Title: Application of conditional independence to gapped quantum many-body systems

Date: Dec 12, 2012 04:00 PM

URL: http://www.pirsa.org/12120035

Abstract: It is widely known in the quantum information community that the states that satisfy strong subadditivity of entropy with equality have the form of quantum Markov chain. Based on a recent strengthening of strong subadditivity of entropy, I will describe how such structure can be exploited in the studies of gapped quantum many-body system. In particular, I will describe a diagrammatic trick to i) give a quantitative statement about the locality of entanglement spectrum ii) perturbatively bound changes of topological entanglement entropy under generic perturbation.

Application of conditional independence to gapped quantum many-body systems

Isaac H. Kim

Institute of Quantum Information and Matter
California Institute of Technology
Pasadena, CA 91105, USA

December 12, 2012

Background

There are some surprising universal properties that arise in the ground state of gapped quantum many-body systems.

- No fine-tuning of the hamiltonian is required.
- The predictions have been confirmed experimentally and numerically.
  - IQHE, FQHE, TI, etc...
  - Numerical calculation of topological entanglement entropy, entanglement spectrum, particle statistics, etc...

Background : General philosophy

Question : Why are these properties stable?

Background : General philosophy

Question : Why are these properties stable?
Answer 1: RG argument, low-energy effective field theory.

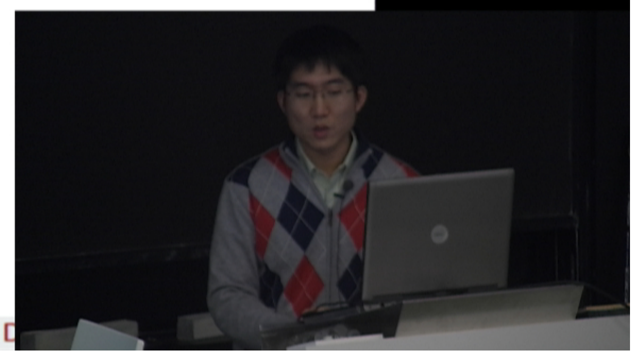
Background : General philosophy

Question : Why are these properties stable?
Answer 1: RG argument, low-energy effective field theory.
Answer 2: It can be mathematically proved! (Bravyi, Hastings, Michalakis 2010)

Background : General philosophy

Question : Why are these properties stable?
Answer 1: RG argument, low-energy effective field theory.
Answer 2: It can be mathematically proved! (Bravyi, Hastings, Michalakis 2010)

- Caveat 1: Some conditions are needed. (TQO-1 and 2)
- Caveat 2: It only answers the stability of statistics and gap.



Background : Origin of the gap stability

According to Bravyi et al.'s work, there are properties of the ground state that protect the phase.

- Locality of the parent hamiltonian.
- Local indistinguishability : Different sectors of the ground state cannot be detected, nor be altered via local operation.
- Local ground state is equal to the global ground state.

Background : Implication of the gap stability

If ground states of two different local hamiltonian can be adiabatically connected, there exists an *almost-local* "hamiltonian" that generates a unitary evolution between those two ground states.(Hastings, Wen 2005)

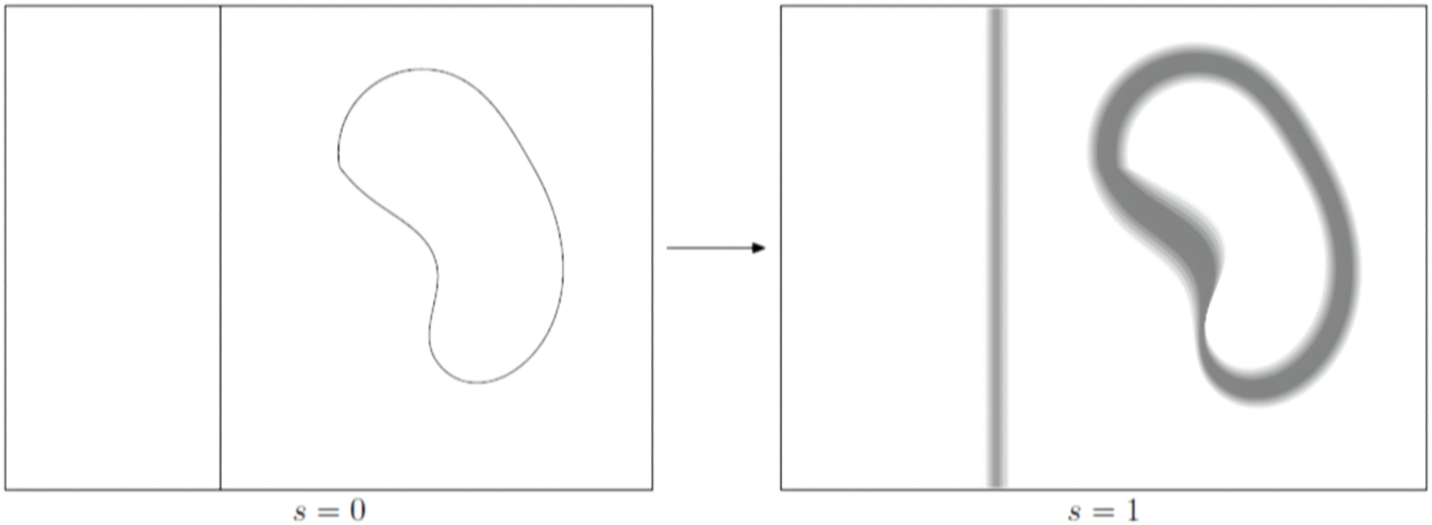
$$H = H_0 + sV, s \in (0, 1]$$

$$|\psi(s)\rangle = U(s)\,|\psi(0)\rangle$$

$$\frac{dU(s)}{ds} = \sum_i h_i(s)$$

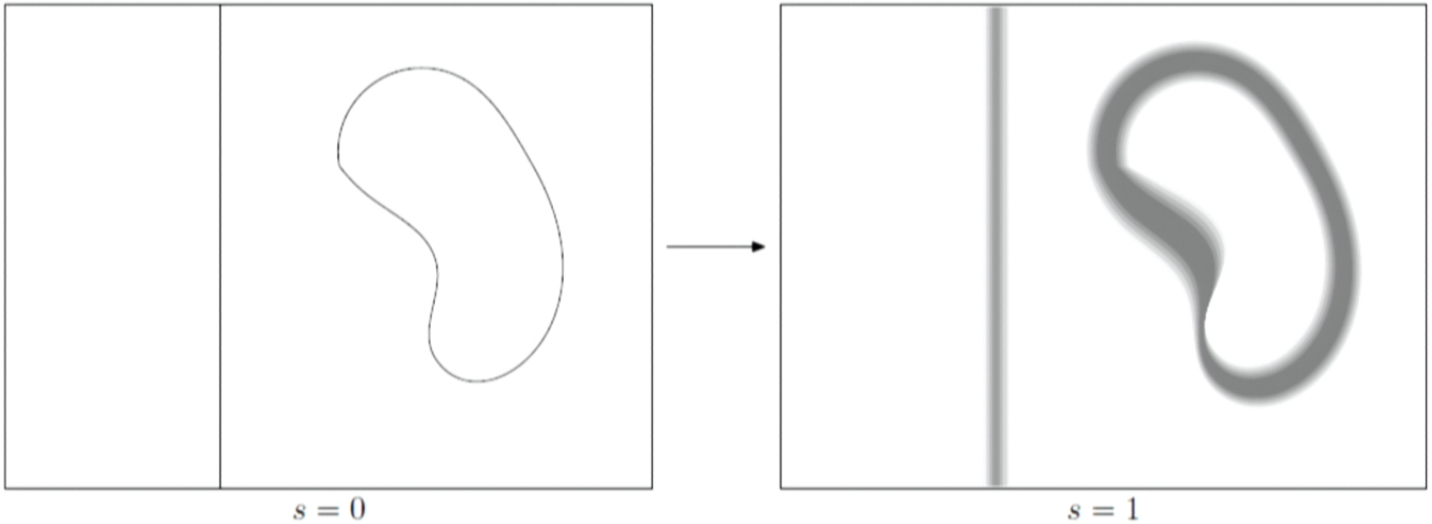
- $h_i(s)$ can be approximated by a strictly local operator with superpolynomially decaying tail.
- By using Trotter-Suzuki expansion, this unitary evolution can be approximated by a finite-depth local unitary circuit. (with small error)

Background : Implication of the gap stability



- Particle statistics is preserved.
- Logical operators are preserved.

Background : Implication of the gap stability



- Particle statistics is preserved.
- Logical operators are preserved.

Background : Unanswered questions

- Stability of topological entanglement entropy γ.
  - $S_A = a|\partial A| - \gamma$
  - γ is a universal constant : Levin and Wen(2006), Kitaev and Preskill (2006)
  - Passed number of numerical tests : Isakov et al. (2011), Jiang et al.(2012), Cincio and Vidal(2012), Selem et al.(2012),
- Locality of entanglement spectrum.
  - $\log \rho_A$ can be described by a local hamiltonian! : Li and Haldane(2008), Dubail et al. (2012), Cirac et al. (2011), Schuch et al. (2012)

Main message of this talk

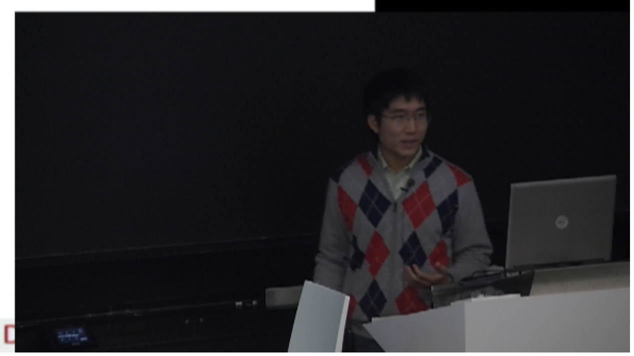
There is another property of the ground state that protects the phase : **conditional independence**

Main message of this talk

There is another property of the ground state that protects the phase : **conditional independence**

1. Conditionally independent states naturally appear in the ground state of topologically ordered systems.

Isaac H. Kim (Caltech) Application of conditional independence to ga



Click on Comment and Share to create, mark-up and send PDF files.

Main message of this talk

There is another property of the ground state that protects the phase : **conditional independence**

1. Conditionally independent states naturally appear in the ground state of topologically ordered systems.
2. Conditional independence can be exploited to produce some nontrivial statements about topological entropy, entanglement spectrum, etc.

What is conditional independence?

One of the most fundamental inequalities in quantum information theory is the strong subadditivity of entropy(Lieb, Ruskai 1972):

$$S_{AB} + S_{BC} - S_B - S_{ABC} \geq 0.$$

- This inequality holds for *any* tripartite quantum state!
- Many results in quantum information theory is based on this inequality.
- It has found applications in other settings
  - Simple proof of c-theorem in 1+1D CFT (Casini and Huerta 2004)
  - Some bounds on topological entanglement entropy (Zhang et al. 2012)

What is conditional independence?

Alternatively,

$$I(A:C|B) \geq 0,$$

where $I(A:C|B)$ is *conditional mutual information.*

$$I(A:C|B) = S_{AB} + S_{BC} - S_B - S_{ABC}$$

Definition : A tripartite state ρ_{ABC} is conditionally independent if the inequality is satisfied with an *equality.*

Click on Comment and Share to create, mark-up and send PDF files.

Why are conditionally independent states interesting?

Short answer : Because they have a special structure, and they seem to appear in many places.

Why are conditionally independent states interesting?

Short answer : Because they have a special structure, and they seem to appear in many places.

Long answer :

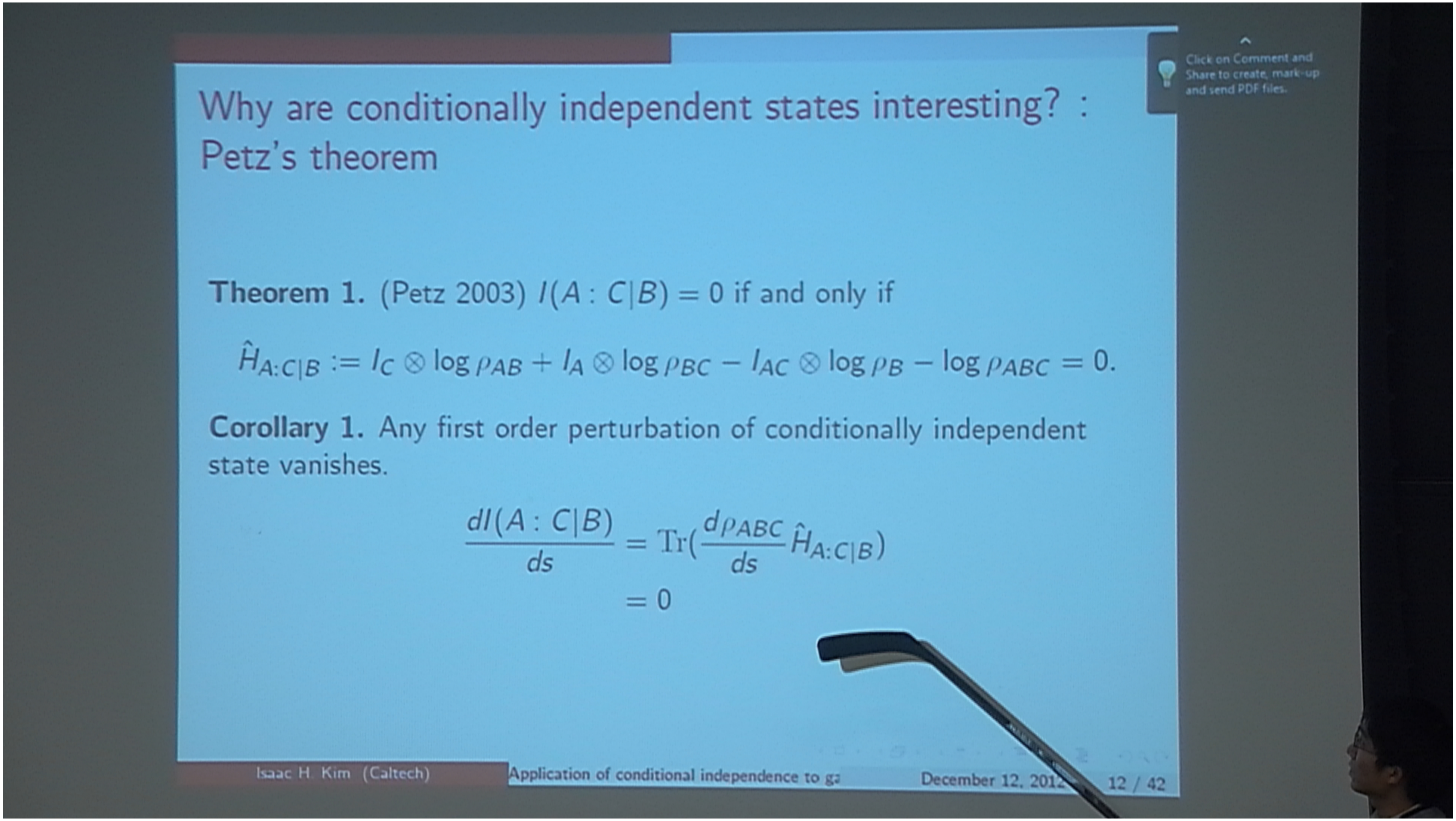
- Such state forms a quantum Markov chain.
    - Given two reduced density matrices ρ_{AB} and ρ_{BC}, one can "glue" them together to reconstruct ρ_{ABC}. (Petz 2002)
    - Such states have a special structure. (Hayden et al. 2003)
    - Reduced density matrices satisfy many nontrivial relations. (Leifer and Poulin 2006)

Why are conditionally independent states interesting?

