

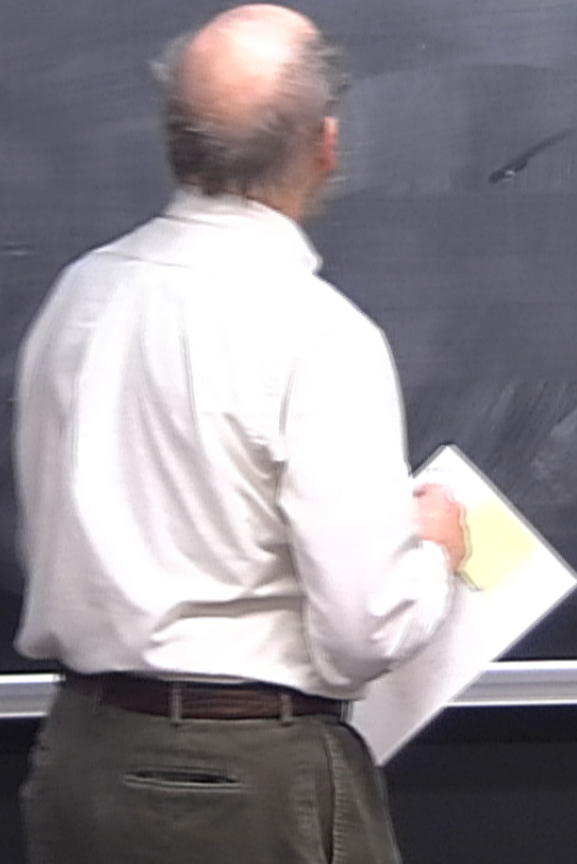
Title: Advanced General Relativity - Lecture 4

Date: Jan 31, 2013 05:00 PM

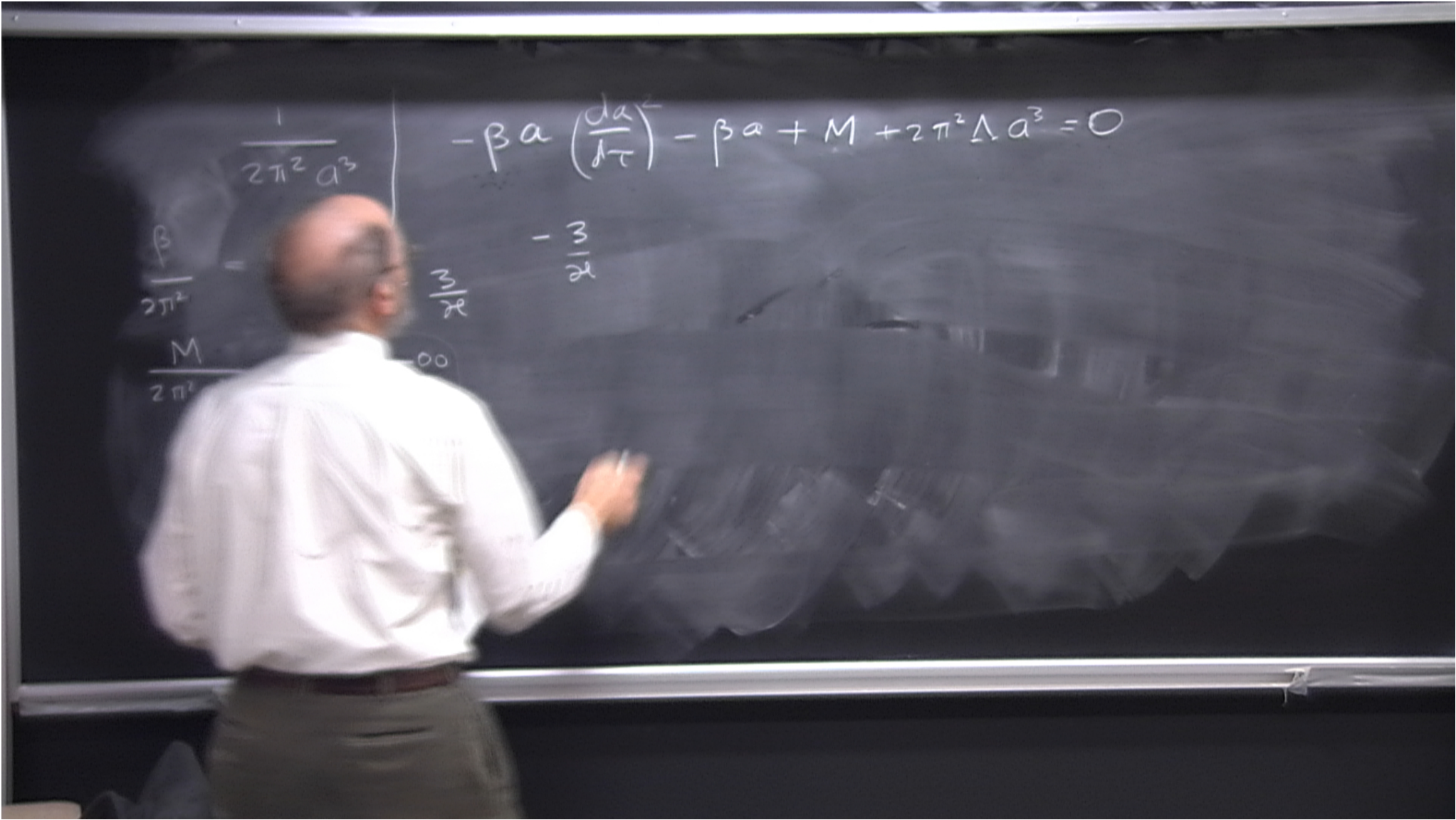
URL: <http://pirsa.org/13010015>

Abstract:

$$-\beta a \left( \frac{da^2}{dt} \right) - \beta a + M + 2\pi^2 \Lambda a^3 = 0$$









$$\frac{1}{2\pi^2 a^3} \left| \frac{\beta}{2\pi^2} = \frac{6\pi^2/\rho\ell}{2\pi^2} = \frac{3}{\rho\ell} \right.$$

$$\frac{M}{2\pi^2 a^3} = \rho = T^{00}$$

= mass density

$$-\beta a \left( \frac{da}{d\tau} \right)^2 - \beta a + M + 2\pi^2 \Lambda a^3 = 0$$

$$-\frac{3}{\rho\ell} \left( \frac{1}{a} \frac{da}{d\tau} \right)^2 - \frac{3}{\rho\ell a^2} + \rho + \Lambda = 0$$

$$\frac{1}{a} \frac{da}{d\tau} \equiv H$$

$$\frac{1}{a}$$



$$\frac{1}{2\pi^2 a^3} \left| \frac{\beta}{2\pi^2} = \frac{6\pi^2/\partial\ell}{2\pi^2} = \frac{3}{\partial\ell} \right.$$

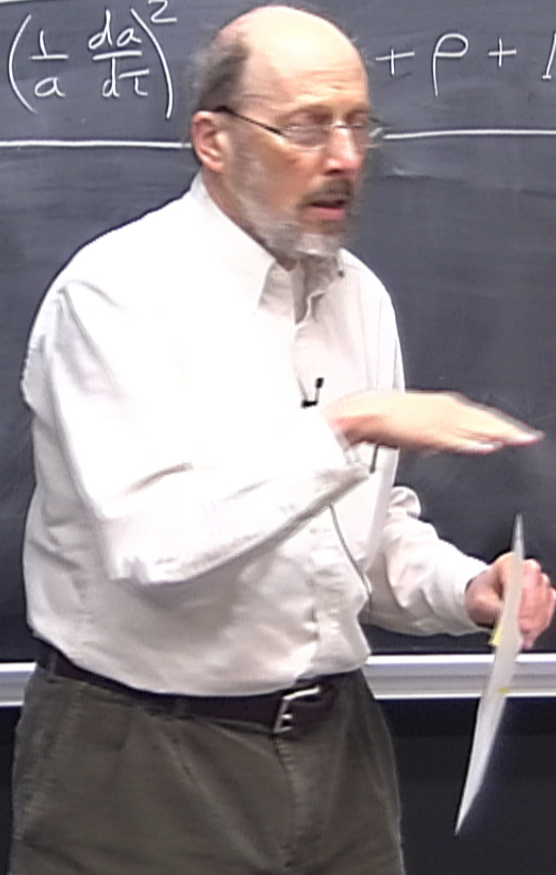
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$$-\beta a \left( \frac{da}{d\tau} \right)^2 - \beta a + M + 2\pi^2 \Lambda a^3 = 0$$

$$\boxed{-\frac{3}{\partial\ell} \left( \frac{1}{a} \frac{da}{d\tau} \right)^2 + \rho + \Lambda = 0}$$

$$\boxed{\frac{1}{a} \frac{da}{d\tau} \equiv H}$$





$$\frac{1}{2\pi^2 a^3}$$

$$\frac{\beta}{2\pi^2} = 6$$

$$\frac{M}{2\pi^2 a^3}$$

$$-\beta a \left(\frac{da}{d\tau}\right)^2 - \beta a + M + 2\pi^2 \Lambda a^3 = 0$$

$$\boxed{-\frac{3}{2} \left(\frac{1}{a} \frac{da}{d\tau}\right)^2 - \frac{3}{2a^2} + \rho + \Lambda = 0}$$

$$\boxed{\frac{1}{a} \frac{da}{d\tau} \equiv H}$$

"Kinetic energy"

Curvature  
P.E. of grav.

energy  
density

"Vacuum  
energy density"



$$\frac{\beta}{2\pi^2} = \frac{6\pi^2/\partial\ell}{2\pi^2} = \frac{3}{\partial\ell}$$

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"Kinetic energy"

Curvature  
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$$\frac{1}{a} \frac{da}{d\tau} \equiv H$$



$$\frac{1}{2\pi^2 a^3}$$

$$\frac{\beta}{2\pi^2} \frac{4\pi^2/8\pi}{2} = \frac{3}{2\pi}$$

$$\frac{M}{2\pi^2 a^3} = T_{00}$$

$$-\beta a \left(\frac{da}{d\tau}\right)^2 - \beta a + M + 2\pi^2 \Lambda a^3 = 0$$

$$\boxed{-\frac{3}{2\pi} \left(\frac{1}{a} \frac{da}{d\tau}\right)^2 - \frac{3}{2\pi a^2} + \rho + \Lambda = 0}$$

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"Kinetic energy"

↑  
Curvature  
P.E. of grav.

