Title: Quasiprobability representations of qubits

Date: Nov 13, 2012 03:30 PM

URL: http://www.pirsa.org/12110070

Abstract: Negativity in a quasi-probability representation is typically interpreted as an indication of nonclassical behavior.

However, this does not preclude bases that are non-negative from having interesting applications---the single-qubit

stabilizer states have non-negative Wigner functions and yet play a fundamental role in many quantum information tasks.

We determine what other sets of quantum states and measurements of a qubit can be non-negative in a quasiprobability

representation, and identify nontrivial groups of unitary transformations that permute such states. These sets of states

and measurements are analogous to the single-qubit stabilizer states. We show that no quasiprobability representation of a

qubit can be non-negative for more than two bases in any plane of the Bloch sphere. Furthermore, there is a single family of

sets of four bases that can be non-negative in an arbitrary quasiprobability representation of a qubit. We provide an

exhaustive list of the sets of single-qubit bases that are nonnegative in some quasiprobability representation and are also

closed under a group of unitary transformations, revealing two families of such sets of three bases. We also show that not

all two-qubit Clifford transformations can preserve non-negativity in any quasiprobability representation that is

non-negative for the computational basis. This is in stark contrast to the qutrit case, in which the discrete Wigner

function is non-negative for all n-qutrit stabilizer states and Clifford transformations. We also provide some evidence

that extending the other sets of non-negative single-qubit states to multiple qubits does not give entangled states.

Quasiprobability representations of qubits

Joel Wallman

School of Physics, The University of Sydney, Australia

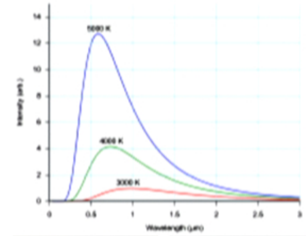
Based on *Phys. Rev. A* **85**, 062121 (2012), arXiv:1203.2652

In the beginning

In the beginning was

In the beginning

In the beginning was $\longrightarrow$ *



Then Planck and Einstein said

"Let there be quanta of energy"

and then there was quantum mechanics.

The Founding Fathers looked at the theory they had made, and said

* *Max Planck: the reluctant revolutionary*, Physics World (December 2000).

The Exodus

After much wandering in the wilderness and worshiping at the altars of Heisenberg and Schrödinger,

The Exodus

After much wandering in the wilderness and worshiping at the altars of Heisenberg and Schrödinger,

The four “commandments”

1. Thou shalt describe an isolated system by some $\rho \in \mathcal{B}(\mathcal{H}_d)$;
2. Thou shalt describe composite systems by some $\rho \in \mathcal{B}(\bigotimes_n \mathcal{H}_{d_n})$;
3. Thou shalt describe the evolution of a system by some CPTP superoperator acting on $\mathcal{B}(\mathcal{H}_d)$; and

The four "commandments"

1. Thou shalt describe an isolated system by some $\rho \in \mathcal{B}(\mathcal{H}_d)$;
2. Thou shalt describe composite systems by some $\rho \in \mathcal{B}(\bigotimes_n \mathcal{H}_{d_n})$;
3. Thou shalt describe the evolution of a system by some CPTP superoperator acting on $\mathcal{B}(\mathcal{H}_d)$; and
4. Thou shalt describe a measurement by a POVM $\mathcal{M} = \{E_k\}$, and predict that the outcome k occurs with probability $\mathrm{tr}(\rho E_k)$

Blessings and curses

And Bohr looked at the theory and said:

"It is very good."

"Let any who seek further explanation be described in uncomplementary terms."

But then a new generation of physicists arose, who knew not the ways of the Fathers and sacrificed

Approaches that please everybody

The main approaches

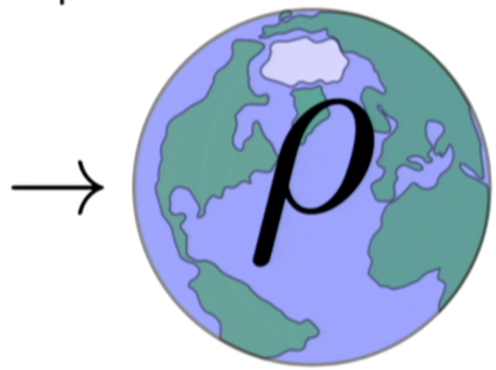
How do the abstract postulates map to the real world?

