Title: Renormalizing TGFTs: a 3d example on SU(2)

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Abstract: I will recall the
main motivations for considering spin foam models in their Group Field Theory (GFT) versions, which are quantum field theories defined on group manifolds. As for any other quantum field theory, a fully consistent definition of the latter must involve renormalization. I will briefly review a specific class of GFTs, called tensorial, for which progress in this direction has recently been possible. A new just-renormalizable model, in three dimensions and on the SU(2) group, will be presented. Interestingly, it includes the geometric constraint of the Boulatov model, and might as such be related to Euclidean quantum gravity in three dimensions. Furthermore, this opens the way to a similar analysis of current 4d gravity spin foam models.

Renormalizing TGFTs: a $3d$ example on SU(2)

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Joint work with Daniele Oriti and Vincent Rivasseau.

Introduction and motivations

TGFTs are an approach to quantum gravity, which can be justified by two complementary logical paths:

- **The Tensor track** [Rivasseau '12]: matrix models, tensor models [Sasakura '91, Ambjorn et al. '91, Gross '92], $1/N$ expansion [Gurau, Rivasseau '10 '11], universality [Gurau '12], renormalization of tensor *field* theories... [Ben Geloun, Rivasseau '11 '12]

- **The Group Field Theory approach to Spin Foams** [Rovelli, Reisenberger '00, ...]
  - Quantization of simplicial geometry.
  - No triangulation independence ⇒ lattice gauge theory limit [Dittrich et al.] or sum over foams.
  - GFT provides a prescription for performing the sum: simplicial gravity path integral = Feynman amplitude of a QFT.
  - Amplitudes are generically divergent ⇒ renormalization?
  - Need for a continuum limit ⇒ many degrees of freedom ⇒ renormalization (phase transition along the renormalization group flow?)

Big question

Can we find a renormalizable TGFT exhibiting a phase transition from discrete geometries to the continuum, and recover GR in the classical limit?

Purpose of this talk

- State of the art: several renormalizable TGFTs
  - $U(1)$ model in 4d: just renormalizable up to φ^6 interactions, asymptotically free [Ben Geloun, Rivasseau '11, Ben Geloun '12]
  - $U(1)$ model in 3d: just renormalizable up to φ^4 interactions, asymptotically free [Ben Geloun, Samary '12]
  - even more renormalizable models [Ben Geloun, Livine '12]
- Question: does this formalism have the potential to accommodate interesting spin foam models (i.e. with geometric content)?

Main message of this talk

Yes it does, at least if:

- non-trivial propagators and a well-behaved class of foams are used;
- key QFT notions are generalized.

This is supported by recent studies of models with gauge invariance: [Oriti, Rivasseau, SC '12], [Samary, Vignes-Tourneret '12], [Oriti, Rivasseau, SC, in preparation].

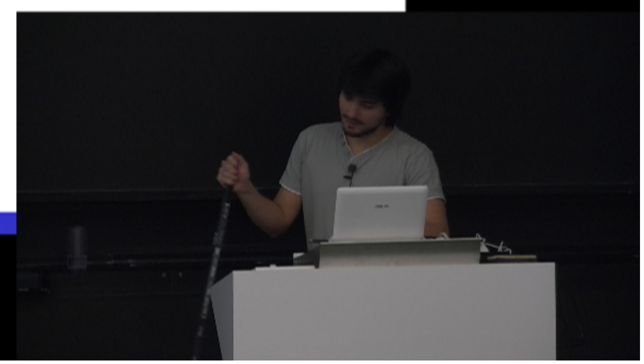
Nice example in this class: a just-renormalizable Boulatov-type model for $\mathrm{SU}(2)$ in $d = 3$!

