Title: Quantum Chaos, Information Gain and Quantum Tomography.

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Abstract: Quantum chaos is the study of quantum systems whose classical description is chaotic.

How does chaos manifest itself in the quantum world? In recent years, attempts have been made to address this question from the perspective of quantum information theory. It is in this spirit that we study the connection between quantum chaos and information gain in the time series of a measurement record used for quantum tomography. The record is obtained as a sequence of expectation values of a Hermitian operator evolving under repeated application of the Floquet operator of the quantum kicked top on a large ensemble of identical systems. We find an increase in information gain and hence higher fidelities in the process when the Floquet maps employed increase in chaoticity. We make predictions for the information gain using random matrix theory in the fully chaotic regime and show a remarkable agreement between the two.

Quantum Chaos, Information Gain and Quantum Tomography

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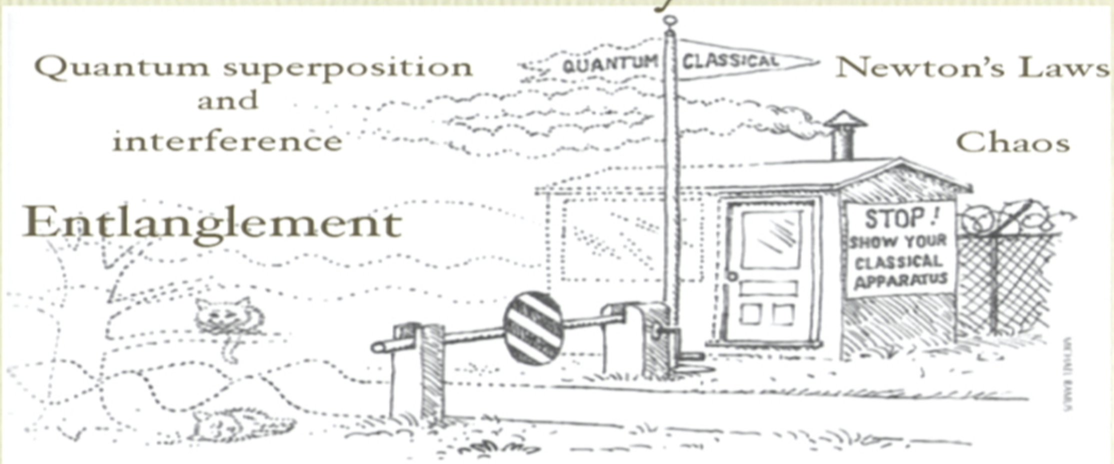
The Big Picture

The universe, as we know, is quantum.

Classical mechanics gives an excellent description of the macroscopic world !

A tale of two theories: Quantum and Classical Physics



Drawing by Michael Ramus, 1991.

How does classical behavior emerge out of quantum mechanics?

Are there any footprints of classical chaos in the quantum world?

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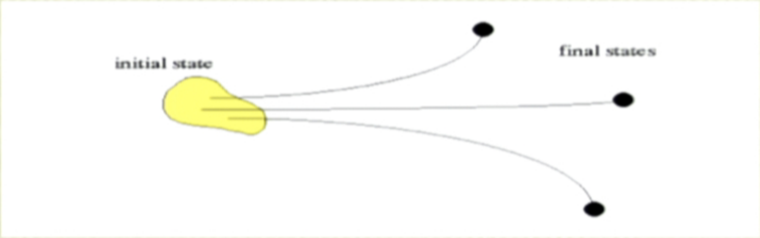
What is Classical Chaos?

- Aperiodic long term behavior in a deterministic system.
- Sensitive dependence to initial conditions.



$$\Delta x_n = e^{\lambda n} \Delta x_0$$

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Quantum Chaos?

- Quantum mechanics preserves the inner product between state vectors. Unitary evolution gives us:

$$\langle\phi(0)|\psi(0)\rangle = \langle\phi\ U^{\dagger}|U\ \psi\rangle = \langle\phi(t)|\psi(t)\rangle$$

- What are other ways to characterize chaos in quantum mechanical systems?
- Quantum Chaology : What are the quantum signatures of classical chaos?
- We have a new quantum signature of chaos.

