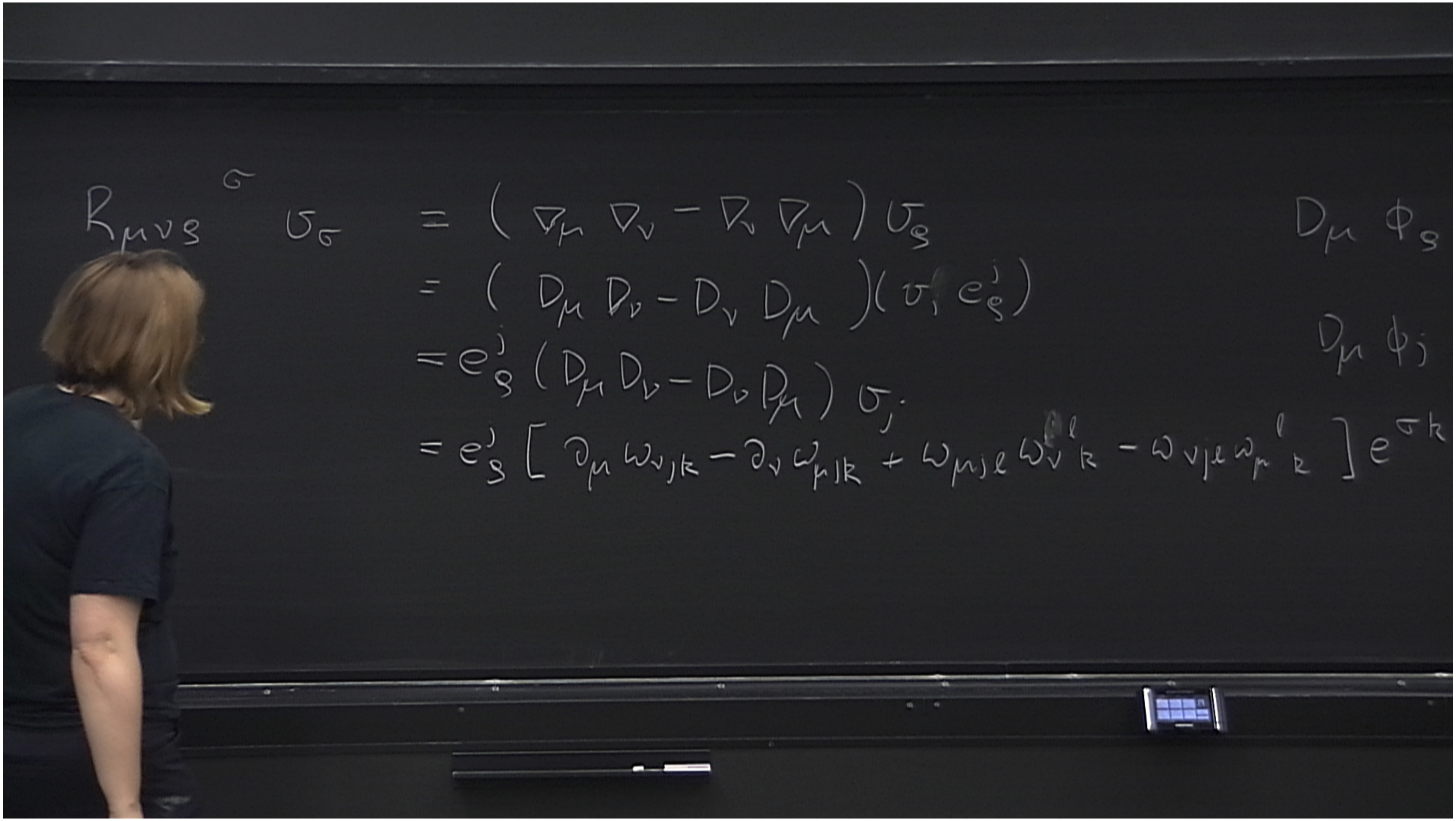

$R_{\mu\nu\rho}{}^{\sigma}\, v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, v_\rho$
$= (D_\mu D_\nu - D_\nu D_\mu)(v_j\, e^j_\rho)$
$= e^j_\rho\, (D_\mu D_\nu - D_\nu D_\mu)\, v_j .$