Boulatov model and its mutations

1. Boulatov model and its mutations
2. A class of dynamical models with gauge symmetry
3. $SU(2)$ model in $d = 3$

Outline

1. Boulatov model and its mutations
2. A class of dynamical models with gauge symmetry
3. $\mathrm{SU}(2)$ model in $d = 3$



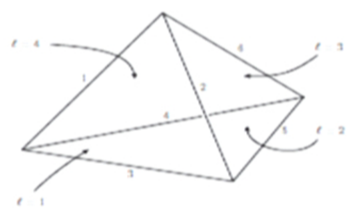
Boulatov model: initial formulation

A model for Euclidean 3d quantum gravity: gauge group SU(2). [Boulatov '92]

- Real scalar field on $\mathrm{SU}(2)^3$: $\varphi(g_1, g_2, g_3)$.
- Gauge invariance:

$$\forall h \in \mathrm{SU}(2), \quad \varphi(hg_1, hg_2, hg_3) = \varphi(g_1, g_2, g_3) \tag{1}$$

  Interpretation: φ as a quantized triangle.
- Action:



$$S_{kin}[\varphi] = \int [\mathrm{d}g_i]^3 \, \varphi(g_1, g_2, g_3)\varphi(g_1, g_2, g_3), \tag{2}$$

$$S_{int}[\varphi] = \lambda \int [\mathrm{d}g_i]^6 \, \varphi(g_1, g_2, g_3)\varphi(g_3, g_5, g_4) \varphi(g_5, g_2, g_6)\varphi(g_4, g_6, g_1) \tag{3}$$

- Partition function:

$$\mathcal{Z} \equiv \int \mathrm{d}\mu_{inv}(\varphi, \overline{\varphi}) \, \mathrm{e}^{-S[\varphi]} = \sum_{\mathcal{G}} \frac{\lambda^{N_{\mathcal{G}}}}{\mathrm{sym}(\mathcal{G})} \mathcal{A}_{\mathrm{PR}}(\mathcal{G}) \tag{4}$$

  $\Rightarrow$ Sum over discrete quantum spacetimes (triangulations dual to 2-complexes), with Ponzano-Regge weights.

Boulatov model: main issues

- Combinatorics / Topology: which triangulations are summed over?
  - all possible topological manifolds;
  - very singular topologies: extended singularities [Gurau '09];
  - the 2-complex does not fully capture the topology of the triangulation [Baratin, Girelli, Oriti '10; Bonzom, Smerlak '12].

- Divergences:
  - a cut-off on large spins needs to be introduced (e.g. heat kernel regularization):

$$\delta(g) = \sum_{j \in \mathbb{N}/2} (2j+1)\chi_j(g) \rightarrow K_\Lambda(g) = \sum_{j \in \mathbb{N}/2} e^{-\Lambda j(j+1)}(2j+1)\chi_j(g) \qquad (5)$$

  - complicated structure of divergences, not captured by topological invariants [Bonzom, Smerlak '12].

Boulatov model: colored version and 1/N expansion

- Four complex fields, with color labels ℓ: φ_ℓ, $\qquad \ell \in \{1, \cdots, 4\}$
- Restrict the interaction to fields with 4 different colors:

$$S[\phi] = \sum_{\ell} \int |\varphi_\ell|^2 + \lambda \int \varphi_1 \varphi_2 \varphi_3 \varphi_4 + c.c \tag{6}$$

$\Rightarrow$ amplitudes unchanged, but restricted class of simplicial complexes summed over: pseudo-manifolds only, full cellular homology... [Gurau '10]

- $1/N$ expansion [Gurau '10 '11; Gurau, Rivasseau '11]: appropriate scaling of λ such that

$$\mathcal{Z}_\Lambda = [K_\Lambda(\mathbf{1})]^2 \mathcal{Z}_0(\lambda\overline{\lambda}) + O([K_\Lambda(\mathbf{1})]^1) \tag{7}$$

$\mathcal{Z}_0$: contains only triangulations of the sphere, associated to melonic graphs [Bonzom, Gurau, Riello, Rivasseau '11]

$1/N$ expansion: unique scaling of λ such that manifolds dominate over singular pseudo-manifolds [Oriti, SC '11 '12]

A class of dynamical models with gauge symmetry

1. Boulatov model and its mutations
2. A class of dynamical models with gauge symmetry
3. SU(2) model in $d = 3$

Structure of a TGFT

- Dynamical variable: rank-d complex field

$$\varphi : (g_1, \ldots, g_d) \ni G^d \mapsto \mathbb{C} \,,$$

with G a (compact) Lie group.

