Title: Inference of effective quantum models in the asymptotic regime

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Abstract: In this talk I will sketch a project which aims at the design of systematic and efficient procedures to infer quantum models from measured data. Progress in experimental control have enabled an increasingly fine tuned probing of the quantum nature of matter, e.g., in superconducting qubits. Such experiments have shown that we not always have a good understanding of how to model the experimentally performed measurements via POVMs. It turns out that the ad hoc postulation of POVMs can lead to inconsistencies. For example, when doing asymptotic state tomography via linear inversion, one sometimes recovers density operators which are significantly not positive semidefinite. Assuming the asymptotic regime, we suggest an alternative procedure where we do not make a priori assumptions on the quantum model, i.e., on the Hilbert space dimension, the prepared states or the measured POVMs. In other words, we simultaneously estimate the dimension of the underlying Hilbert space, the quantum states and the POVMs. We are guided by Occam's razor, i.e., we search for the minimal quantum model consistent with the data.

Inference of effective quantum models in the asymptotic regime

Cyril Stark

(based on arXiv:1209.5737, arXiv:1209.6499, arXiv:1210.1105)

December, 2012

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National Centre of Competence in Research
ETH
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   Hilbert space, states, measurements **unknown**

   Collect measurement data

   Find a quantum model (→ this talk)

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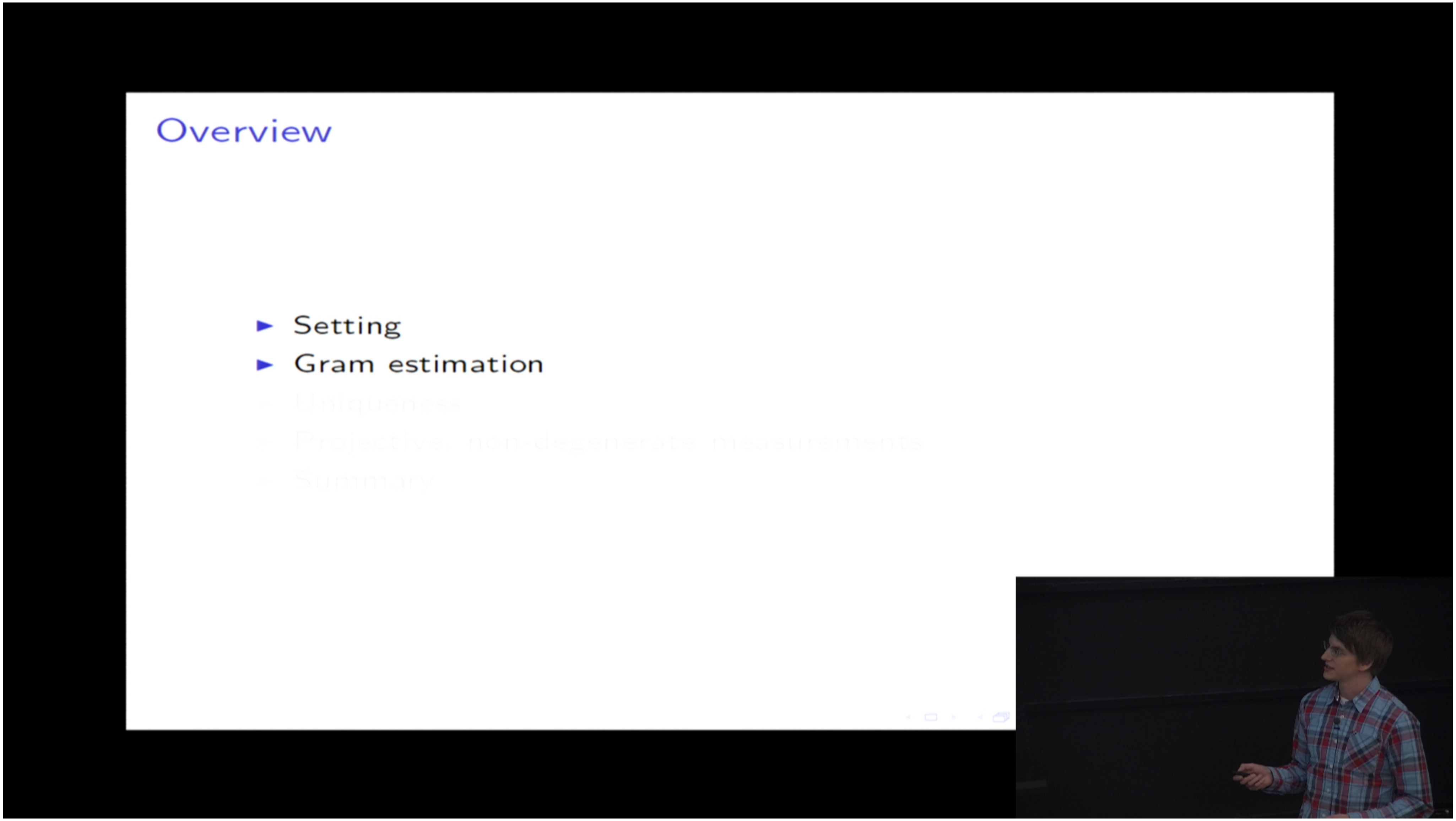
Overview
Setting
Gram estimation
Uniqueness
Projective, non-degenerate measurements
Summary

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- Setting
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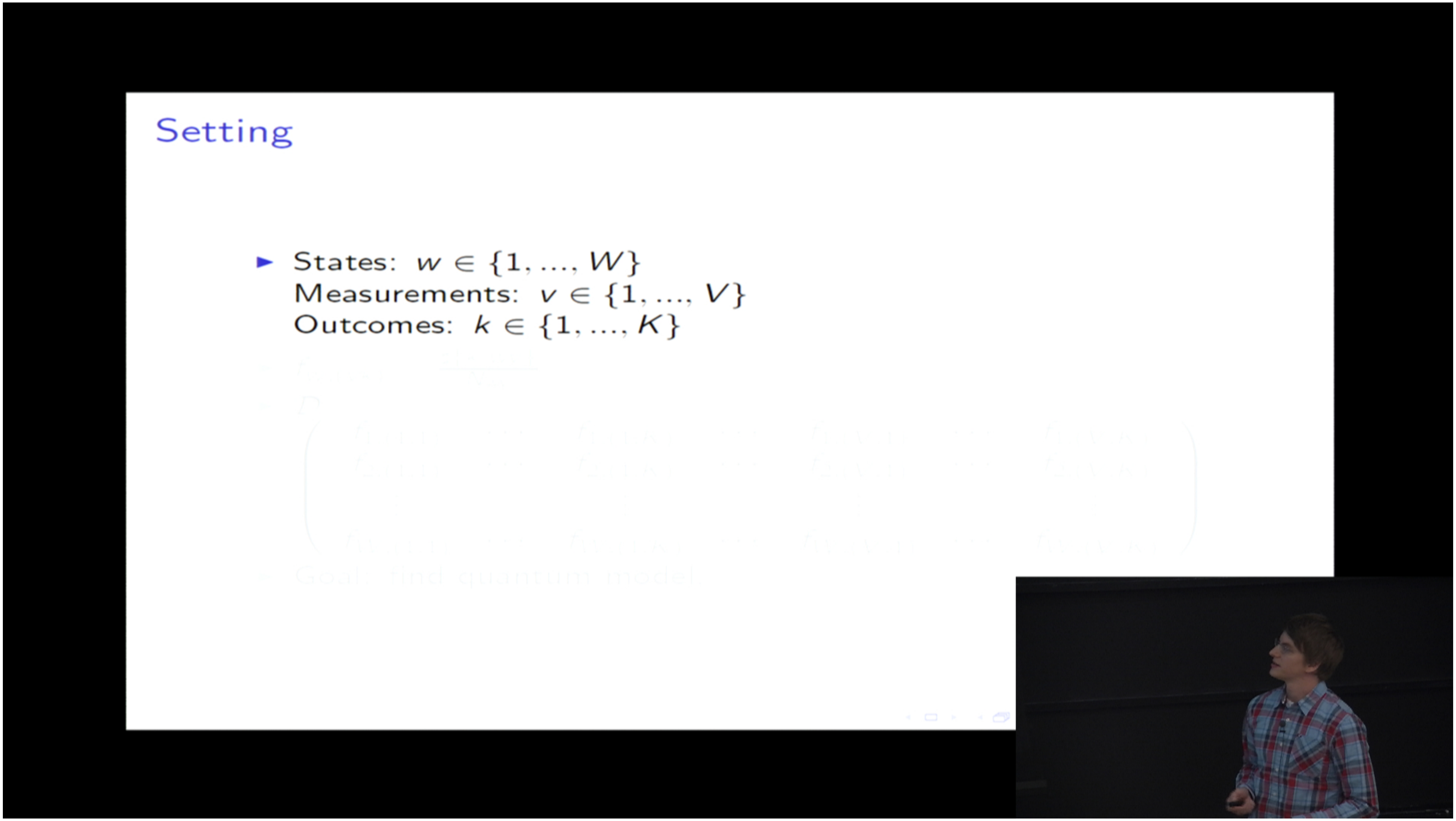
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Some common approaches to find quantum models:[1]