Short answer : Because they have a special structure, and they seem to appear in many places.

Long answer :

- Such state forms a quantum Markov chain.
  - Given two reduced density matrices ρ_{AB} and ρ_{BC}, one can "glue" them together to reconstruct ρ_{ABC}. (Petz 2002)
  - Such states have a special structure. (Hayden et al. 2003)
  - Reduced density matrices satisfy many nontrivial relations. (Leifer and Poulin 2006)
- Conditionally independent states naturally appear in virtually all the known exactly solvable topologically ordered systems.(Hastings, Poulin 2011)

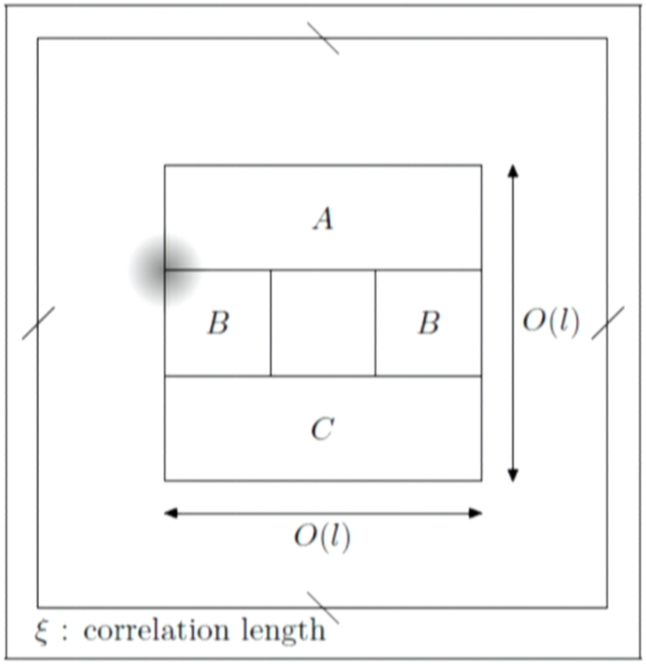

Why are conditionally independent states interesting? : Petz's theorem
Theorem 1. (Petz 2003) $I(A:C|B) = 0$ if and only if
$\hat{H}_{A:C|B} := I_C \otimes \log \rho_{AB} + I_A \otimes \log \rho_{BC} - I_{AC} \otimes \log \rho_B - \log \rho_{ABC} = 0.$
Corollary 1. Any first order perturbation of conditionally independent state vanishes.
$\frac{dI(A:C|B)}{ds} = \mathrm{Tr}(\frac{d\rho_{ABC}}{ds} \hat{H}_{A:C|B})$
$= 0$
Isaac H. Kim (Caltech)
Application of conditional independence to ga
December 12, 2012
12 / 42
Click on Comment and Share to create, mark-up and send PDF files.

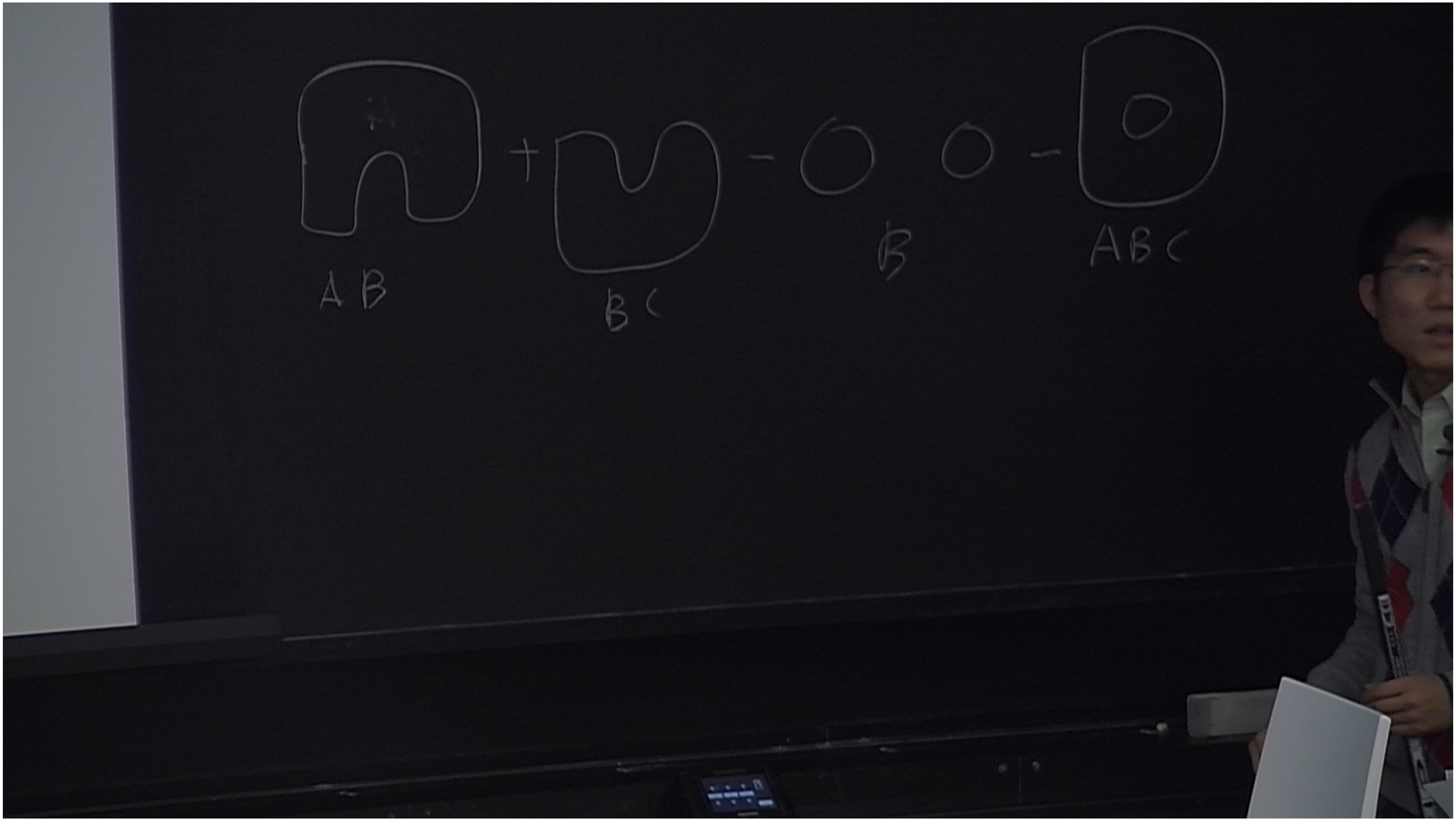
Application 1 : Topological entanglement entropy

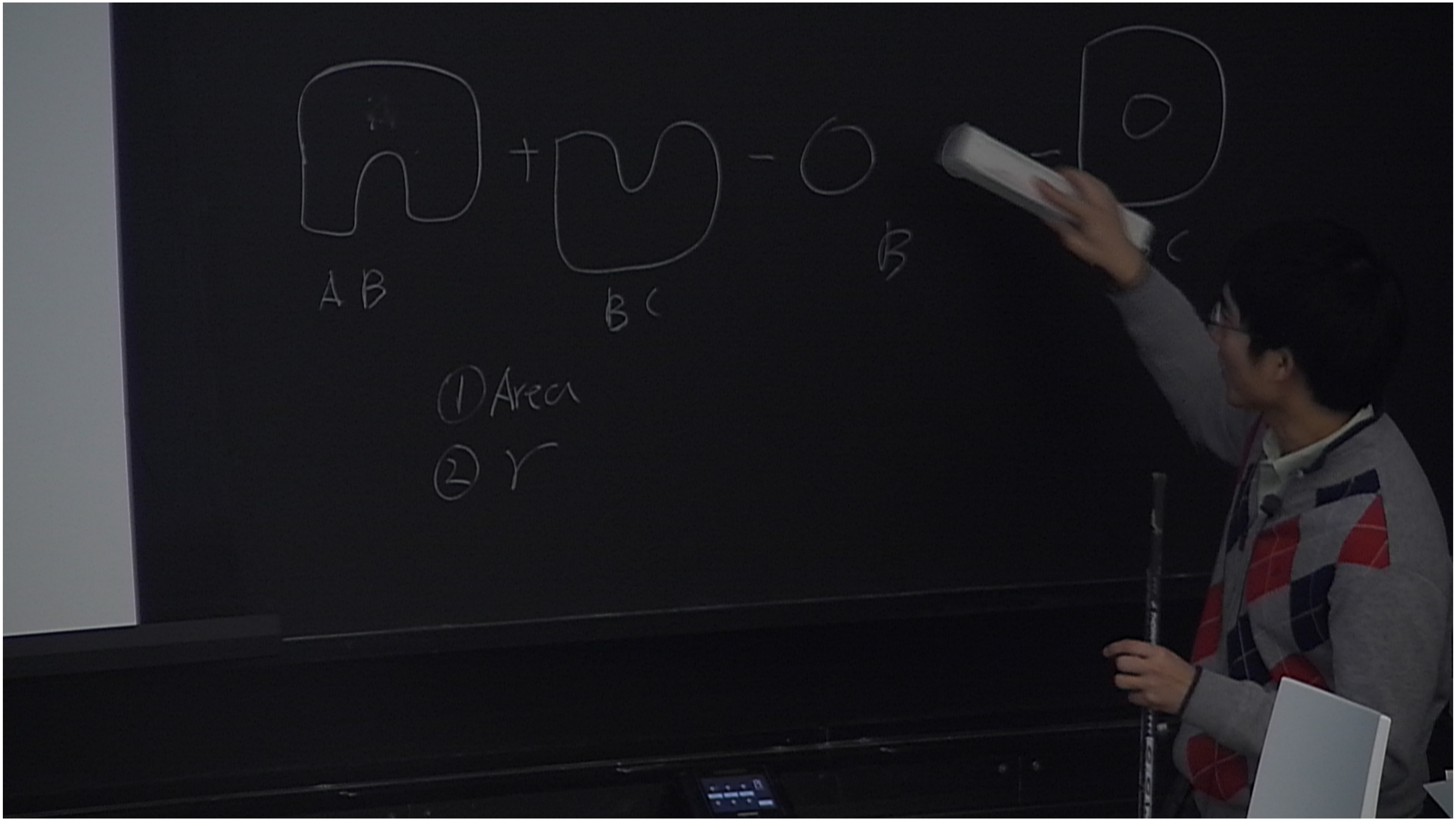
Problem : Consider a unitary transformation generated by a sum of local hamiltonian $H = \sum_i h_i$. What is $\frac{d\gamma}{ds}|_{s=0}$?