Michael Berry 1989 Phys. Scr. 40 335

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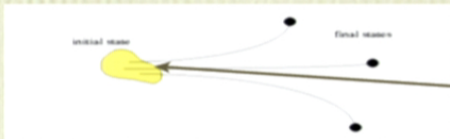
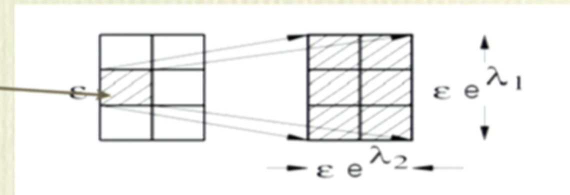
Information Gain: A Classical Analogue





$$H_t = -\sum_j p_j \log p_j$$

Chaotic trajectory acts as an information source

As one follows a single trajectory, one gains information about the initial conditions of the trajectory

K-S Entropy: Rate of information gain about the initial conditions on increasingly finer scales

$H_{KS} > 0$ for a chaotic system

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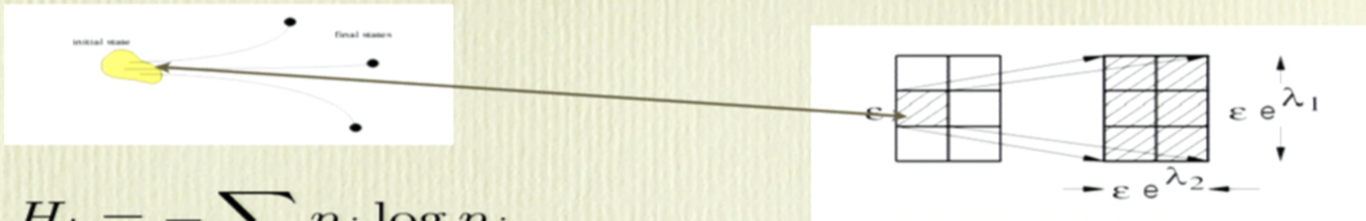
Information Gain: A Classical Analogue



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Continuous Measurement Quantum Tomography

N copies

ρ_0 → Evolution & Measurement → $M(t)$ → Gaussian Estimator → $\bar{\rho}$

Unknown Quantum State

Measured Signal

Reconstructed Quantum State

$$M(t) = N\langle O(t)\rangle + \eta(t)$$

We gain information about the initial state by continuously measuring the system.

Silberfarb *et al.* PRL **95**, 030402 (2005) G. Smith *et al.* PRL **97**, 180403 (2006)

Riofrío *et al.* J. Phys. B. 15, 154007 (2011).

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Reconstruction Algorithm

$$M_i = Tr(O_i \rho_0) + \sigma W_i$$ Measurement at time i

$$\rho_0 = \frac{1}{d} I + \sum_{\alpha=1}^{d^2-1} r_\alpha E_\alpha$$ E_α Orthonormal basis of hermitian matrices

$$M_i = \sum_{\alpha=1}^{d^2-1} r_\alpha Tr(O_i E_\alpha) + \sigma W_i$$

$$\mathbf{r}_{LS} = (\tilde{\mathbf{O}}^T \tilde{\mathbf{O}})^{-1} \tilde{\mathbf{O}}^T \mathbf{M}$$ Least square estimate

where $\tilde{O}_{i\alpha} = \mathrm{Tr}(O_i E_\alpha)$

Reconstruction Algorithm

$$M_i = \sum_{\alpha=1}^{d^2-1} r_\alpha Tr(O_i E_\alpha) + \sigma W_i$$

$$p(M|r) \propto \exp\left\{-\frac{1}{2\sigma^2}\sum_i [M_i - \sum_\alpha \mathcal{O}_{i\alpha} r_\alpha]^2\right\}$$

$$\propto \exp(-\frac{1}{2}(r - r_{ML})^T C^{-1}(r - r_{ML}))$$

Maximum likelihood solution = $r_{ML} =$ $\mathbf{r}_{LS} = (\tilde{\mathbf{O}}^T\tilde{\mathbf{O}})^{-1}\tilde{\mathbf{O}}^T\mathbf{M}$