1. An isolated system is described by some $\rho \in \mathcal{B}(\mathcal{H}_d)$;
2. Composite systems are described by some $\rho \in \mathcal{B}\left(\bigotimes_n \mathcal{H}_{d_n}\right)$;
3. The evolution of a system is described by some CPTP superoperator acting on $\mathcal{B}(\mathcal{H}_d)$; and
4. A measurement is described by a POVM $\mathcal{M} = \{E_k\}$, where the outcome k occurs with probability $\mathrm{tr}(\rho E_k)$

The main approaches

How do the abstract postulates map to the real world?

1. An isolated system is described by some $\rho \in \mathcal{B}(\mathcal{H}_d)$;
2. Composite systems are described by some $\rho \in \mathcal{B}\left(\bigotimes_n \mathcal{H}_{d_n}\right)$;
3. The evolution of a system is described by some CPTP superoperator acting on $\mathcal{B}(\mathcal{H}_d)$; and
4. A measurement is described by a POVM $\mathcal{M} = \{E_k\}$, where the outcome k occurs with probability $\mathrm{tr}(\rho E_k)$



The main approaches

How do the abstract postulates map to the real world?

Two types of cavemen

Bottom-up: derive quantum mechanics from some axioms (axiomatics).

Two types of cavemen

Bottom-up: derive quantum mechanics from some axioms (axiomatics).

Top-down: start with (a subtheory of) quantum mechanics and try to create a “sensible” model that is consistent with it (hidden variables).

I will use a top-down approach:
$\rightarrow$ the goal is to map (a subtheory of) quantum mechanics to distributions over a phase space Λ.

Subtheories of quantum mechanics

A *subtheory* of quantum mechanics is:

1. a set of preparations $\mathcal{R} := \{\rho \in \mathcal{B}(\mathcal{H}_d)\}$;
2. a group $\mathcal{G}$ of unitary operators; and

Subtheories of quantum mechanics

A *subtheory* of quantum mechanics is:

1. a set of preparations $\mathcal{R} := \{\rho \in \mathcal{B}(\mathcal{H}_d)\}$;
2. a group $\mathcal{G}$ of unitary operators; and
3. a collection of POVMs $\mathbb{M} := \{\mathcal{M}\}$.

Subtheories of quantum mechanics

A *subtheory* of quantum mechanics is:

1. a set of preparations $\mathcal{R} := \{\rho \in \mathcal{B}(\mathcal{H}_d)\}$;
2. a group $\mathcal{G}$ of unitary operators; and
3. a collection of POVMs $\mathbb{M} := \{\mathcal{M}\}$.

Assumption: $\mathcal{R}$ and $\mathbb{M}$ are convex, i.e., for $\mathcal{R}$,

$$\rho, \rho' \in \mathcal{R} \Rightarrow \forall a \in [0,1] \;\; a\rho + (1-a)\rho' \in \mathcal{R}$$

Physically: prepare ρ or ρ' based on outcome of a coin toss such that $p(\text{heads}) = a$ and $p(\text{tails}) = 1 - a$, then forgetting the outcome.

Why subtheories?

- If a subtheory can be described in a "sensible" model, then the way you have to distort the model to describe all of QM gives one definition of "nonclassicality";
- Experimentally, people implement subtheories of QM, not all of it, so it is important to know what subtheories are "truly" quantum;
- Quantum computers (in, e.g., the circuit model or MBQC) are based on subtheories:
  - prepare some input/resource state;

Why subtheories?

- If a subtheory can be described in a "sensible" model, then the way you have to distort the model to describe all of QM gives one definition of "nonclassicality";
- Experimentally, people implement subtheories of QM, not all of it, so it is important to know what subtheories are "truly" quantum;
- Quantum computers (in, e.g., the circuit model or MBQC) are based on subtheories:
  - prepare some input/resource state;

Why subtheories?

- If a subtheory can be described in a "sensible" model, then the way you have to distort the model to describe all of QM gives one definition of "nonclassicality";
- Experimentally, people implement subtheories of QM, not all of it, so it is important to know what subtheories are "truly" quantum;
- Quantum computers (in, e.g., the circuit model or MBQC) are based on subtheories:
  - prepare some input/resource state;
  - possibly apply a sequence of transformations from some set of unitary generators; and
  - choose measurements from some set.

