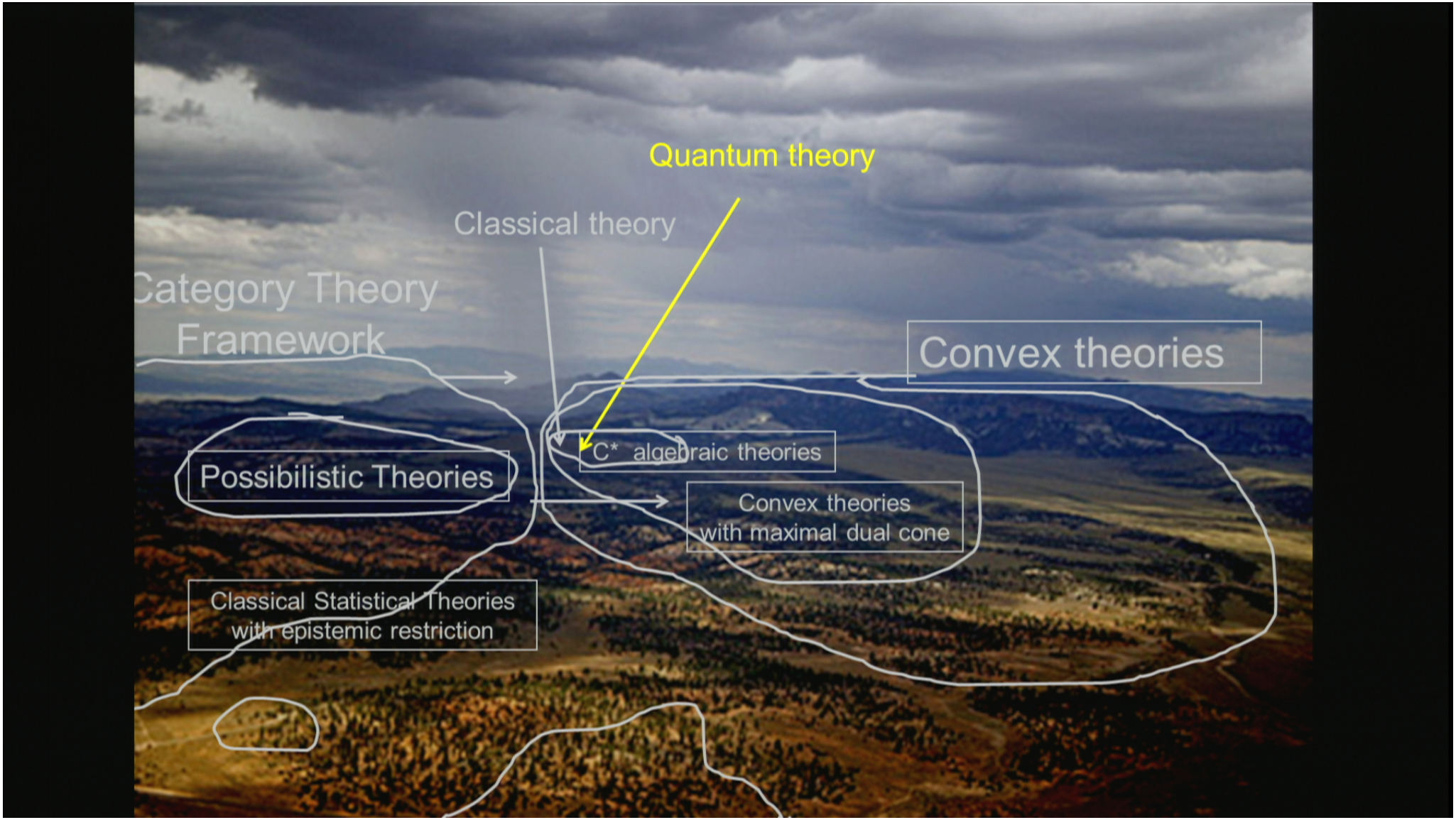
Title: 12/13 PSI - Found Quantum Mechanics Lecture 5

Date: Jan 11, 2013 11:30 AM

URL: http://pirsa.org/13010070

Abstract:


Quantum theory
Classical theory
Category Theory
Framework
Convex theories
Possibilistic Theories
C* algebraic theories
Convex theories
with maximal dual cone
Classical Statistical Theories
with epistemic restriction

Why are preparations represented by density operators?

Why are measurements represented by POVMs?

Why the Born rule? (Why a rule that's linear in the state?)

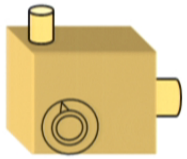
Why is composition of systems represented by tensor product?

Why Hilbert space over the complex field?

Why Hilbert space at all?

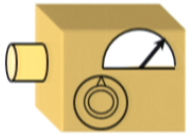
A framework for convex operational theories

See: L. Hardy, quant-ph/0101012



Preparation

P

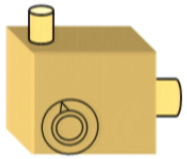


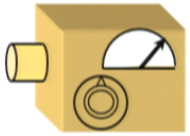
Measurement

M

A framework for convex operational theories

See: L. Hardy, quant-ph/0101012





Preparation
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Measurement
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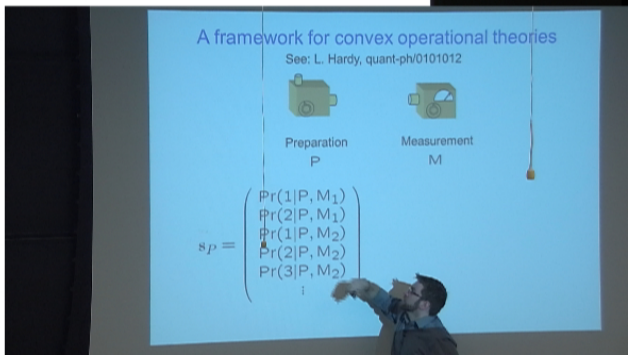
$$\mathbf{s}_P = \begin{pmatrix} \Pr(1|\mathrm{P},\mathrm{M}_1) \\ \Pr(2|\mathrm{P},\mathrm{M}_1) \\ \Pr(1|\mathrm{P},\mathrm{M}_2) \\ \Pr(2|\mathrm{P},\mathrm{M}_2) \\ \Pr(3|\mathrm{P},\mathrm{M}_2) \\ \vdots \end{pmatrix}$$



A framework for convex operational theories

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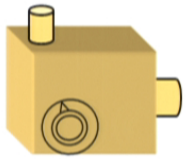