- State tomography[2]
  (assumptions about $\dim(\mathcal{H})$, measurements)
- Falsification of target models, fidelity to target models[3]
  (relation to target model, no explicit expressions)
- Dimension witnessing[4]
  (lower bounds on $\dim(\mathcal{H})$)
- Confidence region estimators for state tomography[5]
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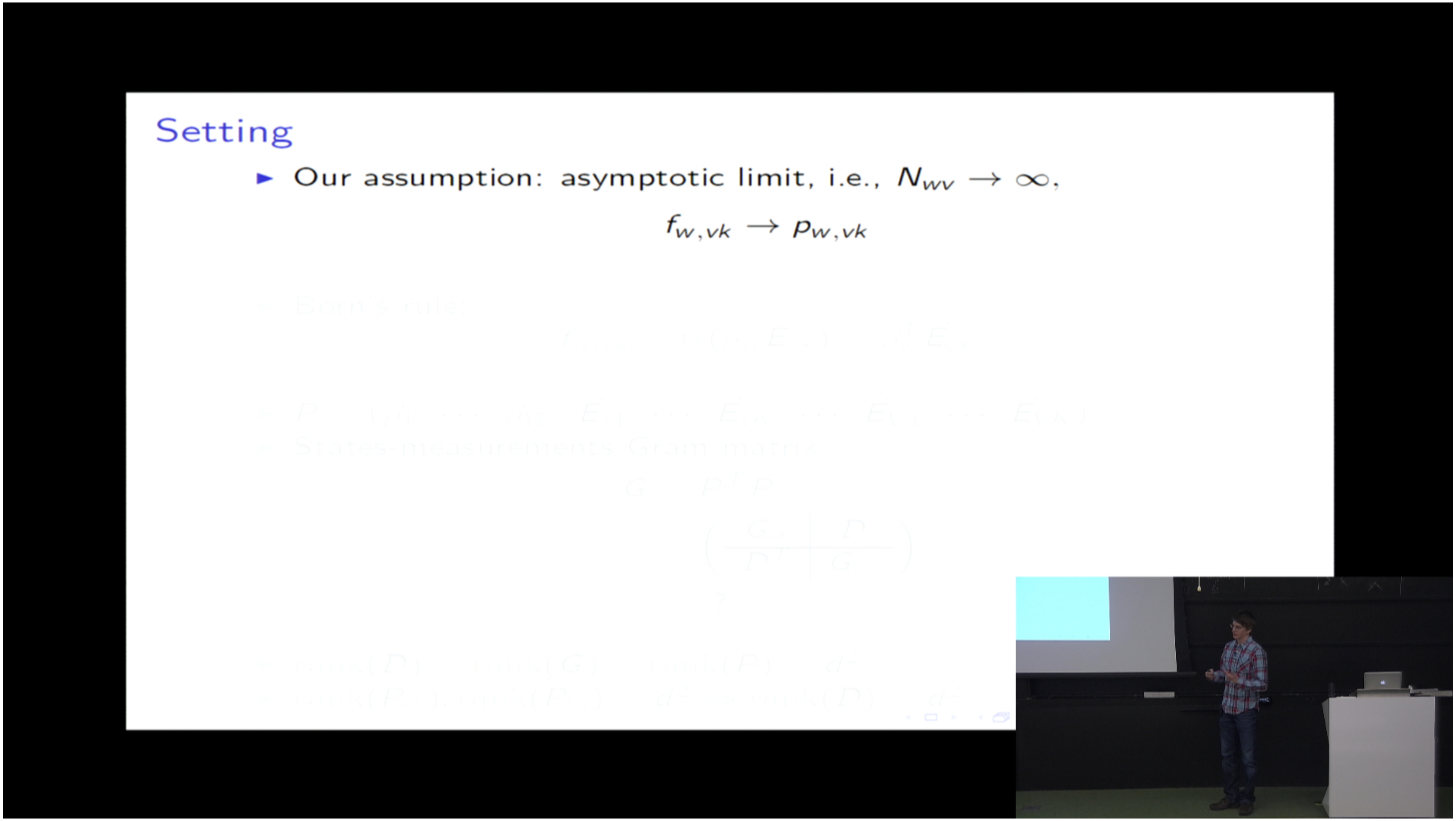
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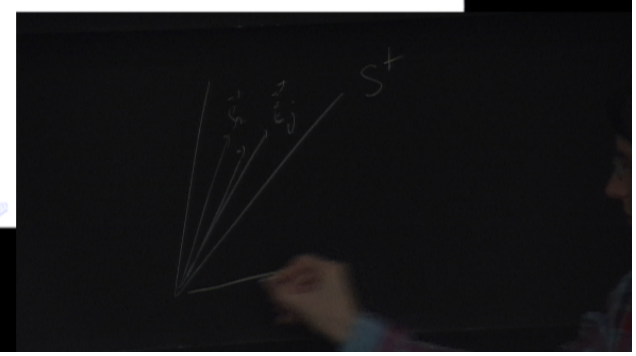

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Gram estimation

- Our goal: data-compatible quantum model.
- Gigantic family of data-compatible quantum descriptions.
- Occam's razor:

$$\begin{aligned} &\text{argmin} && \operatorname{rank} G \\ &\text{subject to} && G \in \{\text{quantum Gram matrices}\}, \\ & && G = \begin{pmatrix} * & \mathcal{D} \\ \mathcal{D}^T & * \end{pmatrix} \end{aligned}$$

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- NP-hard[7]
- Convex relaxtion: hard problem $\mapsto$ similar convex problem
- feasible set $\mapsto$ convex feasible set
  objective function $\mapsto$ convex objective function

[7] [Recht, Fazel, Parillo, 2010]

Feasible set $\mapsto$ convex feasible set

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- Rough replacement:

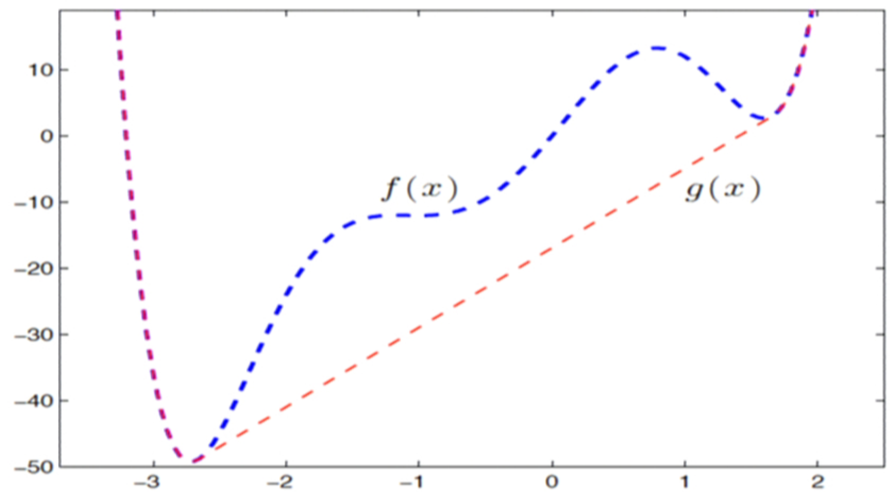
$$S^+ \supset \{\text{quantum Gram matrices}\} \mapsto S^+ \cap B_{\|\cdot\| \le M_{QM}(d_{\max})},$$

where

$$M_{QM}(d_{\max}) := \sup\{\|G\| : G \text{ quantum Gramian, } \dim(\mathcal{H}) \le d_{\max}\}$$

Objective function $\mapsto$ convex objective function

- $\mathrm{rank}(\cdot) \mapsto$ Convex envelope of $\mathrm{rank}(\cdot)$



Convex relaxation

- Relaxed Gram estimation:[8]

$$
\begin{array}{ll}
\text{argmin} & \operatorname{tr} G \\
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- Semidefinite program (SDP)

[8] See also [Fazel, Hindi, Boyd, 2001] (relaxation wrt. $B_{\|\cdot\| \leq M_{\mathrm{QM}}(d_{\max})}$ instead of $S^+ \cap B_{\|\cdot\| \leq M_{\mathrm{QM}}(d_{\max})}$).

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because $\emptyset \neq$ {correct feasible} $\subseteq$ {relaxed feasible}.

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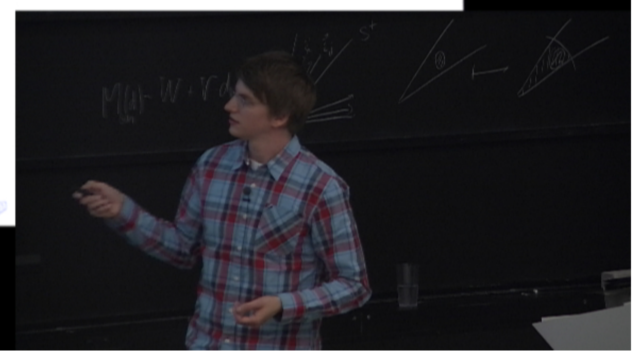

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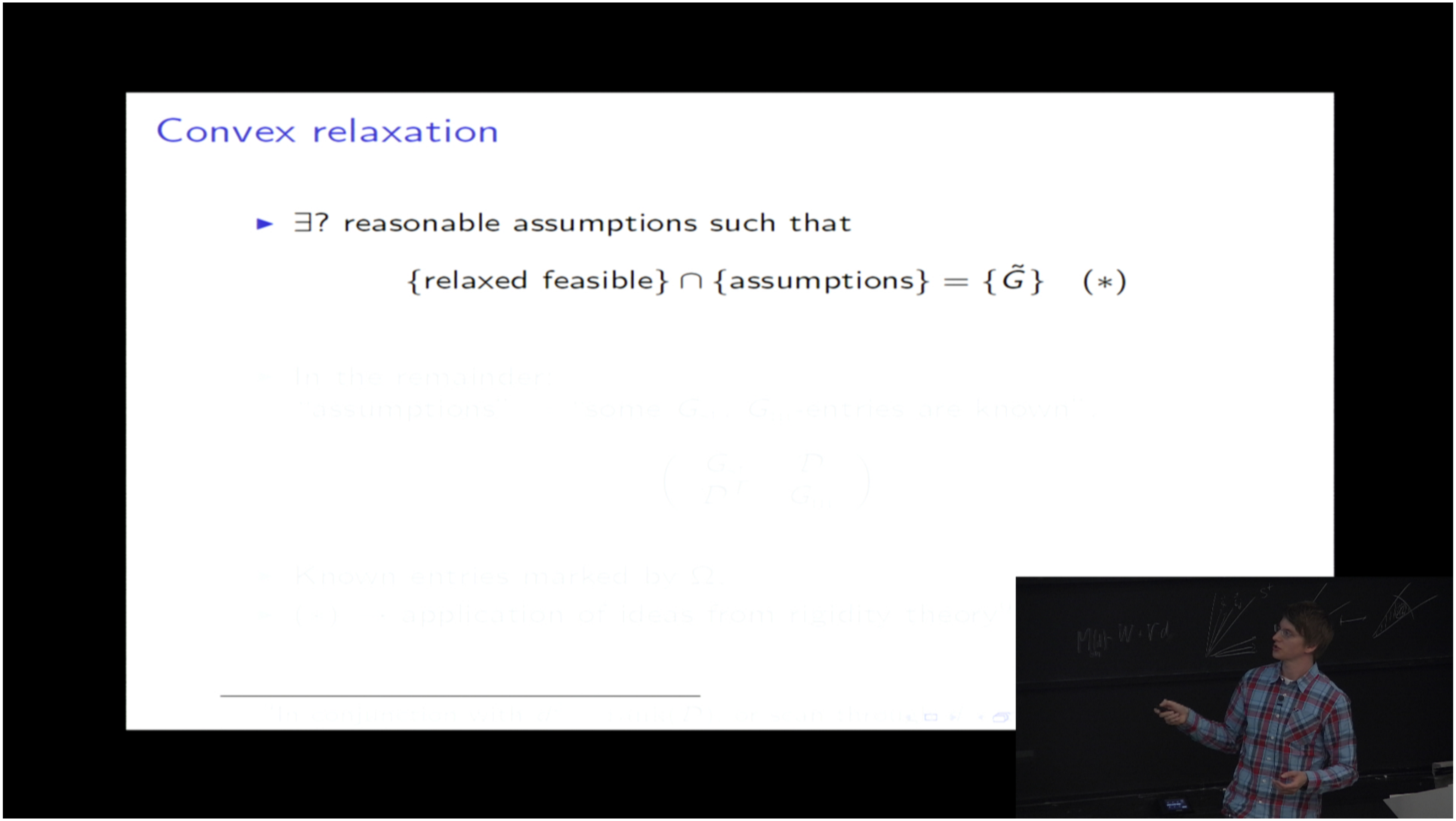
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- (∗) ↔ application of ideas from rigidity theory

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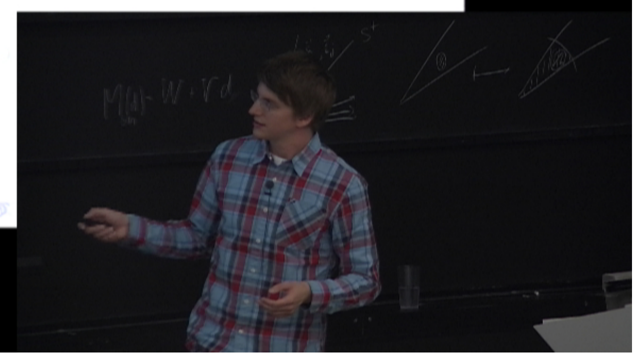

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- $(*) \rightarrow$ application of ideas from rigidity theory[9]

[9] In conjunction with $d^2 = \mathrm{rank}(\mathcal{D})$, or scan through $d \leq d_{\max}$.

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- Q: Does G_Ω uniquely determine G?
- A: G unique[10] $\Leftrightarrow$

$$\mathrm{rank}\left(\vec{\nabla}_P G_{k_1,l_1}(P), \ldots, \vec{\nabla}_P G_{k_{|\Omega|},l_{|\Omega|}}(P)\right) = d^2 n - \frac{1}{2} d^2 (d^2 - 1)$$

[10] Up to discrete symmetries.