- Partition function:

$$\mathcal{Z} = \int \mathrm{d}\mu_C(\varphi, \overline{\varphi}) \, \mathrm{e}^{-S(\varphi, \overline{\varphi})} \,.$$

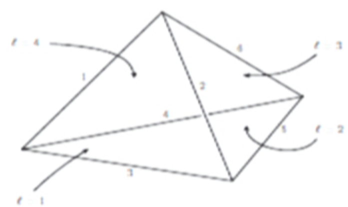
  - $S(\varphi, \overline{\varphi})$ is the interaction part of the action, and should be a sum of local terms.
  - Dynamics + geometrical constraints contained in the Gaussian measure $\mathrm{d}\mu_C$ with covariance C (i.e. 2nd moment):

$$\int \mathrm{d}\mu_C(\varphi, \overline{\varphi}) \, \varphi(g_\ell) \overline{\varphi}(g'_\ell) = C(g_\ell; g'_\ell)$$

Locality I: simplicial interactions

- Natural assumption in d dimensional Spin Foams: elementary building block of space-time = d-simplex.
  In GFT, translates into a φ^{d+1} interaction, e.g. in 3d:

$$S(\varphi, \overline{\varphi}) \propto \int [\mathrm{d}g]^6 \varphi(g_1, g_2, g_3)\varphi(g_3, g_5, g_4)\varphi(g_5, g_2, g_6)\varphi(g_4, g_6, g_1) + \text{ c.c.}$$



Problems:

- Full topology of the simplicial complex not encoded in the 2-complex [Baratin, Girelli, Oriti '10; Bonzom, Smerlak '12];
- (Very) degenerate topologies.

- A way out: add colors [Gurau '09]

$$S(\varphi, \overline{\varphi}) \propto \int [\mathrm{d}g]^6 \varphi_1(g_1, g_2, g_3)\varphi_2(g_3, g_5, g_4)\varphi_3(g_5, g_2, g_6)\varphi_4(g_4, g_6, g_1) + \text{ c.c.}$$

... then **uncolor** [Gurau '11; Bonzom, Gurau, Rivasseau '12] i.e. d auxiliary fields and 1 true dynamical field ⇒ infinite set of tensor invariant effective interactions.

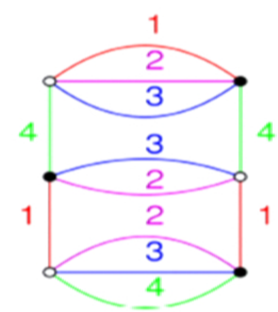
Locality II: tensor invariance

- Instead, **start** from tensor invariant interactions. They provide:
  - a good combinatorial control over topologies: full homology, pseudo-manifolds only etc.
  - analytical tools: $1/N$ expansion, universality theorems etc.

- S is a (finite) sum of connected tensor invariants, indexed by d-colored graphs (d-bubbles):

$$S(\varphi, \overline{\varphi}) = \sum_{b \in \mathcal{B}} t_b I_b(\varphi, \overline{\varphi}) \, .$$

- d-colored graphs are regular (valency d), bipartite, edge-colored graphs.