$(\tilde{\mathbf{O}}^T\tilde{\mathbf{O}})^{-1}$ is the Covariance Matrix C $\tilde{O}_{i\alpha} = \mathrm{Tr}(O_i E_\alpha)$

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Tomography and Chaos

$$M_i = Tr(O_i \rho_0) + \sigma W_i$$

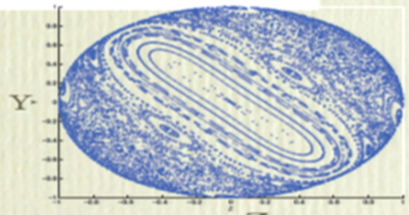
Let $O_i = U^{\dagger^i} O_0 U^i$ -- repeated application of the same map

How does the nature of U affect the tomography process ?

Tomography by the kicked top dynamics

$$H = \alpha F_X + \frac{\lambda}{2F} \sum_{n=-\infty}^{\infty} \delta(t - n\tau) F_z^2.$$ Kicked top floquet map

$$U_1 = e^{-i\lambda F_z^2/2F} e^{-i\alpha F_x}$$ λ is the chaoticity parameter



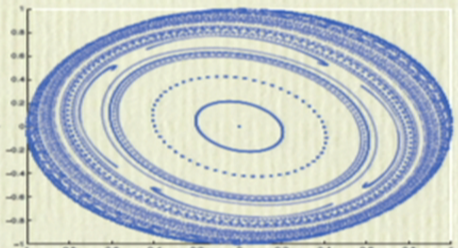
Phase space (mixed) for the kicked top , $\alpha = 1.4$