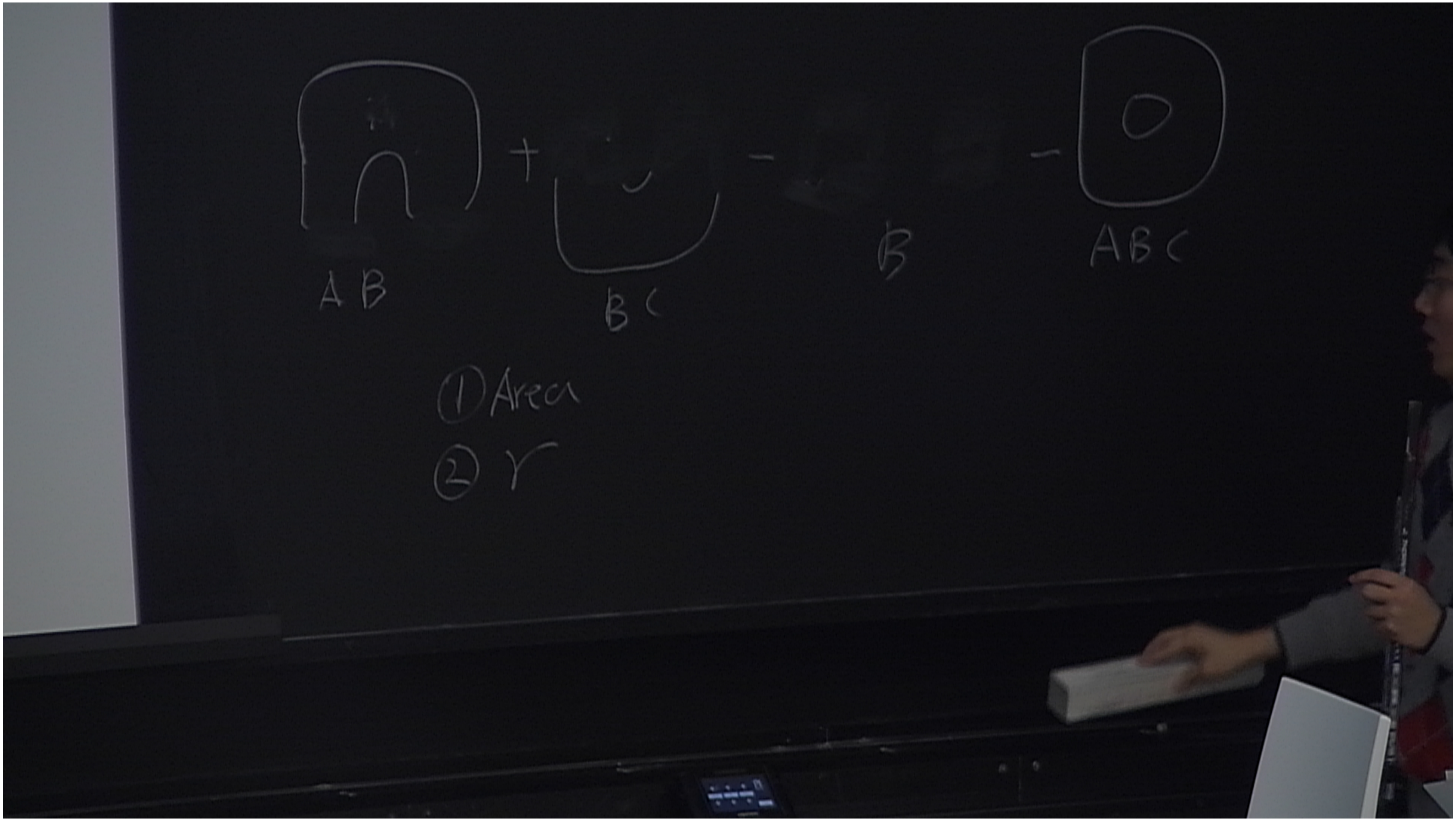
$$S_A = a|\partial A| - \gamma$$

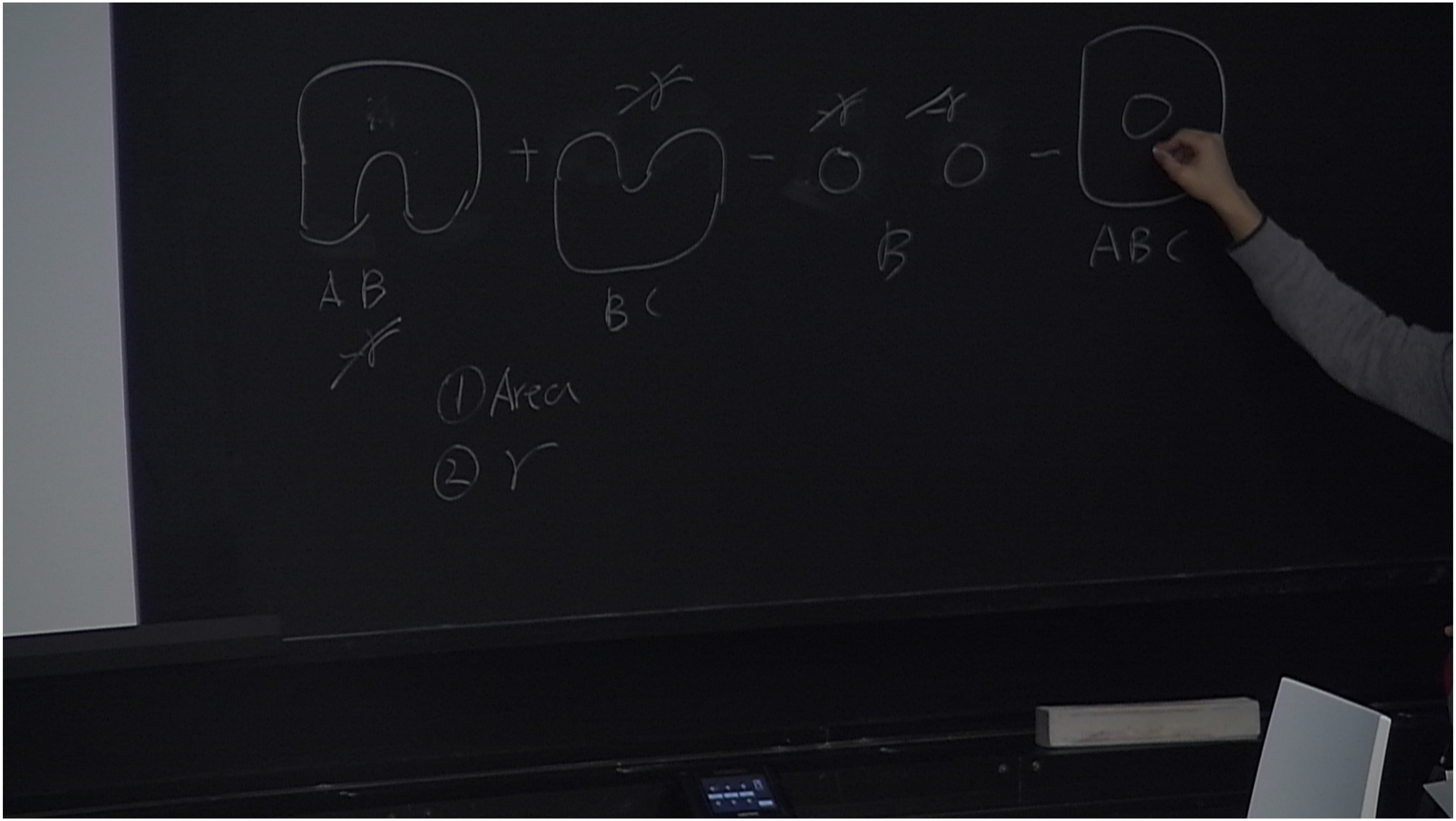
$$I(A:C|B) = S_{AB} + S_{BC} - S_B - S_{ABC}$$
$$= 2\gamma$$

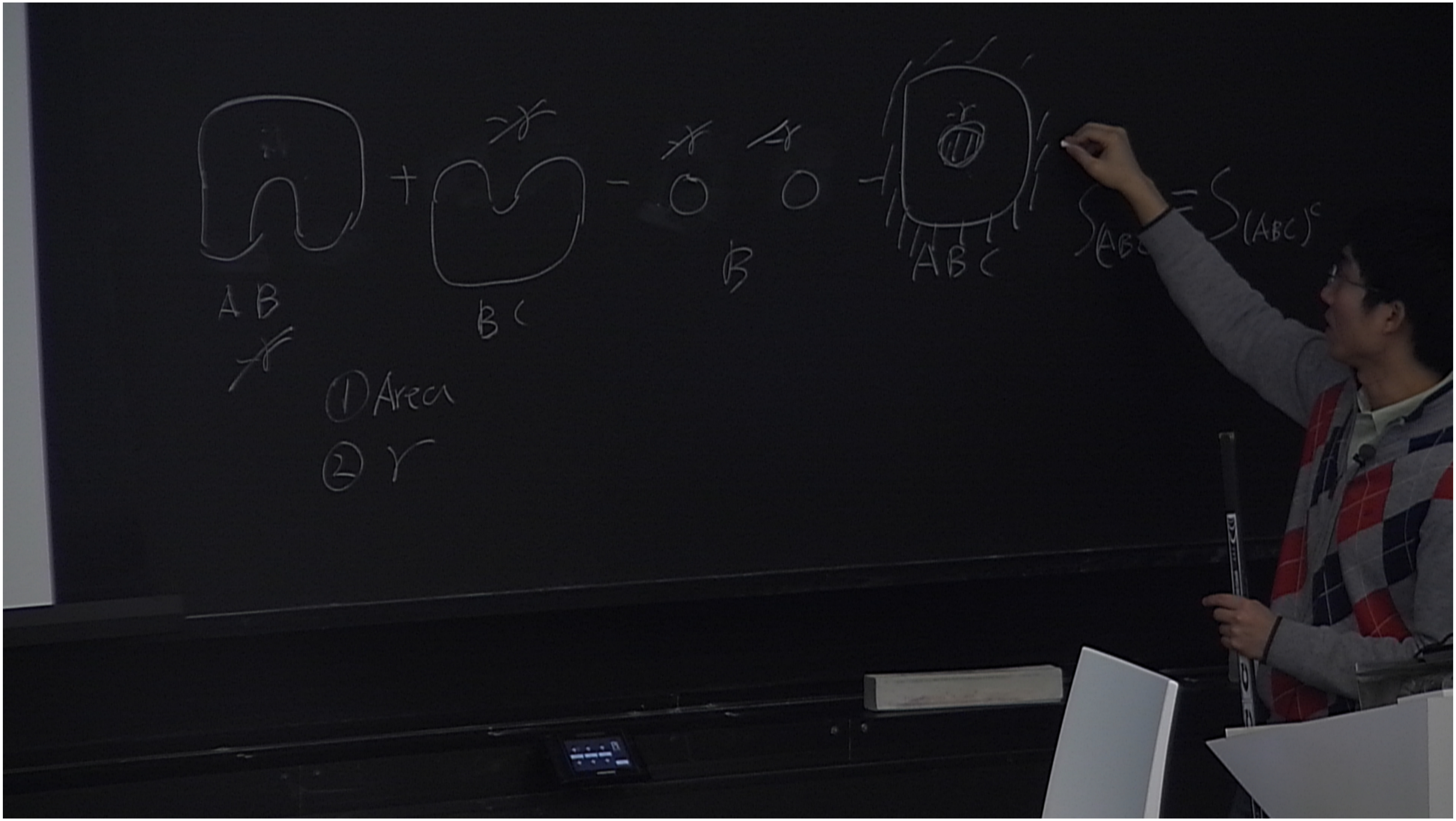



AB
BC
B
ABC

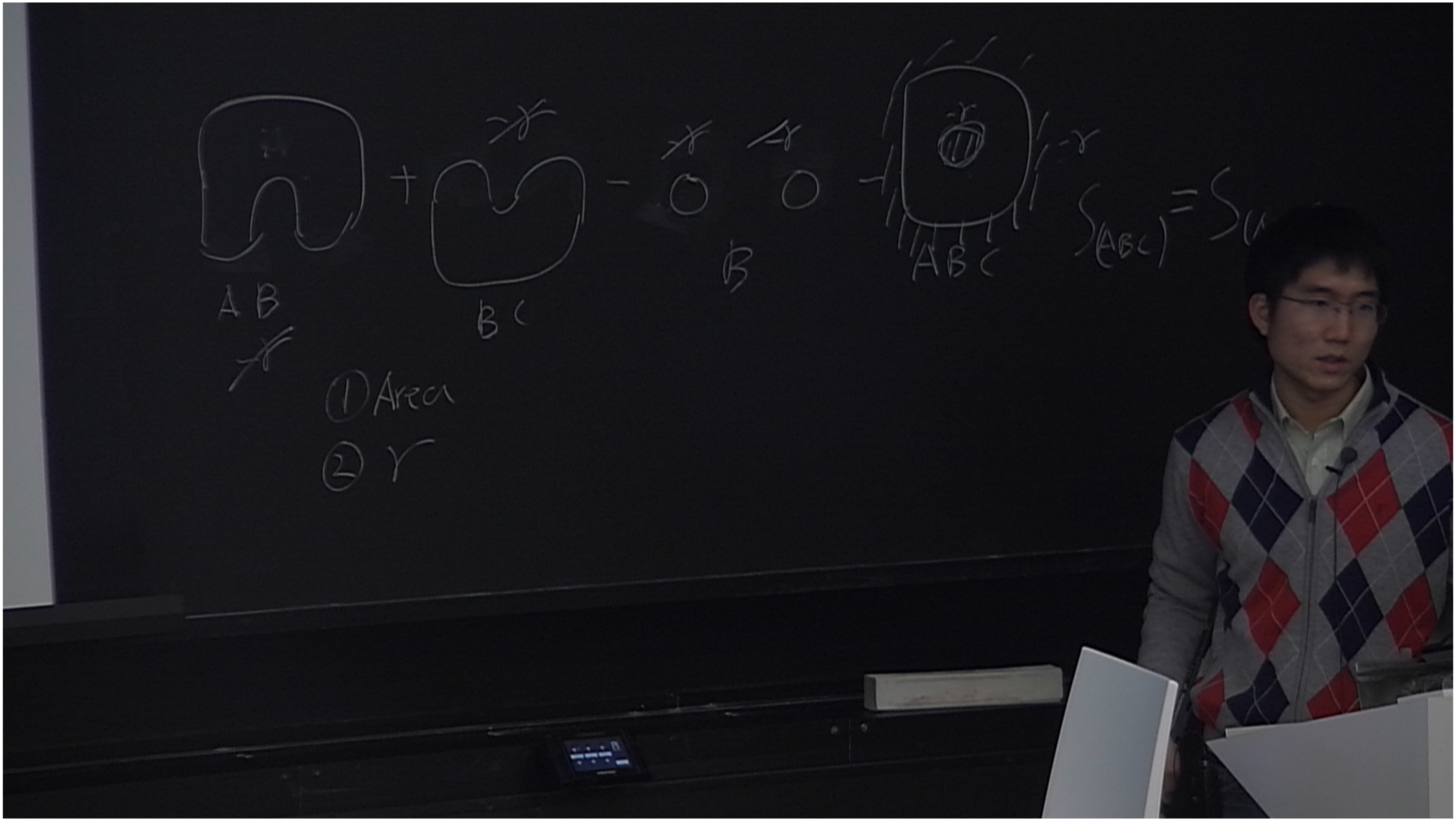

AB
BC
B
C
① Area
② γ


+
−
−
AB
BC
B
ABC
① Area
② γ


AB
BC
B
ABC
① Area
② γ



AB
BC
B
ABC
① Area
② γ



AB
BC
B
ABC
① Area
② γ

Application 1 : Topological entanglement entropy

Simplifying the problem

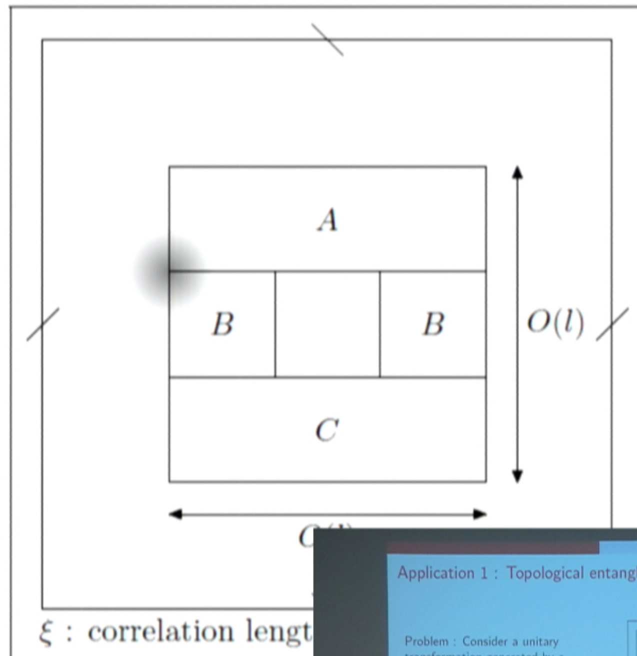
- First order perturbation : effect of perturbation can be decomposed into a sum of local contributions.
  - $\frac{dS_A}{ds} = i\sum_j \mathrm{Tr}([h_j, \rho] \log \rho_A)$.
- Local unitary transformation does not change entanglement entropy
  - $\mathrm{Tr}(-\rho_A \log(\rho_A)) = \mathrm{Tr}(-U_A^\dagger \rho_A U_A \log(U_A^\dagger \rho_A U_A))$.
  - It suffices to only consider the local terms on the boundary!

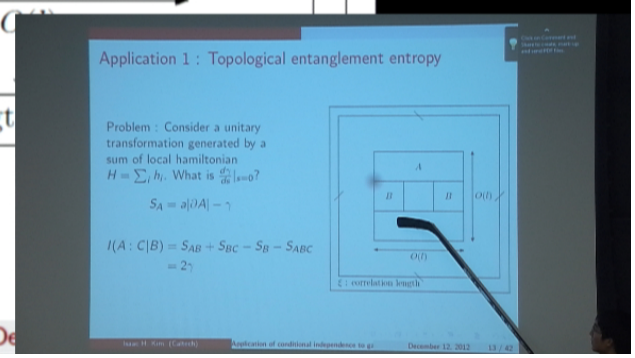
Application 1 : Topological entanglement entropy

Problem : Consider a unitary transformation generated by a sum of local hamiltonian $H = \sum_i h_i$. What is $\frac{d\gamma}{ds}|_{s=0}$?

$$S_A = a|\partial A| - \gamma$$

$$I(A : C|B) = S_{AB} + S_{BC} - S_B - S_{ABC}$$
$$= 2\gamma$$





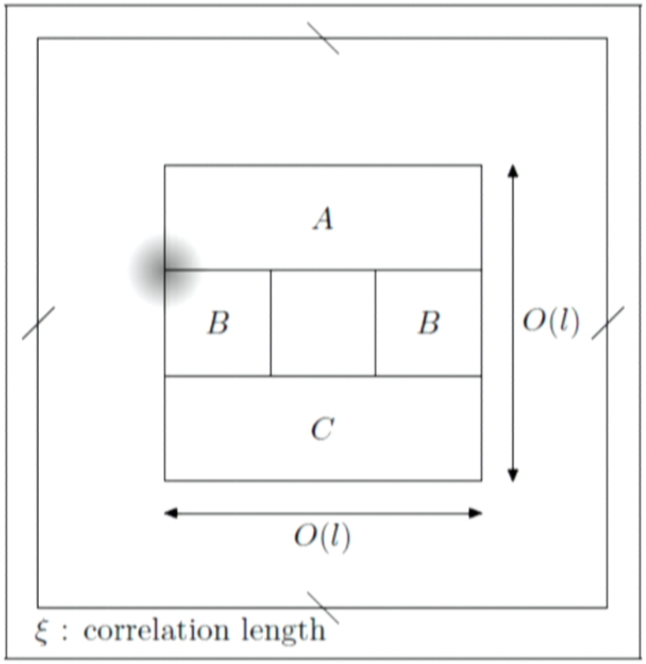
Click on Comment and Share to create, mark-up and send PDF files.