- Correspondence with tensor invariants:
  - white (resp. black) dot $\leftrightarrow$ field (resp. complex conjugate field);
  - edge of color ℓ $\leftrightarrow$ convolution of ℓ-th indices of φ and $\overline{\varphi}$.

$$\int [\mathrm{d}g_i]^{12} \varphi(g_1, g_2, g_3, g_4) \overline{\varphi}(g_1, g_2, g_3, g_5) \varphi(g_8, g_7, g_6, g_5)$$
$$\overline{\varphi}(g_8, g_9, g_{10}, g_{11}) \varphi(g_{12}, g_9, g_{10}, g_{11}) \overline{\varphi}(g_{12}, g_7, g_6, g_4)$$

Gaussian measure I: constraints

- In general, the Gaussian measure has to implement the geometrical constraints:
  - gauge invariance
$$\forall h \in G\,, \quad \varphi(hg_1, \ldots, hg_d) = \varphi(g_1, \ldots g_d)\,; \tag{8}$$
  - simplicity constraints.

$\Rightarrow$ C expected to be a projector, for instance

$$C(g_1, g_2, g_3; g_1', g_2', g_3') = \int \mathrm{d}h \prod_{\ell=1}^{3} \delta(g_\ell h g_\ell'^{-1}) \tag{9}$$

in the Boulatov model.

- But: not always possible in practice...
  - In 4d, with Barbero-Immirzi parameter: simplicity and gauge constraints don't commute $\rightarrow$ C not necessarily a projector.
  - Even when C is a projector, its cut-off version is not $\Rightarrow$ differential operators in radiative corrections e.g. Laplacian in the Boulatov-Ooguri model [Ben Geloun, Bonzom '11].

- Advantage: built-in notion of scale from C with non-trivial spectrum.

Gaussian measure II: non-trivial propagators

We would like to have a TGFT with:

- a built-in notion of scale i.e. a non-trivial propagator spectrum;
- a notion of discrete connection at the level of the amplitudes.

Particular realization that we consider:

- Gauge constraint:

$$\forall h \in G \,, \quad \varphi(hg_1, \dots, hg_d) = \varphi(g_1, \dots g_d) \,, \tag{10}$$

- supplemented by the non-trivial kernel (conservative choice, also justified by [Ben Geloun, Bonzom '11])

$$\left(m^2 - \sum_{\ell=1}^{d} \Delta_\ell \right)^{-1} . \tag{11}$$

This defines the measure $\mathrm{d}\mu_C$:

$$\int \mathrm{d}\mu_C(\varphi, \overline{\varphi}) \, \varphi(g_\ell) \overline{\varphi}(g'_\ell) = C(g_\ell; g'_\ell) = \int_0^{+\infty} \mathrm{d}\alpha \, \mathrm{e}^{-\alpha m^2} \int \mathrm{d}h \prod_{\ell=1}^{d} K_\alpha(g_\ell h g'^{-1}_\ell) \,, \tag{12}$$

where K_α is the heat kernel on G at time α.

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Amplitudes and gauge symmetry

- The amplitude of $\mathcal{G}$ depends on oriented products of group elements along its faces:

$$\mathcal{A}_{\mathcal{G}} = \left[\prod_{e \in L(\mathcal{G})} \int \mathrm{d}\alpha_e \, e^{-m^2 \alpha_e} \int \mathrm{d}h_e \right] \left(\prod_{f \in F(\mathcal{G})} K_{\alpha(f)} \left(\overrightarrow{\prod_{e \in \partial f}} h_e{}^{\epsilon_{ef}} \right) \right)$$
$$\left(\prod_{f \in F_{ext}(\mathcal{G})} K_{\alpha(f)} \left(g_{s(f)} \left[\overrightarrow{\prod_{e \in \partial f}} h_e{}^{\epsilon_{ef}} \right] g_{t(f)}^{-1} \right) \right) ,$$
$$= \left[\prod_{e \in L(\mathcal{G})} \int \mathrm{d}\alpha_e \, e^{-m^2 \alpha_e} \right] \{ \textit{ Regularized Boulatov-like amplitudes } \}$$

where $\alpha(f) = \sum_{e \in \partial f} \alpha_e$, $g_{s(f)}$ and $g_{t(f)}$ are boundary variables, and $\epsilon_{ef} = \pm 1$ when $e \in \partial f$ is the incidence matrix between oriented lines and faces.