$\lambda = 2.5$

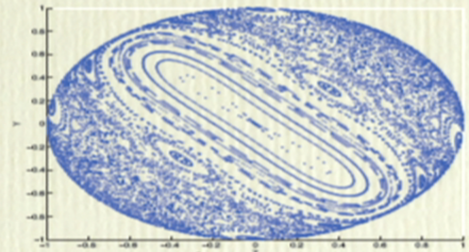
The Kicked Top Phase Space

$$U_1 = e^{-i\lambda F_z^2/2F} e^{-i\alpha F_x}$$ Kicked top floquet map

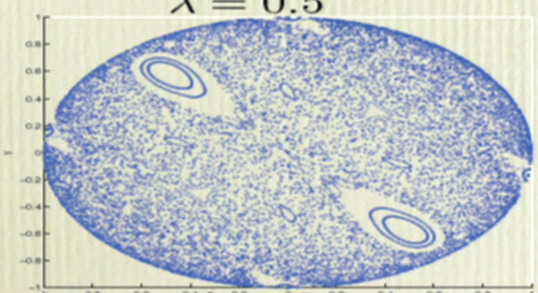
λ is the chaoticity parameter



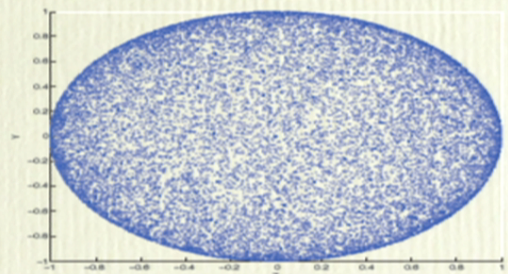
$\lambda = 0.5$



$\lambda = 2.5$



$\lambda = 3.0$



$\lambda = 7.0$

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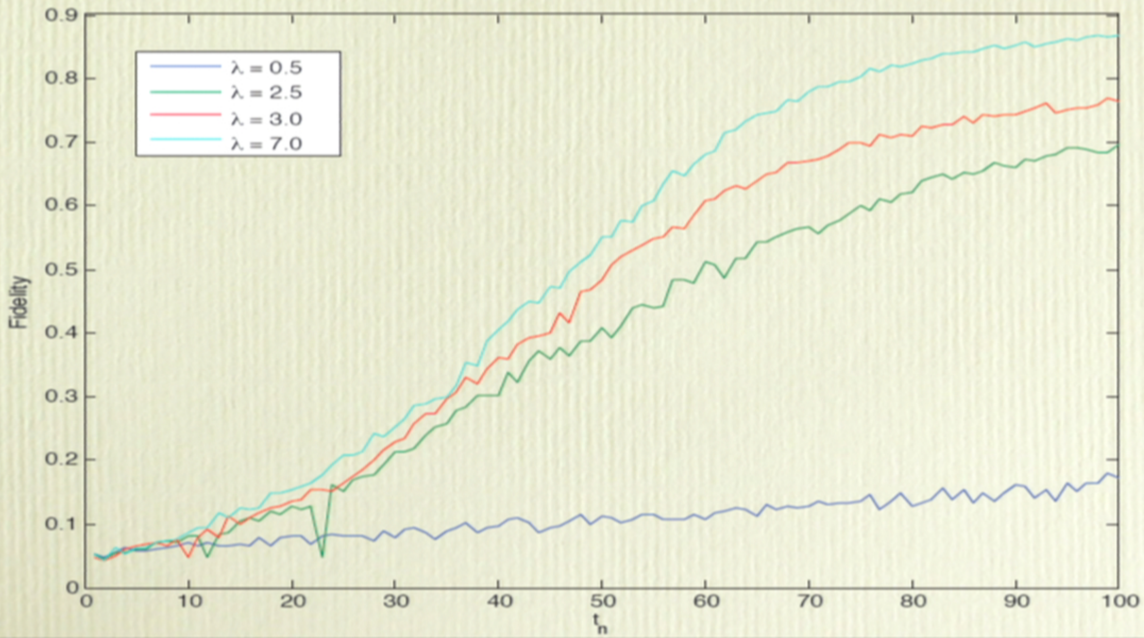
Tomography Fidelity Vs Chaoticity



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Tomography and Information gain

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Fisher Information

For a family of probability densities, $f(x;\theta)$

$$\mathcal{I}(\theta) = \mathrm{E}\left[\left(\frac{\partial}{\partial\theta} \log f(X;\theta) \right)^2 \middle| \theta \right].$$

$E[\ldots|\theta]$ is the expectation with respect to $f(x;\theta)$ given a particular θ

Fisher information tells us how well we can estimate a parameter based on a sequence of data

For n iid observations : $I_n(\theta) = nI(\theta)$

Fisher Information

For example : For a gaussian density

$$f(x;\theta) = N(0,\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{\frac{-x^2}{2\theta}}$$

Fisher information = $\mathcal{I}(\theta) = \mathrm{E}\left[\left(\frac{\partial}{\partial\theta}\log f(X;\theta)\right)^2\middle|\theta\right], \quad = \quad \frac{1}{\theta}$

Fisher Information and parameter estimation

Cramer Rao Bound $\quad V_\theta(T(X_1, ..., X_n)) \geq \frac{1}{I_n(\theta)}$

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Multivariate Fisher Information

For a single variable

$$\mathcal{I}(\theta) = \mathrm{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(X;\theta)\right)^2 \middle| \theta\right]$$

Multivariable

$$(\mathcal{I}(\theta))_{i,j} = \mathrm{E}\left[\left(\frac{\partial}{\partial\theta_i}\ln f(X;\theta)\right)\left(\frac{\partial}{\partial\theta_j}\ln f(X;\theta)\right)\middle|\theta\right].$$

For a multivariate gaussian

$$P(r|M) \propto \exp(-\frac{1}{2}(r - r_{ML})^T C^{-1}(r - r_{ML}))$$

$$I(\theta) = C^{-1}$$

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Multivariate Fisher Information

For a multivariate gaussian $$I(\theta) = C^{-1}$$

$$Tr\langle\{(\rho_0 - \bar{\rho})^2\}\rangle = \sum_{\alpha}\langle(\Delta r_\alpha)^2\rangle \geq \left[I(\theta)^{-1}\right]_{\alpha\alpha} = Tr(C)$$

$$\epsilon \geq Tr(C)$$

Thus, we get a single number to capture the information gain in tomography

$1/Tr(C)$ is the collective Fisher information.