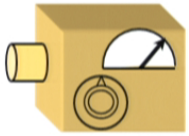

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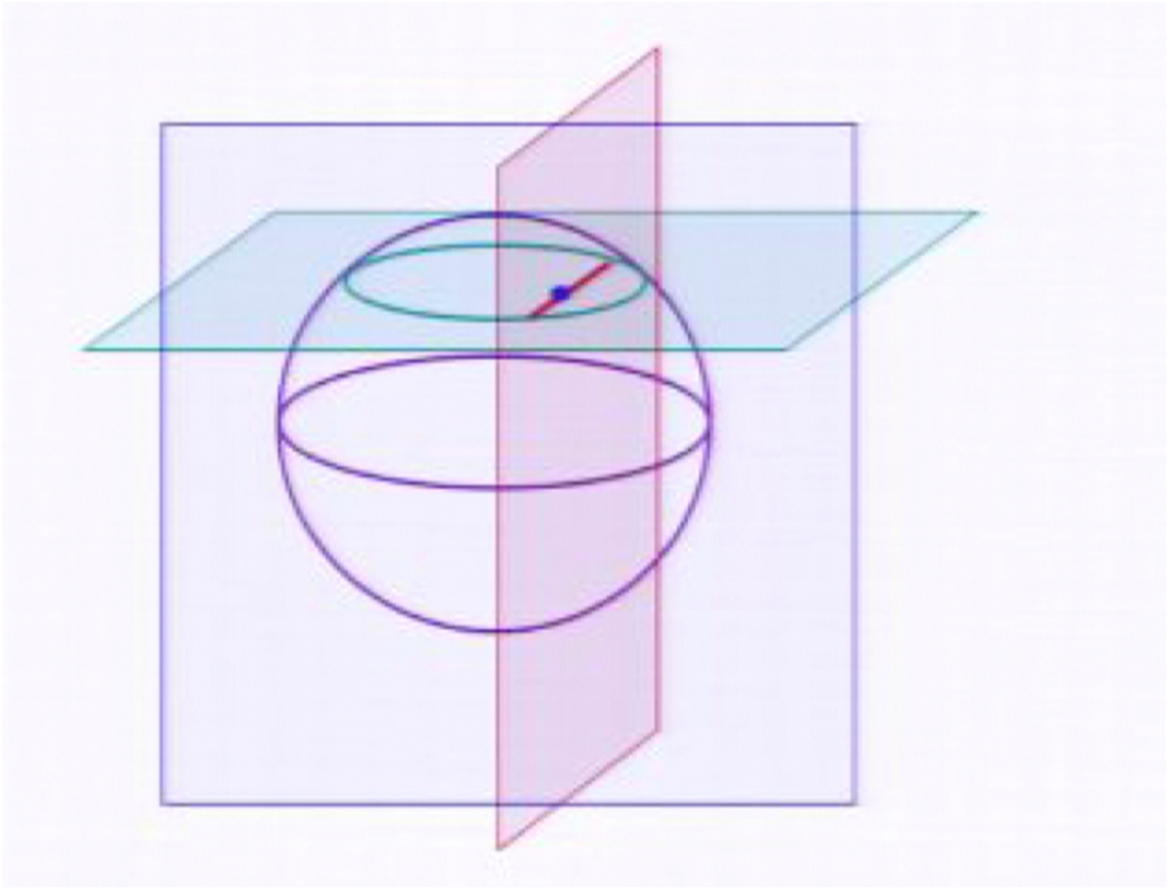
$$Pr(k|\mathrm{P},\mathrm{M}) = \mathbf{r}_{M,k} \cdot \mathbf{s}_P$$

State tomography for a single qubit



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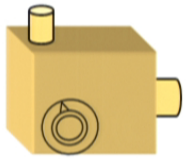

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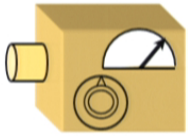

Measurement

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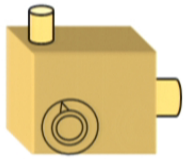
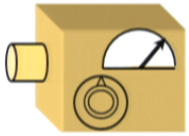
Suppose there are K fiducial measurements (pass-fail mmts from which one can infer the statistics for all mmts)

A framework for convex operational theories

See: L. Hardy, quant-ph/0101012





Preparation P

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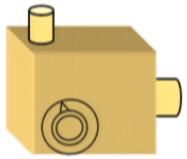
$$\mathbf{s}_P = \begin{pmatrix} \Pr(\text{pass}|\mathrm{P},\mathrm{M}_1) \\ \Pr(\text{pass}|\mathrm{P},\mathrm{M}_2) \\ \vdots \\ \Pr(\text{pass}|\mathrm{P},\mathrm{M}_K) \end{pmatrix}$$

A framework for convex operational theories

See: L. Hardy, quant-ph/0101012



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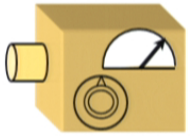

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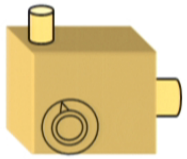
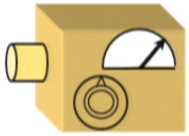
"operational state"

A framework for convex operational theories

See: L. Hardy, quant-ph/0101012





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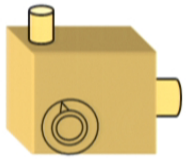
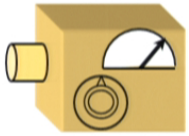
"operational state"

$$Pr(k|\mathrm{P},\mathrm{M}) = f_{M,k}(\mathbf{s}_P)$$

A framework for convex operational theories

See: L. Hardy, quant-ph/0101012





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"operational state"

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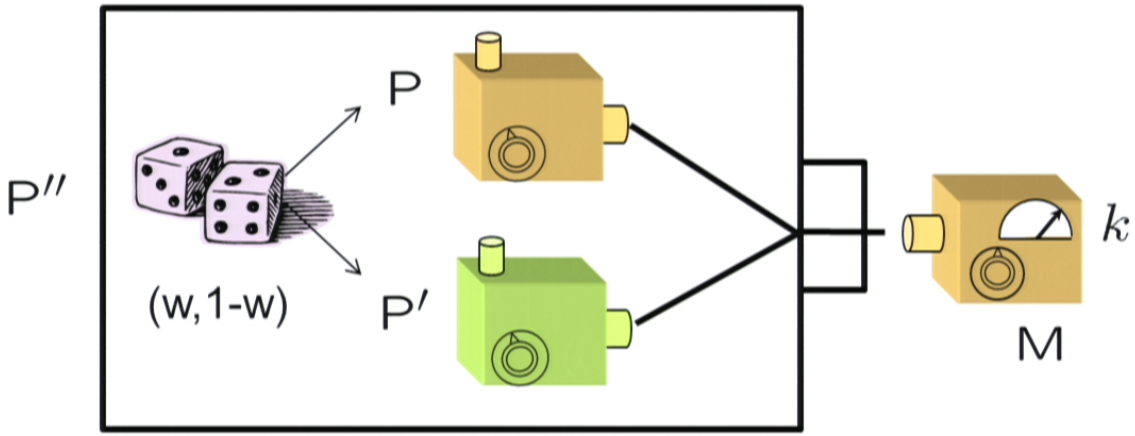
What can we say about *f*?