- A gauge symmetry associated to vertices ($h_e \mapsto g_{t(e)} h_e g_{s(e)}^{-1}$) allows to impose $h_e = \mathbf{1}$ along a maximal tree of (dotted) lines.

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New notion of connectedness

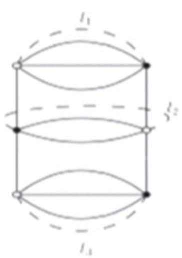
Spin Foam wisdom: lines $\rightarrow$ faces; faces $\rightarrow$ bubbles.

Amplitudes depend on holonomies along faces, built from group elements associated to lines $\Rightarrow$ new notion of connectedness: incidence relations between lines and faces instead of incidence relations between vertices and lines.

Definition

- A **subgraph** $\mathcal{H} \subset \mathcal{G}$ is a subset of (dotted) lines of $\mathcal{G}$.
- **Connected components** of $\mathcal{H}$ are the subsets of lines of the maximal factorized rectangular blocks of its ϵ_{ef} incidence matrix.

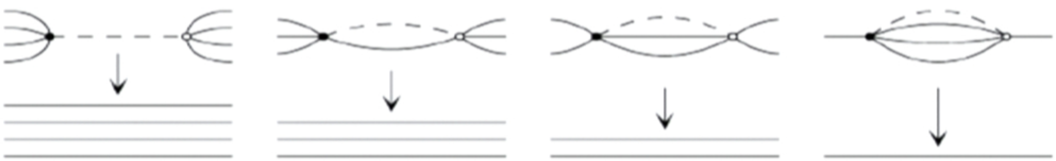
Equivalently, two lines of $\mathcal{H}$ are elementarily connected if they have a common internal face in $\mathcal{H}$, and we require transitivity.



- $\mathcal{H}_1 = \{l_1\}$, $\mathcal{H}_{12} = \{l_1, l_2\}$ are connected;
- $\mathcal{H}_{13} = \{l_1, l_3\}$ has two connected components (despite the fact that there is a single vertex!).

Contraction of a subgraph

- The contraction of a line is implemented by so-called dipole moves, which in $d = 4$ are:



Definition: k-dipole = line appearing in exactly $(k-1)$ closed faces of length 1.

- The contraction of a subgraph $\mathcal{H} \subset \mathcal{G}$ is obtained by successive contractions of its lines.

Net result

The contraction of a subgraph $\mathcal{H} \subset \mathcal{G}$ amounts to delete all the internal faces of $\mathcal{H}$ and reconnect its external legs according to the pattern of its external faces.

$\Rightarrow$ well-suited for coarse-graining / renormalization steps!

Remark Would be interesting to analyze these moves in a coarse-graining context [Dittrich et al.].

General form

- Dynamical variable: rank-3 complex field

$$\varphi : (g_1, g_2, g_3) \ni \mathrm{SU}(2)^3 \mapsto \mathbb{C}\,.$$

- Partition function:

$$\mathcal{Z}_\Lambda = \int \mathrm{d}\mu_{C^\Lambda}(\varphi, \overline{\varphi})\, \mathrm{e}^{-S(\varphi,\overline{\varphi})}\,.$$

- $S(\varphi, \overline{\varphi})$ is a sum of tensor invariants:

$$S(\varphi, \overline{\varphi}) = \sum_{b \in \mathcal{B}} t_b I_b(\varphi, \overline{\varphi})\,,$$

  with maximum valency: v_{max}.