$$D_\mu \phi_s = \nabla_\mu \phi_s$$

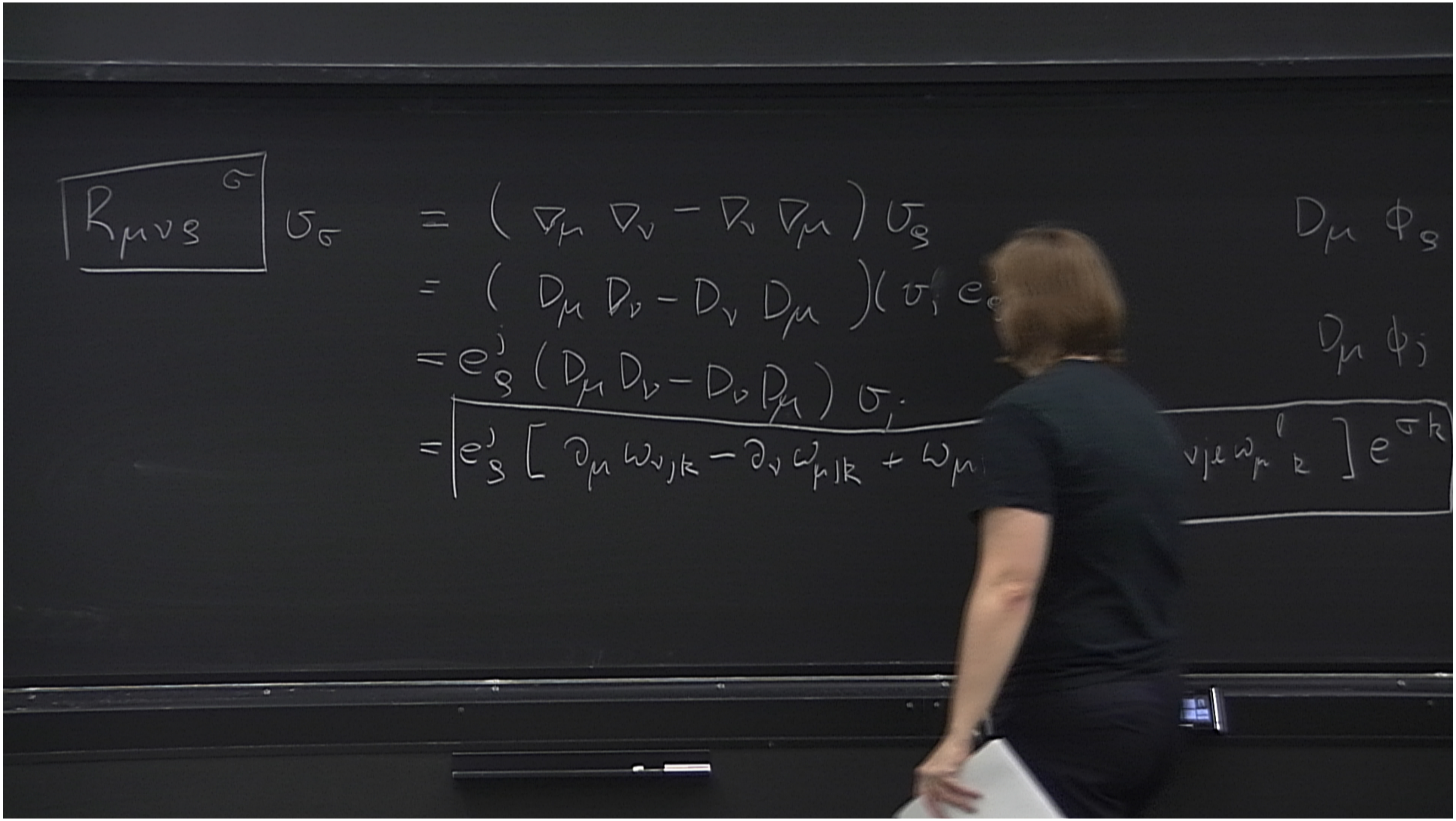
$$D_\mu \phi_j = \partial_\mu \phi_j + \omega_\mu{}^k{}_j \, \phi_k$$



$R_{\mu\nu\rho}{}^{\sigma}\, v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, v_\rho$
$= (D_\mu D_\nu - D_\nu D_\mu)(v_j e^j_\rho)$
$= e^j_\rho (D_\mu D_\nu - D_\nu D_\mu)\, v_j$
$= e^j_\rho [\, \partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu jl}\, \omega_\nu{}^l{}_k -$
ϕ_ρ
ϕ_j



$R_{\mu\nu\rho}{}^{\sigma} \, v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v_\rho$
$= (D_\mu D_\nu - D_\nu D_\mu)(v_j e^j_\rho)$
$= e^j_\rho (D_\mu D_\nu - D_\nu D_\mu) v_j$
$= e^j_\rho [\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu jl} \omega_\nu{}^l{}_k - \omega_{\nu jl} \omega_\mu{}^l{}_k] e^{\sigma k}$
$D_\mu \phi_\rho$
$D_\mu \phi_j$