↑  
energy  
density

↑  
"vacuum  
energy density"

$$a(\tau_i)/a(\tau_e)$$



$$\frac{\beta}{2\pi^2} = \frac{6\pi^2/\partial\ell}{2\pi^2} = \frac{3}{\partial\ell}$$

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$H^2$  "Kinetic energy"  
 $a(\tau_1)/a(\tau_2)$

Curvature  
 P.E of grav.

energy  
 density

"Vacuum  
 energy density"

$$\frac{1}{a} \frac{da}{d\tau} \equiv H$$



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$H^2$  "Kinetic energy"  
 $a(\tau_1)/a(\tau_2)$

Curvature  
 P.E of grav.

energy density  
 "Vacuum energy density"



# Gravitational Kinematics (Taketani II)

EP S locally like  $M^4$

CAUTION

DO NOT TOUCH THE BOARD OR THE BOARDER. PLEASE CONTACT US FOR ASSISTANCE AT THE BOARD.

IT IS PROHIBITED TO WRITE ON THE BOARD WITH OTHER THAN CHALK.

PLEASE BE CAREFUL



# Gravitational Kinematics (Taketani II)

EP Spacetime locally like  $M^4$

( $\alpha$ ) topology  $\mathbb{R}^4$

( $\beta$ ) vector space

$$M = \sum P_x^M$$

CAUTION

NE BLAKE THE BOARD FROM WRITING BOARD.  
PLEASE HANDLE ON THE HANDLE OF THE BOARD.

IT IS IMPORTANT TO USE  
YOUR OWNERS PRESENCE ONLY

ATTENTION: PRESENCE ONLY



# Gravitational Kinematics (Taketani II)

EP Spacetime locally like  $M^4$

- a) topology  $\mathbb{R}^4$
- b) vector space
- c) metric

$$\sum p_i^m = \sum p_s^m$$



# Gravitational Kinematics (Taketani II)

EP Spacetime locally like  $M^4$

- ( $\alpha$ ) topology  $\mathbb{R}^4$   $\longrightarrow$  topological ( $C^0$ ) manifold
- ( $\beta$ ) vector space  $\longrightarrow$
- ( $\gamma$ ) metric  $(-+++)$
- ( $\delta$ ) causal structure

$$= \mathbb{R}^M = \sum P_S^M$$

CAUTION

DO NOT REACH OR CLIMB OVER THE BOARD. PLEASE REMAIN AT THE BOTTOM OF THE BOARD.

IF AN INDIVIDUAL IS IN DISTRESS, CALL 911 IMMEDIATELY.

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# Gravitational Kinematics (Taketani II)

EP Spacetime locally like  $M^4$

- ( $\alpha$ ) topology  $\mathbb{R}^4$   $\longrightarrow$  topological ( $C^0$ ) manifold
- ( $\beta$ ) vector space  $\longrightarrow$  differentiable structure
- ( $\gamma$ ) metric  $(-+++)\eta_m$   $\longrightarrow$  metric field (line)
- ( $\delta$ ) causal structure



$$\xi p_i^M = \sum p_{\xi}^M$$

CAUTION

DO NOT REACH OR CLIMB OVER THE CHALKBOARD.  
PLEASE CONTACT AN INSTRUCTOR OR THE STAFF.

IF AN INDIVIDUAL IS IN DISTRESS,  
CALL 911 IMMEDIATELY.

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# Gravitational Kinematics (Taketani II)

EP Spacetime locally like  $M^4$

( $\alpha$ ) topology  $\mathbb{R}^4$   $\longrightarrow$  topological ( $C^0$ ) manifold (4d)

( $\beta$ ) vector space  $\longrightarrow$  differentiable structure

( $\gamma$ ) metric  $(-+++)$   $\longrightarrow$  metric field (line elt)  $ds^2$

( $\delta$ ) causal structure  $\longrightarrow$  causal structure  $(\pm \begin{matrix} F \\ \times \\ P \end{matrix})$  T-orientation

$$\sum P_i^M = \sum P_F^M$$





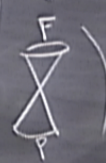
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( $\gamma$ ) metric  $(-+++)$   $\longrightarrow$  metric field (line elt)  $ds^2 = g_{ab}$

( $\delta$ ) causal structure  $\longrightarrow$  causal structure ( $\pm$  ) T-orientation

$$\sum P_i^M = \sum P_F^M$$





differential structure  
 (7) metric  $(-+++)^n$  → metric field (line elt)  $ds^2 = g_{ab}$   
 (8) causal structure → causal structure  $(\pm \begin{matrix} \uparrow \\ \downarrow \end{matrix})$  T-orientation

Reading Wald (AppA, §2.1) Schutz (151-156, §6.1, §6.2 (part))





"contiguity" neighborhood  $\leftrightarrow$  convergence  
(nhbd)

CAUTION  
TO AVOID OR LOWER THE WRITING BOARD,  
PLEASE GRAB BY THE HANDLE OF THE BOARD.  
IT IS NECESSARY TO APPLY  
YOUR WEIGHT PROPERLY ONLY  
AFTER PROPERING BOARD



"Contiguity"

neighborhood  
(nhbd)

↔ convergence



CAUTION

TO AVOID OR LOWER THE BRASSING RISK,  
PLEASE CENTER BY THE MIDDLE OF THE BOARD.

IT IS ESSENTIAL TO APPLY  
YOUR WEIGHTS PROPERLY ONLY

APPROX PROBABLY ONLY



"Contiguity"

neighborhood  
(nhbd)

↔ convergence



CAUTION  
DO NOT LEAN ON THE BOARD  
OR WRITE ON THE BOARD  
OR REMOVE THE BOARD  
OR REMOVE THE BOARD



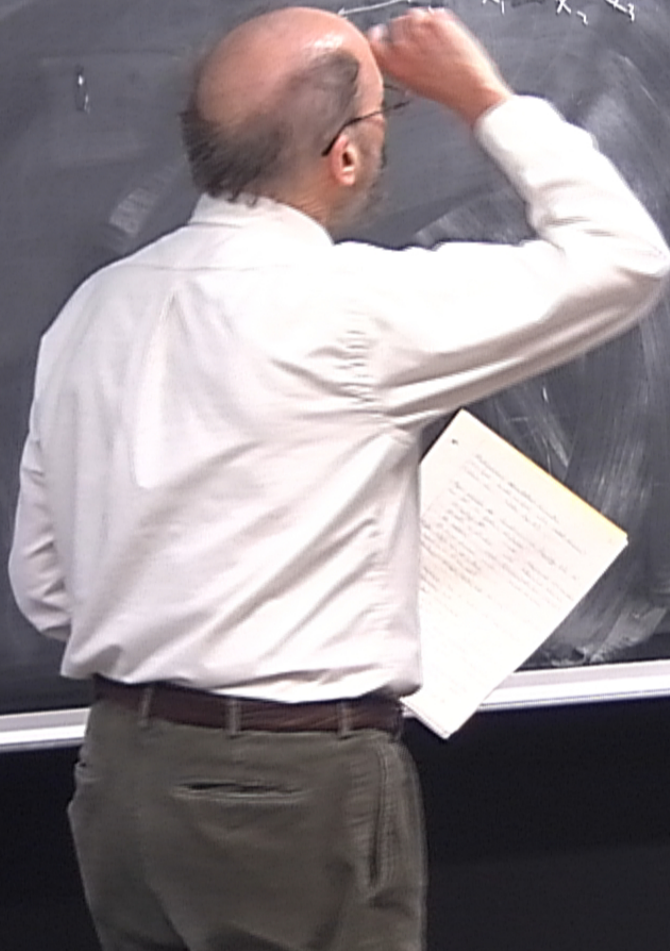
"contiguity"

neighborhood (nhbd)  $\leftrightarrow$  convergence



$x_1 \rightarrow x_2 \rightarrow \dots$

$$x_n \rightarrow y \Leftrightarrow (\forall \epsilon) (\exists N) (\forall n \geq N \Rightarrow x_n \in U)$$



**CAUTION**  
TO AVOID OR REDUCE THE RISK OF INJURY,  
PLEASE CENTER IN THE MIDDLE OF THE BOARD.  
IT IS ESSENTIAL TO APPLY  
THE WEIGHTS PROPERLY ONLY.  
JOYCE PERFORMING BOARD



"Contiguity"

neighborhood  
(nhbd)