Example: stabilizer states and Clifford transformations

Let $\mathcal{P} := \langle X, Y, Z \rangle$ be the group of Pauli matrices and define the N-qubit Clifford group

$$\mathcal{C}_N := \left\{ U \in U(2^N) : U\mathcal{P}^{\otimes N}U^\dagger = \mathcal{P}^{\otimes N} \right\} .$$

Then the convex N-qubit stabilizer subtheory is:

- the set of all convex combinations of the states $\mathcal{C}_N \left|0\right\rangle \left\langle 0\right|^{\otimes N} \mathcal{C}_N^\dagger$;
- the group of transformations $\mathcal{C}_N$; and
- the set of all convex combinations of the observables $\mathcal{P}^{\otimes N}$.

Example: stabilizer states and Clifford transformations

Let $\mathcal{P} := \langle X, Y, Z \rangle$ be the group of Pauli matrices and define the N-qubit Clifford group

$$\mathcal{C}_N := \left\{ U \in U(2^N) : U\mathcal{P}^{\otimes N}U^\dagger = \mathcal{P}^{\otimes N} \right\} .$$

Then the convex N-qubit stabilizer subtheory is:

- the set of all convex combinations of the states $\mathcal{C}_N \left|0\right\rangle \left\langle 0\right|^{\otimes N} \mathcal{C}_N^\dagger$;
- the group of transformations $\mathcal{C}_N$; and
- the set of all convex combinations of the observables $\mathcal{P}^{\otimes N}$.

A "sensible" assumption

Prepare a system in the state ρ;
Measure the POVM $\mathcal{M} = \{E_k\}$;
The outcome "k" occurs with probability $\mathrm{tr}\,(\rho E_k)$.

Note that nothing depends on the specific representation of ρ or E_k, e.g., on:

- whether the POVM is implemented directly or through coarse-graining some other POVM $\mathcal{M}' = \{E'_j\}$

A “sensible” assumption

Prepare a system in the state ρ;
Measure the POVM $\mathcal{M} = \{E_k\}$;
The outcome “k” occurs with probability $\text{tr}\,(\rho E_k)$.

Note that nothing depends on the specific representation of ρ or E_k, e.g., on:

- whether the POVM is implemented directly or through coarse-graining some other POVM $\mathcal{M}' = \{E'_j\}$

A "sensible" assumption

Prepare a system in the state ρ;
Measure the POVM $\mathcal{M} = \{E_k\}$;
The outcome "k" occurs with probability $\mathrm{tr}(\rho E_k)$.

Note that nothing depends on the specific representation of ρ or E_k, e.g., on:

- whether ρ is a proper or improper mixture

improper mixture
(tracing out the environment)

Explaining this independence

Such models are quasiprobability representations †, defined by two dual Hermitian frames $\{F(\lambda)\}$ and $\{G(\lambda)\}$ such that:

---

† C. Ferrie and J. Emerson, *J. Phys. A* **41**, 352001 (2008), R. W. Spekkens, *Phys. Rev. Lett.* **101**, 020401 (2008).

Unitary transformations

Unitaries correspond to deterministic evolution of quantum states

Unitary transformations

Unitaries correspond to deterministic evolution of quantum states

Non-negative bases

Restricting a quasiprobability representation to the subtheory for which it is non-negative gives an ontological model for that subtheory.

Question: what subtheories can be non-negative in some quasiprobability representation?

Non-negative bases

Restricting a quasiprobability representation to the subtheory for which it is non-negative gives an ontological model for that subtheory.

Question: what subtheories can be non-negative in some quasiprobability representation?

Properties of non-negative bases

- Two elements of non-negative bases are orthogonal iff the corresponding probability distributions are disjoint;
- For any λ, exactly one element of any non-negative basis assigns non-zero probability $q(\lambda)$ to λ,