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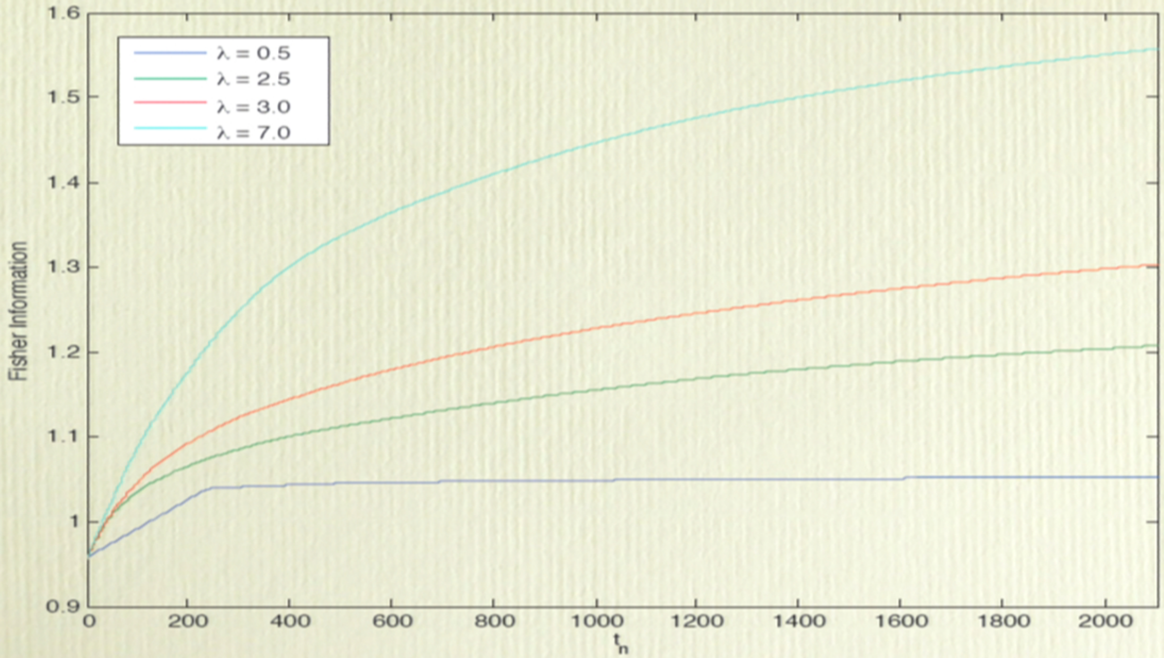
Fisher Information vs Chaoticity



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Information content in the Covariance matrix

- Eigenvectors of the inverse of the covariance matrix give specify the direction in the operator space that have been measured. They are the "observables" that we measure.
- The corresponding eigenvalues give the uncertainty with which we measure those "observables" given a signal to noise ratio.
- Entropy of the eigenvalues of the inverse of the covariance matrix captures the collective information gain in tomography
- We expect high fidelity reconstruction for higher entropy values in the presence of noise.

Multivariate Fisher Information

For a multivariate gaussian $I(\theta) = C^{-1}$

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For a single variable

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Multivariable

$$\left(\mathcal{I}\left(\theta\right)\right)_{i,j} = \mathrm{E}\left[\left(\frac{\partial}{\partial\theta_i} \ln f(X;\theta) \right) \left(\frac{\partial}{\partial\theta_j} \ln f(X;\theta) \right) \Bigg| \theta \right].$$

For a multivariate gaussian

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$$I(\theta) = C^{-1}$$

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Random Matrix theory and Quantum Chaos

How well our analysis agrees with
the Random Matrix theory ?

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The Random Matrix conjecture of Quantum Chaos.

When Hamiltonian or Floquet map is completely chaotic, then its statistical properties can be described by a Random Matrix picked from an appropriate ensemble.

Signatures of chaos/integrability: Level statistics, Level repulsion are described by the Random Matrix theory.

O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett 52, 1 (1984.)
F. Haake, Quantum Signatures of Chaos (Spring-Verlag, Berlin, 1991.)

Choice of the appropriate ensemble of Random Matrix

Appropriate ensemble is given by fundamental symmetries of the Hamiltonian e.g Time reversal invariance

The kicked top is Time reversal Invariant.

$$U = e^{-i\lambda \frac{F_z^2}{2F}} e^{-i\alpha F_x}$$

$$TU_1T^{-1} = U^{-1} = e^{i\alpha F_x} e^{i\lambda \frac{F_z^2}{2F}}$$

where, T is an anti-unitary operator.