Operational states form a convex set



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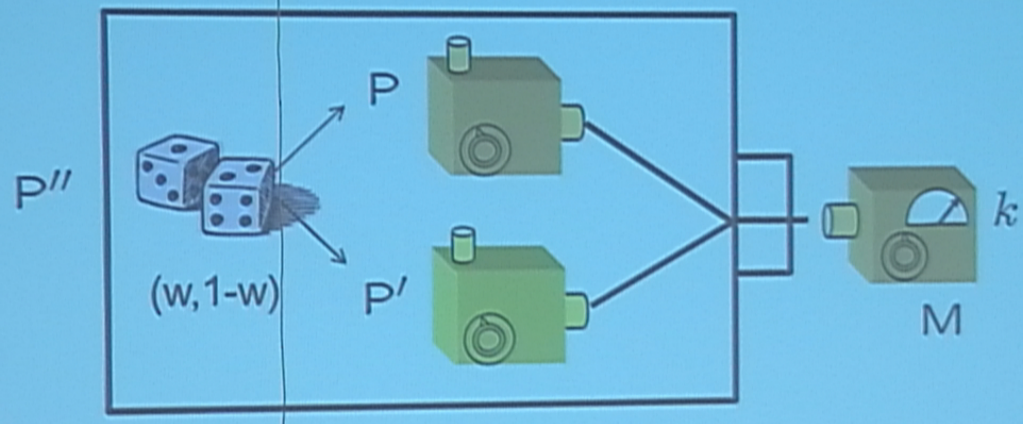
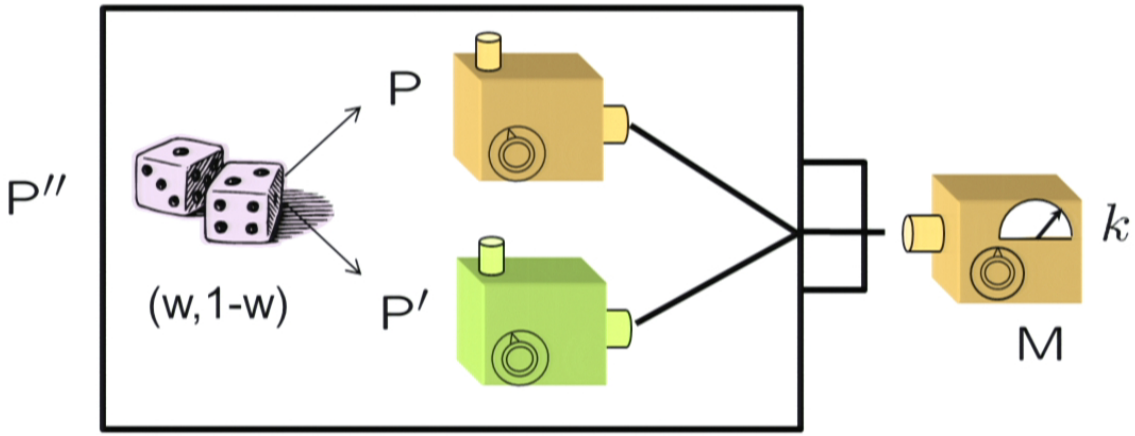


$$\forall M, k : p(k|\mathsf{P}'', \mathsf{M}) = w\, p(k|\mathsf{P}, \mathsf{M}) + (1-w)\, p(k|\mathsf{P}', \mathsf{M})$$

$$f(\mathsf{s}_{\mathsf{P}''}) = w\, f(\mathsf{s}_{\mathsf{P}}) + (1-w)\, f(\mathsf{s}_{\mathsf{P}'})$$

Also true for fiduc[illegible]mmts, so $\mathsf{s}_{\mathsf{P}''} = w\, \mathsf{s}_{\mathsf{P}} + (1-w)\, \mathsf{s}_{\mathsf{P}'}$

Operational states form a convex set



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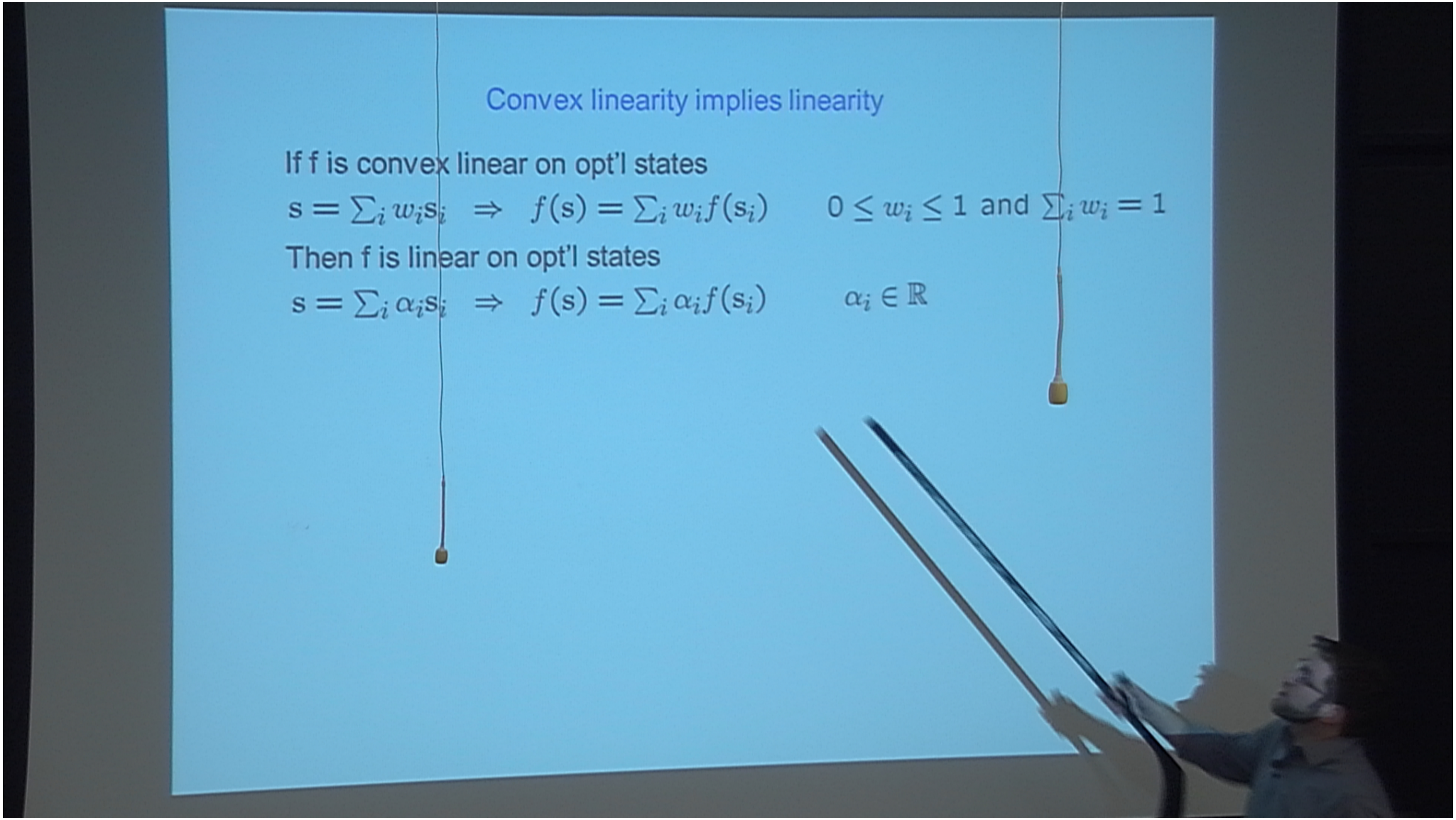
Also true for fiducial mmts, so $\mathbf{s}_{\mathsf{P}''} = w\, \mathbf{s}_{\mathsf{P}} + (1-w)\, \mathbf{s}_{\mathsf{P}'}$

Closed under convex combination –> a convex set

$f(w\, \mathbf{s}_{\mathsf{P}} + (1-w)\, \mathbf{s}_{\mathsf{P}'}) = w\, f(\mathbf{s}_{\mathsf{P}}) + (1-w)\, f(\mathbf{s}_{\mathsf{P}'})$ Convex linear


Convex linearity implies linearity
If f is convex linear on opt'l states
$s = \sum_i w_i s_i \Rightarrow f(s) = \sum_i w_i f(s_i)$ $0 \le w_i \le 1$ and $\sum_i w_i = 1$
Then f is linear on opt'l states
$s = \sum_i \alpha_i s_i \Rightarrow f(s) = \sum_i \alpha_i f(s_i)$ $\alpha_i \in \mathbb{R}$

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Note that the fiducial mmts are clearly represented by linear functions

$$\mathbf{s}_P = \begin{pmatrix} \Pr(\text{pass}|\mathrm{P}, \mathrm{M}_1) \\ \Pr(\text{pass}|\mathrm{P}, \mathrm{M}_2) \\ \vdots \\ \Pr(\text{pass}|\mathrm{P}, \mathrm{M}_K) \end{pmatrix} \qquad \mathbf{r}_{\mathrm{M}_i,\text{pass}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