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Phase diagrams

- Goal: answer uniqueness-question before knowing P
- Possible for generic P.[11]
- Universality $\longrightarrow$ phase diagrams valid for generic states-measurement configurations P

[11] See also [Cucuringu, Singer, 2010] (no proof)

Phase diagrams, examples (2D)

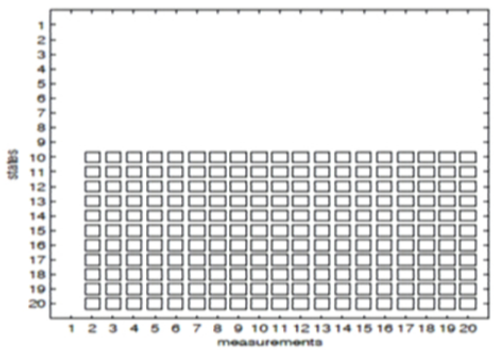


Figure: Pure states.

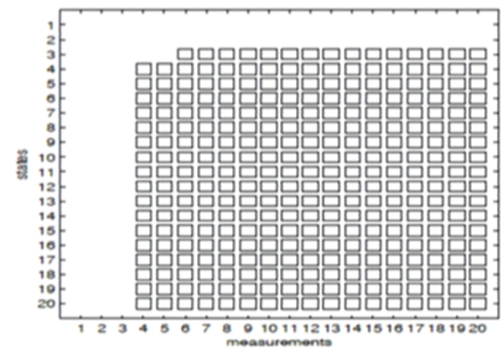


Figure: Projective measurements.

Phase diagrams, examples (4D)

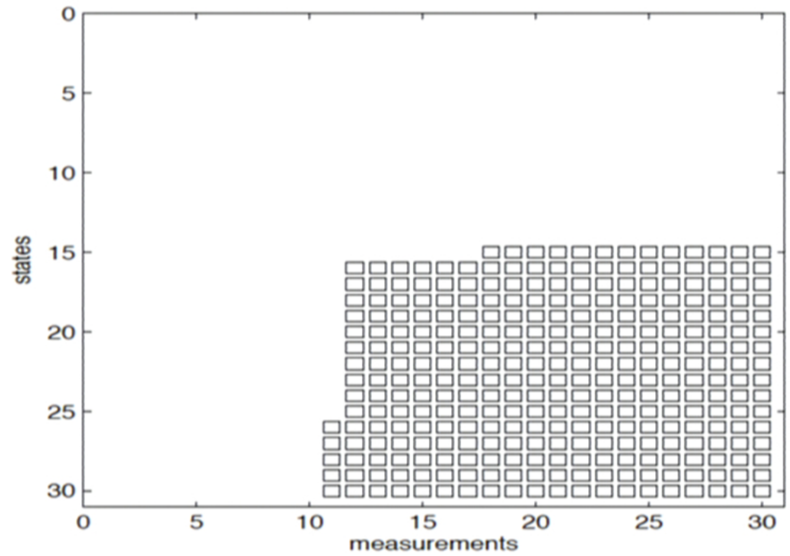
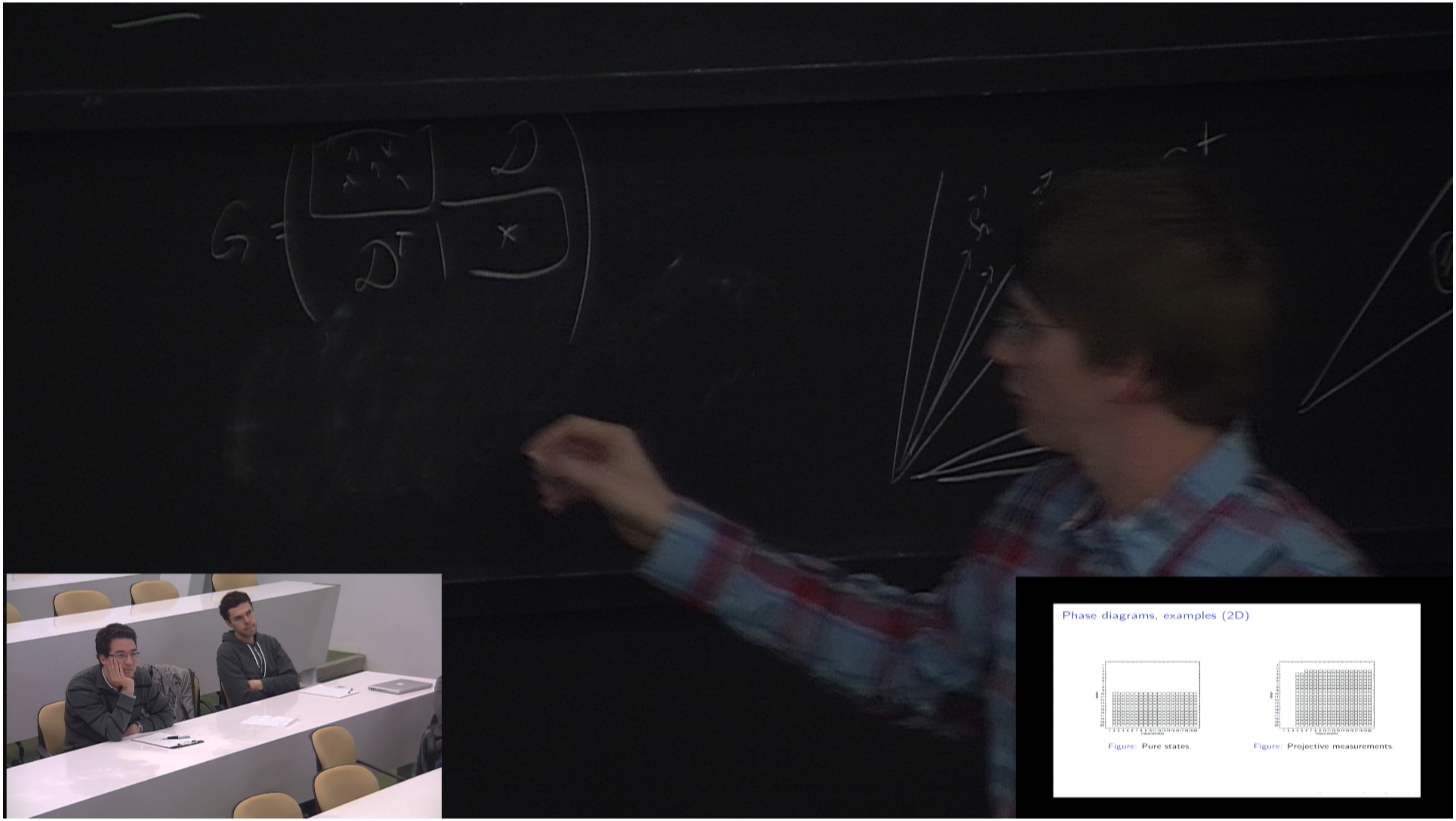


Figure: Pure states and projective, non-degenerate measurements.