↔ convergence



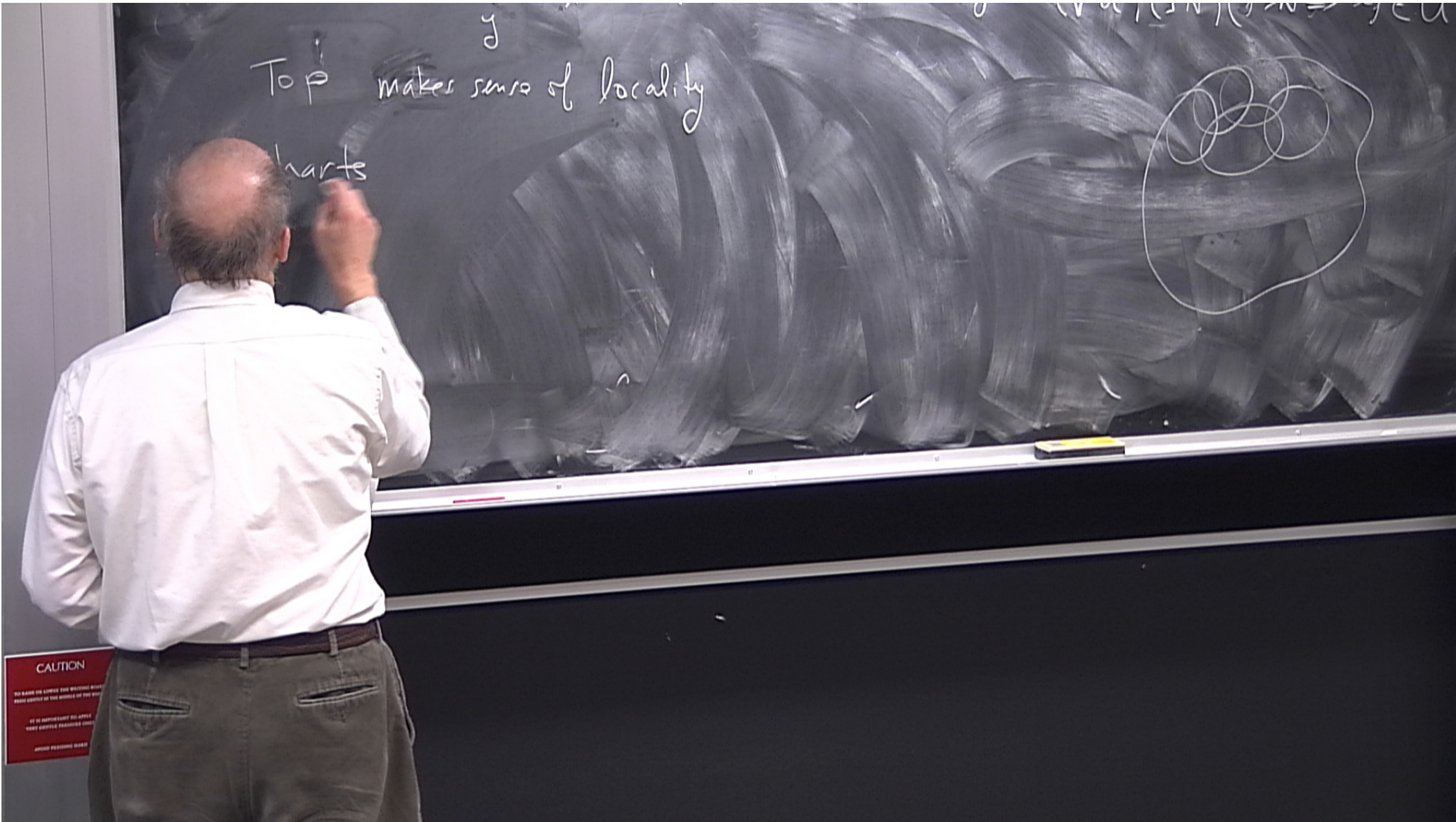
$$x_n \rightarrow y \Leftrightarrow (\forall \epsilon) (\exists N) (\forall n) (n > N \Rightarrow |x_n - y| < \epsilon)$$

Top makes sense of locality

CAUTION

TO AVOID OR LIMIT THE RISK OF INJURY,  
PLEASE CENTER ON THE MIDDLE OF THE BOARD.  
  
DO NOT OVEREXPOSE TO LIGHT  
FROM ANY SOURCE PRODUCED ONCE  
THE BOARD IS ON.







Top makes sense of locality  
Charts



**CAUTION**

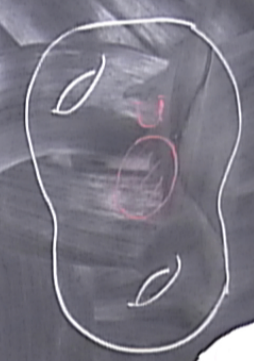
TO AVOID OR LIMIT THE RISKING BURNS,  
PLEASE CENTER ON THE MIDDLE OF THE BOARD.

DO NOT MANIPULATE THE BOARD  
WITH CHESTS PROVIDED ONLY

JOYCE PRODUCTIONS BOARD



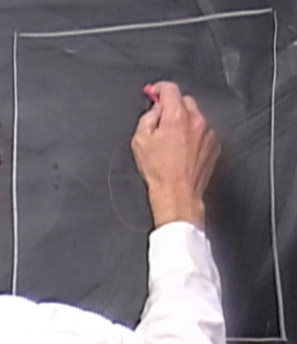
Top makes sense of locality  
Charts



CAUTION  
DO NOT STAND ON LIMBS AND WRISTING BRACKETS  
PLEASE CENTER IN THE MIDDLE OF THE BOARD  
DO NOT PARTICIPATE IN JESTS  
YOUR SAFETY IS OUR ONLY CONCERN  
ATTENTION PLEASE