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Proof: $\mathbf{s} = \sum_i \alpha_i \mathbf{s}_i$

$\mathbf{s} + \sum_{j \in I_-} |\alpha_j| \mathbf{s}_j = \sum_{i \in I_+} |\alpha_i| \mathbf{s}_i$

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Consider a coarse-graining of all the outcomes of a fiducial mmt. $1 = \sum_i \alpha_i$

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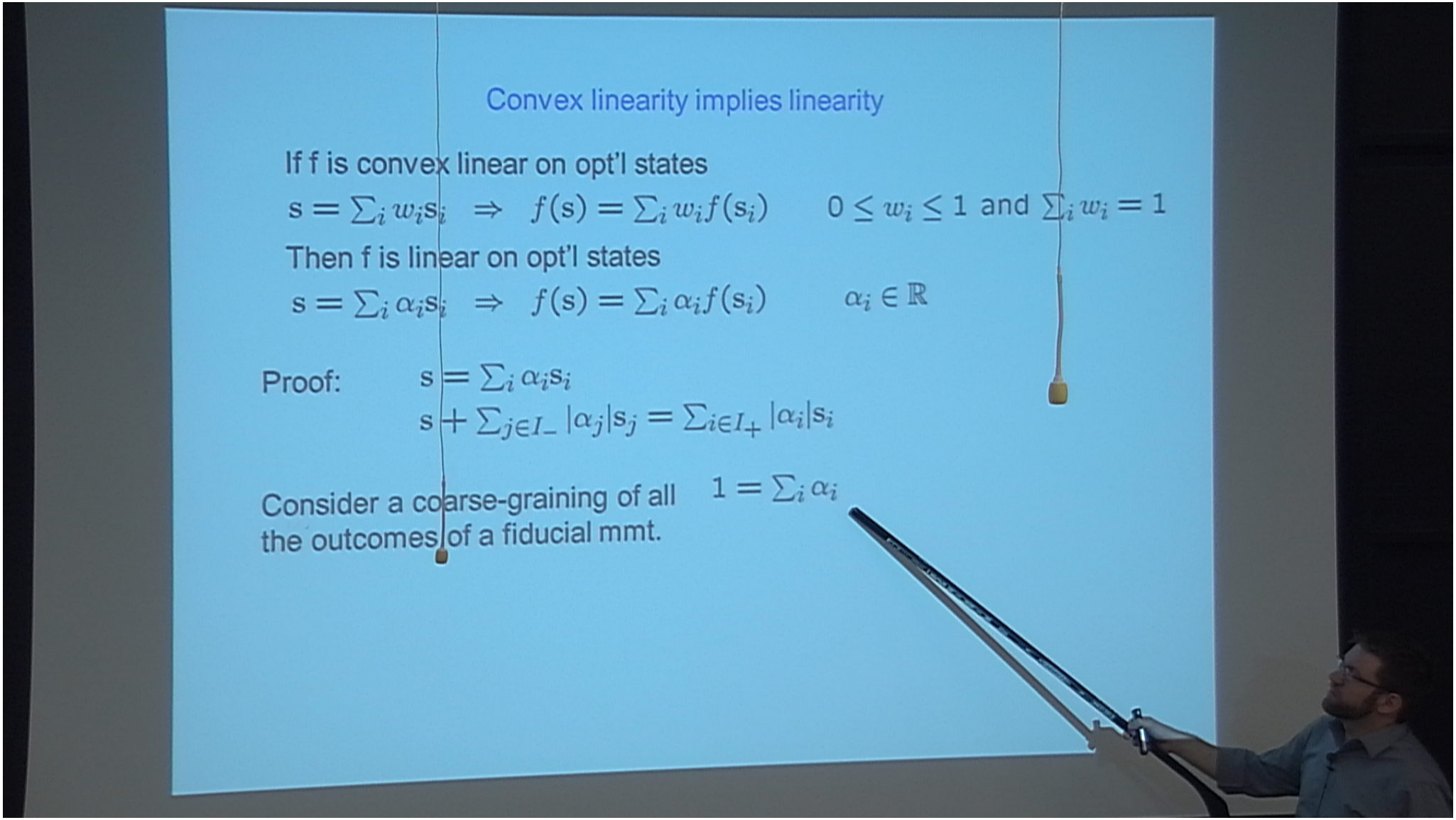
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$1 = \sum_i \alpha_i$

$1 + \sum_{j \in I_-} |\alpha_j| = \sum_{i \in I_+} |\alpha_i| \equiv \mathcal{N}$

Thus: $\frac{1}{\mathcal{N}} \mathrm{s} + \sum_{j \in I_-} \frac{|\alpha_j|}{\mathcal{N}} \mathrm{s}_j = \sum_{i \in I_+} \frac{|\alpha_i|}{\mathcal{N}} \mathrm{s}_i$

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Proof:

$$\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i$$

$$\mathrm{s} + \sum_{j \in I_-} |\alpha_j| \mathrm{s}_j = \sum_{i \in I_+} |\alpha_i| \mathrm{s}_i$$

Consider a coarse-graining of all the outcomes of a fiducial mmt.

$$1 = \sum_i \alpha_i$$

$$1 + \sum_{j \in I_-} |\alpha_j| = \sum_{i \in I_+} |\alpha_i| \equiv \mathcal{N}$$

Thus:

$$\tfrac{1}{\mathcal{N}} \mathrm{s} + \sum_{j \in I_-} \tfrac{|\alpha_j|}{\mathcal{N}} \mathrm{s}_j = \sum_{i \in I_+} \tfrac{|\alpha_i|}{\mathcal{N}} \mathrm{s}_i$$

$$\tfrac{1}{\mathcal{N}} f(\mathrm{s}) + \sum_{j \in I_-} \tfrac{|\alpha_j|}{\mathcal{N}} f(\mathrm{s}_j) = \sum_{i \in I_+} \tfrac{|\alpha_i|}{\mathcal{N}} f(\mathrm{s}_i)$$

Convex linearity implies linearity

If f is convex linear on opt'l states

$\mathrm{s} = \sum_i w_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i w_i f(\mathrm{s}_i) \qquad 0 \le w_i \le 1 \text{ and } \sum_i w_i = 1$

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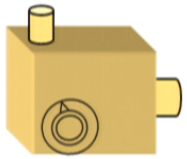
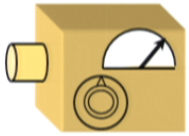
Therefore $\exists \mathbf{r} : f(\mathbf{s}) = \mathbf{r} \cdot \mathbf{s}$

A convex operational theory





Preparation

P

Measurement

M

$\mathbf{s}_P \in S$

"operational states"

S = Convex set

$\mathbf{r}_{M,k} \in R$

"operational effects"

R = Interval of positive cone

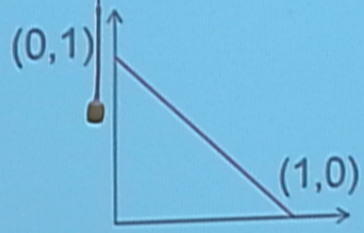
$$Pr(k|\mathrm{P},\mathrm{M}) = \mathbf{r}_{M,k} \cdot \mathbf{s}_P$$