Application 1 : Topological entanglement entropy

Problem : Consider a unitary transformation generated by a sum of local hamiltonian $H = \sum_i h_i$. What is $\frac{d\gamma}{ds}|_{s=0}$?

$$S_A = a|\partial A| - \gamma$$

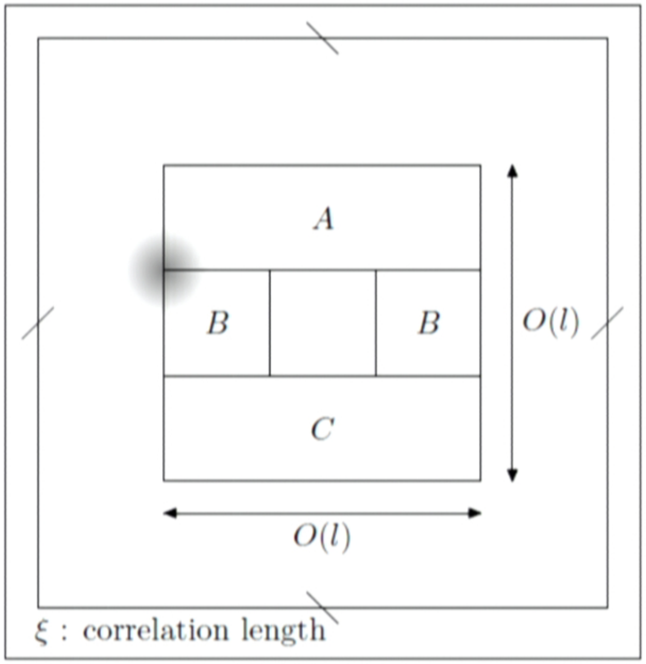
$$I(A:C|B) = S_{AB} + S_{BC} - S_B - S_{ABC}$$
$$= 2\gamma$$

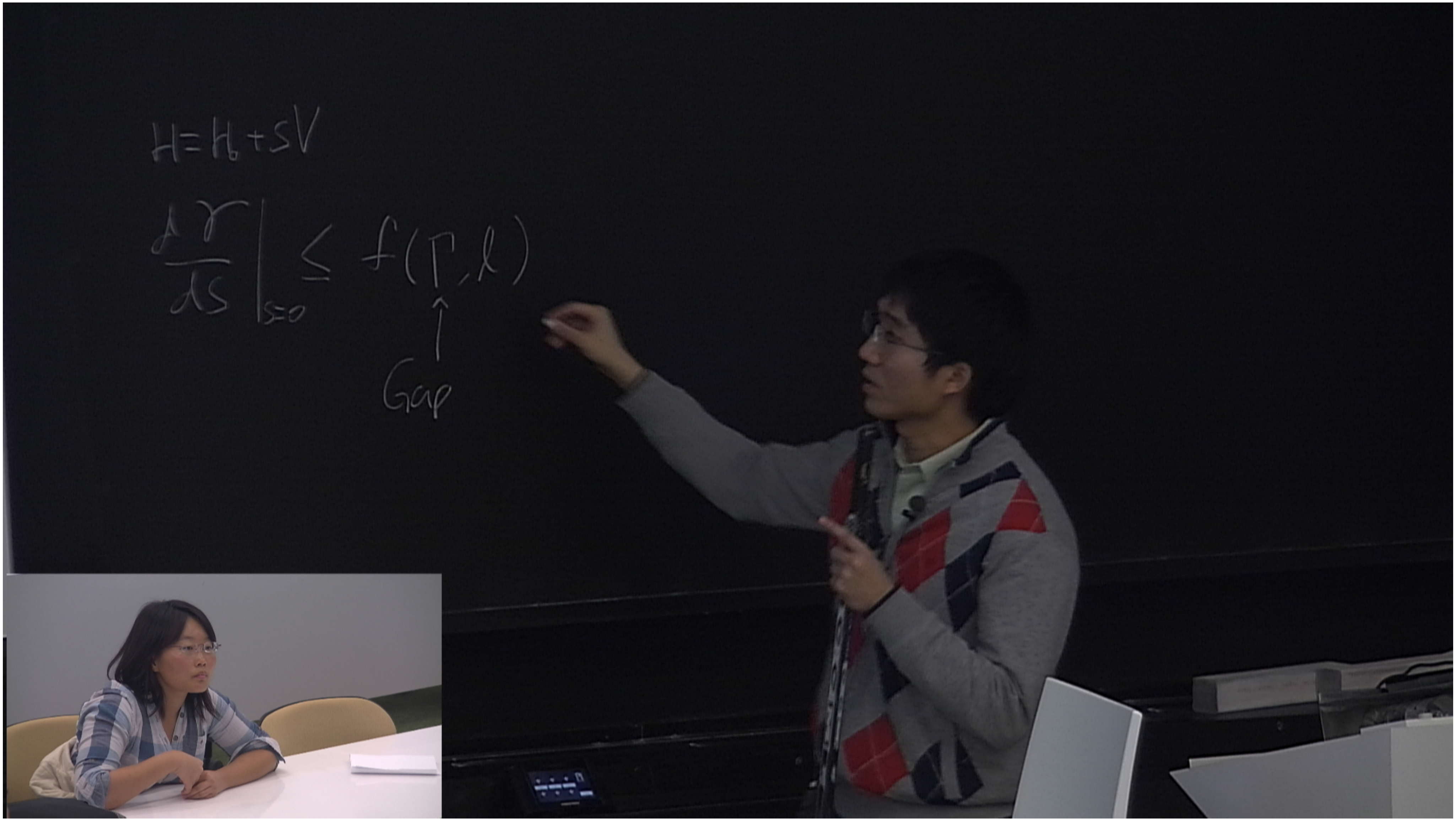


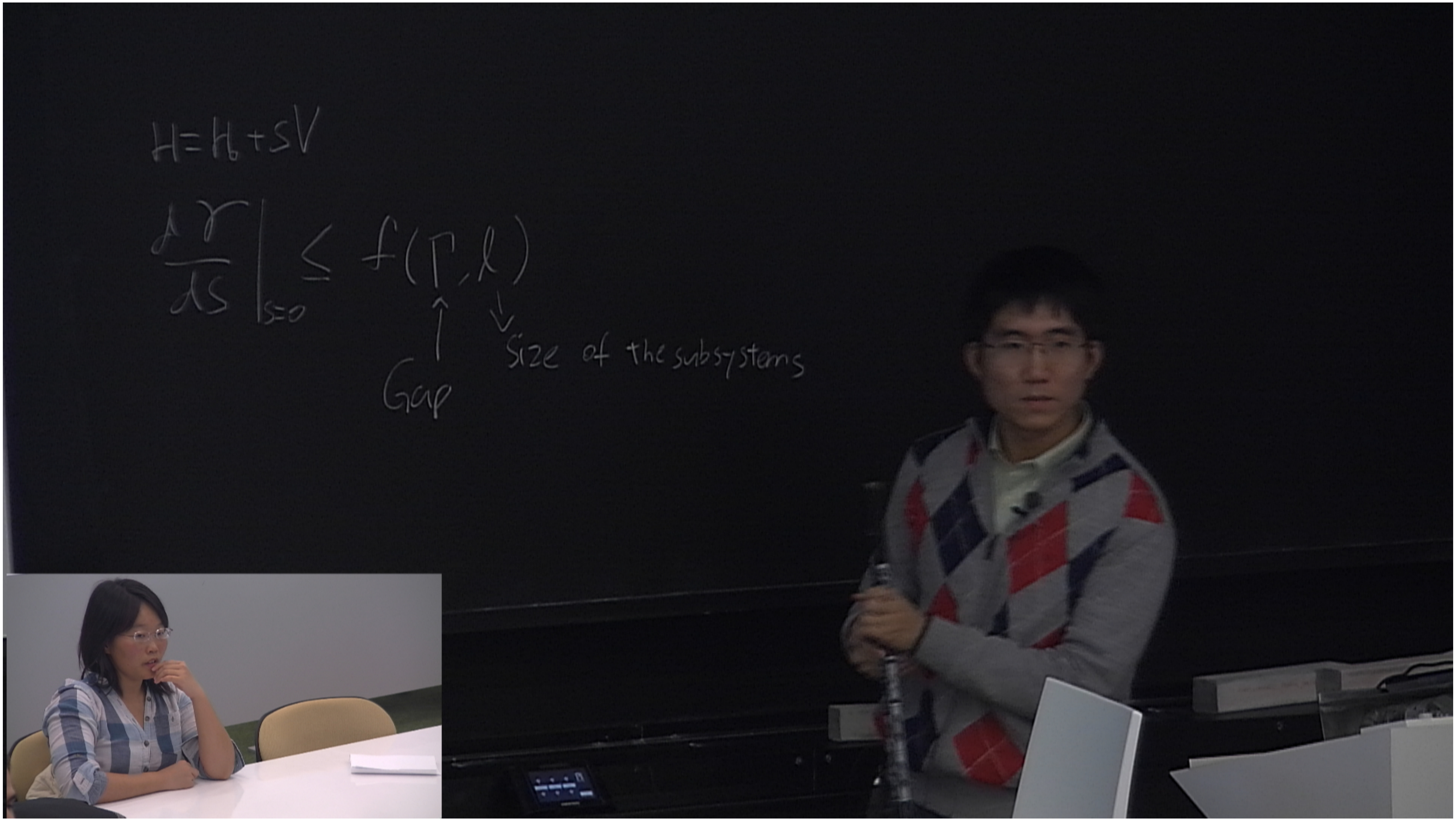
Application 1 : Topological entanglement entropy