Phase diagrams, examples (2D)
Figure: Pure states.
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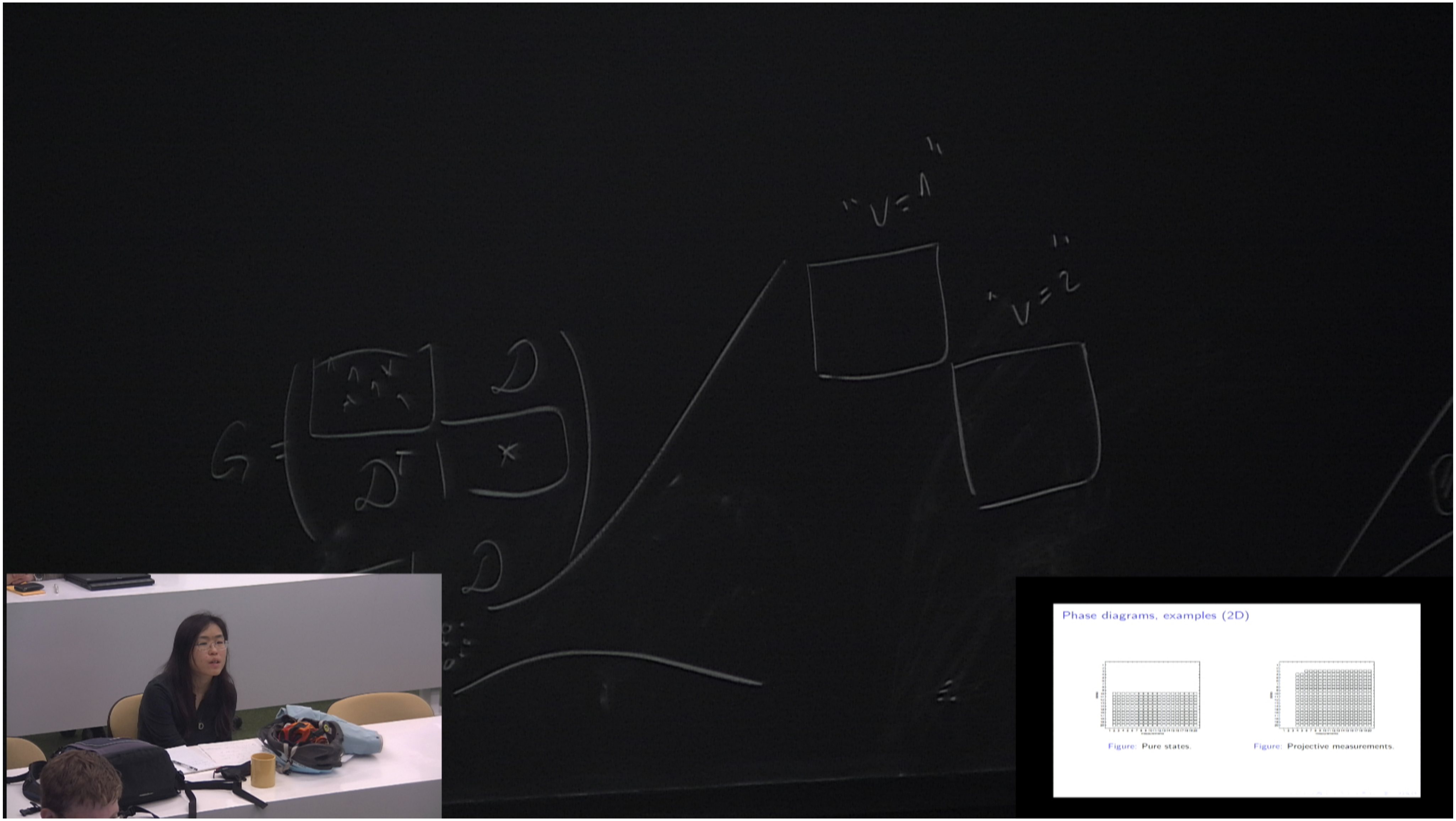
"V=1"
"V=2"
Phase diagrams, examples (2D)
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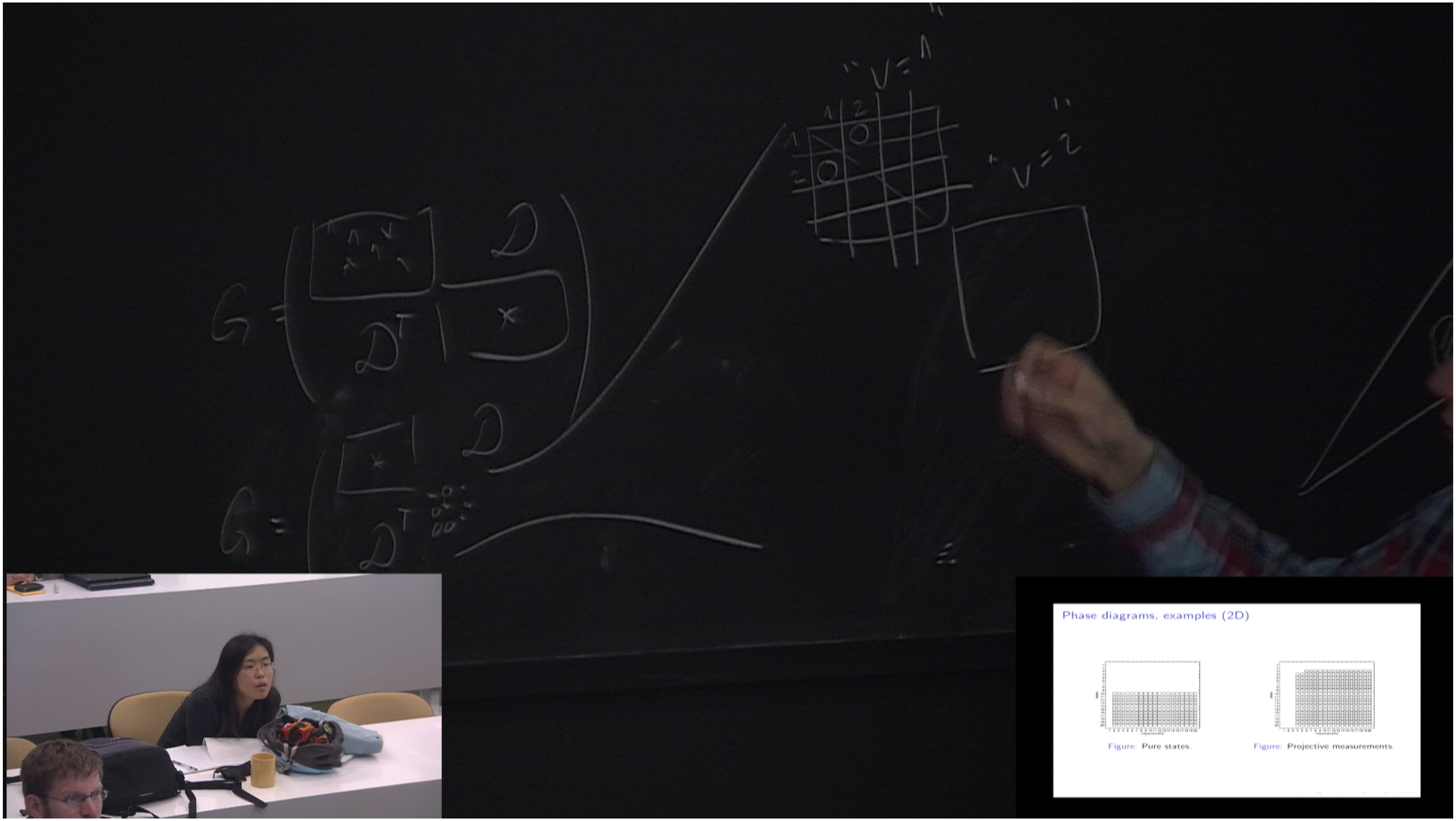
Figure: Projective measurements.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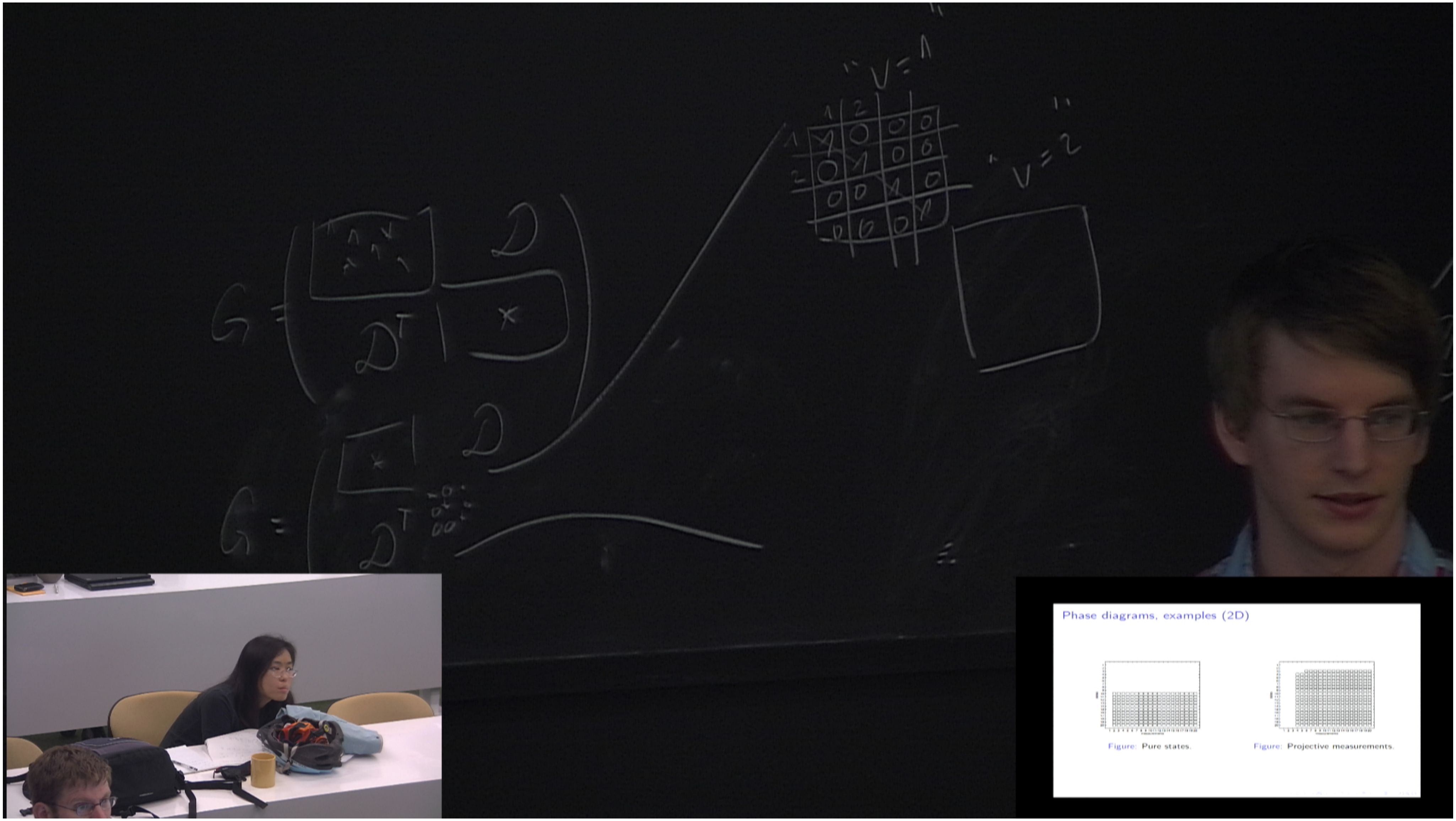
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Overview