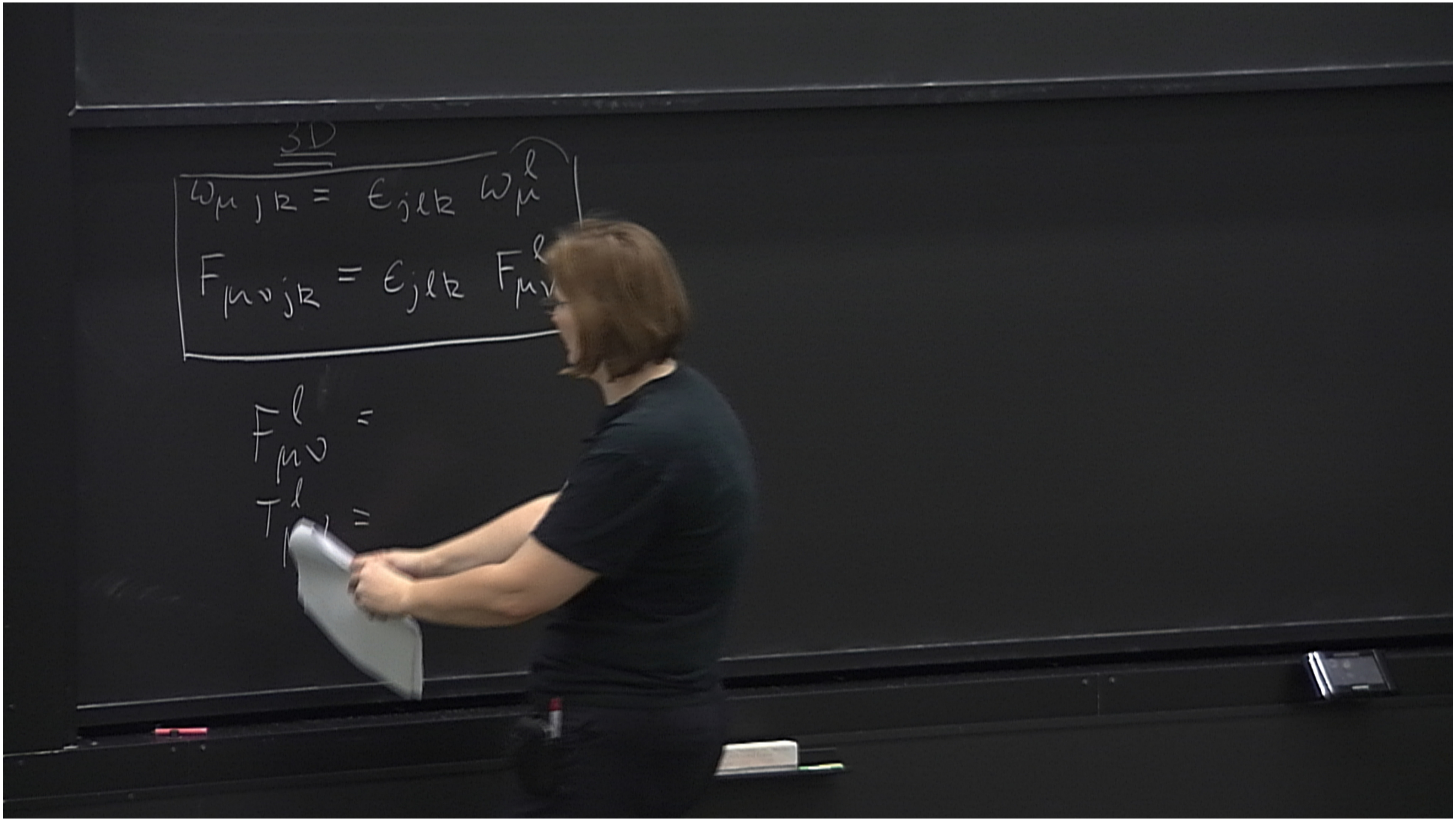

$\boxed{R_{\mu\nu\rho}{}^{\sigma}}\, v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, v_\rho$
$= (D_\mu D_\nu - D_\nu D_\mu)(v_j\, e$
$= e^j_\rho\, (D_\mu D_\nu - D_\nu D_\mu)\, v_j$
$= e^j_\rho\, [\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_\mu$
$\omega_{\nu jl}\, \omega_\mu{}^l{}_k]\, e^{\sigma k}$
$D_\mu \phi_\rho$
$D_\mu \phi_j$