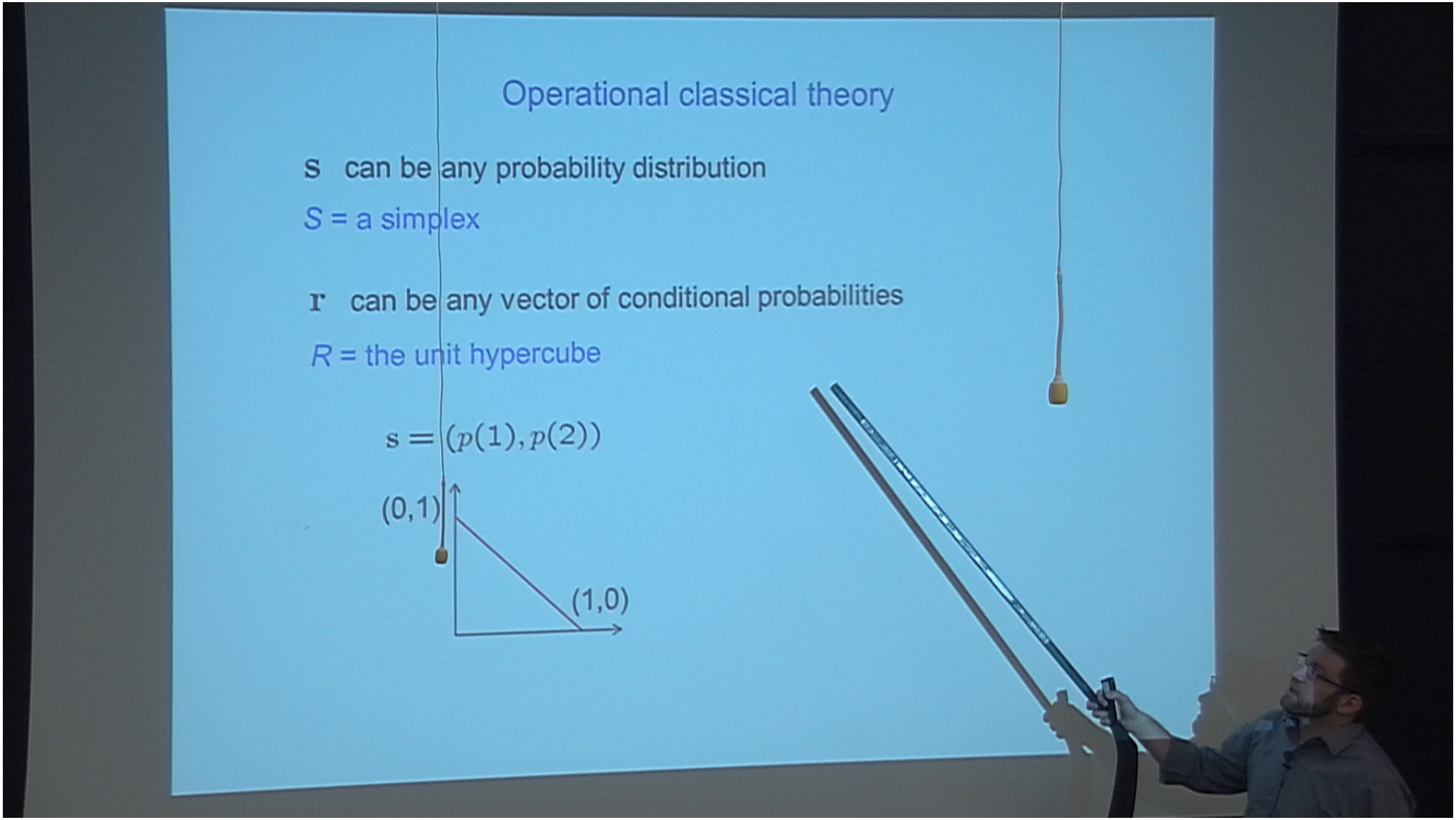
Operational classical theory

s can be any probability distribution

S = a simplex

$$s = (p(1), p(2))$$




Operational classical theory
s can be any probability distribution
S = a simplex
r can be any vector of conditional probabilities
R = the unit hypercube
s = (p(1), p(2))
(0,1)
(1,0)

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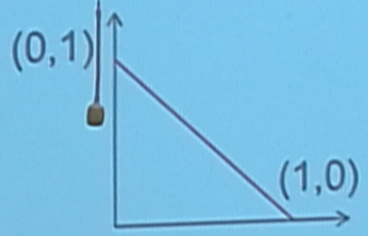
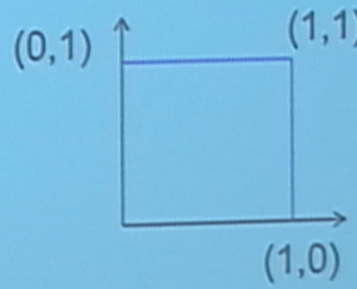


$\mathbf{r} = (p(\text{pass}|1), p(\text{pass}|2))$



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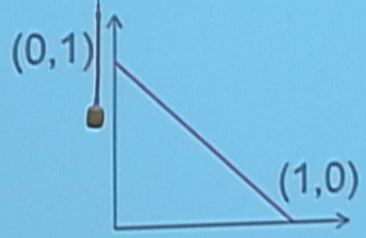

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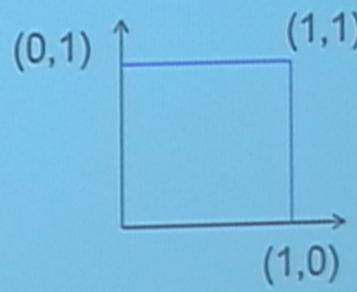

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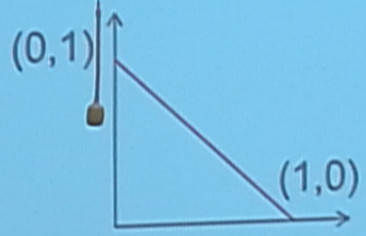

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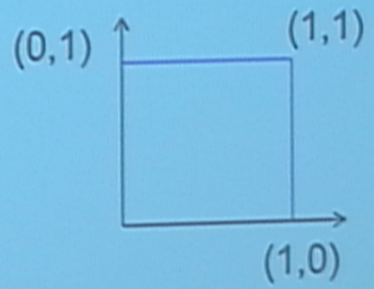

$$Pr(\text{pass}) = \mathbf{r} \cdot \mathbf{s} = \sum_i p(i)p(\text{pass}|i)$$

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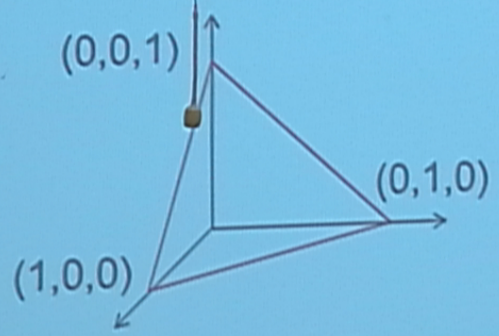