- Setting
- Gram estimation
- Uniqueness
- Projective, non-degenerate measurements
- Summary

Projective, non-degenerate measurements

- Recall that

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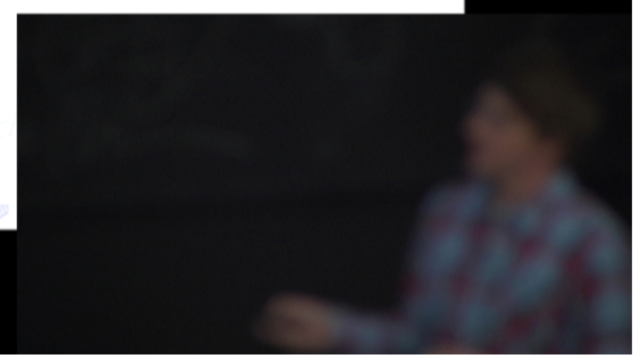

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Theorem (Converse)

Assume $\dim(\mathcal{H}) = d$ *and* $(*)$. *Then,* $\rho_{vk} = E_{vk}$, *and* $G_{\mathrm{st}} = \mathcal{D} = G_{\mathrm{m}}$.

Representative density matrices and POVM elements

- arXiv:1210.1105

Summary

- Gram estimation guided by Occam's razor
  $\rightsquigarrow$ rank minimization (NP-hard)
- Convex relaxation
- Observation: $G_{\text{estimated}} \notin \{\text{quantum Gram matrices}\}$ possible.
  $\rightsquigarrow$ Uniqueness analysis (of independent interest)
- Special case: projective, non-degenerate measurements
- Representative density matrices and POVMs via a heuristic procedure