Coplanar bases

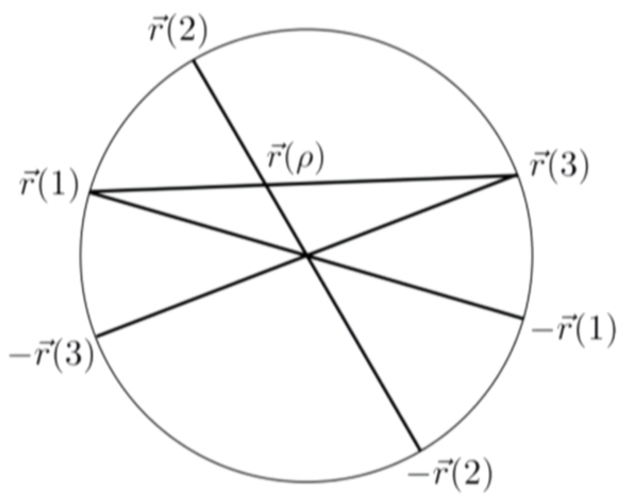
Theorem 1: in *any* quasiprobability representation of a qubit there are at most two non-negative bases in any plane of the Bloch sphere.

Coplanar bases

Theorem 1: in *any* quasiprobability representation of a qubit there are at most two non-negative bases in any plane of the Bloch sphere.

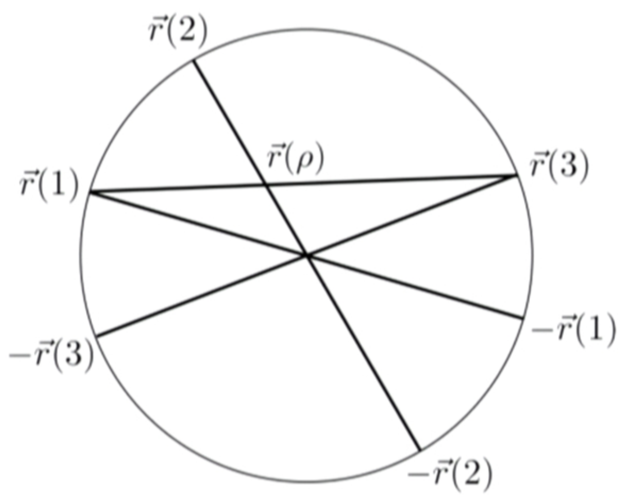
Coplanar bases

Theorem 1: in *any* quasiprobability representation of a qubit there are at most two non-negative bases in any plane of the Bloch sphere.



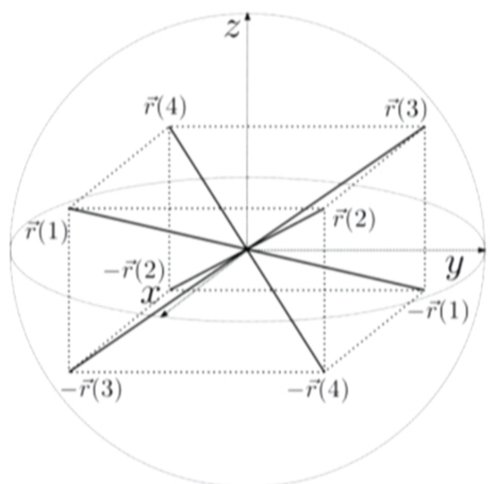
Coplanar bases

Theorem 1: in *any* quasiprobability representation of a qubit there are at most two non-negative bases in any plane of the Bloch sphere.



Maximum number of non-negative qubit bases

Theorem 2: in *any* quasiprobability representation of a qubit, any four non-negative bases must correspond to the vertices of a right cuboid.



Non-negative subtheories

Question: what subtheories containing a nontrivial unitary group $\mathcal{G}$ can be non-negative in some quasiprobability representation?

Elements of $\mathcal{G}$ must permute non-negative bases, so correspond to point groups of 1, 2, or 3 pairs of antipodal points (Theorem 1) or $\mathcal{P}$ (Theorem 2).

Non-negative subtheories

Question: what subtheories containing a nontrivial unitary group $\mathcal{G}$ can be non-negative in some quasiprobability representation?

Elements of $\mathcal{G}$ must permute non-negative bases, so correspond to point groups of 1, 2, or 3 pairs of antipodal points (Theorem 1) or $\mathcal{P}$ (Theorem 2).

Note that $\Gamma_{U^{-1}} = \Gamma_U^{-1}$ implies that if Γ_U and $\Gamma_{U^{-1}}$ are both non-negative, they must be permutation matrices:

Non-negative subtheories

Question: what subtheories containing a nontrivial unitary group $\mathcal{G}$ can be non-negative in some quasiprobability representation?

Elements of $\mathcal{G}$ must permute non-negative bases, so correspond to point groups of 1, 2, or 3 pairs of antipodal points (Theorem 1) or $\mathcal{P}$ (Theorem 2).