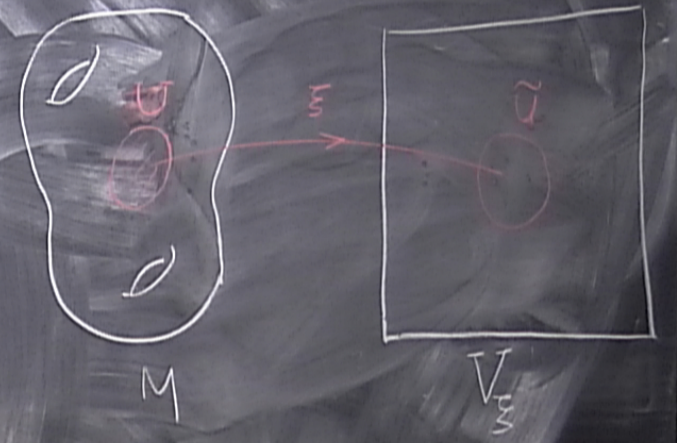
Top makes sense of locality  
Charts





Top makes sense of locality

Charts

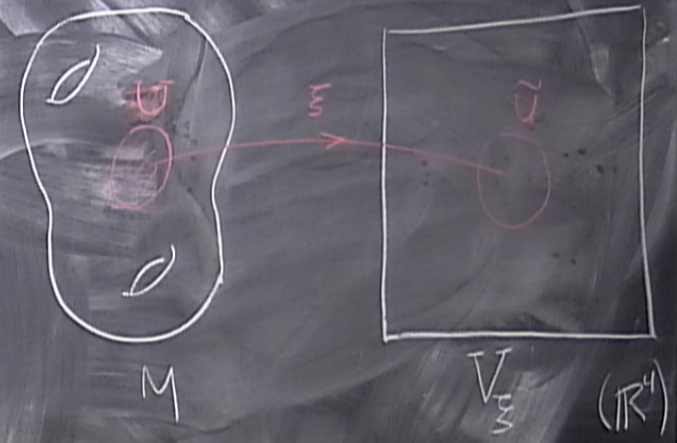




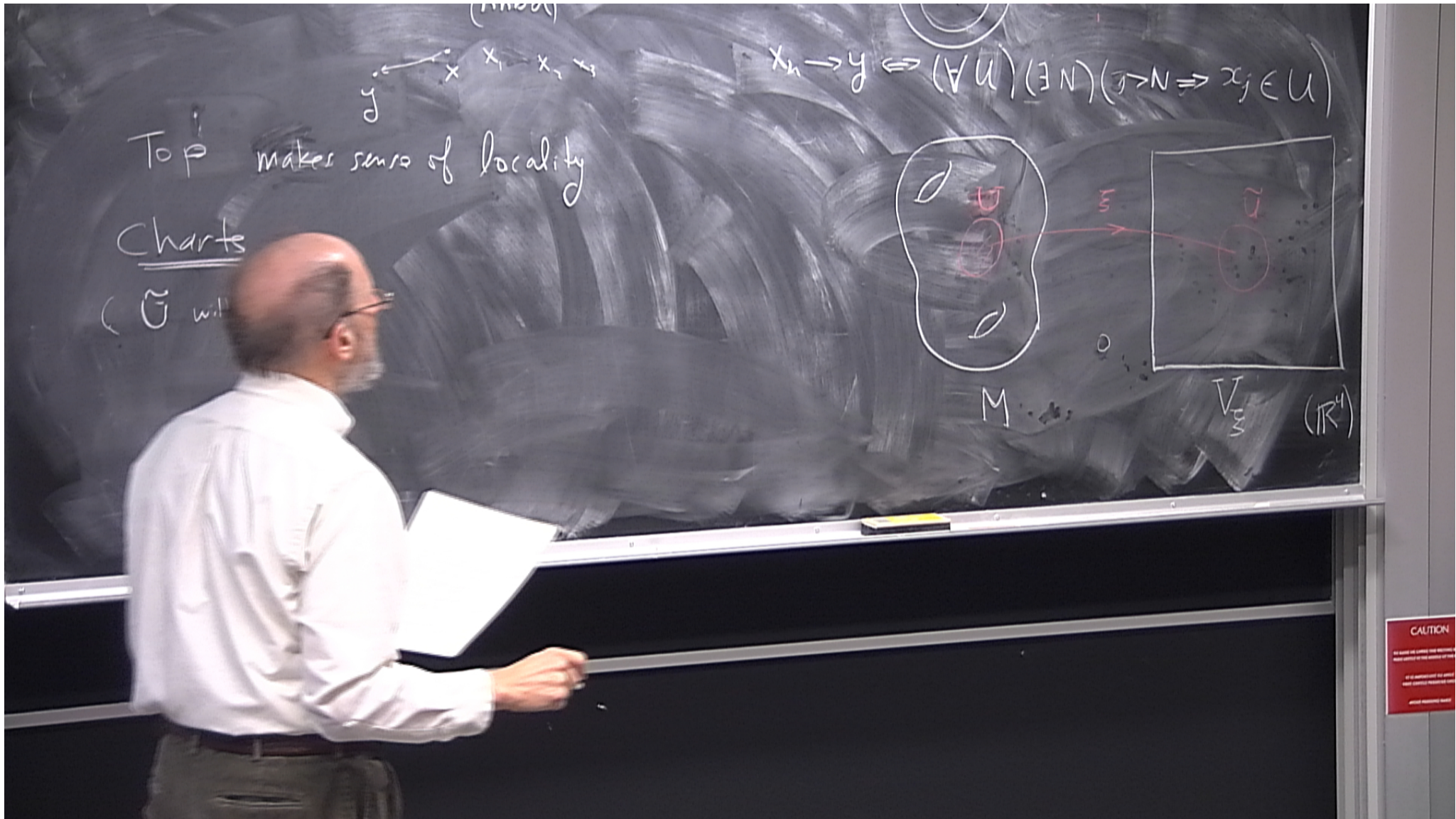
Top makes sense of locality

Charts

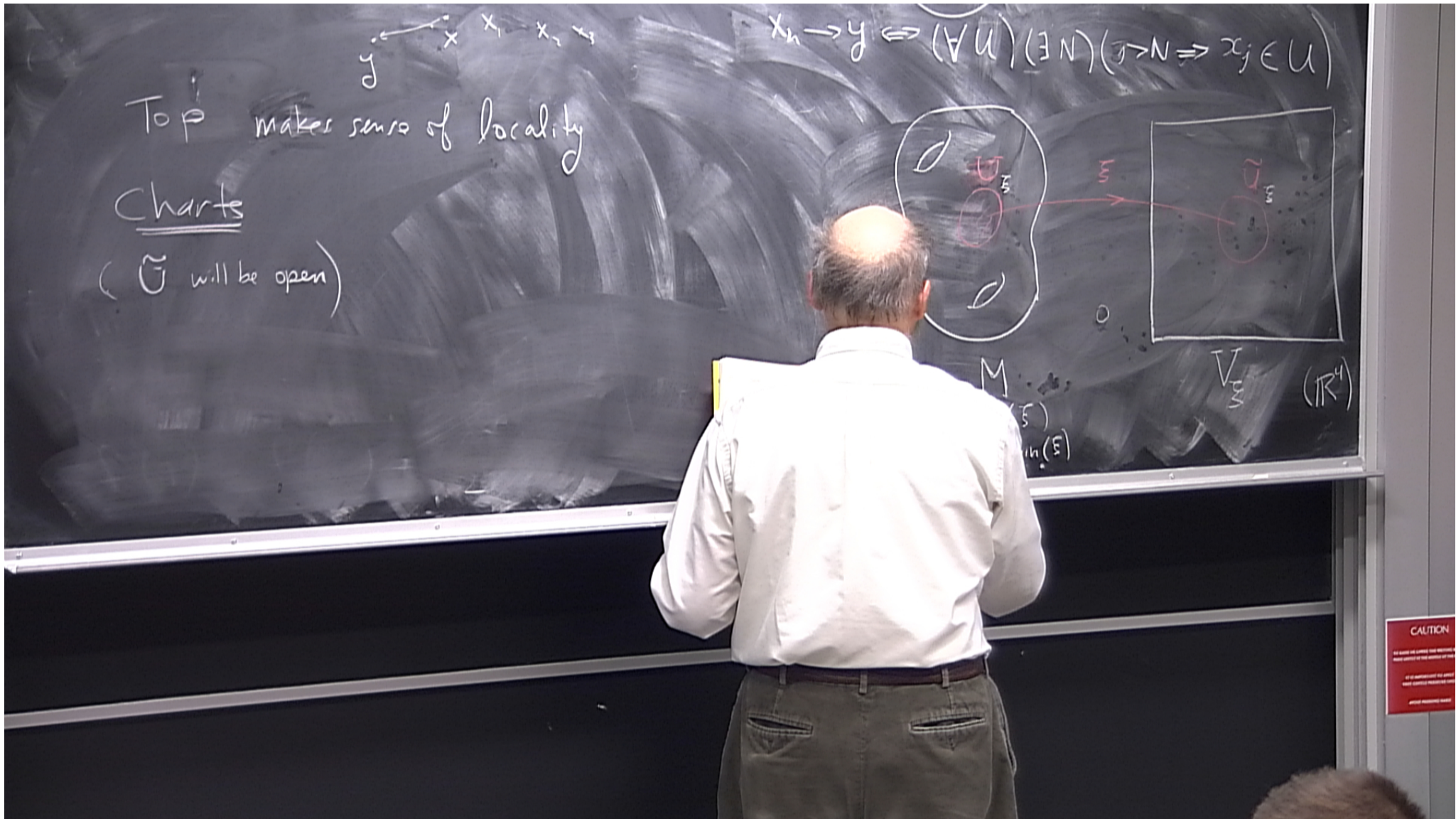
( $\tilde{U}$  will be open)











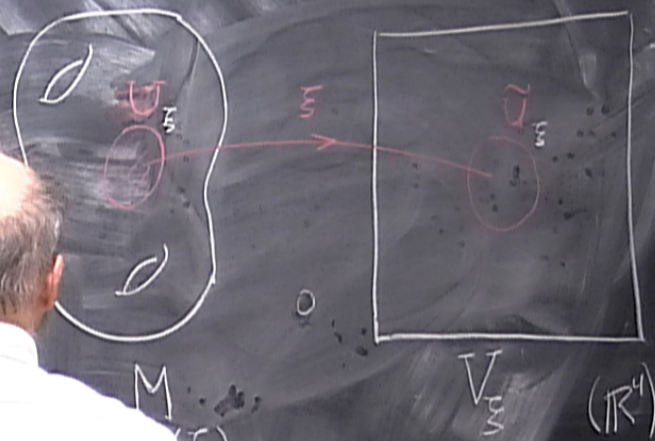
Top makes sense of locality

Charts

( $\tilde{U}$  will be open)

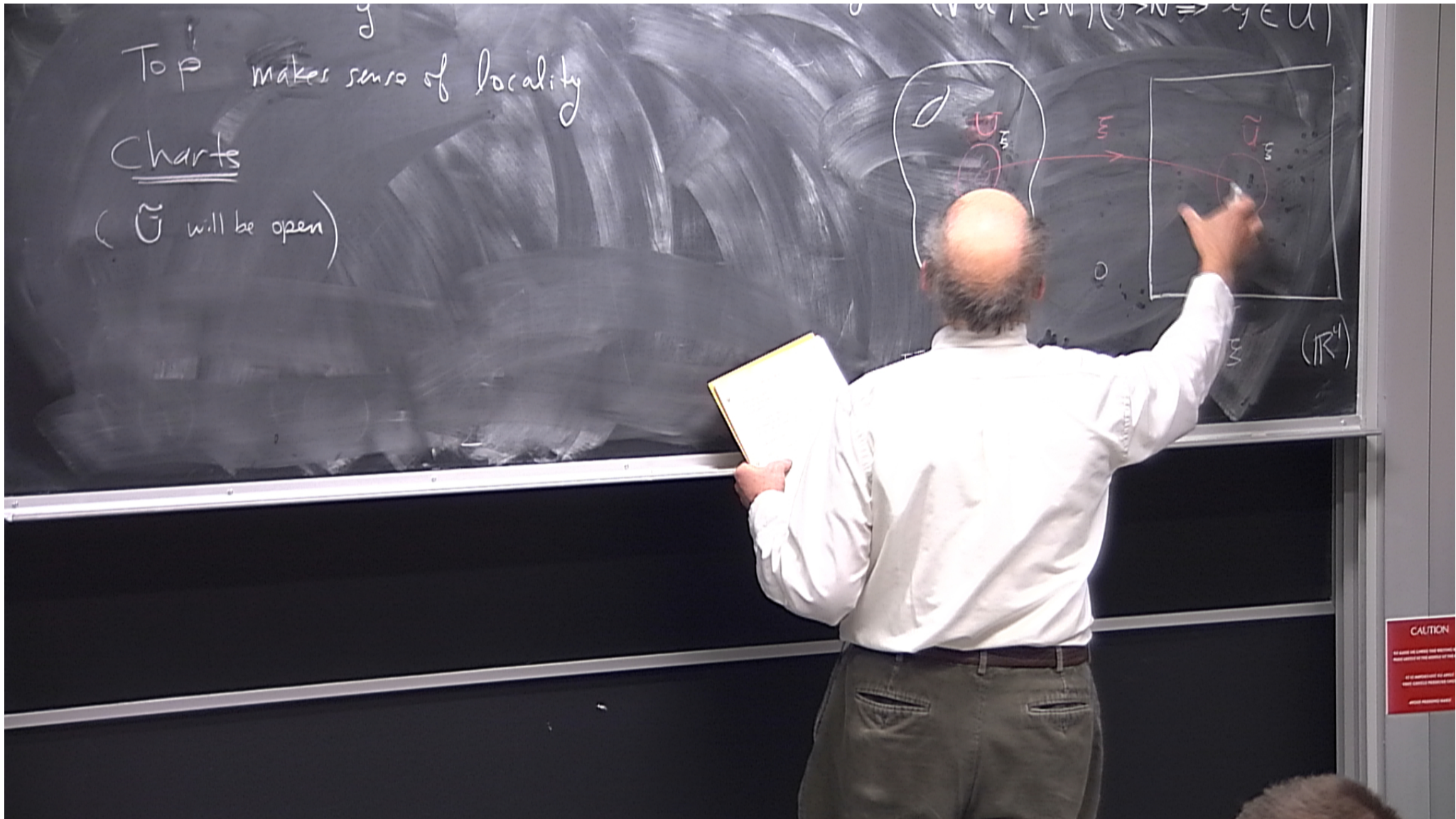


$$x_1 \rightarrow y \Leftrightarrow (\forall U) (\exists N) (\exists N') (\exists N'') x_j \in U$$