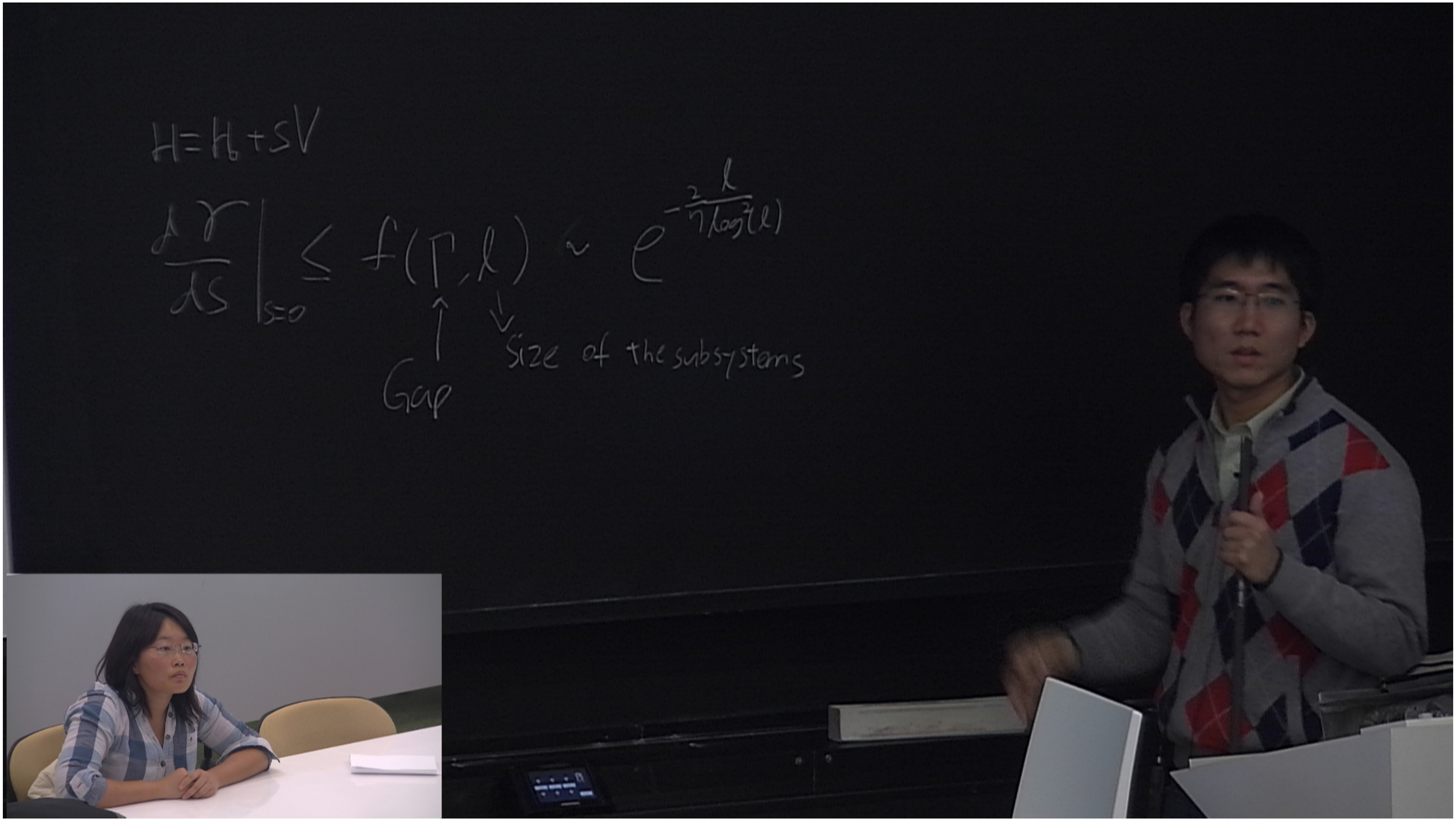
Observation 1

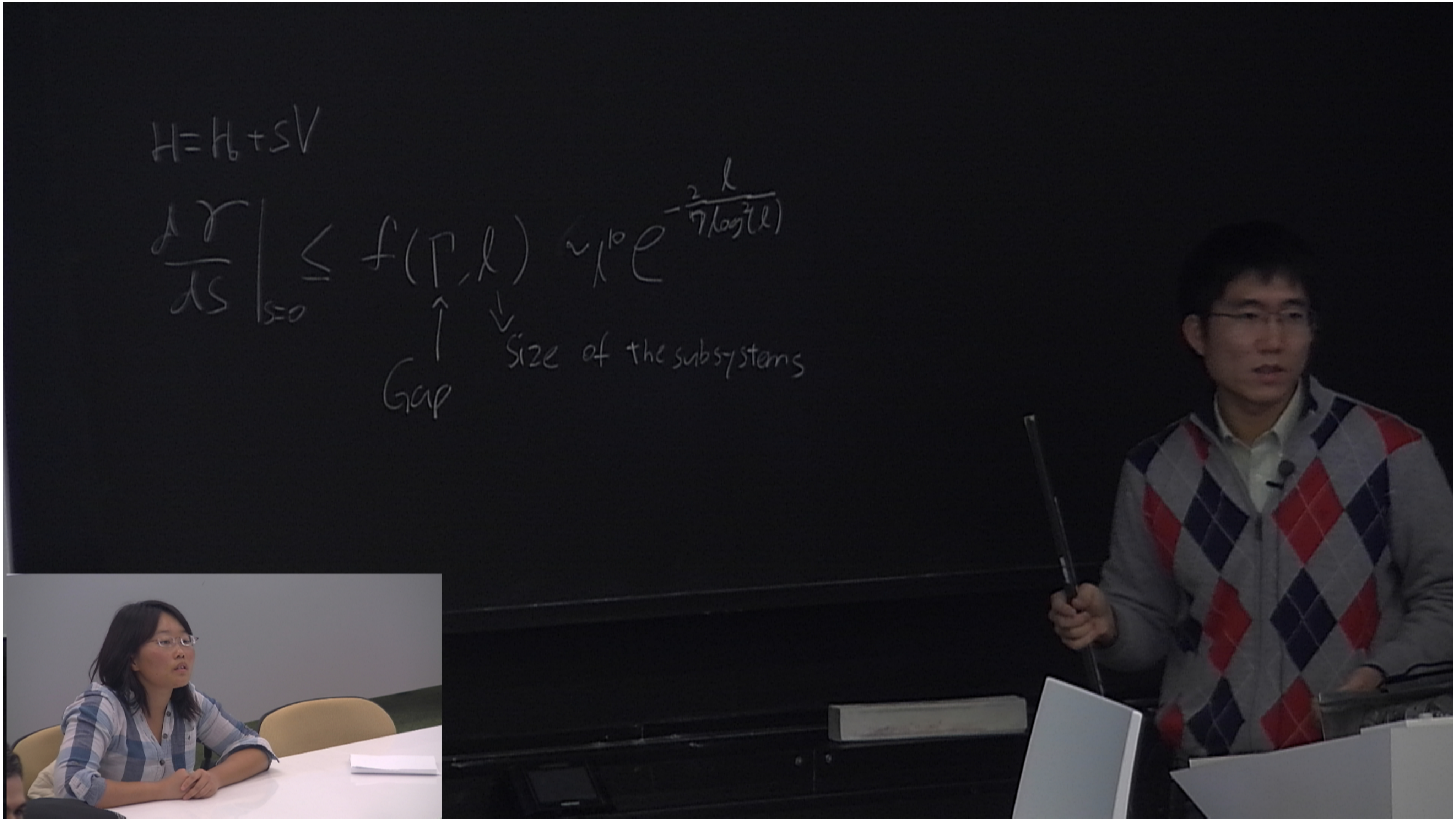
- $I(A : C|B) = I(C : A|B)$.
  - A, C : *target parties* ('T')
  - B : *reference party* ('R')




$H = H_0 + sV$
$\left.\frac{d\gamma}{ds}\right|_{s=0} \leq f(\Gamma, \ell)$
Gap

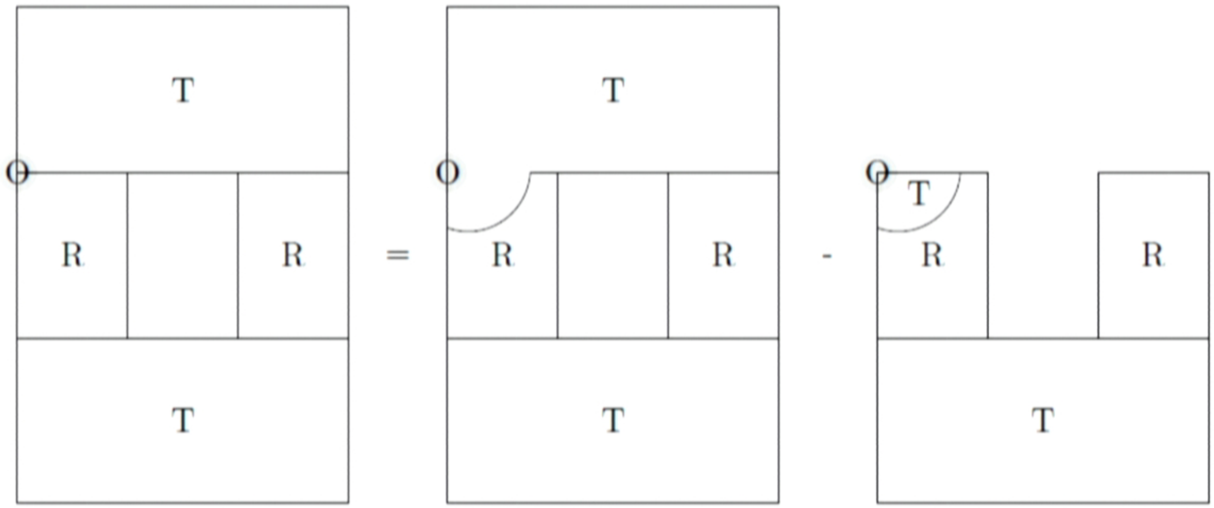

$H=H_0+sV$
$\frac{d\gamma}{ds}\Big|_{s=0} \leq f(\Gamma, \ell)$
Gap
Size of the subsystems


H=H0+sV
Size of the subsystems
Gap


H=H0+sV
dγ/ds |s=0 ≤ f(Γ, l) ~ l^10 e^(-2 l / (7 log^2(l)))
Gap
Size of the subsystems

Application 1 : Topological entanglement entropy

Isolation move : Isolate the unitary away from the reference party.



Click on Comment and Share to create, mark-up and send PDF files.

Why are conditionally independent states interesting? : Petz's theorem

Theorem 1. (Petz 2003) $I(A:C|B) = 0$ if and only if

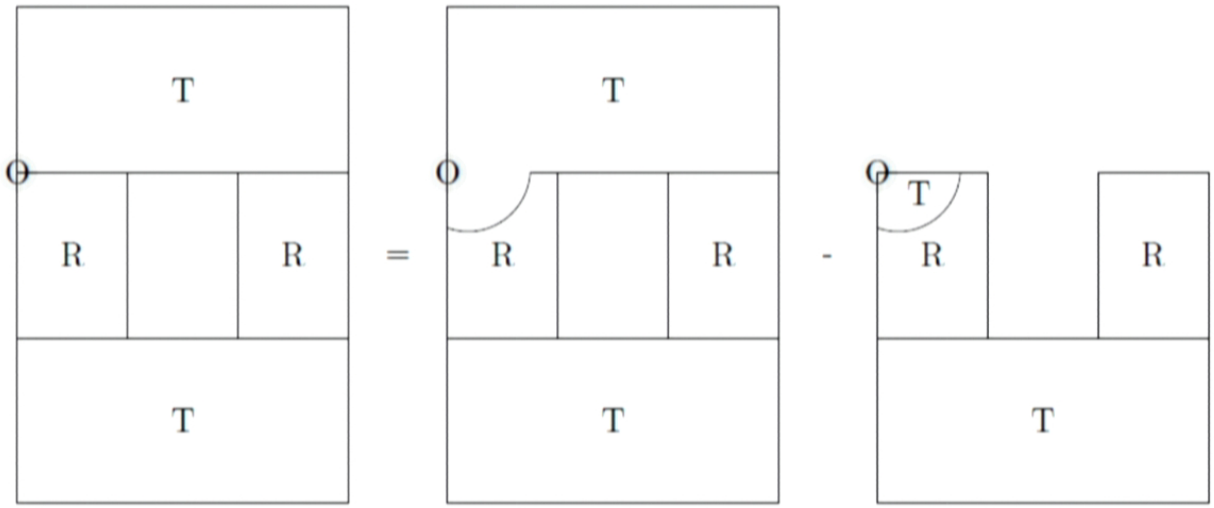
$$\hat{H}_{A:C|B} := I_C \otimes \log \rho_{AB} + I_A \otimes \log \rho_{BC} - I_{AC} \otimes \log \rho_B - \log \rho_{ABC} = 0.$$

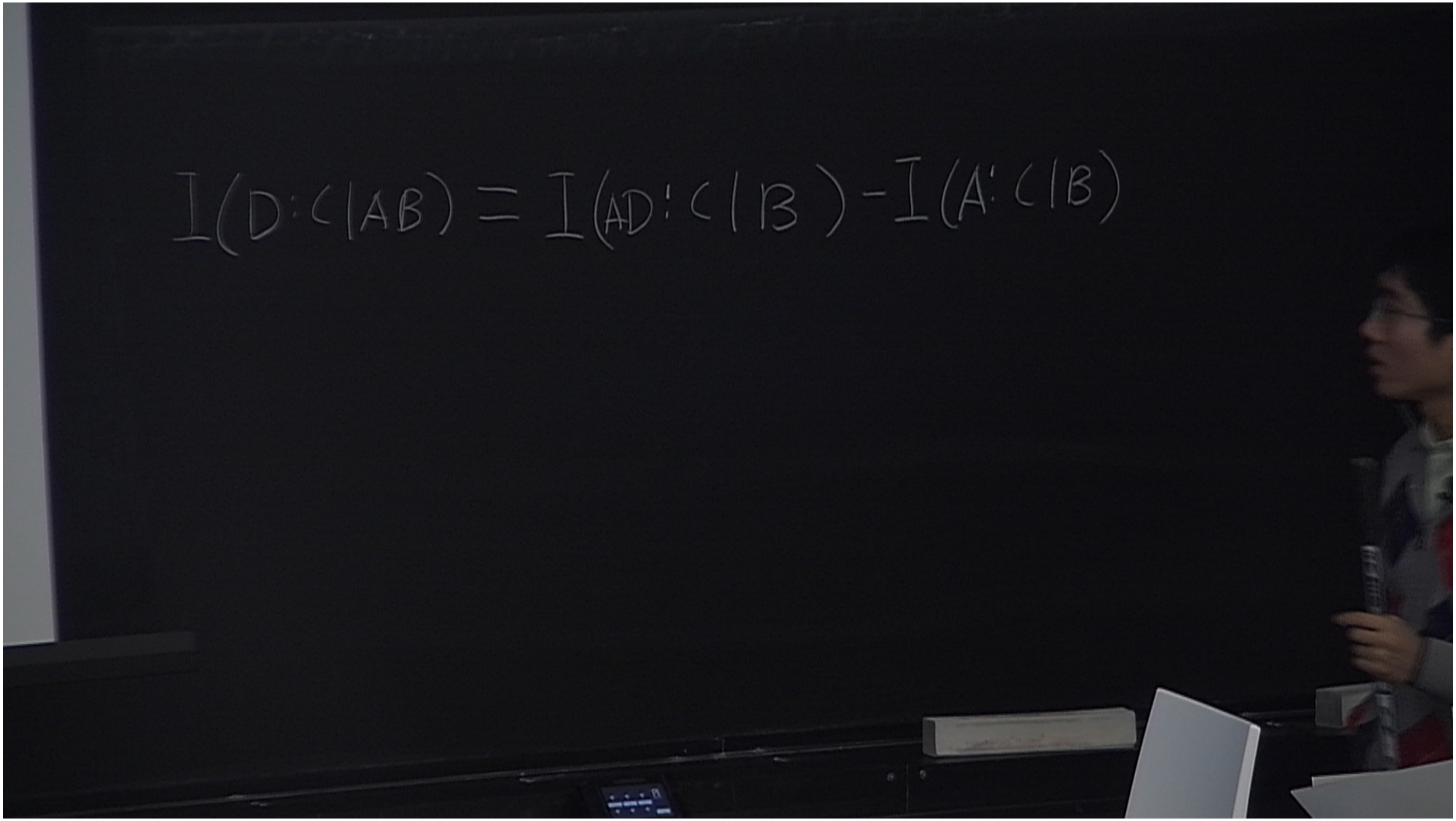
Corollary 1. Any first order perturbation of conditionally independent state vanishes.

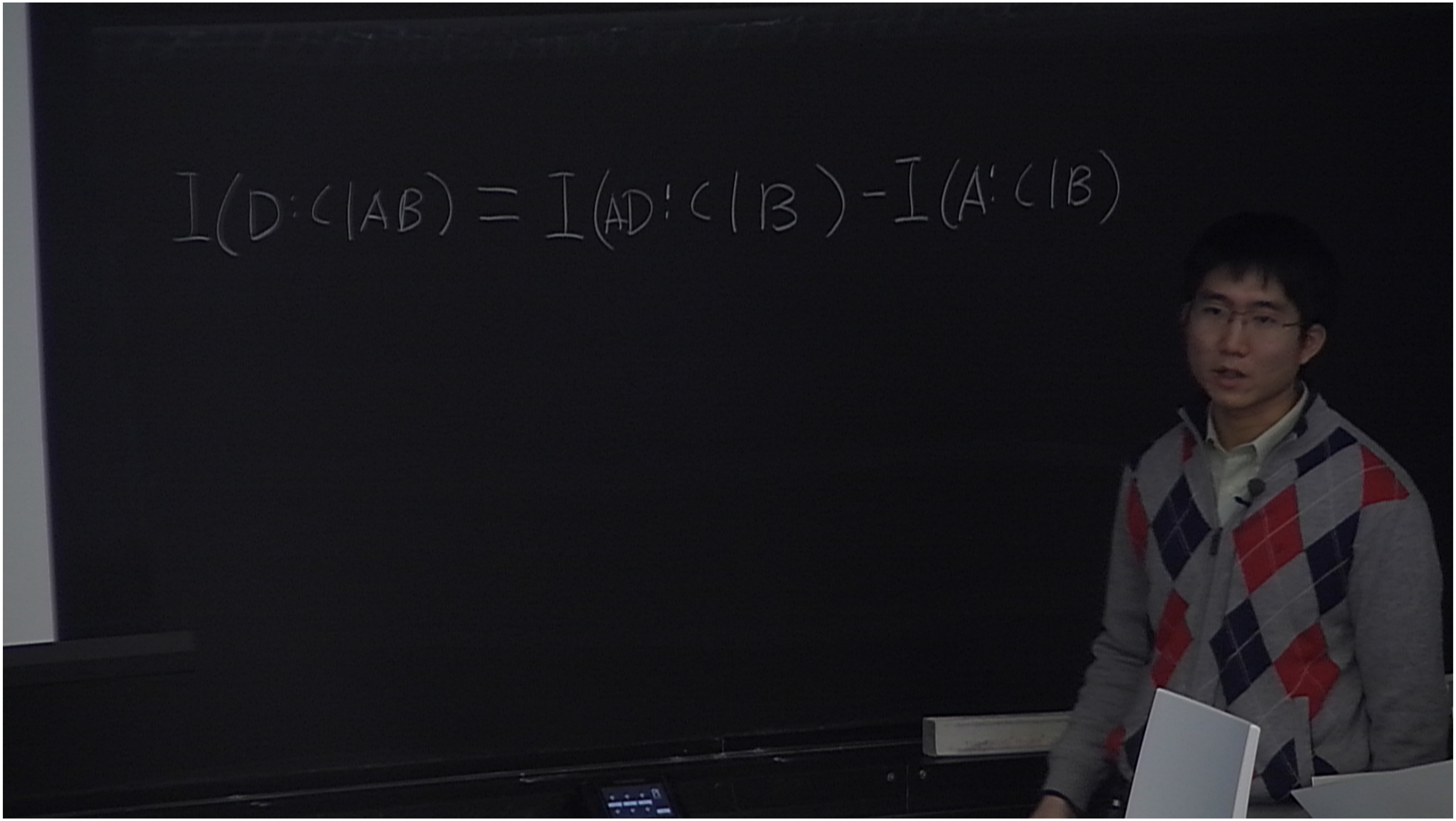
$$\frac{dI(A:C|B)}{ds} = \mathrm{Tr}\left(\frac{d\rho_{ABC}}{ds}\hat{H}_{A:C|B}\right)$$
$$= 0$$