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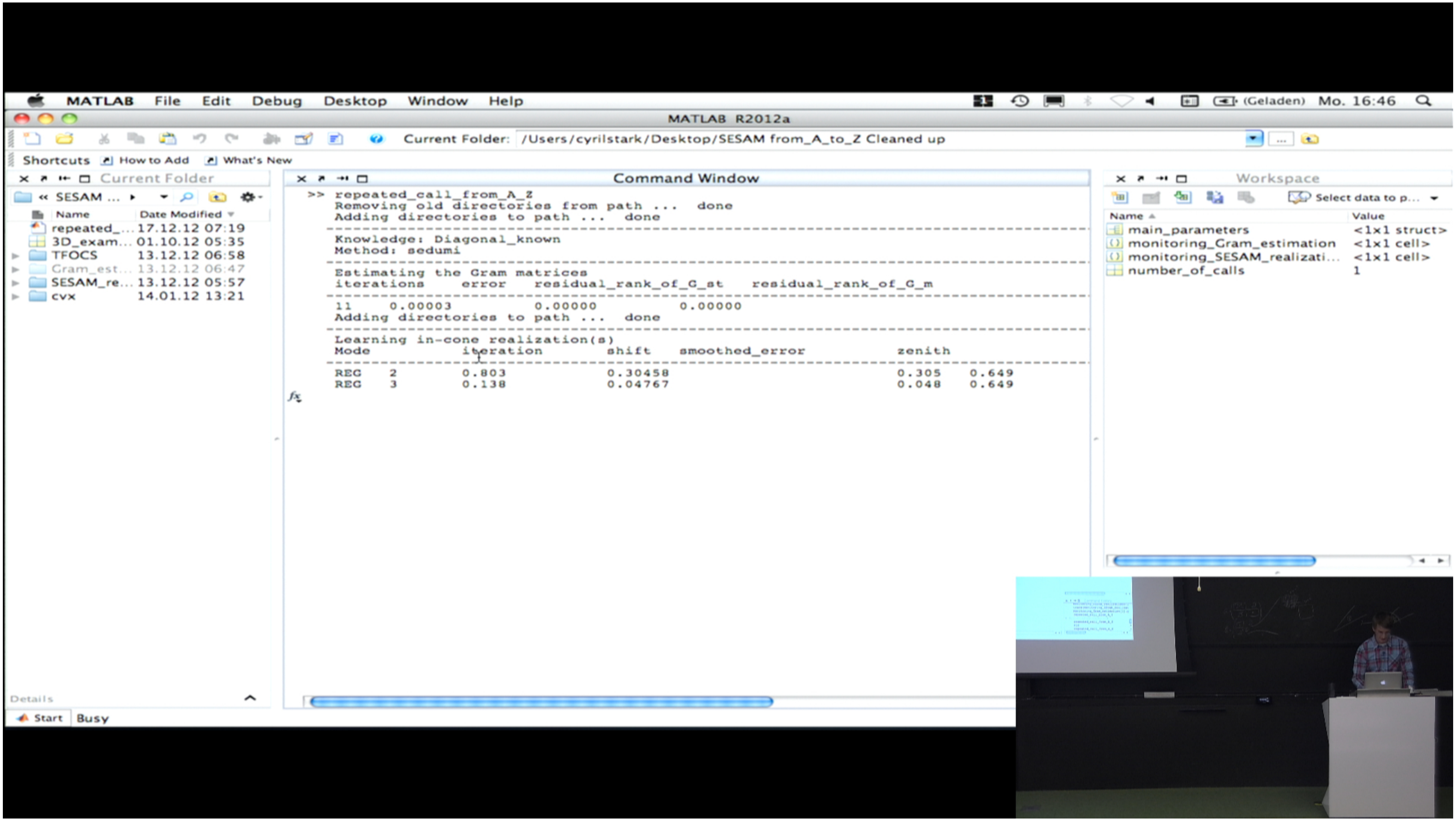
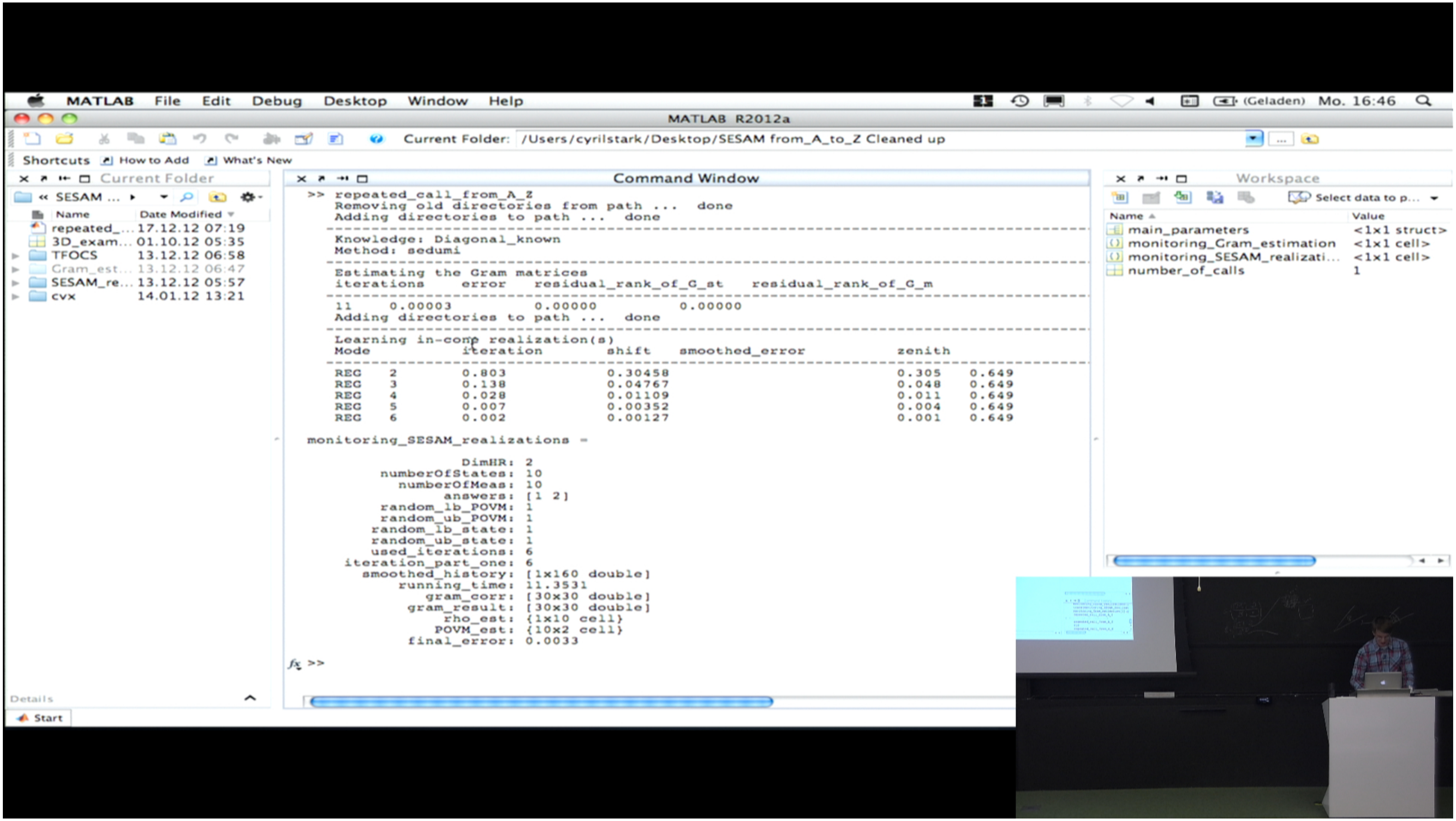

MATLAB File Edit Debug Desktop Window Help
(Geladen) Mo. 16:46
MATLAB R2012a
Current Folder: /Users/cyrilstark/Desktop/SESAM from_A_to_Z Cleaned up
Shortcuts How to Add What's New
Current Folder
SESAM ...
Name Date Modified
repeated_... 17.12.12 07:19
3D_exam... 01.10.12 05:35
TFOCS 13.12.12 06:58
Gram_est... 13.12.12 06:47
SESAM_re... 13.12.12 05:57
cvx 14.01.12 13:21
Command Window
>> repeated_call_from_A_Z
Removing old directories from path ... done
Adding directories to path ... done
Knowledge: Diagonal_known
Method: sedumi
Estimating the Gram matrices
iterations error residual_rank_of_G_st residual_rank_of_G_m
11 0.00003 0.00000 0.00000
Adding directories to path ... done
Learning in-cone realization(s)
Mode iteration shift smoothed_error zenith
REG 2 0.803 0.30458 0.305 0.649
REG 3 0.138 0.04767 0.048 0.649
Workspace
Select data to p...
Name Value
main_parameters <1x1 struct>
monitoring_Gram_estimation <1x1 cell>
monitoring_SESAM_realizati... <1x1 cell>
number_of_calls 1
Details
Start Busy