- $\mathrm{d}\mu_{C^\Lambda}$ with covariance:

$$C^\Lambda(g_1, g_2, g_3; g_1', g_2', g_3') = \int_\Lambda^{+\infty} \mathrm{d}\alpha \mathrm{e}^{-m^2\alpha} \int \mathrm{d}h K_\alpha(g_1 h g_1'^{-1}) K_\alpha(g_2 h g_2'^{-1}) K_\alpha(g_3 h g_3'^{-1})$$

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Strategy: multi-scale analysis

1) Decompose amplitudes according to slices of "momenta" (Schwinger parameter);
2) Replace high divergent subgraphs by effective local vertices;
3) Iterate.

$\Rightarrow$ Effective multi-series (1 effective coupling per interaction at each scale).

Can be reshuffled into a renormalized series (1 renormalized coupling per interaction).

Advantages of the effective series:

- Physically transparent, in particular for overlapping divergences;
- No "renormalons": $|\mathcal{A}_{\mathcal{G}}| \leq K^n$.

Decomposition of propagators

- The Schwinger parameter α determines a momentum scale, which can be sliced in a geometric way. One fixes $M > 1$ and decomposes the propagators as

$$C = \sum_i C_i \,, \tag{13}$$

$$C_0(g_\ell; g'_\ell) = \int_1^{+\infty} \mathrm{d}\alpha \, \mathrm{e}^{-\alpha m^2} \int \mathrm{d}h \prod_{\ell=1}^{3} K_\alpha(g_\ell h g'^{-1}_\ell) \tag{14}$$

$$C_i(g_\ell; g'_\ell) = \int_{M^{-2i}}^{M^{-2(i-1)}} \mathrm{d}\alpha \, \mathrm{e}^{-\alpha m^2} \int \mathrm{d}h \prod_{\ell=1}^{3} K_\alpha(g_\ell h g'^{-1}_\ell) \,. \tag{15}$$

- A natural regularization is provided by a cut-off on i: $i \leq \rho$.

$$C^\Lambda = \sum_{i \leq \rho} C_i \,, \tag{16}$$

with: $\Lambda = M^{-2\rho}$

- The amplitude of a connected graph $\mathcal{G}$ is decomposed over scale attributions $\mu = \{i_e\}$ where i_e runs over all integers (smaller than ρ) for every line e:

$$\mathcal{A}_\mathcal{G} = \sum_\mu \mathcal{A}_{\mathcal{G},\mu} \,.$$

Abelian power-counting

Theorem

(i) If G has dimension D, there exists a constant K such that the following bound holds:

$$|\mathcal{A}_{\mathcal{G},\mu}| \leq K^{L(\mathcal{G})} \prod_{(i,k)} M^{\omega[\mathcal{G}_i^{(k)}]} , \tag{17}$$

where the **degree of divergence** ω is given by

$$\omega(\mathcal{H}) = -2L(\mathcal{H}) + 3(F(\mathcal{H}) - r(\mathcal{H})) \tag{18}$$

and $r(\mathcal{H})$ is the rank of the ϵ_{ef} incidence matrix of $\mathcal{H}$.

(ii) These bounds are optimal when $\mathcal{H}$ is contractible.

- Subgraphs with $\omega < 0$ are **convergent** i.e. have finite contributions when $\rho \to \infty$.
- Subgraphs with $\omega \geq 0$ are **divergent** and need to be renormalized. Traciality (or at the very least contractiblity) of divergent subgraphs is therefore needed for renormalizability to hold.

Classification of graphs

Question: what are the divergent graphs in this model?

Notations:

- $n_{2k}(\mathcal{H})$ = number of vertices with valency $2k$ in $\mathcal{H}$;
- $N(\mathcal{H})$ = number of external legs attached to vertices of $\mathcal{H}$;
- $\mathcal{H}/\mathcal{T}$ = contraction of $\mathcal{H}$ along a tree of lines (gauge-fixing).