$M$   
 $(\epsilon)$   
 $U_\epsilon$   
 $V_\epsilon$   
 $(\mathbb{R}^4)$





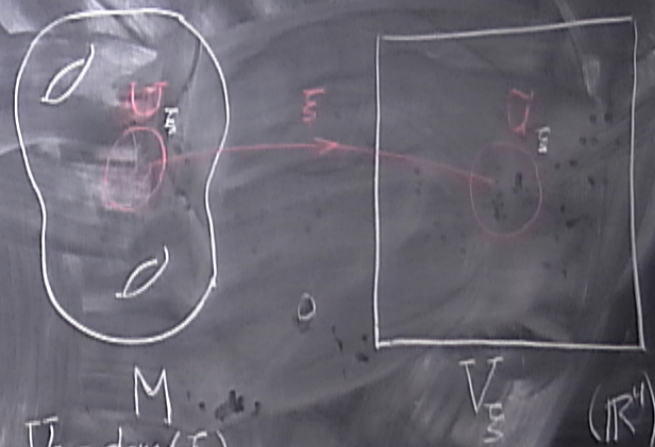


Top makes sense of locality

## Charts

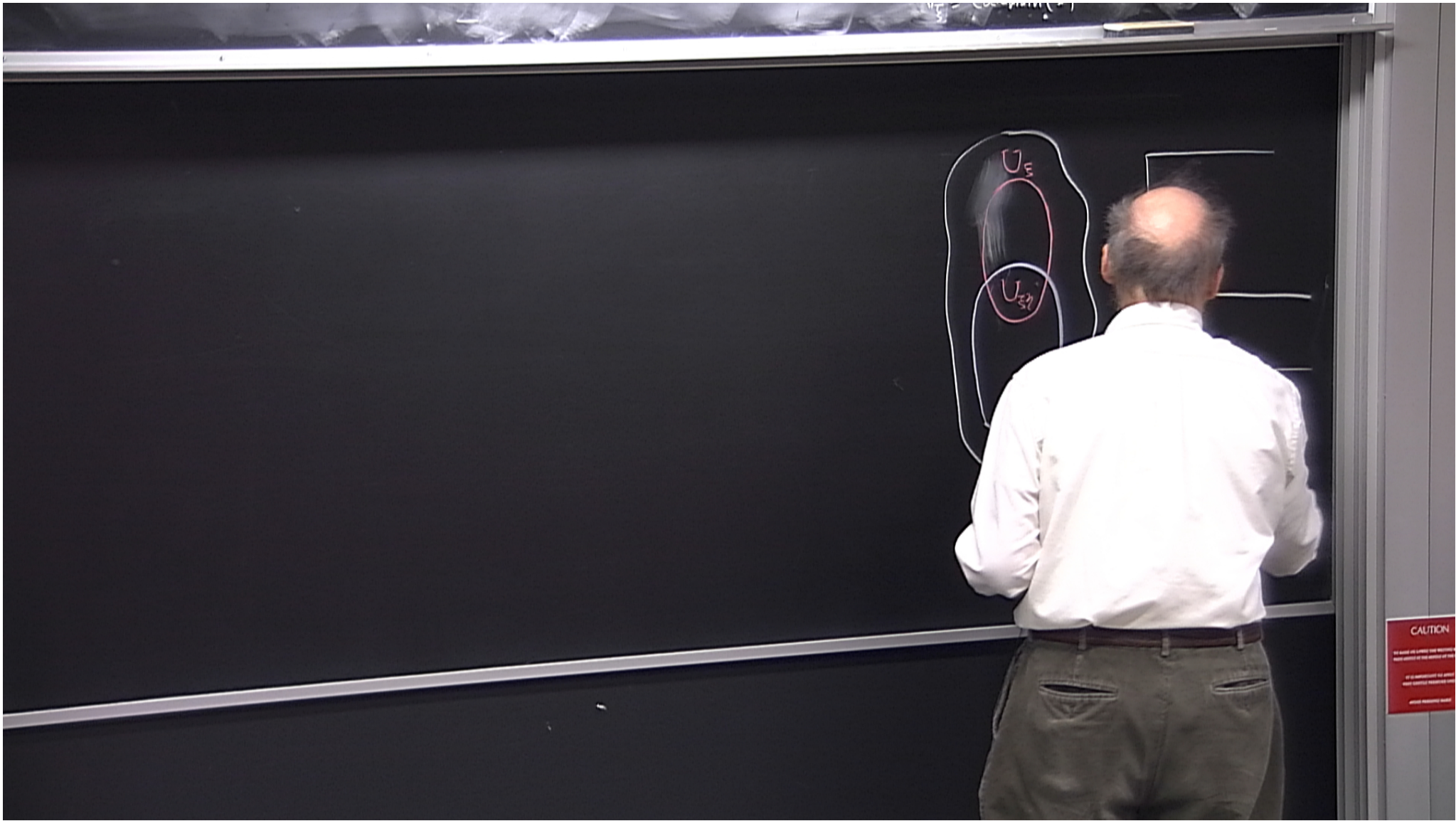
( $\bar{U}$  will be open)