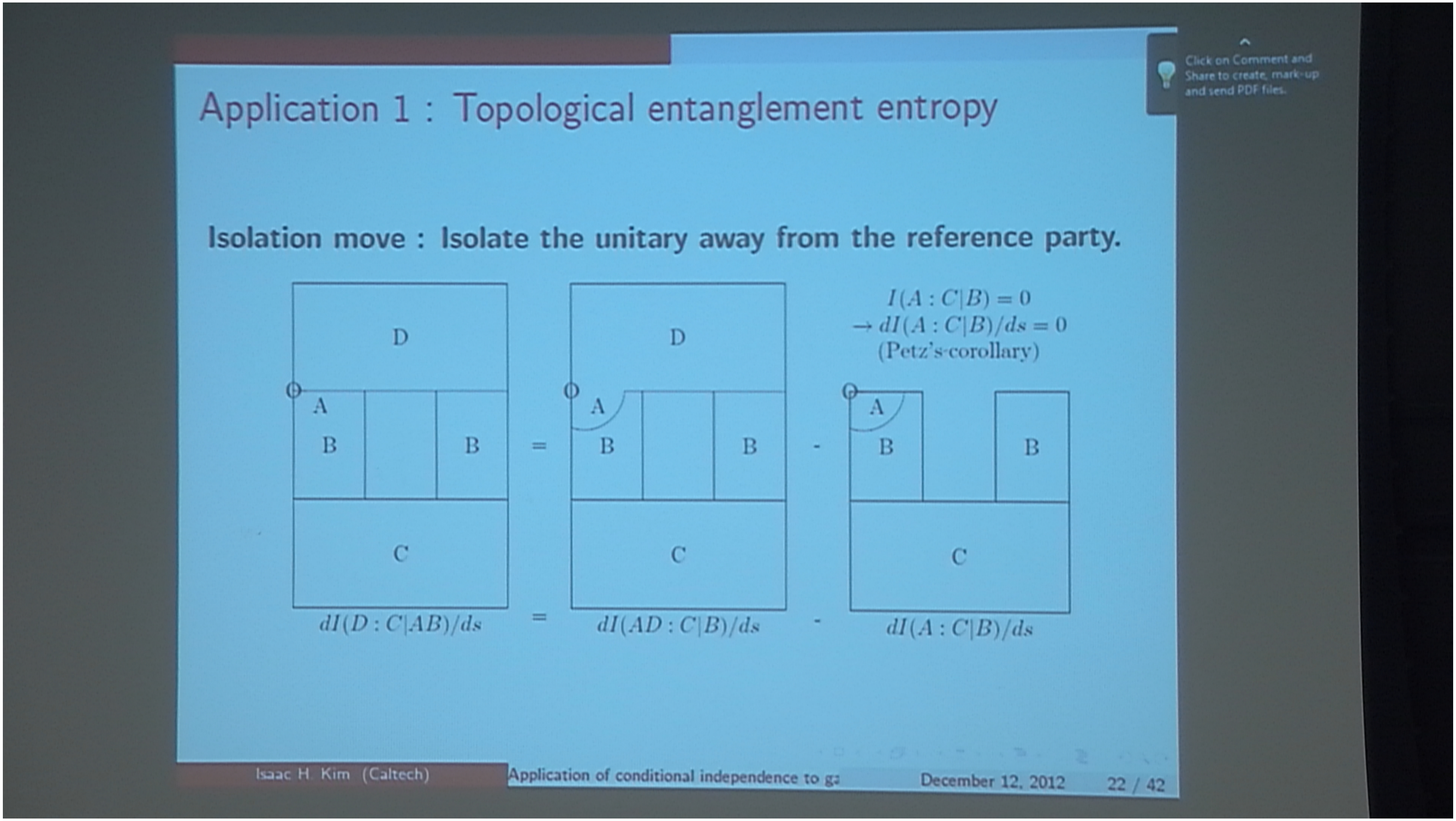
Application 1 : Topological entanglement entropy

Isolation move : Isolate the unitary away from the reference party.



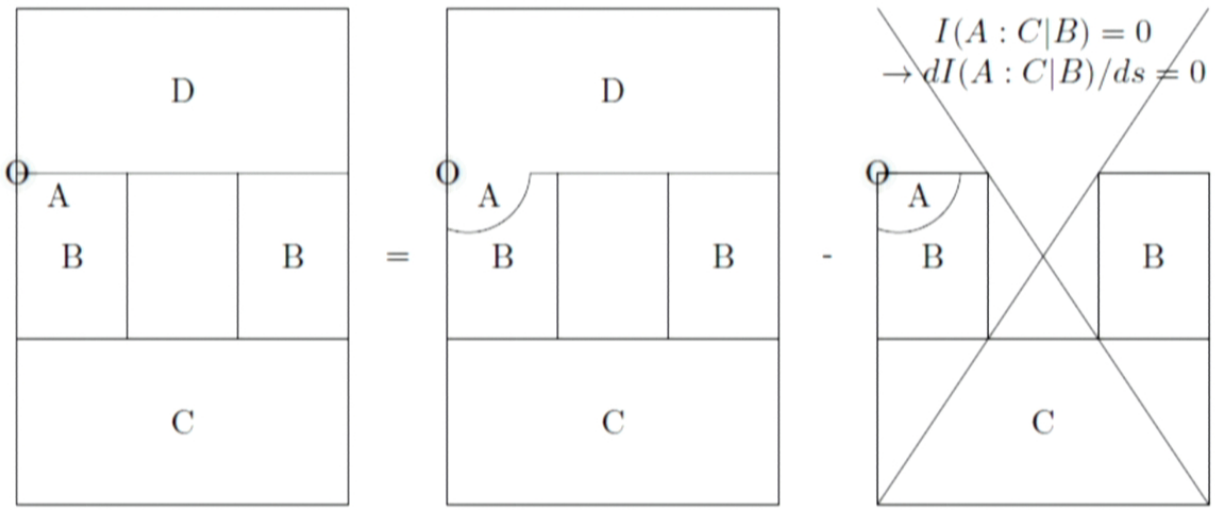

I(D:C|AB) = I(AD:C|B) - I(A:C|B)


I(D:C|AB) = I(AD:C|B) - I(A:C|B)


Application 1 : Topological entanglement entropy
Isolation move : Isolate the unitary away from the reference party.
$I(A:C|B) = 0$
$\rightarrow dI(A:C|B)/ds = 0$
(Petz's corollary)
D
A
B
B
C
$dI(D:C|AB)/ds$
=
$dI(AD:C|B)/ds$
-
$dI(A:C|B)/ds$
Isaac H. Kim (Caltech)
Application of conditional independence to ga
December 12, 2012
22 / 42
Click on Comment and Share to create, mark-up and send PDF files.

Application 1 : Topological entanglement entropy

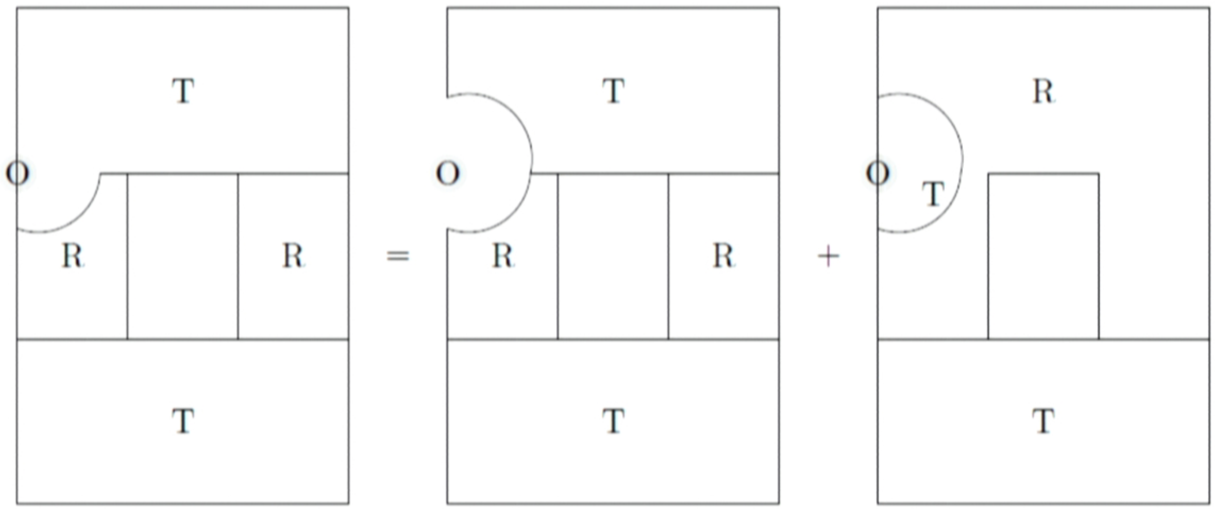
Isolation move : Isolate the unitary away from the reference party.



$$dI(D:C|AB)/ds = dI(AD:C|B)/ds - dI(A:C|B)/ds$$

Application 1 : Topological entanglement entropy

Separation move : Separate the unitary away from the reference party.



A segway to the next point...

When exact conditional independence is satisfied, entanglement spectrum can be "canceled out."

$$I(A:C|B) = 0 \longleftrightarrow \hat{H}_{A:C|B} = 0$$

What happens if conditional mutual information is approximately 0? There are many motivations to study such scenario.

- Condensed matter theorists : This is a more realistic assumption.
- Quantum information theorist : Very few is known about the structure of states that are approximately conditional independent.

A segway to the next point...

When exact conditional independence is satisfied, entanglement spectrum can be "canceled out."

$$I(A:C|B) = 0 \longleftrightarrow \hat{H}_{A:C|B} = 0$$

What happens if conditional mutual information is approximately 0? There are many motivations to study such scenario.

- Condensed matter theorists : This is a more realistic assumption.
- Quantum information theorist : Very few is known about the structure of states that are approximately conditional independent.

A segway to the next point...

Main Result 2: Operator extension of strong subadditivity

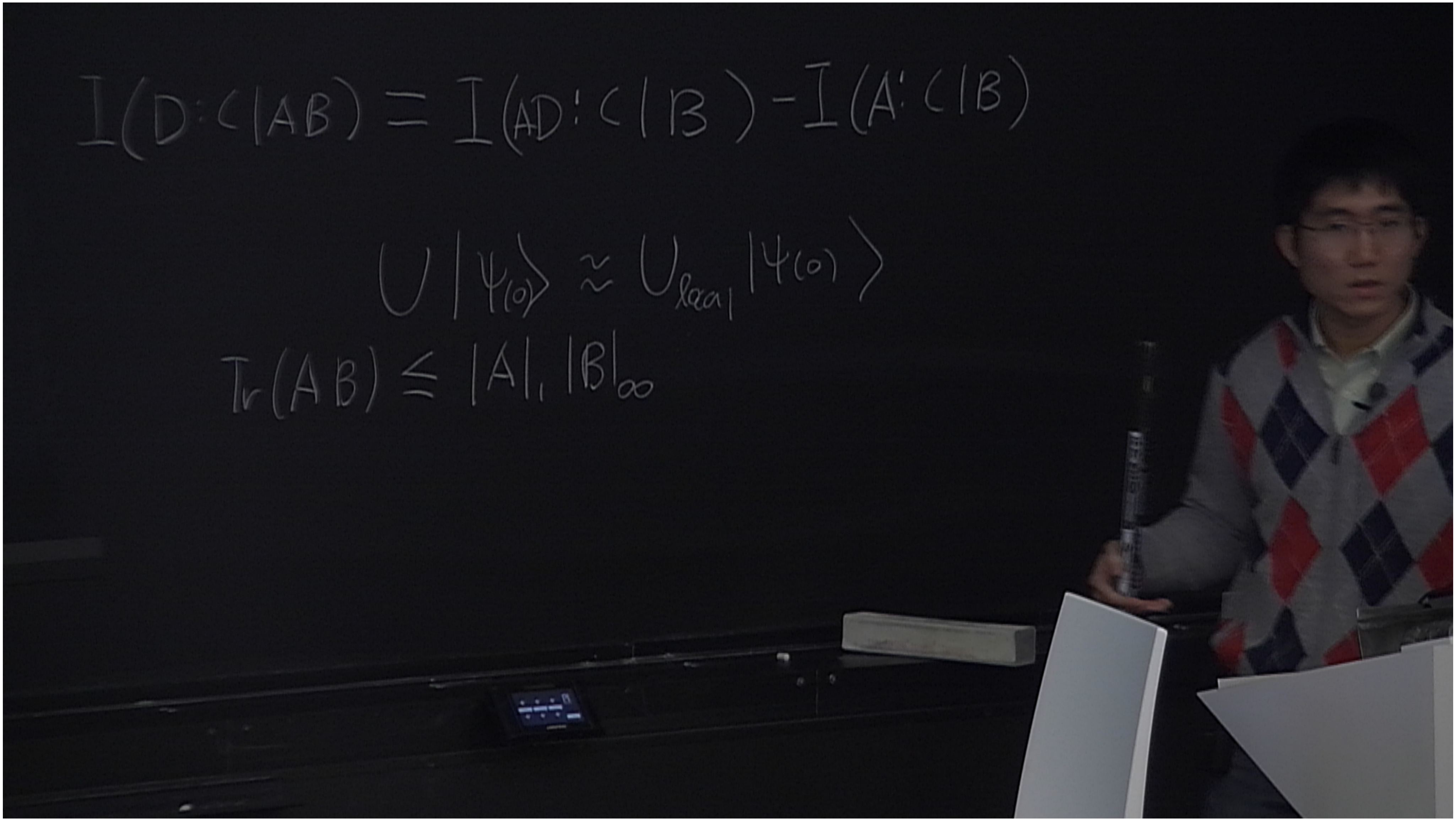
$$\mathrm{Tr}_{AB}(\rho_{ABC}\hat{H}_{A:C|B}) \geq 0.$$

Corollary 2:

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_C) \leq I(A:C|B)\|O_C\|.$$

Proof :

$$\begin{aligned}\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_C) &\leq \|O_C\||\mathrm{Tr}_{AB}(\rho_{ABC}\hat{H}_{A:C|B})|_1 \\ &= \|O_C\|I(A:C|B).\end{aligned}$$


I(D:C|AB) = I(AD:C|B) - I(A:C|B)
U|ψ(0)⟩ ≈ U_local|ψ(0)⟩
Tr(AB) ≤ |A|_1 |B|_∞

A segway to the next point...