$$\boxed{R_{\mu\nu\rho}{}^{\sigma}}\, v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, v_\rho \qquad D_\mu \phi_\rho$$

$$= (D_\mu D_\nu - D_\nu D_\mu)(v_j e^j_\rho) \qquad D_\mu \phi_j$$

$$= e^j_\rho (D_\mu D_\nu - D_\nu D_\mu)\, v_j$$

$$= \boxed{e^j_\rho \underbrace{\left[\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu j l}\, \omega_\nu{}^l{}_k - \omega_{\nu j l}\, \omega_\mu{}^l{}_k \right]}_{F_{\mu\nu jk}} e^{\sigma k}}$$

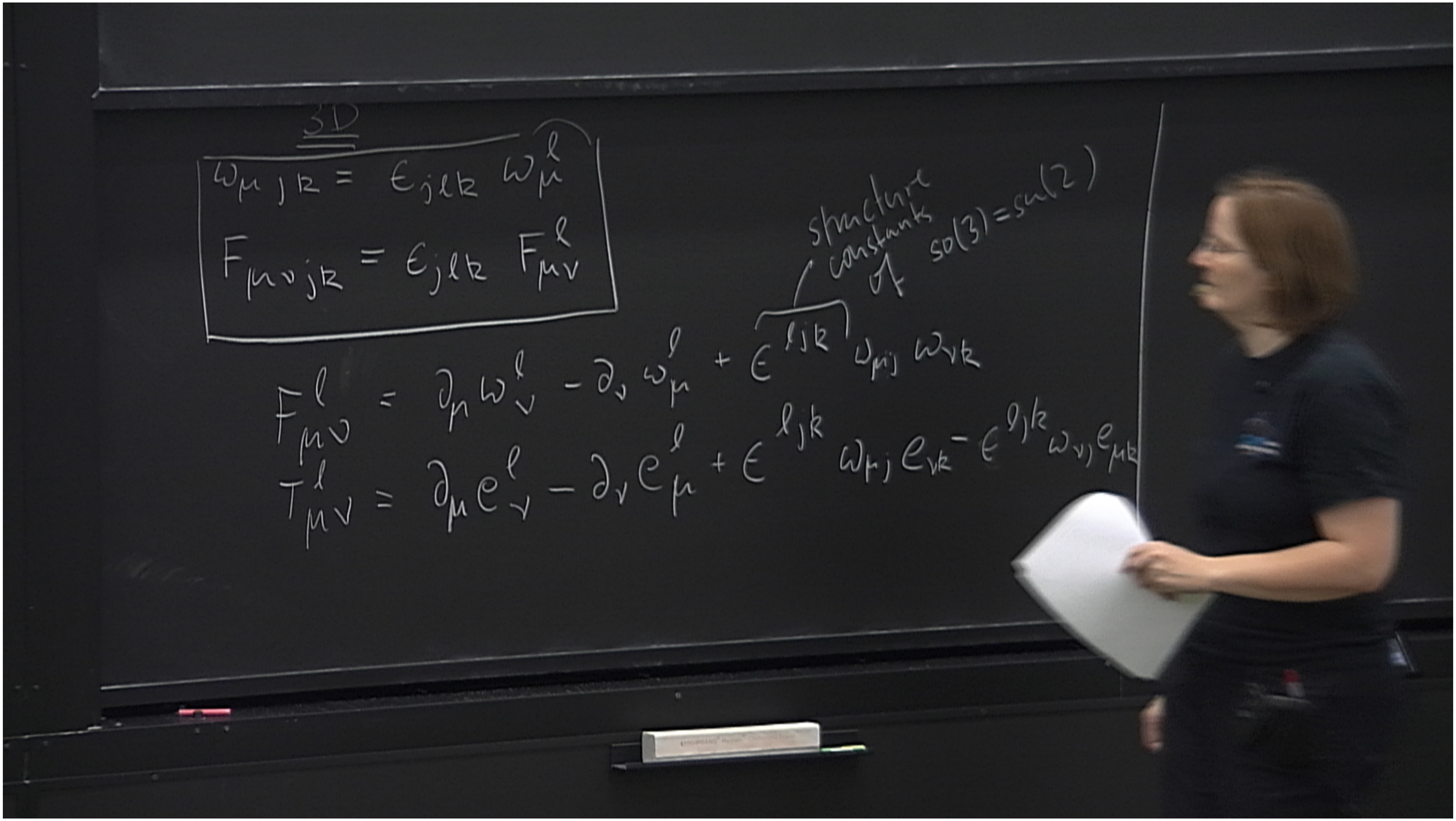

3D
$\omega_{\mu jk} = \epsilon_{jlk}\, \omega^l_\mu$
$F_{\mu\nu jk} = \epsilon_{jlk}\, F^l_{\mu\nu}$
$F^l_{\mu\nu} =$
$T^l_{\mu\nu} =$