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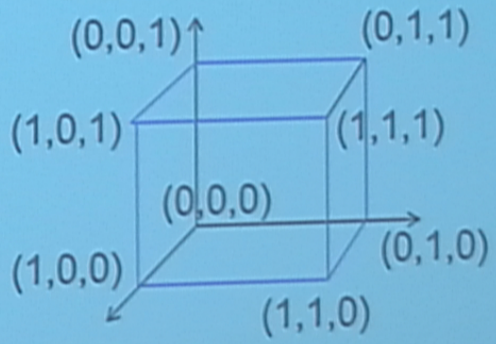
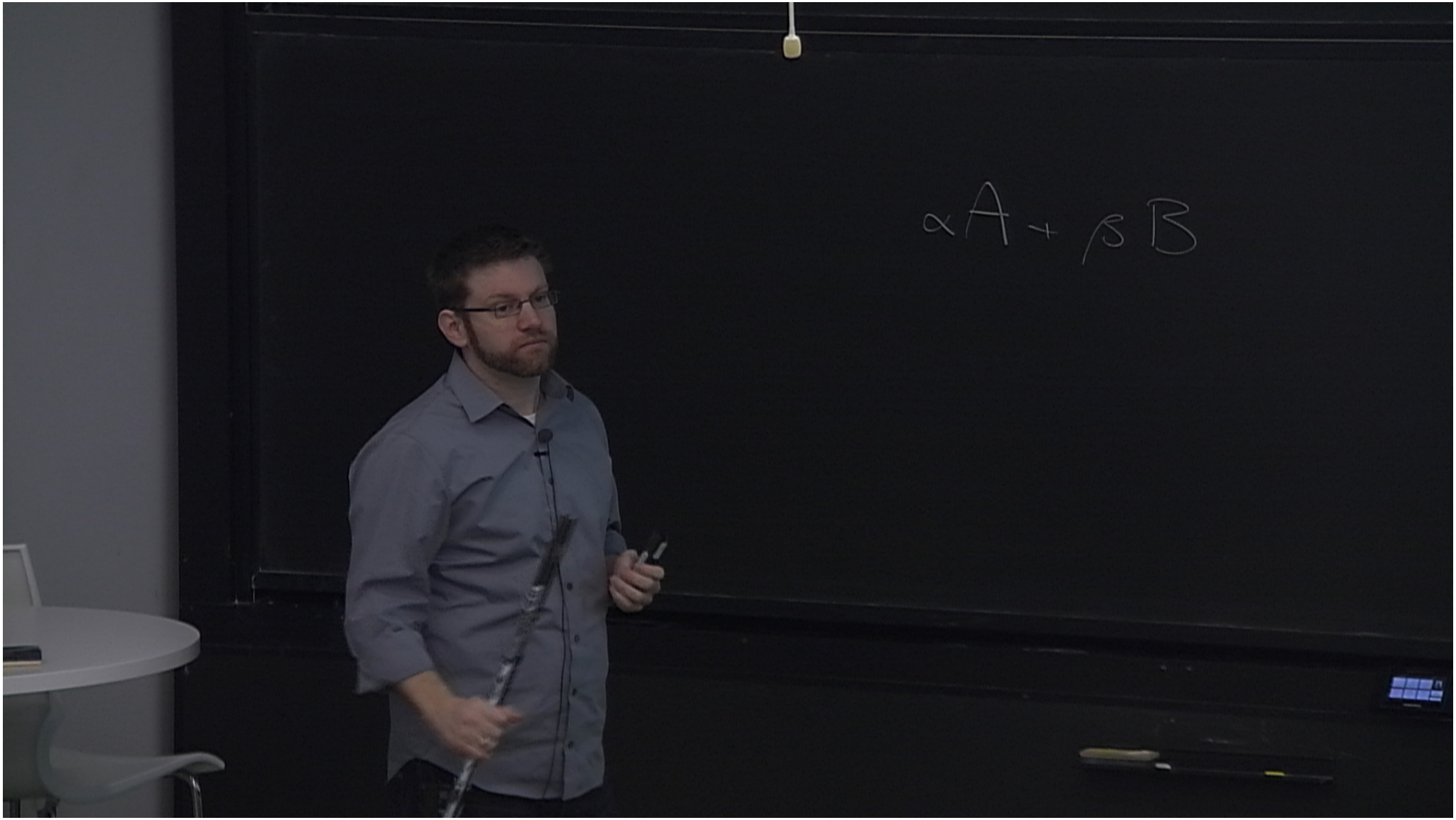


Operational quantum theory

Recall: The Hermitian operators on a Hilbert space of dimension d form a real Euclidean vector space of dimension d^2

s can be any unit-trace positive operator ρ positive, $\mathrm{Tr}(\rho) = 1$

S = the convex set of such operators


αA + βB

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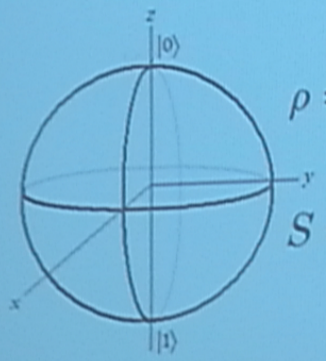

$$\rho = \frac{1}{2}(s_0 I + \mathbf{s}' \cdot \sigma)$$

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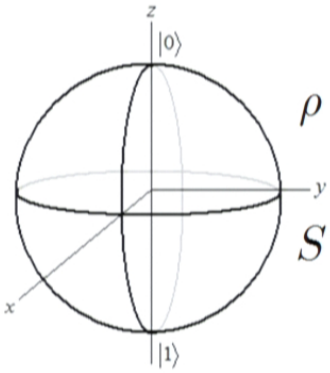

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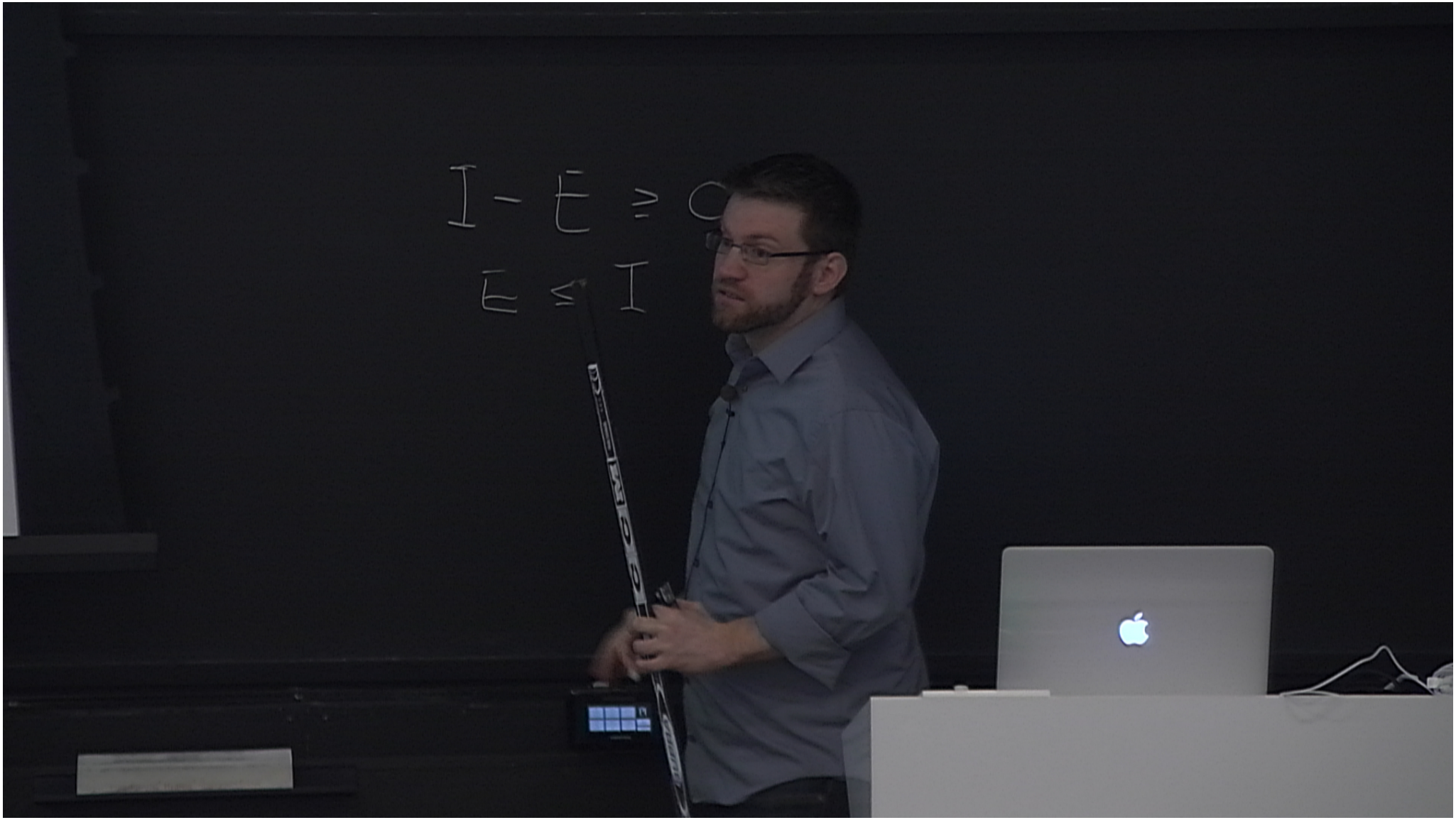