Atlas =  $\{ \xi, \eta \}$

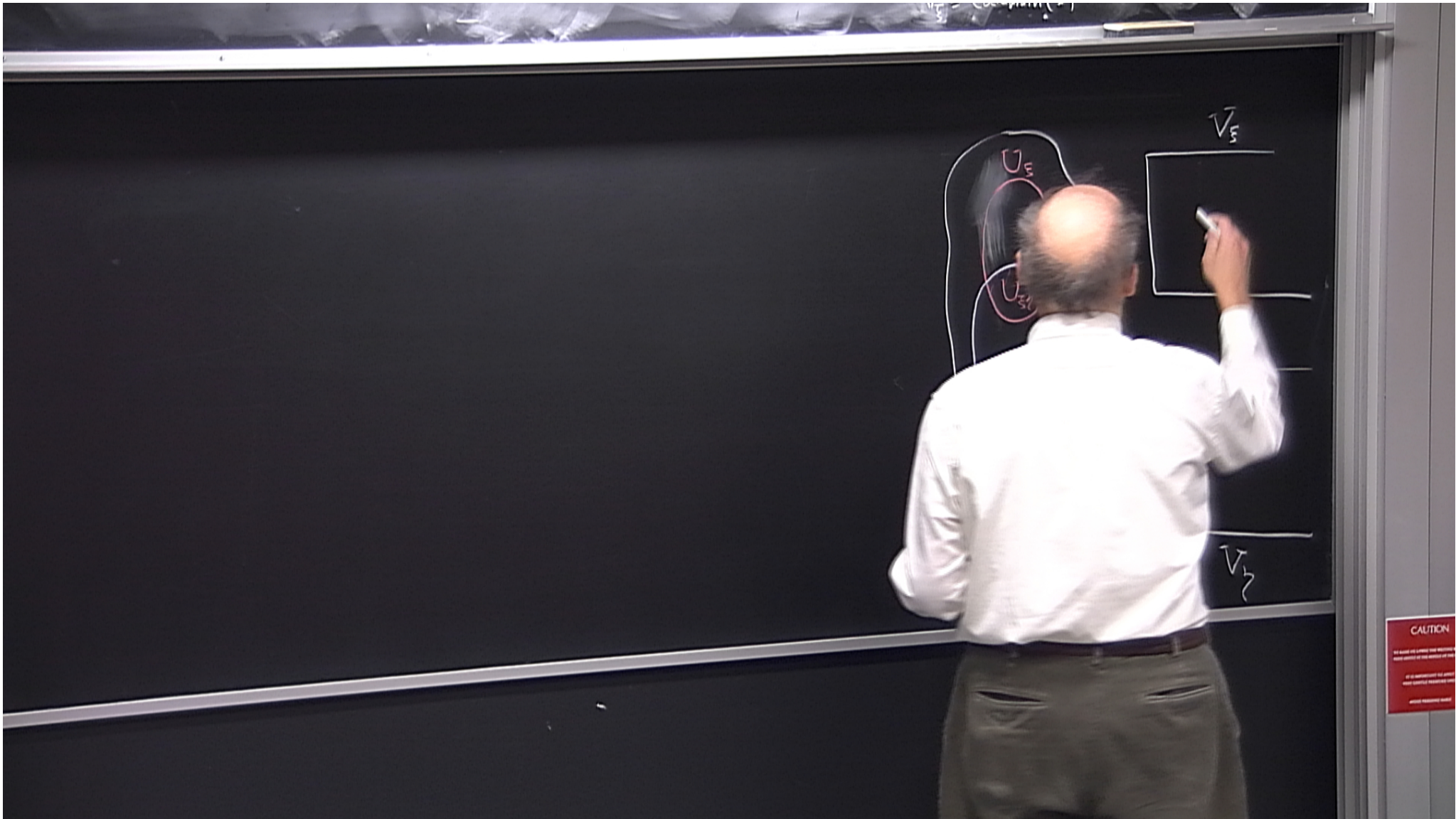


$M$   
 $U_\xi = \text{dom}(\xi)$   
 $V_\xi = \text{codomain}(\xi)$

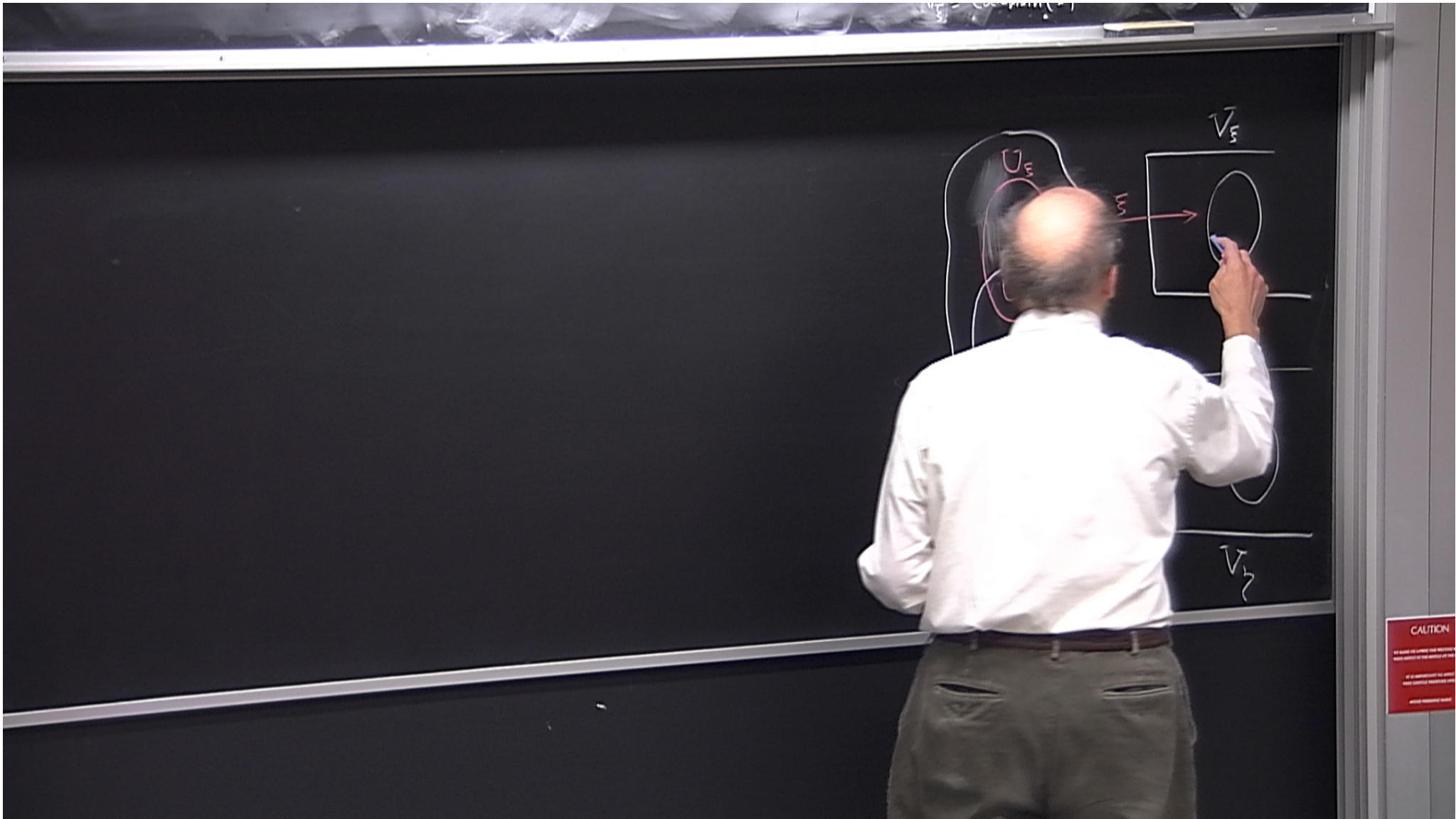








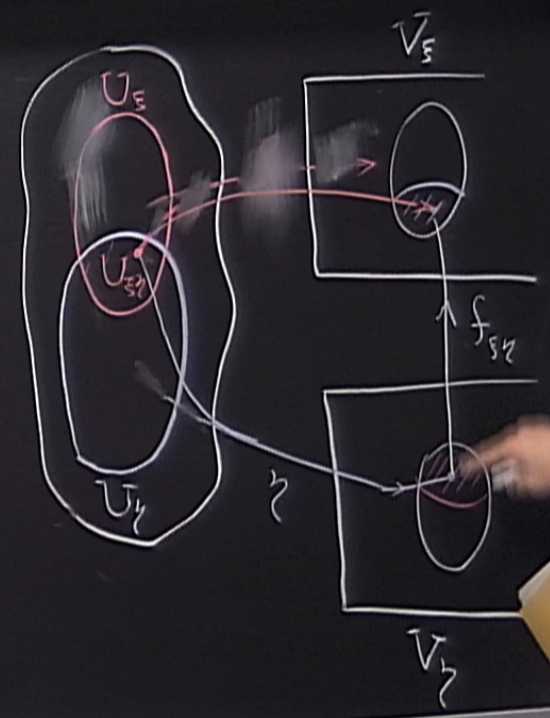






$$f_{\xi\eta} \circ \gamma = f_{\xi}$$

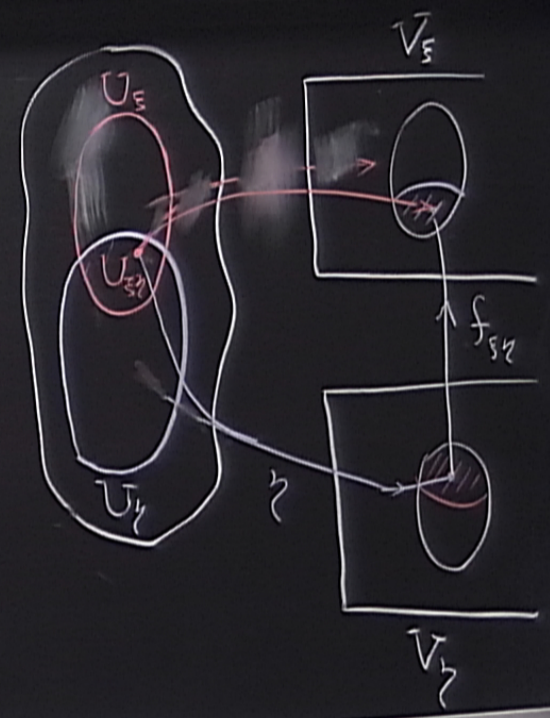
$$f_{\xi\eta} = f_{\eta\xi}^{-1}$$





$$f_{\xi\eta} \circ \gamma = \xi$$

$$f_{\xi\eta} = f_{\eta\xi}^{-1}$$

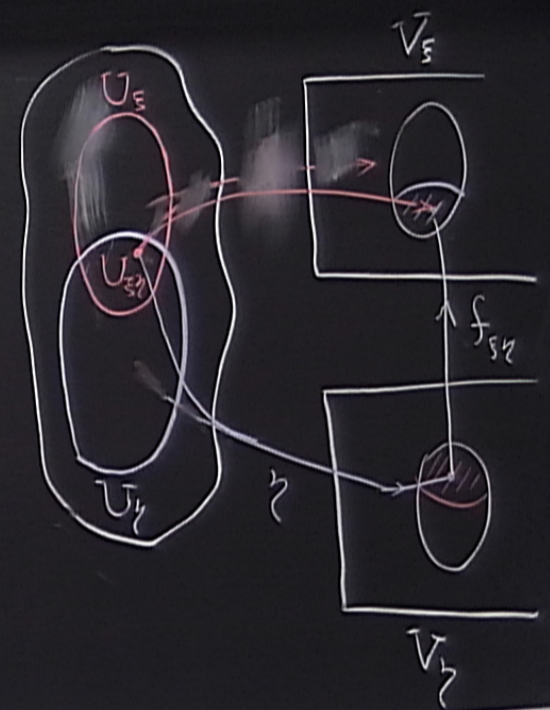




$$f_{\xi\eta} \circ \gamma = \xi$$

$$f_{\xi\eta} = f_{\eta\xi}^{-1}$$

A chart  $\xi$  for a set  $M$  is a bijection of some of  $M$  onto an open subset  $\tilde{U}_\xi$  of



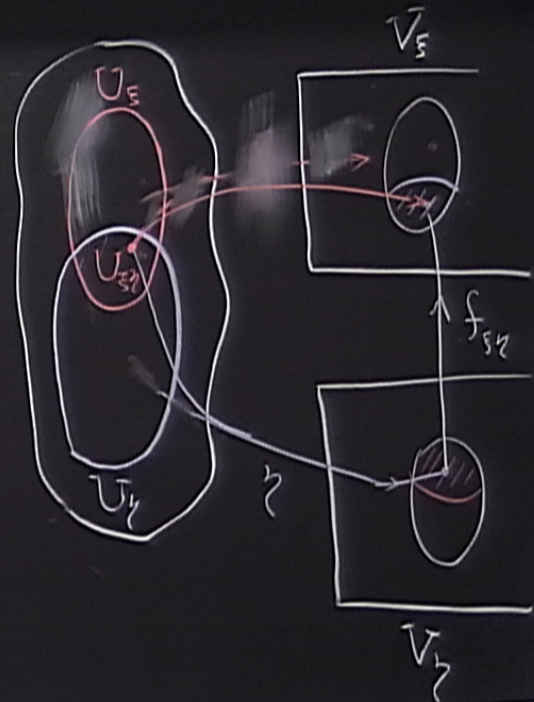


$$f_{\xi\eta} \circ \gamma = \xi$$

$$f_{\xi\eta} = f_{\eta\xi}^{-1}$$

Def A chart  $\xi$  for a set  $M$  is a bijection of some subset  $U_\xi$  of  $M$  onto an open subset  $\tilde{U}_\xi$  of an  $n$ -dimensional vector space  $\tilde{V}_\xi$ .