3D

$$\boxed{\begin{aligned}\omega_{\mu\, jk} &= \epsilon_{jlk}\, \omega_\mu^l \\ F_{\mu\nu\, jk} &= \epsilon_{jlk}\, F_{\mu\nu}^l\end{aligned}}$$

$$F_{\mu\nu}^l = \partial_\mu \omega_\nu^l - \partial_\nu \omega_\mu^l + \epsilon^{ljk}\, \omega_{\mu j}\, \omega_{\nu k}$$

$$T_{\mu\nu}^l =$$


3D
$\omega_{\mu jk} = \epsilon_{jlk}\, \omega^l_\mu$
$F_{\mu\nu jk} = \epsilon_{jlk}\, F^l_{\mu\nu}$
structure constants of so(3)=su(2)
$F^l_{\mu\nu} = \partial_\mu \omega^l_\nu - \partial_\nu \omega^l_\mu + \epsilon^{ljk}\, \omega_{\mu j}\, \omega_{\nu k}$
$T^l_{\mu\nu} = \partial_\mu e^l_\nu - \partial_\nu e^l_\mu + \epsilon^{ljk}\, \omega_{\mu j}\, e_{\nu k} - \epsilon^{ljk}\, \omega_{\nu j}\, e_{\mu k}$

structure constants of $so(3) = su(2)$

$$\ldots + \overbrace{\epsilon^{ljk}} \omega_{\mu j}\, \omega_{\nu k}$$

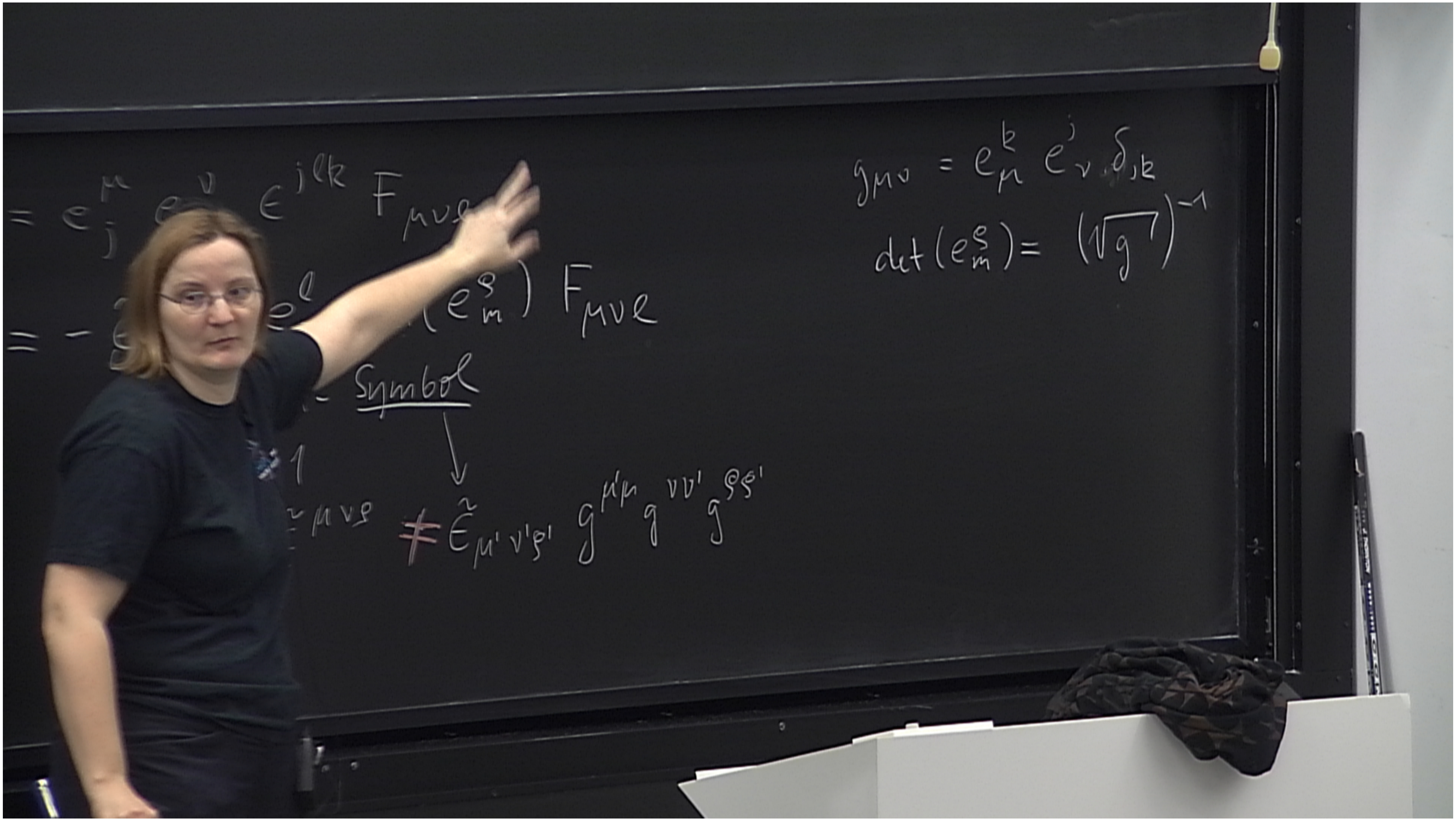
$$\ldots + \epsilon^{ljk}\, \omega_{\mu j}\, e_{\nu k} - \epsilon^{ljk}\, \omega_{\nu j}\, e_{\mu k}$$