Main Result 2: Operator extension of strong subadditivity

$$\mathrm{Tr}_{AB}(\rho_{ABC}\hat{H}_{A:C|B}) \geq 0.$$

This isn't quite strong enough to prove the stability of topological entanglement entropy, but it has other applications.

Click on Comment and Share to create, mark-up and send PDF files.

A segway to the next point...

Main Result 2: Operator extension of strong subadditivity

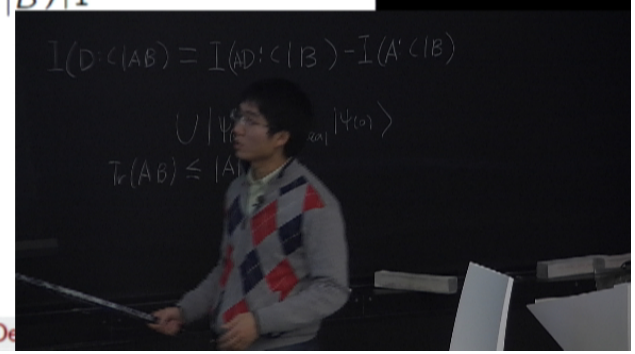
$$\mathrm{Tr}_{AB}(\rho_{ABC}\hat{H}_{A:C|B}) \geq 0.$$

Corollary 2:

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_C) \leq I(A:C|B)\|O_C\|.$$

Proof :

$$\begin{aligned}\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_C) &\leq \|O_C\||\mathrm{Tr}_{AB}(\rho_{ABC}\hat{H}_{A:C|B})|_1 \\ &= \|O_C\|I(A:C|B).\end{aligned}$$



A segway to the next point...

Main Result 2: Operator extension of strong subadditivity

$$\mathrm{Tr}_{AB}(\rho_{ABC}\hat{H}_{A:C|B}) \geq 0.$$

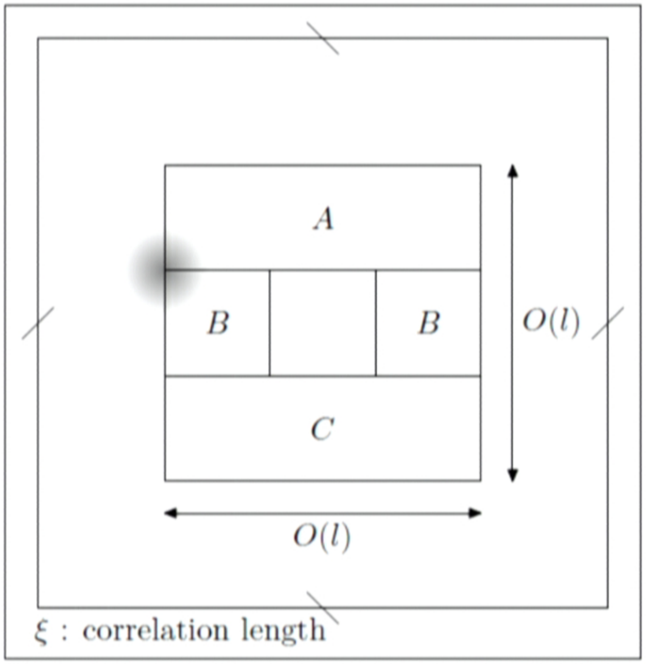
This isn't quite strong enough to prove the stability of topological entanglement entropy, but it has other applications.

Click on Comment and Share to create, mark-up and send PDF files.

Click on Comment and Share to create, mark-up and send PDF files.

Application 2 : Entanglement spectrum

Problem : If area law is satisfied approximately, do entanglement spectrum cancel out each other? Answer : Yes!(with some caveats)



Application 2 : Entanglement spectrum

Easy case:
For an operator $O_C \in \mathcal{B}(\mathcal{H}_C)$,

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_C) \leq \|O_C\| I(A:C|B)$$

Application 2 : Entanglement spectrum

Less trivial case:
For an operator $O_B \in \mathcal{B}(\mathcal{H}_B)$,

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_B) \nleq \|O_B\| I(A:C|B)$$

Application 2 : Entanglement spectrum

Less trivial case:
For an operator $O_B \in \mathcal{B}(\mathcal{H}_B)$,

$$\mathrm{Tr}(\rho_{ABC} \hat{H}_{A:C|B} O_B) \not\lesssim \|O_B\| I(A:C|B)$$

Also, what if $I(A:C|B) = 2\gamma$ is not small?

Click on Comment and Share to create, mark-up and send PDF files.

Application 2 : Entanglement spectrum

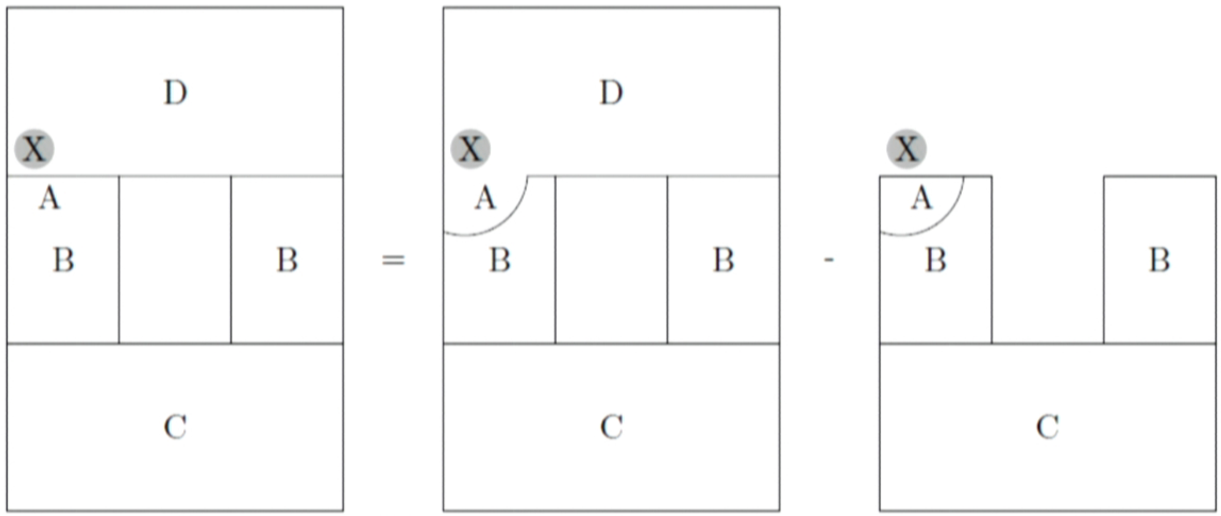
Less trivial case:
For an operator $O_B \in \mathcal{B}(\mathcal{H}_B)$,

$$\mathrm{Tr}(\rho_{ABC}\hat{H}_{A:C|B}O_B) \not\leq \|O_B\| I(A:C|B)$$

Also, what if $I(A:C|B) = 2\gamma$ is not small?
Apply deformation moves!

Application 2 : Entanglement spectrum

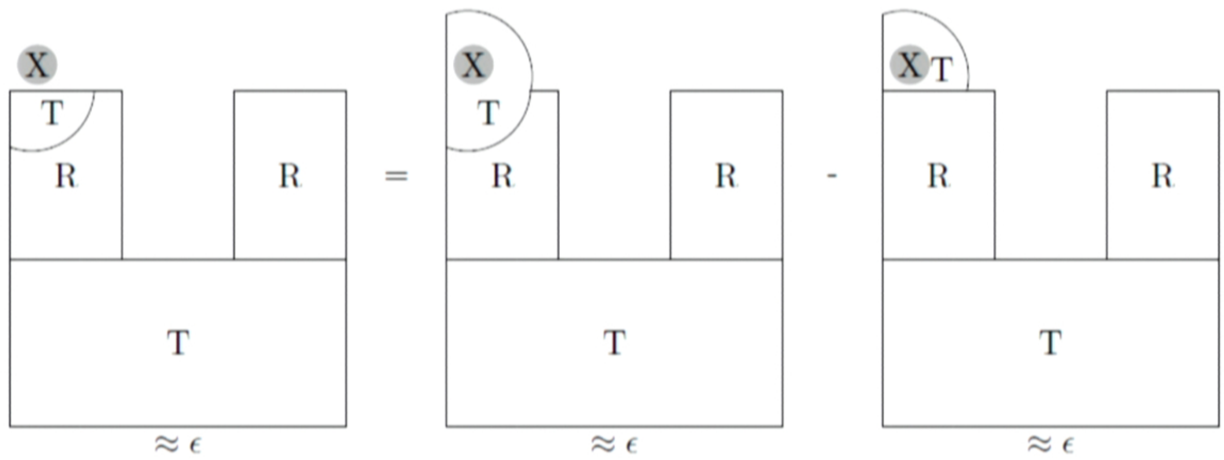
Isolation move revisited

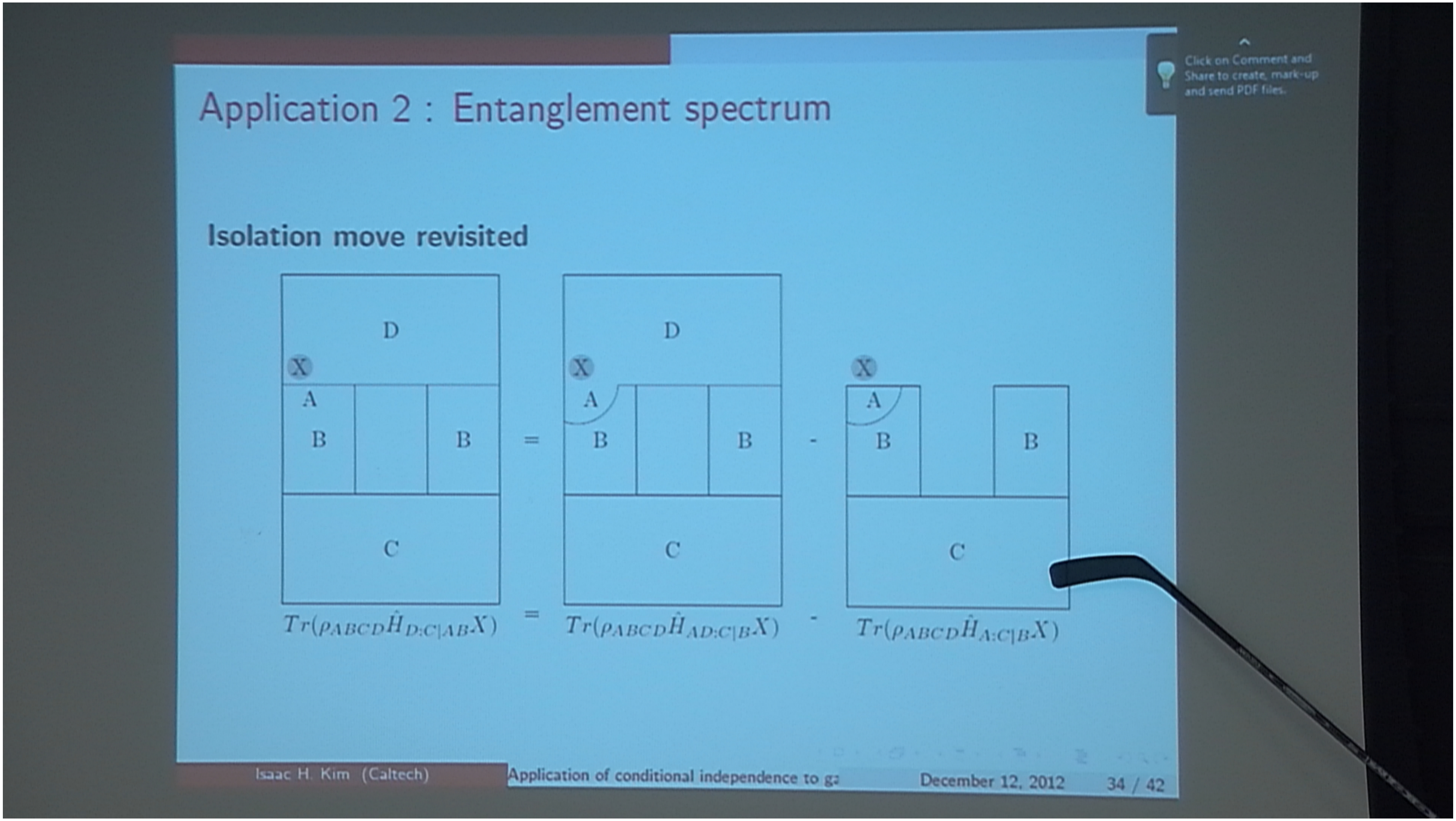


$$Tr(\rho_{ABCD}\hat{H}_{D:C|AB}X) = Tr(\rho_{ABCD}\hat{H}_{AD:C|B}X) - Tr(\rho_{ABCD}\hat{H}_{A:C|B}X)$$

Application 2 : Entanglement spectrum

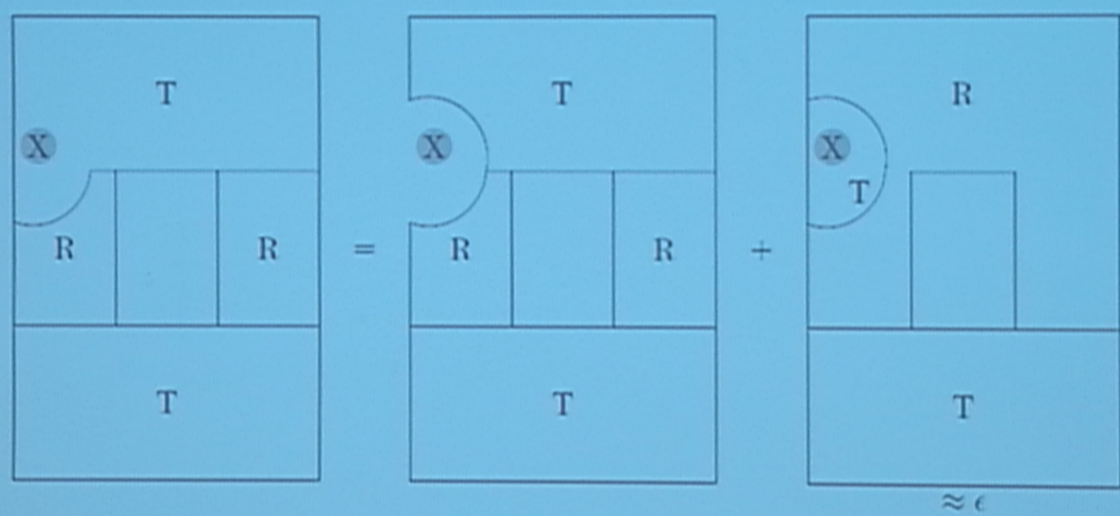
Absorption move




Application 2 : Entanglement spectrum
Isolation move revisited
D
X
A
B
B
=
D
X
A
B
B
-
X
A
B
B
C
C
C
$Tr(\rho_{ABCD}\hat{H}_{D:C|AB}X)$ = $Tr(\rho_{ABCD}\hat{H}_{AD:C|B}X)$ - $Tr(\rho_{ABCD}\hat{H}_{A:C|B}X)$
Isaac H. Kim (Caltech)
Application of conditional independence to ga
December 12, 2012
34 / 42
Click on Comment and Share to create, mark-up and send PDF files.