Def An atlas for  $M$  is a collection  $\mathcal{A}$  of charts which cover  $M$  (ie



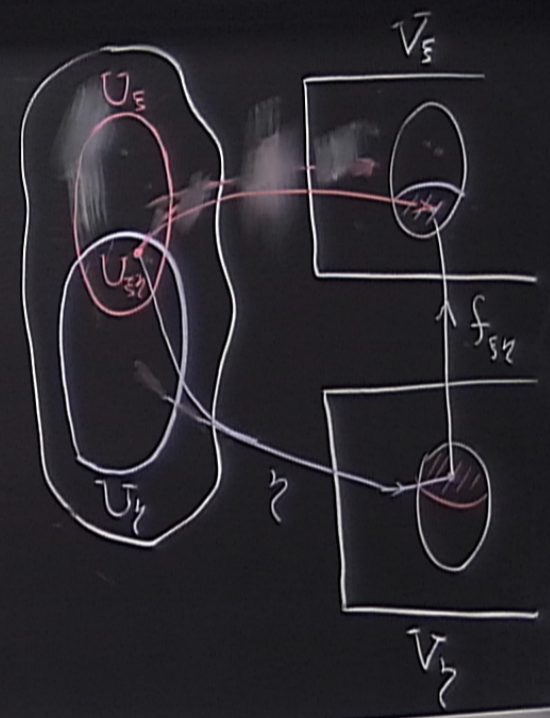


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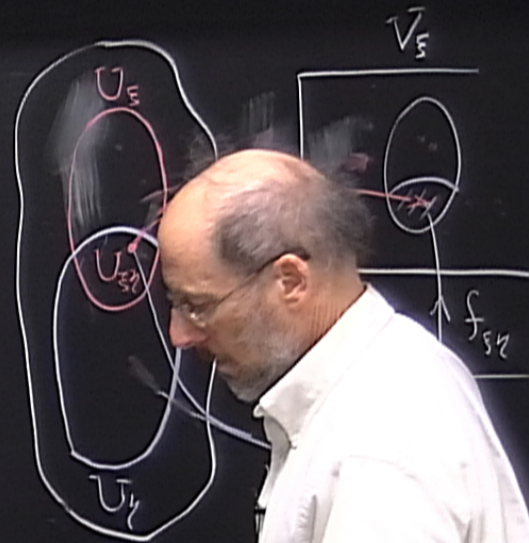


$$f_{\xi\eta} \circ \gamma = \xi$$

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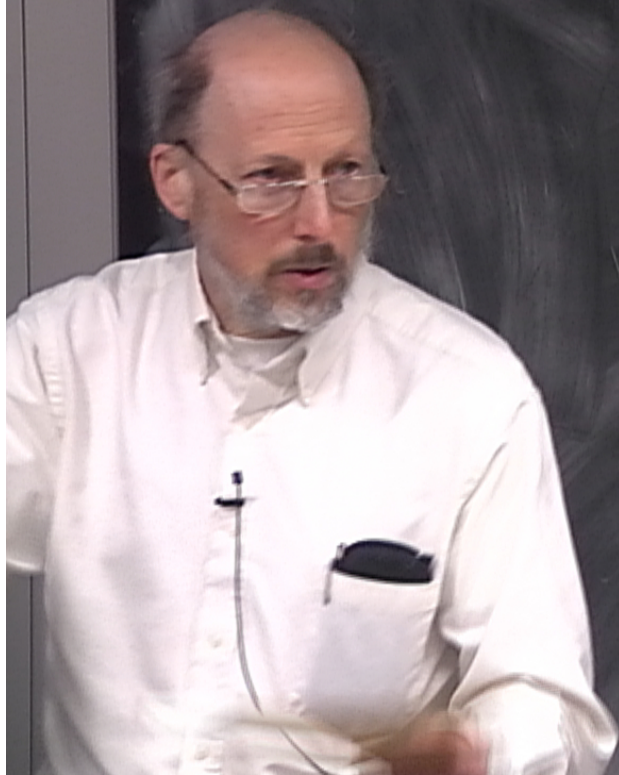
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Atlases  $\mathcal{a}, \mathcal{a}'$  are equivalent iff  $\mathcal{a} \cup \mathcal{a}'$  is an atlas for  $M$





Atlases  $\mathcal{A}, \mathcal{A}'$  are equivalent iff  $\mathcal{A} \cup \mathcal{A}'$  is an atlas for  $M$   
Then  $\mathcal{A}$  &  $\mathcal{A}'$  define the same manifold

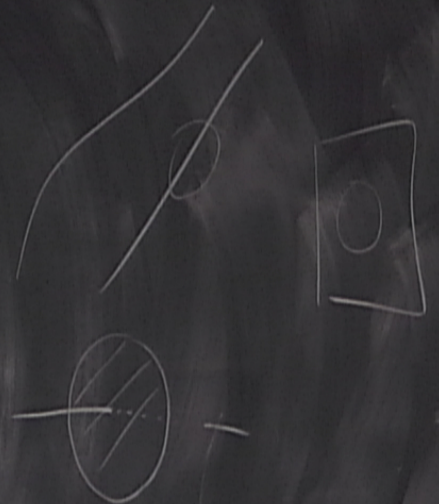
Def A manifold is a pointset  $M$  provided with an atlas  $\mathcal{A}$   
(Equivalent atlases define the same manifold)



Atlases  $\mathcal{A}, \mathcal{A}'$  are equivalent iff  $\mathcal{A} \cup \mathcal{A}'$  is an atlas for  $M$

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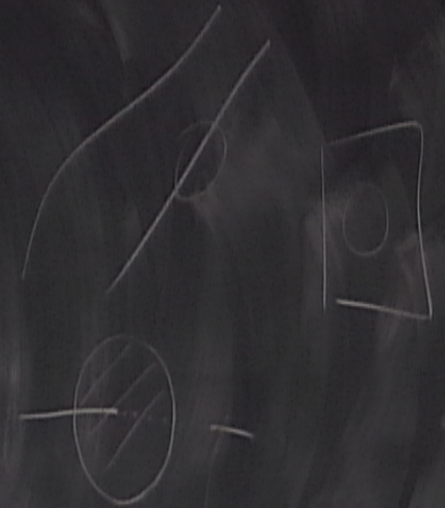


Atlases  $\mathcal{A}, \mathcal{A}'$  are equivalent iff  $\mathcal{A} \cup \mathcal{A}'$  is an atlas for  $M$

Then  $\mathcal{A}$  &  $\mathcal{A}'$  define the same manifold

Def A manifold is a pointset  $M$  provided with an atlas  $\mathcal{A}$

(Figure 11.1 defines the same manifold)





Atlases  $\mathcal{A}, \mathcal{A}'$  are equivalent iff  $\mathcal{A} \cup \mathcal{A}'$  is an atlas for  $M$

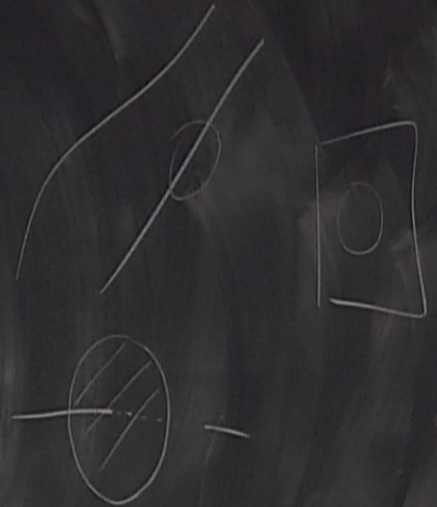
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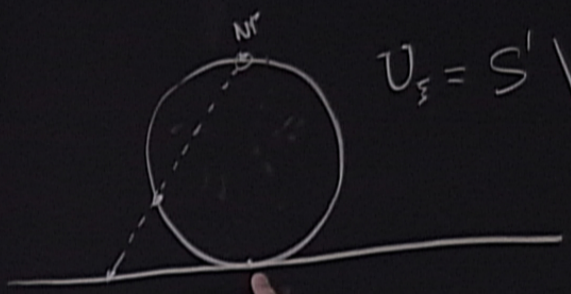
If  $\dim \mathcal{U}_3$  is  $n$  Then  $M$  is an  $n$ -manifold ( $n=4$ )

(Assume  $M$  is connected)





giving partitioning

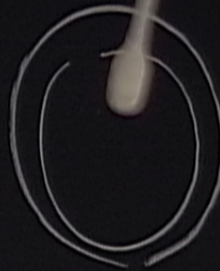
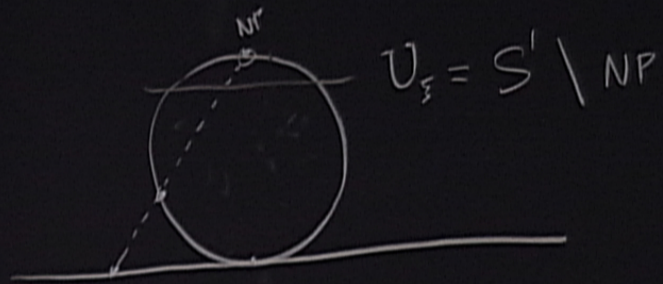


$$U_{\mathbb{F}} = S' \setminus NP$$

Atlases  $\mathcal{A}, \mathcal{A}'$  are  $\mathbb{R}^n$   
Then  $\mathcal{A}$  &  $\mathcal{A}'$  define the same  
pointset  $M$