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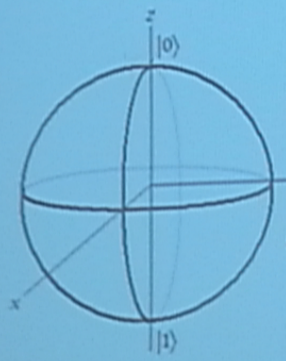

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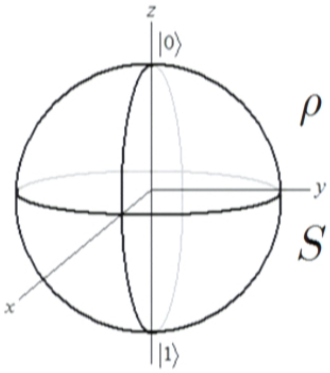

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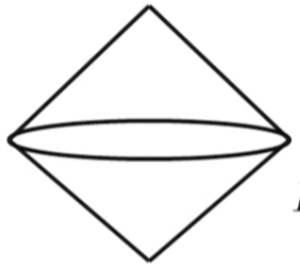

$$E = r_0 I + \mathbf{r}' \cdot \sigma$$

$$R = \{\mathbf{r} \mid 0 \leq r_0 \leq 1, \ |\mathbf{r}'| \leq r_0, 1 - r_0\}$$

$$Pr(\text{pass}) = \mathbf{r} \cdot \mathbf{s} = \mathrm{Tr}(\rho E)$$

A little bit of axiomatics

Suppose one takes as given that

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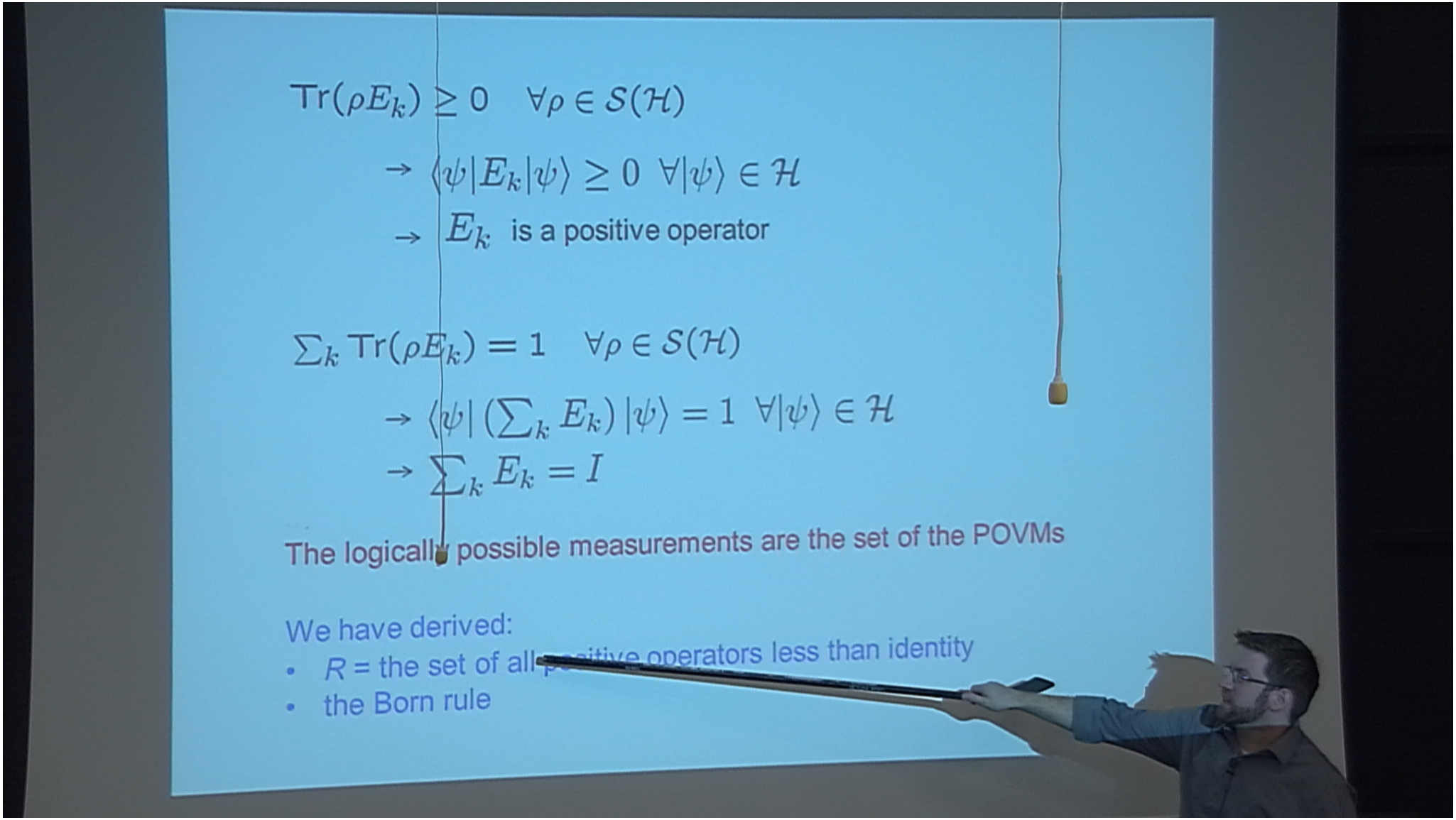
$\mathbf{r}_k \cdot \mathbf{s} = (E_k, \rho) = \mathrm{Tr}(E_k \rho)$ ← the form of the Born rule


$\mathrm{Tr}(\rho E_k) \geq 0 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$
$\rightarrow \langle\psi|E_k|\psi\rangle \geq 0 \ \ \forall|\psi\rangle \in \mathcal{H}$
$\rightarrow E_k$ is a positive operator
$\sum_k \mathrm{Tr}(\rho E_k) = 1 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$
$\rightarrow \langle\psi| \left(\sum_k E_k\right) |\psi\rangle = 1 \ \ \forall|\psi\rangle \in \mathcal{H}$
$\rightarrow \sum_k E_k = I$
The logically possible measurements are the set of the POVMs
We have derived:
• R = the set of all positive operators less than identity
• the Born rule