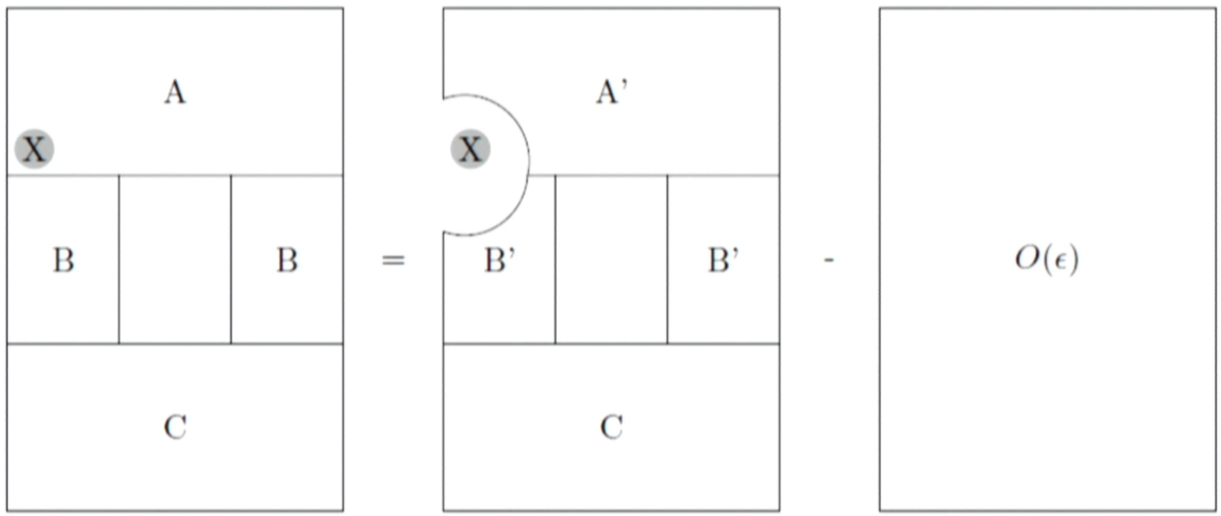
Application 2 : Entanglement spectrum

Separation move revisited



Application 2 : Entanglement spectrum

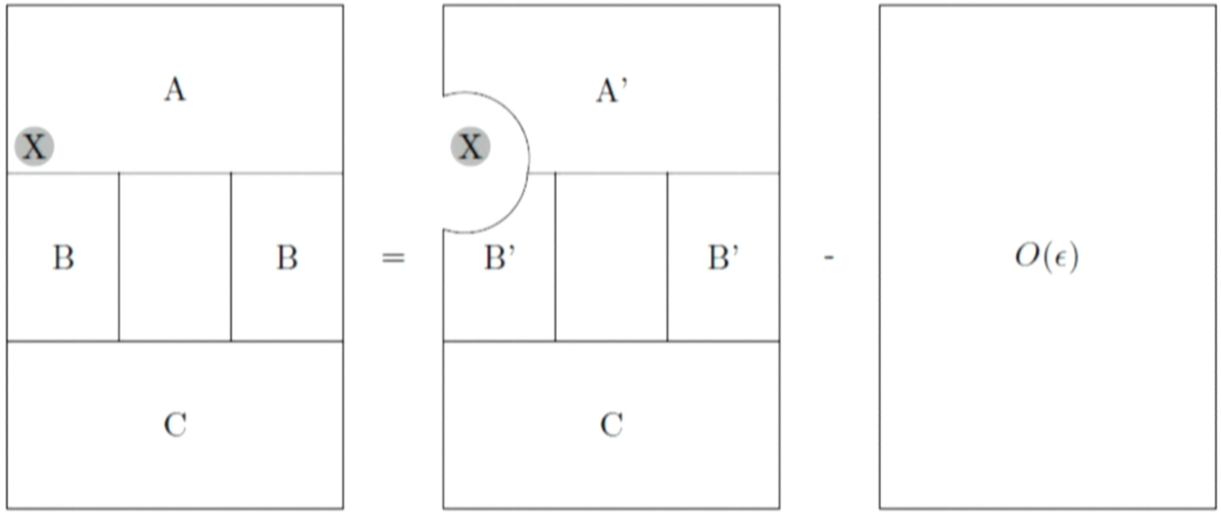
After all these moves... $(\langle \cdots \rangle = \mathrm{Tr}(\rho \cdots))$



$$\langle \hat{H}_{A:C|B} X \rangle \approx \langle \hat{H}_{A':C|B'} X \rangle$$

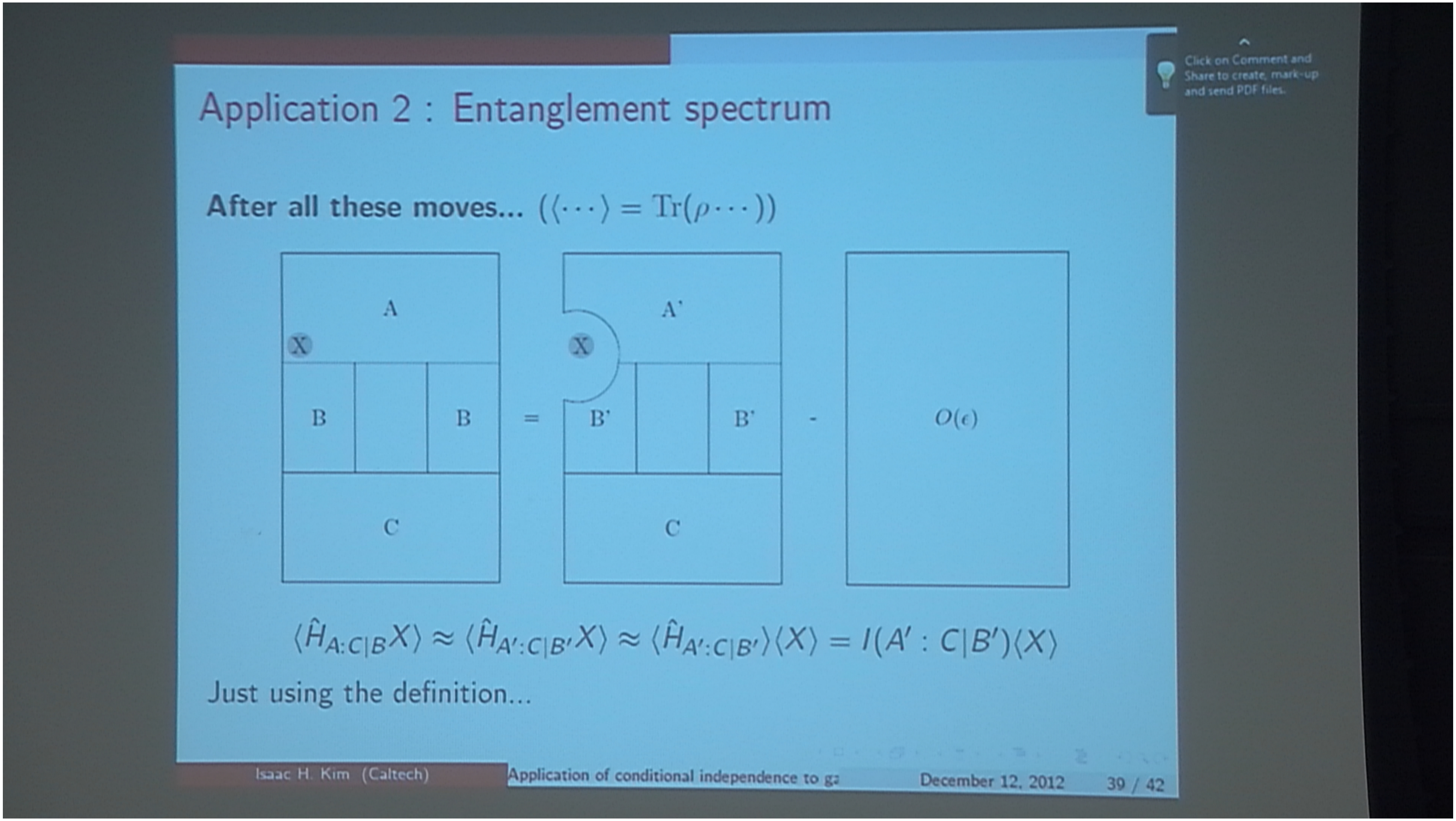
Application 2 : Entanglement spectrum

After all these moves... $(\langle \cdots \rangle = \mathrm{Tr}(\rho \cdots))$



$$\langle \hat{H}_{A:C|B} X \rangle \approx \langle \hat{H}_{A':C|B'} X \rangle \approx \langle \hat{H}_{A':C|B'} \rangle \langle X \rangle$$

Exponential clustering theorem : Hastings and Koma(2006), Nachtergaele et al.(2006).


Application 2 : Entanglement spectrum
After all these moves... $(\langle \cdots \rangle = \mathrm{Tr}(\rho \cdots))$
A
X
B
B
C
=
A'
X
B'
B'
C
-
$O(\epsilon)$
$\langle \hat{H}_{A:C|B} X \rangle \approx \langle \hat{H}_{A':C|B'} X \rangle \approx \langle \hat{H}_{A':C|B'} \rangle \langle X \rangle = I(A' : C|B') \langle X \rangle$
Just using the definition...
Isaac H. Kim (Caltech)
Application of conditional independence to ga
December 12, 2012
39 / 42
Click on Comment and Share to create, mark-up and send PDF files.

Application 2 : Entanglement spectrum

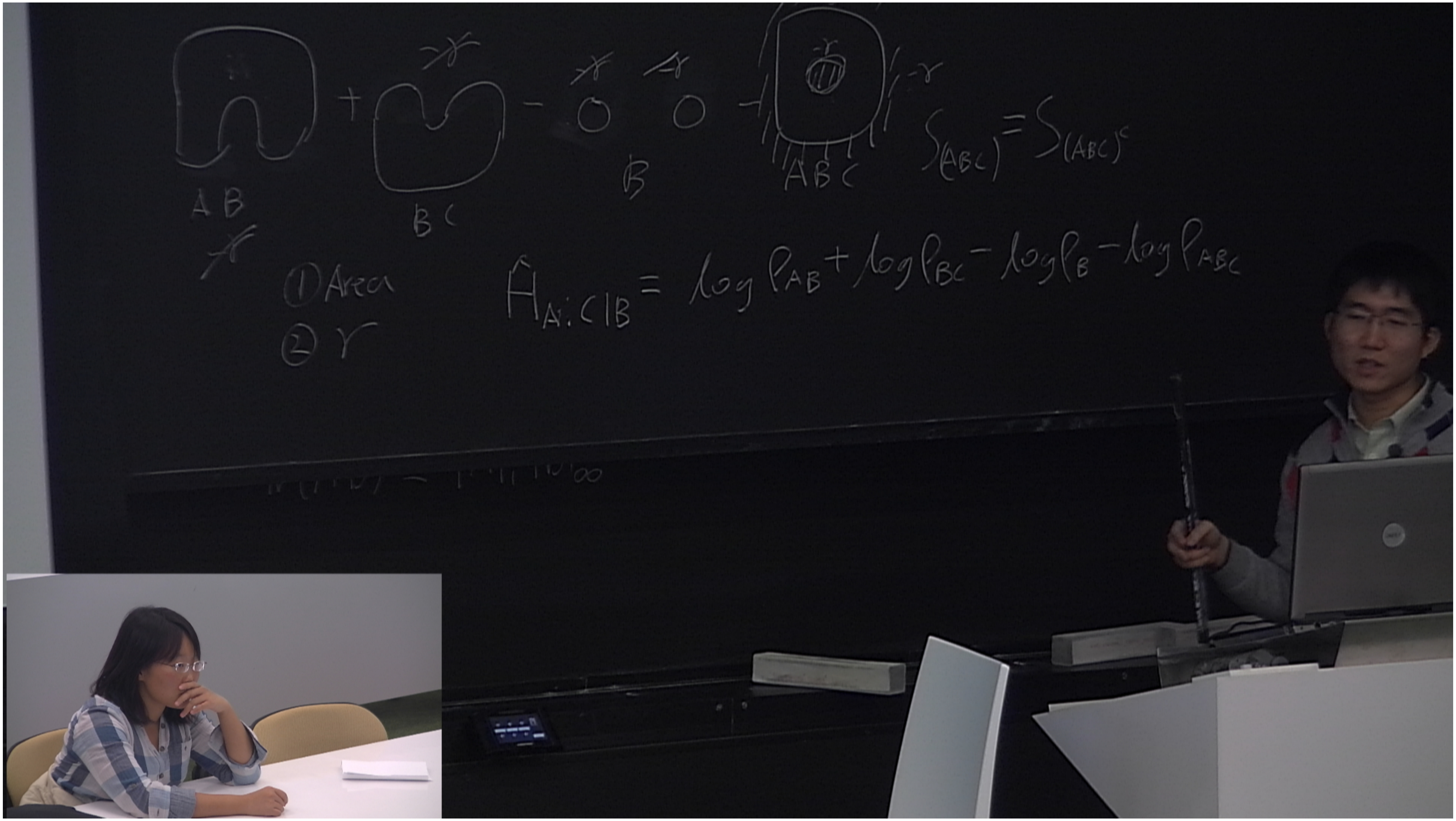
Main Result 3: If $S_A = a|\partial A| - \gamma + \epsilon(|\partial A|)$,

$$|\langle \hat{H}_{A:C|B}, O\rangle| \leq \|O\| \mathcal{O}(|\partial A|^2 \epsilon(|\partial A|)).$$

$$\langle O_1, O_2\rangle = \langle O_1 O_2\rangle - \langle O_1\rangle\langle O_2\rangle.$$

for any O not overlapping with the boundary.

- We cannot prove that $\hat{H}_{A:C|B}$ is 0, but it is pretty close to being 0!
- We made no assumption about the property of the parent hamiltonian!


AB
BC
B
ABC
$S_{(ABC)} = S_{(ABC)^c}$
① Area
② γ
$\hat{H}_{A:C|B} = \log \rho_{AB} + \log \rho_{BC} - \log \rho_B - \log \rho_{ABC}$

Conclusion

- Area law of entanglement entropy implies (approximate) conditional independence.
- Conditional independence implies i) first-order perturbative stability of topological entanglement entropy ii) local "cancelation" of entanglement spectrum.
- Structure of approximately conditionally independent state will have applications in quantum information theory for obvious reasons, but it will also benefit condensed matter theorists too.
  - Are there other extensions of strong subadditivity?