Note that $\Gamma_{U^{-1}} = \Gamma_U^{-1}$ implies that if Γ_U and $\Gamma_{U^{-1}}$ are both non-negative, they must be permutation matrices: i.e., deterministic evolution of quantum states arises from deterministic evolution on phase space.

Example: single-qubit stabilizer subtheory

The discrete Wigner function is defined over 4 points by the operators:

$$F(a,b) = \frac{1}{4}\left(\mathbf{1} + \vec{d}(a,b)\cdot\vec{\sigma}\right)$$

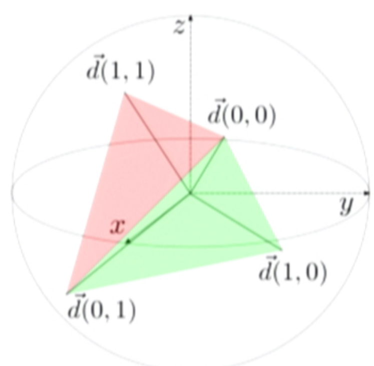
where $\vec{d}(a,b) = (-^{a}, -^{b}, -^{a+b})$. $G(a,b) = 2F(a,b)$.

Quasiprobability for $\rho \equiv \vec{r}(\rho)$ is $\mu_\rho(a,b) = \frac{1}{4}\left(1 + \vec{d}(a,b)\cdot\vec{r}(\rho)\right)$.

Example: single-qubit stabilizer subtheory

How about Clifford transformations?

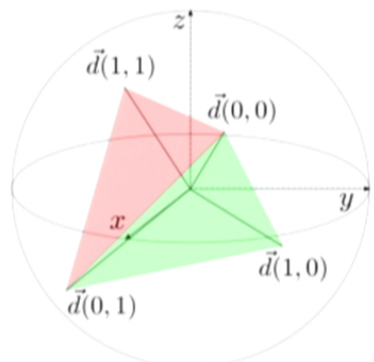
Any non-negative unitary permutes the $\{F(a,b)\}$.



Example: single-qubit stabilizer subtheory

How about Clifford transformations?

Any non-negative unitary permutes the $\{F(a,b)\}$.

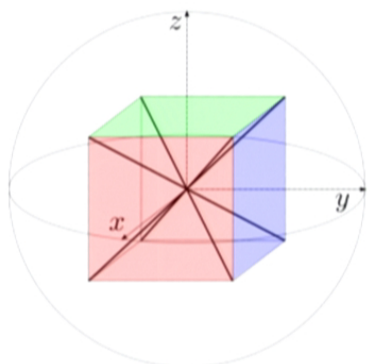


Symmetry group is the tetrahedral group:

Example: single-qubit stabilizer subtheory

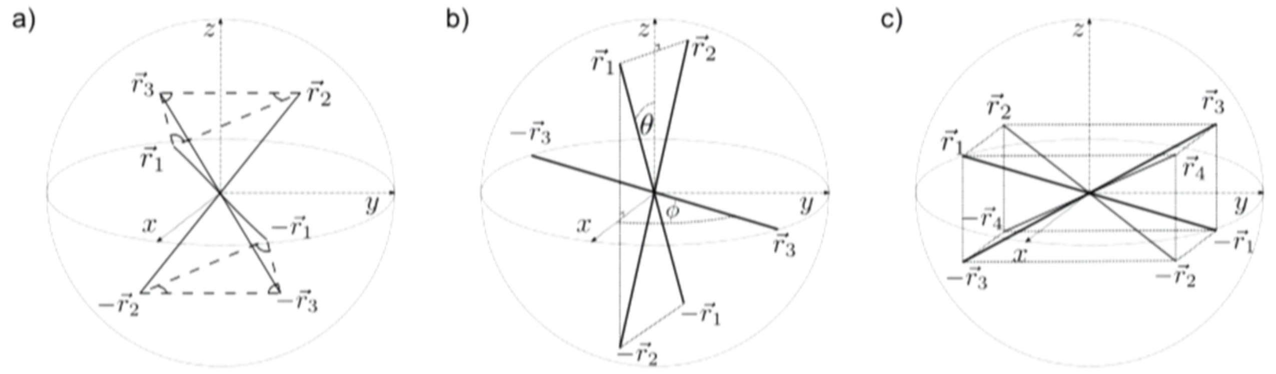
How about Clifford transformations?