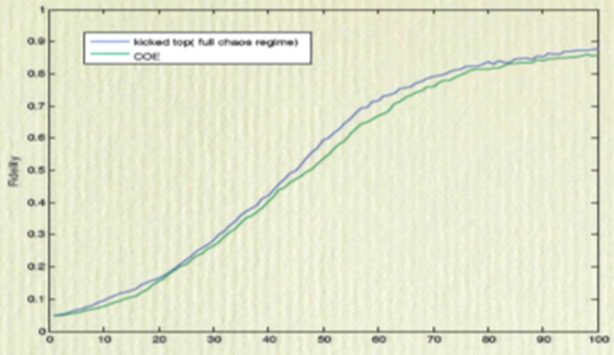
For the kicked top: $T = e^{i\alpha F_x} K$

and : $T^2 = 1$

Thus, the appropriate ensemble for the kicked top is the OE.

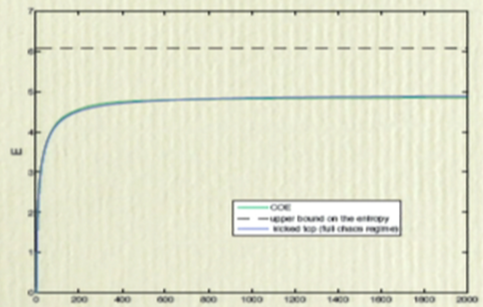
RMT predictions for the kicked top

Fidelity



(a)

Shannon Entropy



(b)

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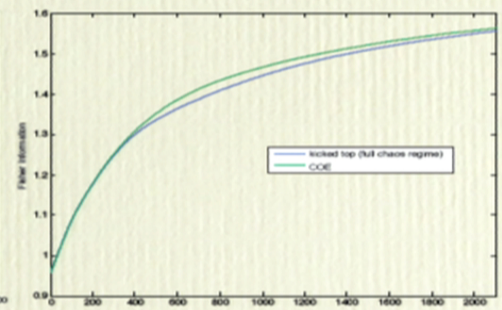

(c)

Fisher Information

Conclusion

- We have a new quantum signature of classical chaos!
- Quantum chaos intimately related to information gain and consequences are manifested in the fidelities of reconstruction.
- Information gain in chaotic systems has a remarkable agreement with the Random Matrix theory.

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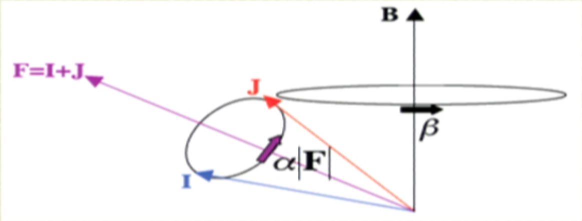
The System: Description of the dynamics



$$H = AI \cdot J + \sum_{n=-\infty}^{\infty} \delta(t - n\tau) B J_z$$

- Floquet map:

$$U_\tau = e^{-i\alpha I \cdot J} e^{-i\beta J_z} \equiv e^{-i\alpha F^2/2} e^{-i\beta J_z}$$

Rotation of J about Z axis by an angle β followed by a precession of I and J about F by an angle $\alpha|F|$

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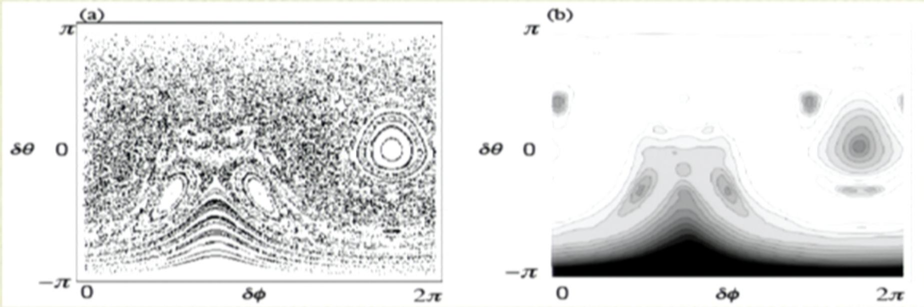
Evidence of Entanglement as a signature of chaos: Dynamics

Spin size $I = J = 150$

We consider $F_z = 0$ subspace.