$$R = e^{\mu}_{j}\, e^{\nu}_{k}\, \epsilon^{jlk}\, F_{\mu\nu l}$$

$$= - \underbrace{\tilde{\epsilon}^{\mu\nu\rho}}_{\text{Levi-Civita-Symbol}}\, e^{l}_{\rho}\, \det(e^{\rho}_{m})\, F_{\mu\nu l}$$

$$\tilde{\epsilon}^{123} = 1$$

$$\tilde{\epsilon}^{\mu\nu\rho} \neq \tilde{\epsilon}_{\mu'\nu'\rho'}\, g^{\mu'\mu}\, g^{\nu\nu'}\, g^{\rho\rho'}$$



$g_{\mu\nu} = e^k_\mu e^j_\nu \delta_{jk}$
$\det(e^\rho_m) = (\sqrt{g})^{-1}$
Symbol

$$= e^{\mu}_{j}\, e^{\nu}_{k}\, \epsilon^{jlk}\, F_{\mu\nu l}$$

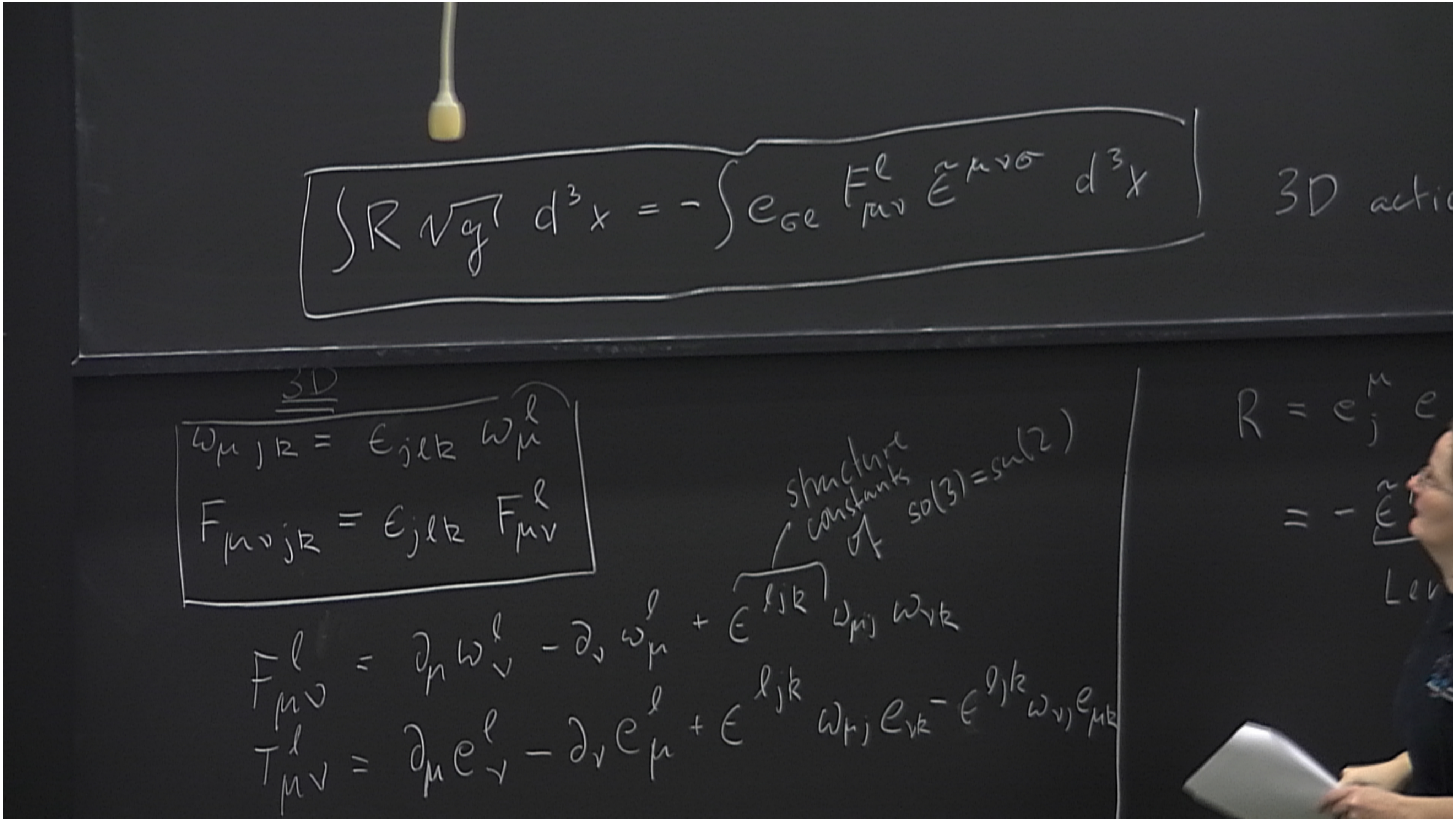
$$= -\underbrace{\tilde{\epsilon}^{\mu\nu\sigma}}_{\text{Levi-Civita-Symbol}}\, e^{l}_{\sigma}\, \det(e^{\rho}_{m})\, F_{\mu\nu l}$$

$$\tilde{\epsilon}^{123} = 1$$

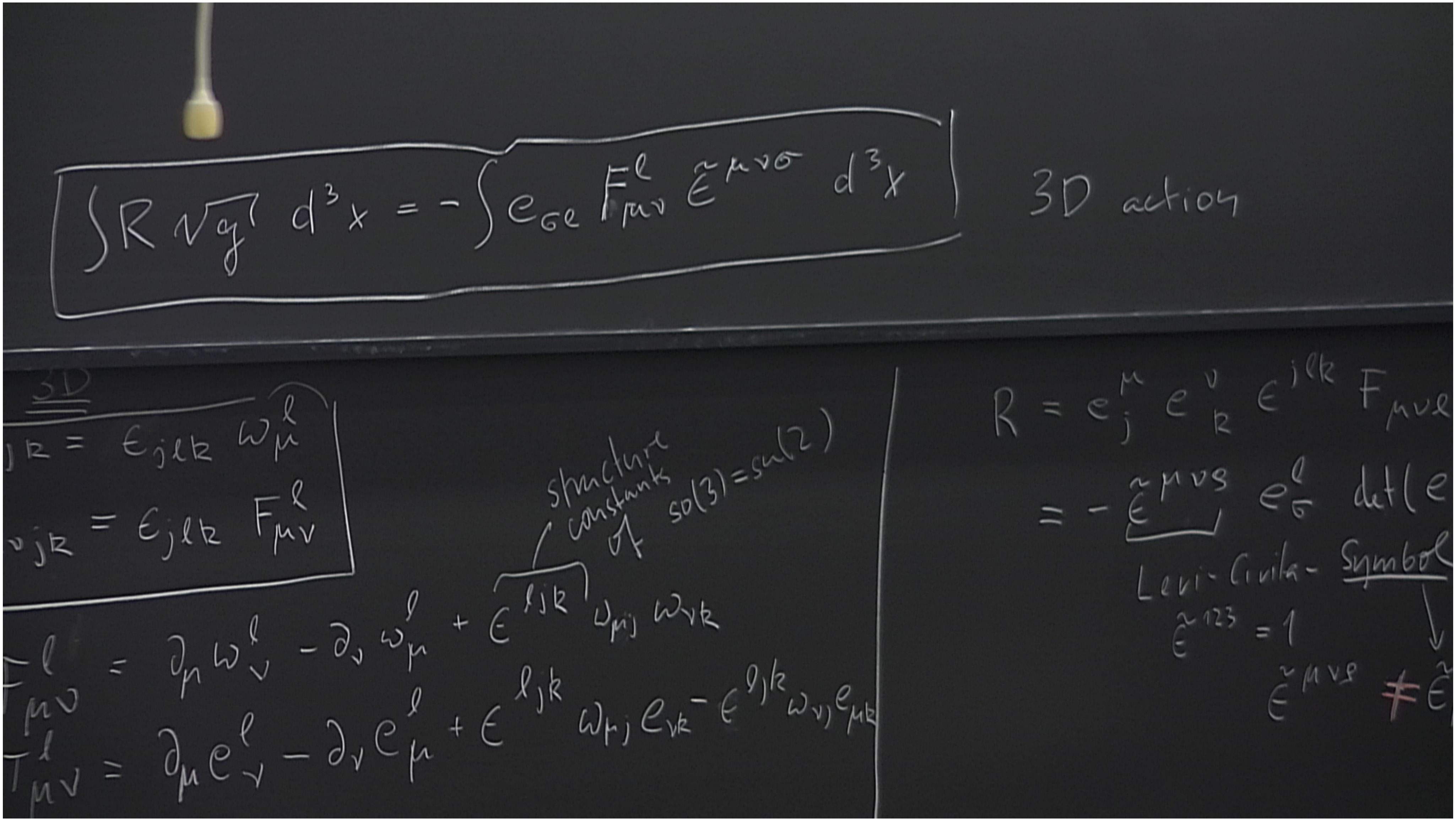
$$\tilde{\epsilon}^{\mu\nu\rho} \neq \tilde{\epsilon}_{\mu'\nu'\rho'}\, g^{\mu'\mu}\, g^{\nu\nu'}\, g^{\rho\rho'}$$