Proposition

Let $\mathcal{H}$ be a non-vacuum subgraph. Then:

$$\omega(\mathcal{H}) = 3 - \frac{N}{2} \quad (19)$$

$$- \sum_{k=1}^{v_{max}/2} (6-2k) n_{2k} \quad (20)$$

$$+ 3\,\rho(\mathcal{H}/\mathcal{T}), \quad (21)$$

with

$$\rho(\mathcal{G}) \leq 0 \quad \text{and} \quad \rho(\mathcal{G}) = 0 \Leftrightarrow \mathcal{G} \text{ is a melopole}. \quad (22)$$

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Classification of graphs

Just renormalizability $\Rightarrow v_{max} = 6$.

$$\omega(\mathcal{H}) = 3 - \frac{N}{2} - 2n_2 - n_4 + 3\rho(\mathcal{H}/\mathcal{T}) \quad (23)$$

N	n_2	n_4	ρ	ω
6	0	0	0	0
4	0	0	0	1
4	0	1	0	0
2	0	0	0	2
2	0	1	0	1
2	0	2	0	0
2	1	0	0	0

Table: Classification of non-vacuum divergent graphs for $d = D = 3$. All of them are melonic.

2-point divergences $\Rightarrow$ mass and wave-function renormalization.

The φ^6 just renormalizable model

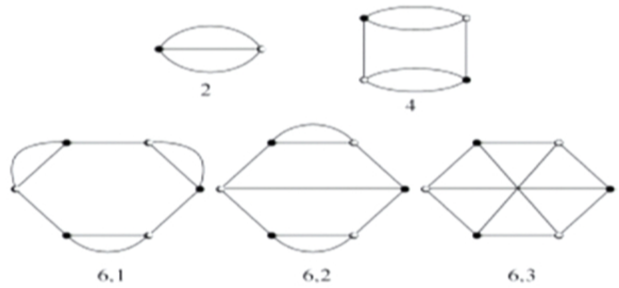

Figure: Possible bubble interactions.

$$\mathcal{Z}_\Lambda = \int \mathrm{d}\mu_{C^\Lambda}(\varphi, \overline{\varphi})\, \mathrm{e}^{-S_\Lambda(\varphi, \overline{\varphi})} \,. \qquad (24)$$

$$S_\Lambda = t_4^\Lambda S_4 + t_{6,1}^\Lambda S_{6,1} + t_{6,2}^\Lambda S_{6,2} + CT_m^\Lambda S_m + CT_\varphi^\Lambda S_\varphi \,, \qquad (25)$$

with:

$$S_m(\varphi, \overline{\varphi}) = \int [\mathrm{d}g]^3 \, \varphi(g_1, g_2, g_3) \overline{\varphi}(g_1, g_2, g_3) \,, \qquad (26)$$

$$S_\varphi(\varphi, \overline{\varphi}) = \int [\mathrm{d}g]^3 \, \varphi(g_1, g_2, g_3) \left(-\sum_{l=1}^{3} \Delta_\ell \right) \overline{\varphi}(g_1, g_2, g_3) \,. \qquad (27)$$

Conclusions and outlook

Summary:

- Introducing geometric constraints is possible in renormalizable TGFTs.
- Interesting interplay between spin foam constraints, tensorial structures and QFT formalism.
- Just-renormalizable $\mathrm{SU}(2)$ model in $d = 3$.

What's next?

- TGFTs are new types of field theories which deserves to be studied on their own. An interesting question: is asymptotic freedom generic?
- Flow of the $\mathrm{SU}(2)$ model in $3d$: asymptotic freedom? exact relation to Ponzano-Regge?
- Generalization to 4d gravity models: EPRL, FK, BO, etc.
  - geometry: interplay between simplicity constraints and tensor invariance?
  - Is there a natural notion of scale in these models?
  - Propagator with or without Laplacian (or other differential operator)?
  - Renormalizability?
  - Asymptotic freedom?
  - Phase transitions? Interpretation?