Initial state: Product state of spin coherent state associated with each spin

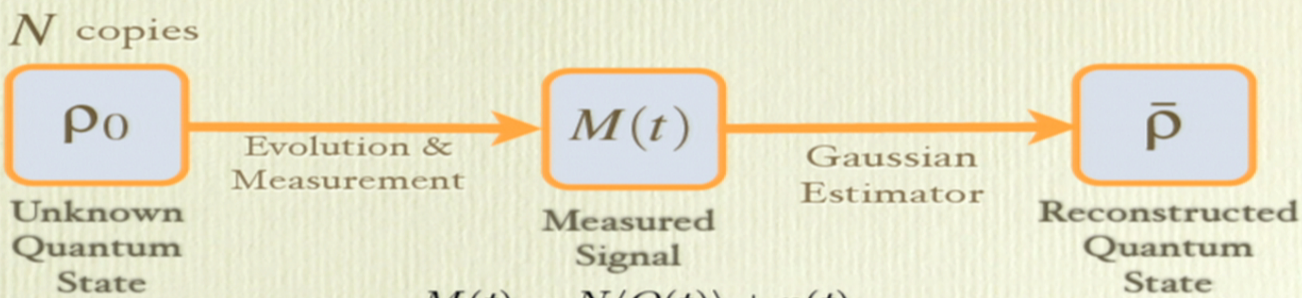


(a) (b)

(a) The classical phase space

(b)Long time average entanglement as a function of the initial position of the coherent states

Continuous Measurement Quantum Tomography



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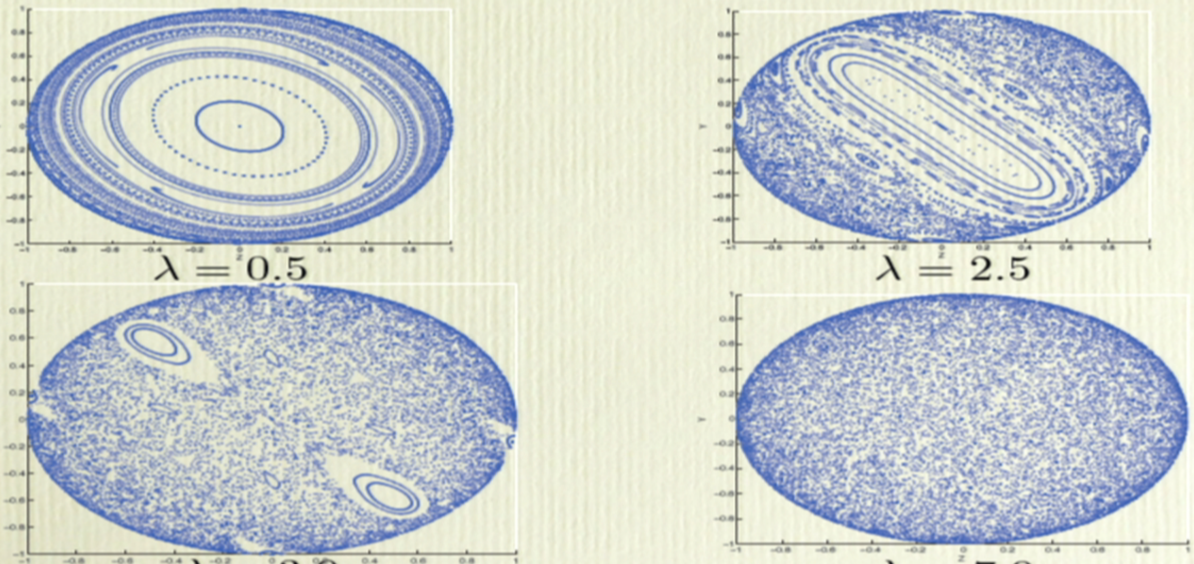
λ is the chaoticity parameter