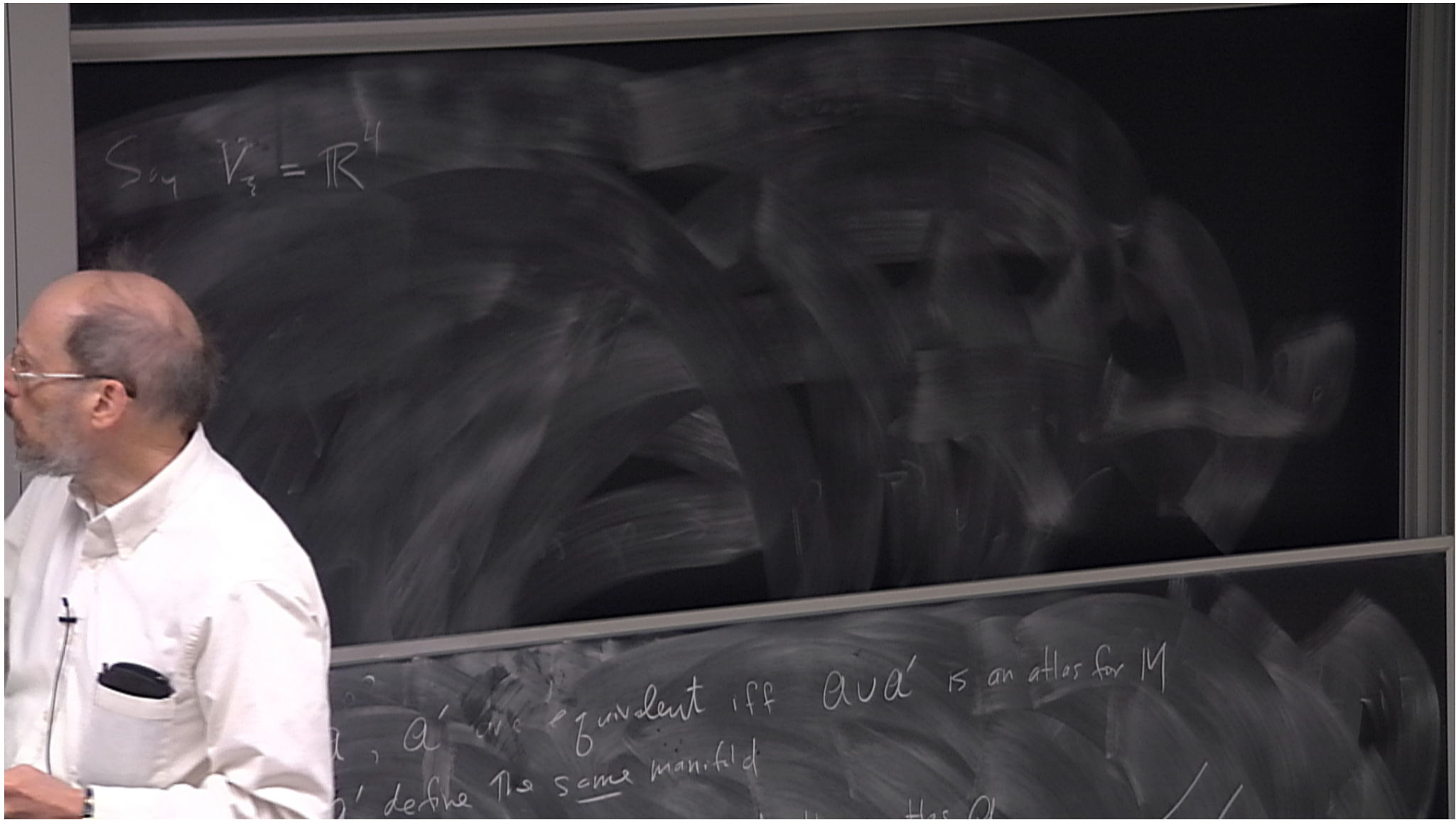
$\mathcal{A}'$  is an atlas for  $M$





Atlases  $\mathcal{A}, \mathcal{A}'$  are equivalent  
 Then  $\mathcal{A}$  &  $\mathcal{A}'$  define the same manifold  
 atlas for  $M$





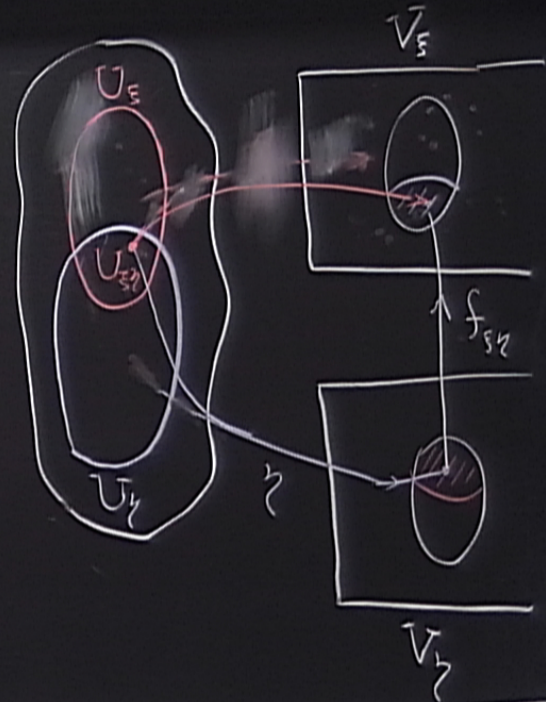


$$f_{\xi\eta} \circ \gamma = \xi$$

$$f_{\xi\eta} = f_{\eta\xi}^{-1}$$

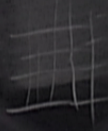
Def A chart  $\xi$  for a set  $M$  is a bijection of some subset  $U_\xi$  of  $M$  onto an open subset  $\tilde{U}_\xi$  of an  $n$ -dimensional vector space  $\tilde{V}_\xi$ .

Def An atlas for  $M$  is a collection  $\mathcal{A}$  of charts which cover  $M$  (ie  $\bigcup \{U_\xi \mid \xi \in \mathcal{A}\} = M$ ) and st. the transition maps  $f_{\xi\eta}$  are all continuous.





Say  $V_x = \mathbb{R}^4$   $(x^0, x^1, x^2, x^3)$




$P \in M$

$\xi(P) \in \mathbb{R}^4$

$\xi(P) = (x^0(P), x^1(P), \dots)$

Atlases  $\mathcal{a}, \mathcal{a}'$  are equivalent iff  
 $\mathcal{a}'$  define the same manifold



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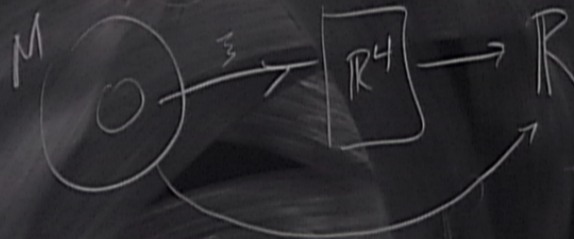


Chart on  $\mathbb{R}^4$  ( $\rightarrow V_z = \mathbb{R}^4$ )

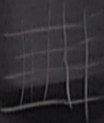
is equiv to 4 coord fns

$x^0, \dots, x^3$  on  $M$

$x^i: M \rightarrow \mathbb{R}$

Atlases  $a, a'$  are equivalent iff  $a \circ a'$  is an atlas  
 $a'$  define the same manifold



Say  $V_3 = \mathbb{R}^4$   $(x^0, x^1, x^2, x^3)$  

$P \in M$

$\xi(P) \in \mathbb{R}^4$

$\xi(P) = (x^0(P), x^1(P), \dots)$

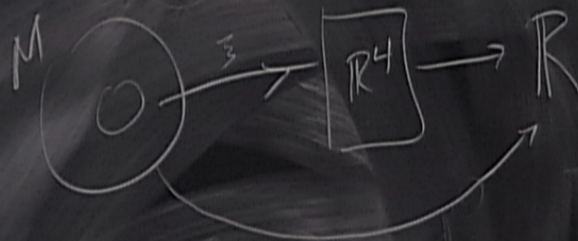


Chart on  $\mathbb{R}^4$  (st  $V_3 = \mathbb{R}^4$ )

is equiv to 4 coord fns

$x^0, \dots, x^3$  on  $M$

$x^i: M \rightarrow \mathbb{R}$

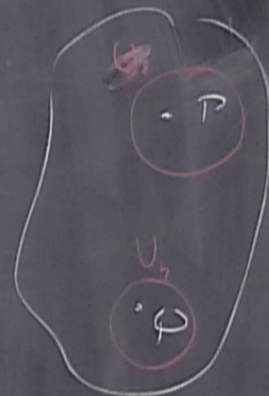
local coord system

Atlases  $\mathcal{a}, \mathcal{a}'$  are equivalent iff  $\mathcal{a} \cup \mathcal{a}'$  is an atlas for  $M$   
 $\mathcal{a}'$  define the same manifold



$\exists$  countable atlas for  $M$  (paracompact)

$\forall P \neq Q$  in  $M \exists$  atlas with  $P$  and  $Q$  in disjoint charts





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Summary  $M$  is a paracompact Hausdorff  $C^\infty$ -manifold  
(without boundary) of dim 4 (usually)