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REG 5 0.007 0.00352 0.004 0.649
REG 6 0.002 0.00127 0.001 0.649
monitoring_SESAM_realizations =
DimHR: 2
numberOfStates: 10
numberOfMeas: 10
answers: [1 2]
random_lb_POVM: 1
random_ub_POVM: 1
random_lb_state: 1
random_ub_state: 1
used_iterations: 6
iteration_part_one: 6
smoothed_history: [1x160 double]
running_time: 11.3531
gram_corr: [30x30 double]
gram_result: [30x30 double]
rho_est: {1x10 cell}
POVM_est: {10x2 cell}
final_error: 0.0033
fx >>
Workspace
Select data to p...
Name Value
main_parameters <1x1 struct>
monitoring_Gram_estimation <1x1 cell>
monitoring_SESAM_realizati... <1x1 cell>
number_of_calls 1
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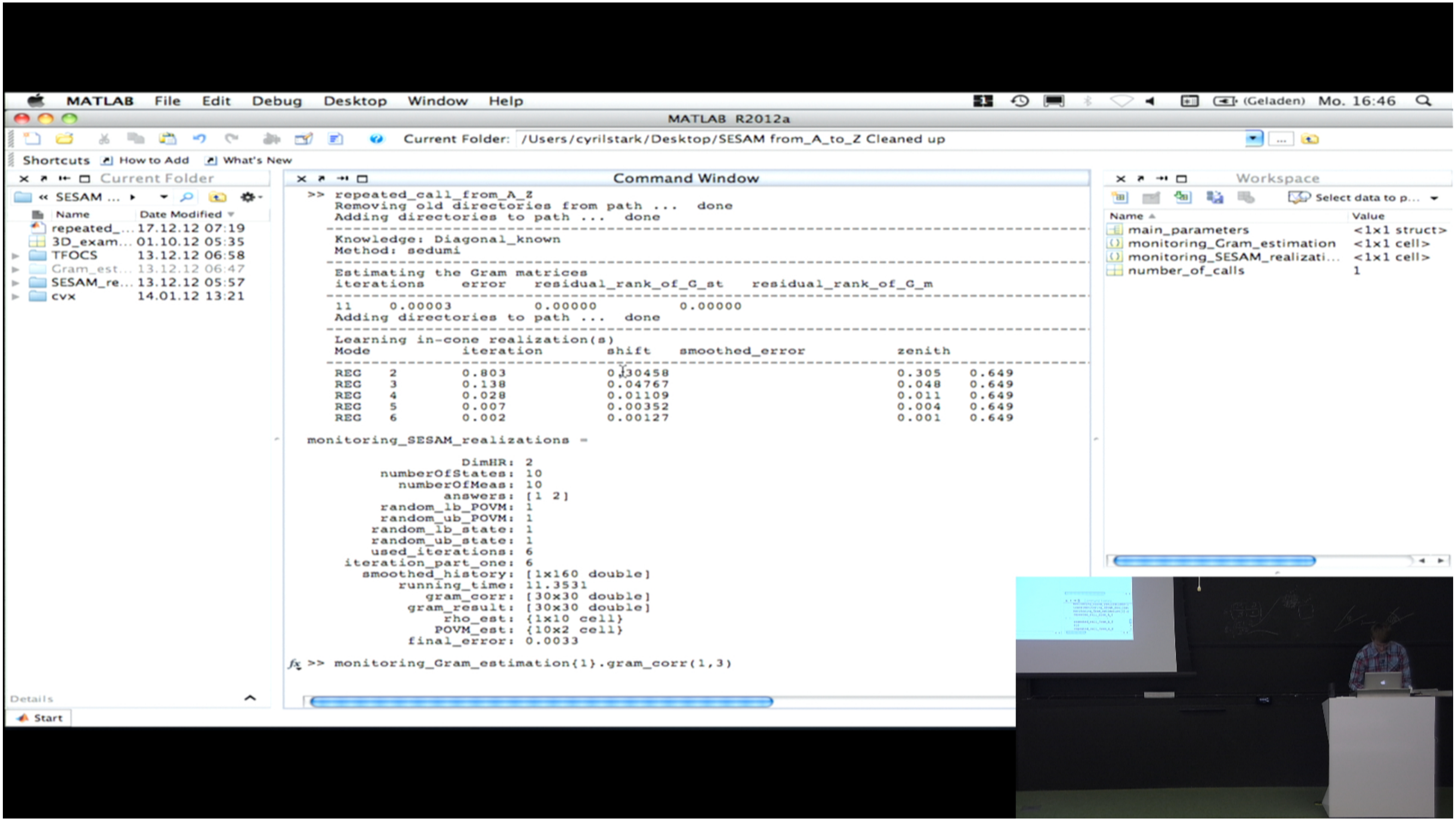
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rho_est: {1x10 cell}
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final_error: 0.0033
>> monitoring_Gram_estimation{1}.gram_corr(1,3)
Workspace
Select data to p...
Name Value
main_parameters <1x1 struct>
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number_of_calls 1
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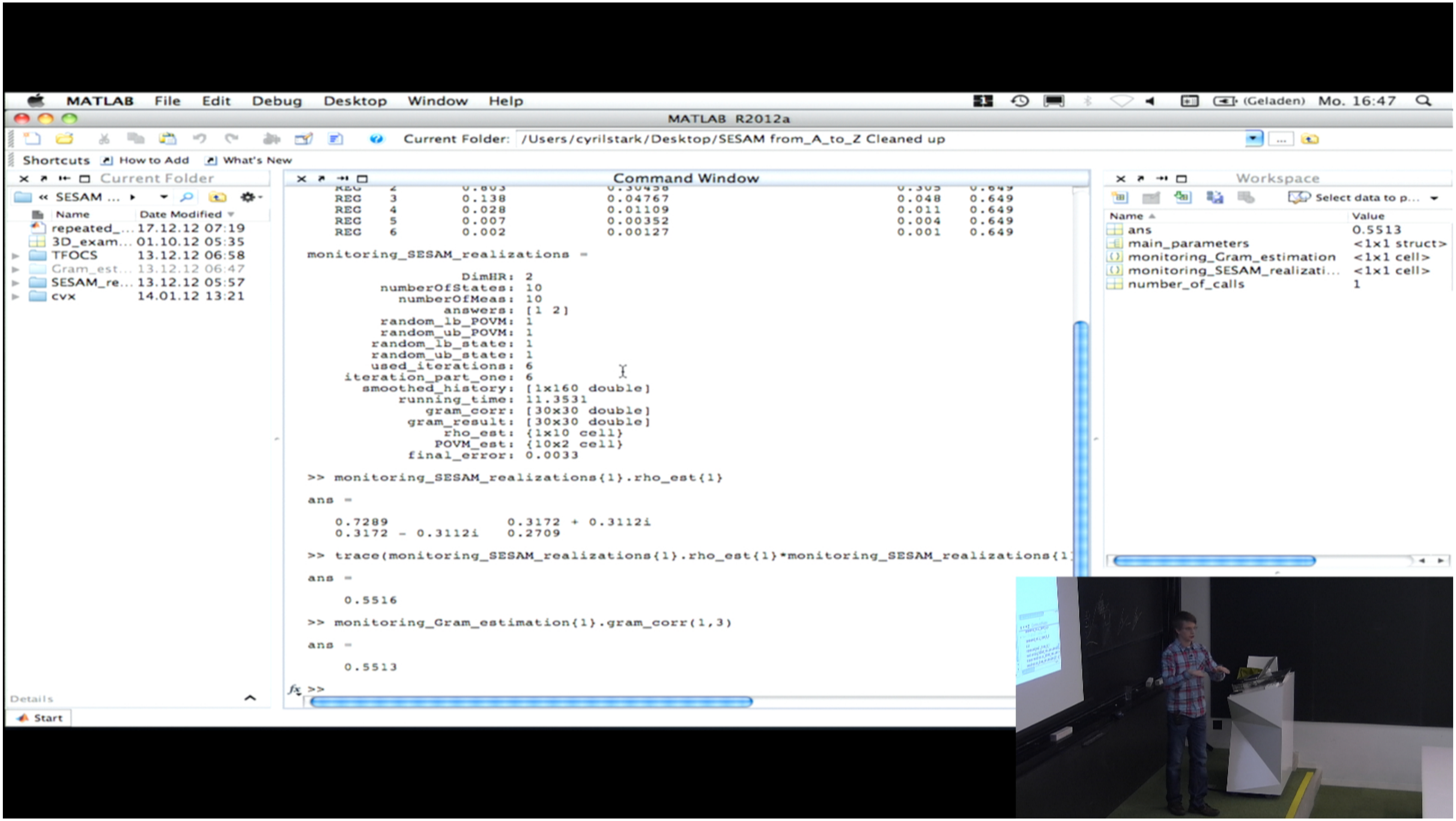
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>> monitoring_SESAM_realizations{1}.rho_est{1}
ans =
0.7289 0.3172 + 0.3112i
0.3172 - 0.3112i 0.2709
>> trace(monitoring_SESAM_realizations{1}.rho_est{1}*monitoring_SESAM_realizations{1
ans =
0.5516
>> monitoring_Gram_estimation{1}.gram_corr(1,3)
ans =
0.5513
>>
Workspace
Select data to p...
Name Value
ans 0.5513
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