Any non-negative unitary permutes the $\{F(a,b)\}$.

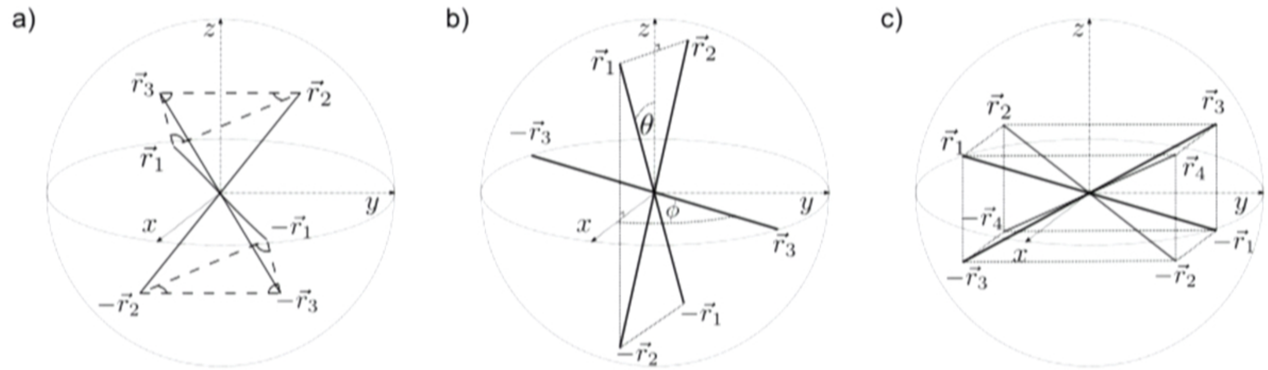


Symmetry group is the octahedral group: can implement all Clifford gates.

Non-negative subtheories with nontrivial $\mathcal{G}$



Non-negative subtheories with nontrivial $\mathcal{G}$



Can construct quasiprobability representations that are non-negative for these subtheories

Non-negative qubit subtheories

A single qubit quasiprobability representation is defined by two frames $\{F(\lambda)\}$ and $\{G(\lambda)\}$

Can define an extension to N qubits using the frames $\{F(\lambda)\}^{\otimes N}$ and $\{G(\lambda)\}^{\otimes N}$.

However, this enforces "local causality": for any bipartite measurement $E_1 \otimes E_2$,

$$
\begin{aligned}
\xi\left(E_1 \otimes E_2|\lambda\right) &= \operatorname{tr}\left[\left(E_1 \otimes E_2\right)\left(G(\lambda_1) \otimes G(\lambda_2)\right)\right] \\
&= \operatorname{tr}\left[E_1 G(\lambda_1)\right] \operatorname{tr}\left[E_2 G(\lambda_2)\right]
\end{aligned}
$$

Non-negative qubit subtheories

The same argument applies to preparations:

$$\mu_{\rho_1 \otimes \rho_2}(\lambda) = \mathrm{tr}\left[\rho_1 F(\lambda_1)\right] \mathrm{tr}\left[\rho_2 F(\lambda_2)\right]$$

Non-negative qubit subtheories

The same argument applies to preparations:

$$\mu_{\rho_1 \otimes \rho_2}(\lambda) = \mathrm{tr}\left[\rho_1 F(\lambda_1)\right] \mathrm{tr}\left[\rho_2 F(\lambda_2)\right]$$

Preparations of product states correspond to independent distributions over phase space.

The PBR theorem then gives a nontrivial restriction on the set of states and measurements that can be non-negative.

Non-negative qubit subtheories

Question: can we avoid these problems using a different quasiprobability representation for multiple systems?

Non-negative qubit subtheories

Question: can we avoid these problems using a different quasiprobability representation for multiple systems?

To define a quasiprobability representation we need a phase space Λ_N for N systems and frames $\{F^{(N)}(\lambda)\}$ and $\{G^{(N)}(\lambda)\}$.

Non-negative qubit subtheories

Question: can we avoid these problems using a different quasiprobability representation for multiple systems?

To define a quasiprobability representation we need a phase space Λ_N for N systems and frames $\{F^{(N)}(\lambda)\}$ and $\{G^{(N)}(\lambda)\}$.