$\lambda = 0.5$ $\lambda = 2.5$ $\lambda = 3.0$ $\lambda = 7.0$

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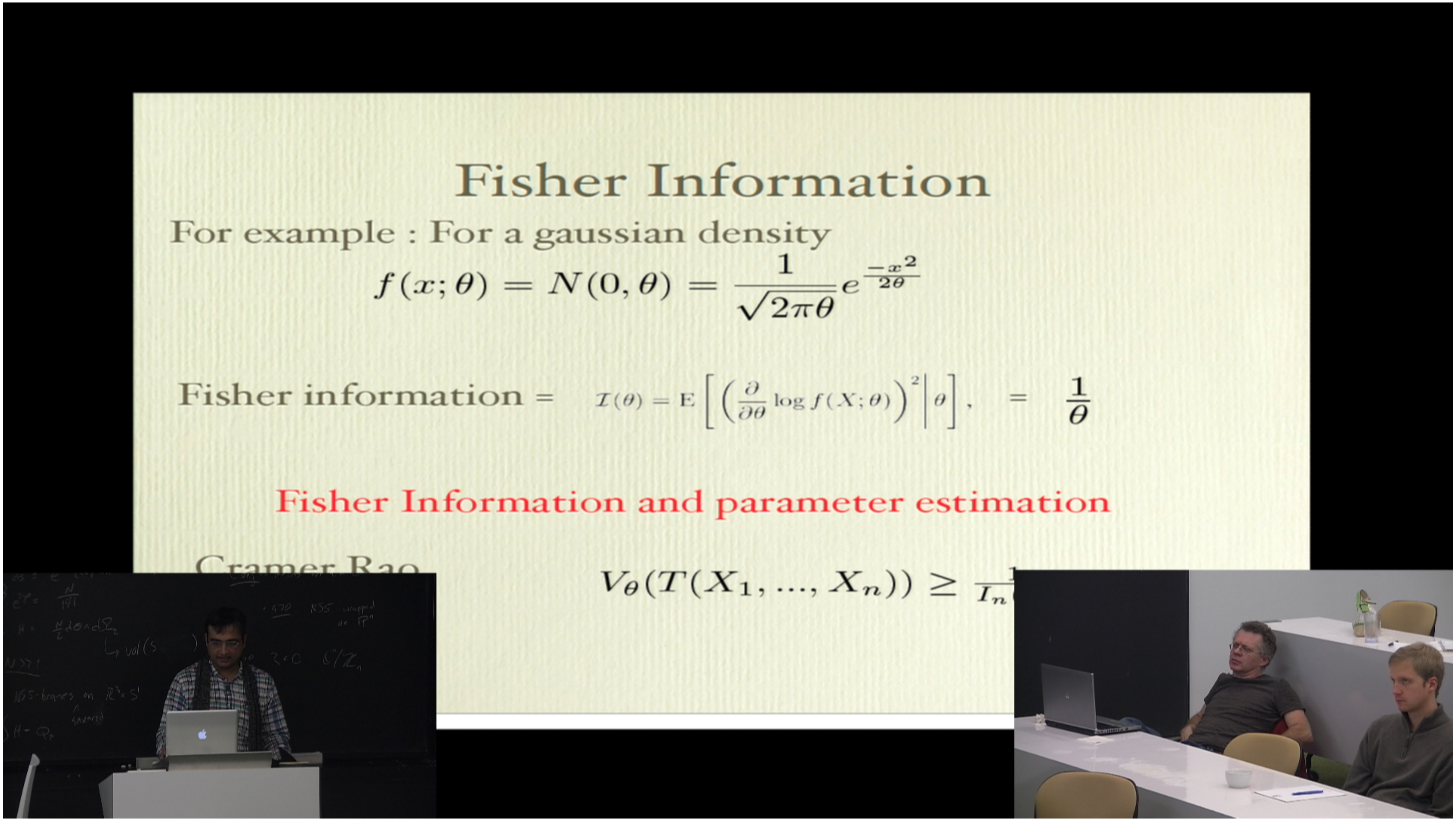
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$V_\theta(T(X_1, ..., X_n)) \geq$