Operational formulation of quantum theory

Every preparation P is associated with a density operator ρ

Every logically possible measurement is physically possible

↳ Every measurement M is associated with a positive operator-valued measure $\{E_k\}$. The probability of M yielding outcome k given a preparation P is $Pr(k|\mathrm{P},\mathrm{M}) = \mathrm{Tr}(\rho E_k)$

Every transformation is associated with a trace-preserving completely-positive linear map $\rho \to \rho' = \mathcal{T}(\rho)$

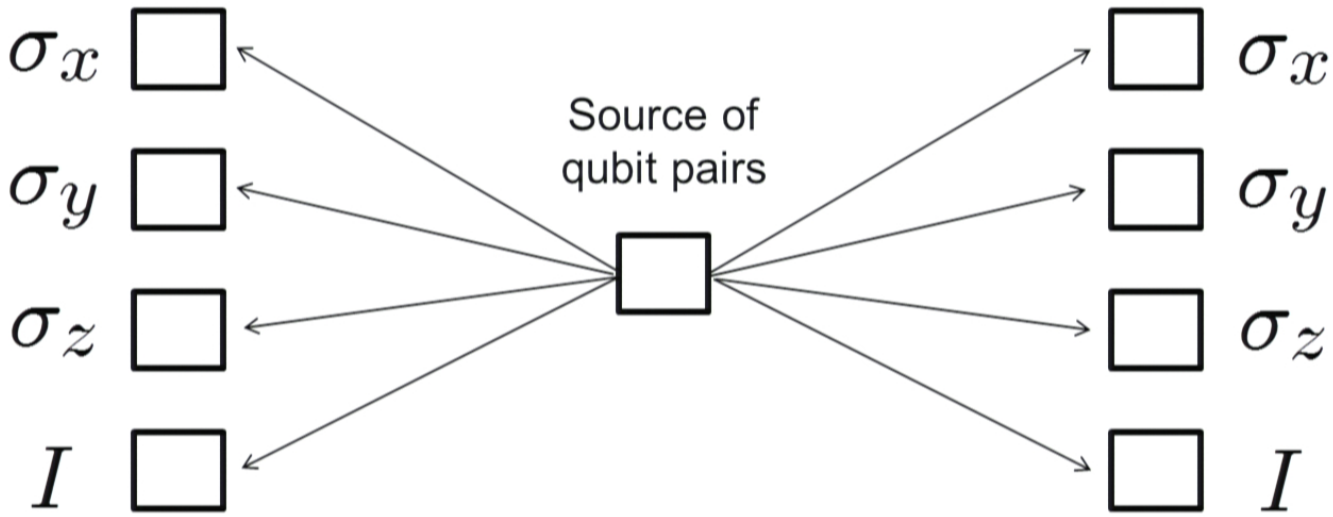
Every measurement outcome k is associated with a trace-nonincreasing completely-positive linear map T_k such that

$$\rho \to \rho_k = \frac{\mathcal{T}_k(\rho)}{\mathrm{Tr}[\mathcal{T}_k(\rho)]} \quad \text{where} \quad \mathcal{T}_k^\dagger(I) = E_k$$

Real versus complex field

	real case	complex case
Pure preparations	rays in $\mathbb{R}^d$	rays in $\mathbb{C}^d$
Complete repeatable measurements	Bases for $\mathbb{R}^d$	Bases for $\mathbb{C}^d$
Reversible transformations	Special orthogonal (rotation)	Unitary
Mixed preparations	Positive unit-trace real matrix	Positive unit-trace complex matrix
Composition rule	Tensor product	Tensor product

State tomography for two qubits

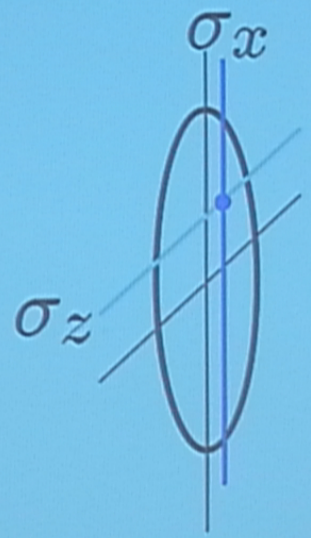


We need 4^2 -1 = 15 parameters

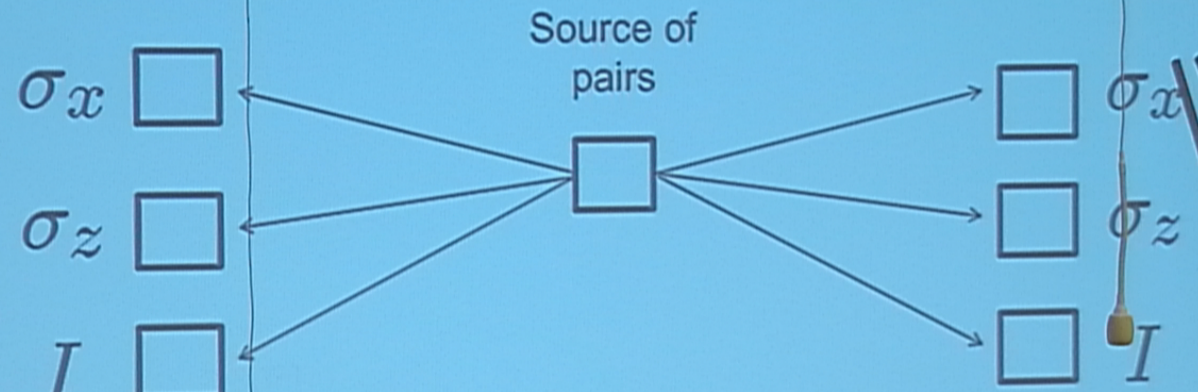
We obtain 4^2 -1 = 15 parameters

The mixed state of two qubits can be determined from local measurements

State tomography for a single real-amplitude qubit



State tomography for two real-amplitude qubits



We need $4(4+1)/2 - 1 = 9$ parameters

We obtain $3^2 - 1 = 8$ parameters

$\sigma_y \otimes \sigma_y$ must be accessed globally

The mixed state of two real-amplitude qubits
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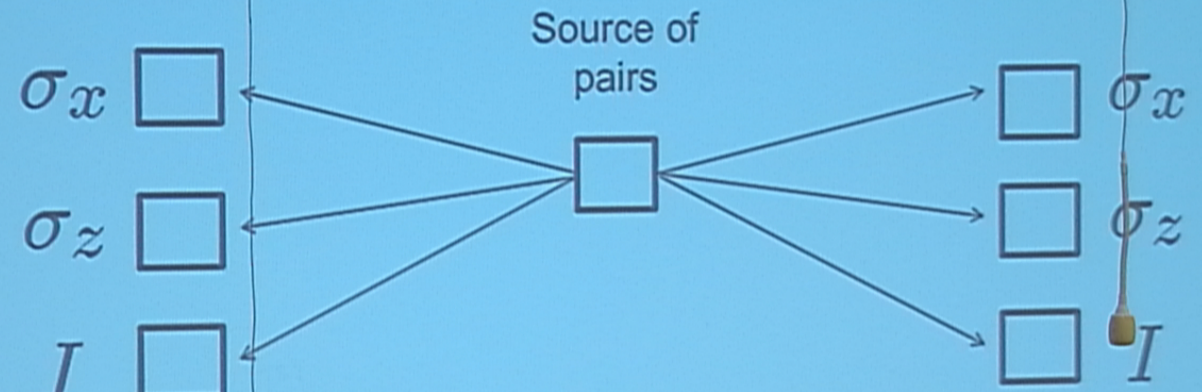
State tomography for two real-amplitude qubits



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