$$g_{\mu\nu} = e^{k}_{\mu}\, e^{j}_{\nu}\, \delta_{jk}$$

$$\det(e^{\rho}_{m}) = \left(\sqrt{g}\right)^{-1}$$



$\int R \sqrt{g}\, d^3x = -\int e_{\sigma l} F^l_{\mu\nu} \tilde{\epsilon}^{\mu\nu\sigma}\, d^3x$
3D actic
3D
$\omega_{\mu jk} = \epsilon_{jlk} \omega^l_\mu$
$F_{\mu\nu jk} = \epsilon_{jlk} F^l_{\mu\nu}$
structure constants of $so(3) = su(2)$
$F^l_{\mu\nu} = \partial_\mu \omega^l_\nu - \partial_\nu \omega^l_\mu + \epsilon^{ljk} \omega_{\mu j} \omega_{\nu k}$
$T^l_{\mu\nu} = \partial_\mu e^l_\nu - \partial_\nu e^l_\mu + \epsilon^{ljk} \omega_{\mu j} e_{\nu k} - \epsilon^{ljk} \omega_{\nu j} e_{\mu k}$
$R = e^\mu_j\, e$
$= -\tilde{\epsilon}$
Lev


$\int R\sqrt{g}\,d^3x = -\int e_{\sigma\ell}\,F^{\ell}_{\mu\nu}\,\tilde{\epsilon}^{\mu\nu\sigma}\,d^3x$
3D action
3D
$= \epsilon_{j\ell k}\,\omega^{\ell}_{\mu}$
$= \epsilon_{j\ell k}\,F^{\ell}_{\mu\nu}$
structure constants of $so(3)=su(2)$
$F^{\ell}_{\mu\nu} = \partial_\mu \omega^{\ell}_{\nu} - \partial_\nu \omega^{\ell}_{\mu} + \epsilon^{\ell jk}\,\omega_{\mu j}\,\omega_{\nu k}$
$T^{\ell}_{\mu\nu} = \partial_\mu e^{\ell}_{\nu} - \partial_\nu e^{\ell}_{\mu} + \epsilon^{\ell jk}\,\omega_{\mu j}\,e_{\nu k} - \epsilon^{\ell jk}\,\omega_{\nu j}\,e_{\mu k}$
$R = e^{\mu}_{j}\,e^{\nu}_{k}\,\epsilon^{j\ell k}\,F_{\mu\nu\ell}$
$= -\tilde{\epsilon}^{\mu\nu\sigma}\,e^{\ell}_{\sigma}\,\det(e$
Levi-Civita-Symbol
$\tilde{\epsilon}^{123} = 1$
$\tilde{\epsilon}^{\mu\nu\sigma} \neq \tilde{\epsilon}$