Theorem 3: any quasiprobability representation that is non-negative for a locally-tomographic subtheory gives a locally causal model for that subtheory.

Non-negative qubit subtheories

Question: can we avoid these problems using a different quasiprobability representation for multiple systems?

To define a quasiprobability representation we need a phase space Λ_N for N systems and frames $\{F^{(N)}(\lambda)\}$ and $\{G^{(N)}(\lambda)\}$.

Theorem 3: any quasiprobability representation that is non-negative for a locally-tomographic subtheory gives a locally causal model for that subtheory.

Non-negative qubit subtheories

Question: can we avoid these problems using a different quasiprobability representation for multiple systems?

To define a quasiprobability representation we need a phase space Λ_N for N systems and frames $\{F^{(N)}(\lambda)\}$ and $\{G^{(N)}(\lambda)\}$.

Theorem 3: any quasiprobability representation that is non-negative for a locally-tomographic subtheory gives a locally causal model for that subtheory.

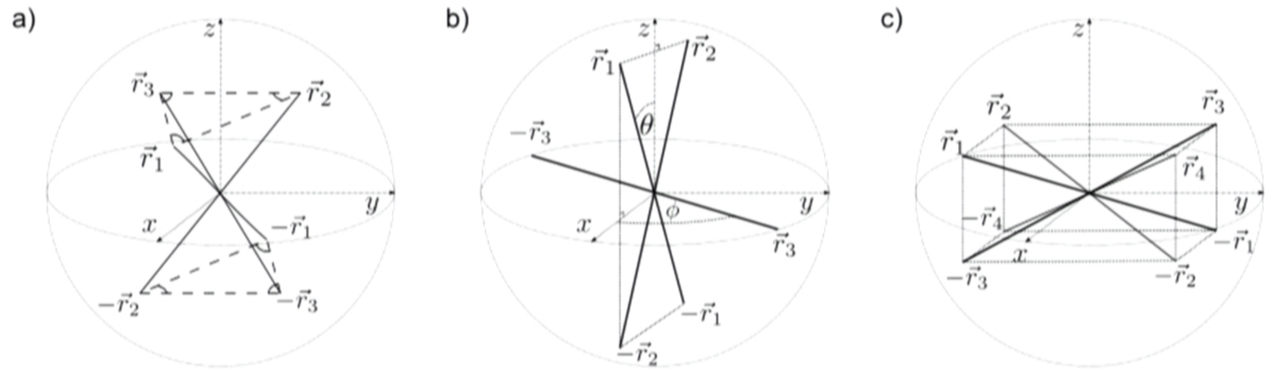
N-qubit stabilizer subtheory

Theorem 3 implies that the N-qubit stabilizer subtheory is negative in any quasiprobability representation for $N > 2$.

Non-negative qubit subtheories with entangling gates

Question: are there any extensions of the other subtheories to multiple qubits with entangling gates?

Non-negative subtheories with nontrivial $\mathcal{G}$



Can construct quasiprobability representations that are non-negative for these subtheories except in (a) and (b) when bases become too close to coplanar.

Non-negative qubit subtheories

Question: can we avoid these problems using a different quasiprobability representation for multiple systems?

To define a quasiprobability representation we need a phase space Λ_N for N systems and frames $\{F^{(N)}(\lambda)\}$ and $\{G^{(N)}(\lambda)\}$.

Theorem 3: any quasiprobability representation that is non-negative for a locally-tomographic subtheory gives a locally causal model for that subtheory.

Non-negative qubit subtheories with entangling gates

$\mathcal{P}$ is a set of Hermitian observables, including the trivial measurement.

Non-negative qubit subtheories with entangling gates

$\mathcal{P}$ is a set of Hermitian observables, including the trivial measurement.

Replace $\mathcal{P}$ by another set of Hermitian observables, $\mathcal{Q} := \{\pm\mathbf{1}, \pm\omega_b\}$, where the $\pm$ eigenvalues of ω_b is the bth non-negative basis.

Non-negative qubit subtheories with entangling gates

$\mathcal{P}$ is a set of Hermitian observables, including the trivial measurement.

Replace $\mathcal{P}$ by another set of Hermitian observables, $\mathcal{Q} := \{\pm \mathbf{1}, \pm\omega_b\}$, where the $\pm$ eigenvalues of ω_b is the *b*th non-negative basis.

Theorem 4: The only set of tomographically complete $\mathcal{Q} \subset \mathcal{B}(\mathcal{H}_2)$ such that $\mathcal{N}(\mathcal{Q})$ is non-trivial and $\mathcal{N}(\mathcal{Q}^{\otimes N})$ contains entangling gates for some N and is $\mathcal{P}$.

Non-negative qubit subtheories with entangling gates

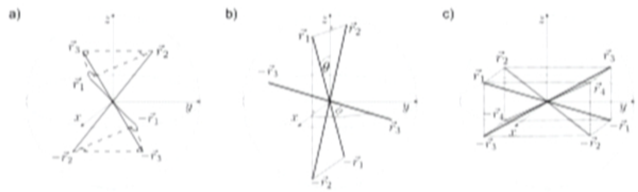
$\mathcal{P}$ is a set of Hermitian observables, including the trivial measurement.

Replace $\mathcal{P}$ by another set of Hermitian observables, $\mathcal{Q} := \{\pm\mathbf{1}, \pm\omega_b\}$, where the $\pm$ eigenvalues of ω_b is the bth non-negative basis.

Theorem 4: The only set of tomographically complete $\mathcal{Q} \subset \mathcal{B}(\mathcal{H}_2)$ such that $\mathcal{N}(\mathcal{Q})$ is non-trivial and $\mathcal{N}(\mathcal{Q}^{\otimes N})$ contains entangling gates for some N and is $\mathcal{P}$.

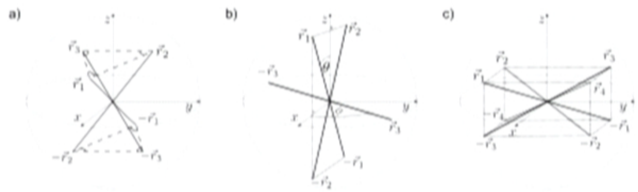
Summary

Question: what subtheories that contain a nontrivial unitary group $\mathcal{G}$ can be non-negative in some quasiprobability representation?



Summary

Question: what subtheories that contain a nontrivial unitary group $\mathcal{G}$ can be non-negative in some quasiprobability representation?



If all you need is ρ, E and U, you can only have four non-negative bases. It is possible to make *all* preparations have non-negative distributions, but this requires making at least one projector in all but at most four bases negative.

Summary

In quasiprobability representations, there are some interesting relations between different “foundational” assumptions.

Summary

In quasiprobability representations, there are some interesting relations between different “foundational” assumptions.

In particular, local tomography implies local causality and has some implications for preparations.

Also negativity is not sufficient for a quantum computational speedup.

Summary

Entanglement between qubits seems “more” quantum than entanglement between qutrits.

The N-qutrit stabilizer subtheory is non-negative in the Wigner function,

Summary

Entanglement between qubits seems “more” quantum than entanglement between qutrits.

The N-qutrit stabilizer subtheory is non-negative in the Wigner function, but the *N-qubit* version is negative in *any* quasiprobability representation for $N > 1$.

Summary

1. An isolated system is described by some $\rho \in \mathcal{B}(\mathcal{H}_d)$;
2. Composite systems are described by some $\rho \in \mathcal{B}\left(\bigotimes_n \mathcal{H}_{d_n}\right)$;
3. The evolution of a system is described by some CPTP superoperator acting on $\mathcal{B}(\mathcal{H}_d)$; and
4. A measurement is described by a POVM $\mathcal{M} = \{E_k\}$, where the outcome k occurs with probability $\mathrm{tr}(\rho E_k)$

Summary

1. An isolated system is described by some $\rho \in \mathcal{B}(\mathcal{H}_d)$;
2. Composite systems are described by some $\rho \in \mathcal{B}(\bigotimes_n \mathcal{H}_{d_n})$;
3. The evolution of a system is described by some CPTP superoperator acting on $\mathcal{B}(\mathcal{H}_d)$; and
4. A measurement is described by a POVM $\mathcal{M} = \{E_k\}$, where the outcome k occurs with probability $\mathrm{tr}(\rho E_k)$

Should we all convert to the church of the larger Hilbert space?

Summary

1. An isolated system is described by some $\rho \in \mathcal{B}(\mathcal{H}_d)$;
2. Composite systems are described by some $\rho \in \mathcal{B}(\bigotimes_n \mathcal{H}_{d_n})$;
3. The evolution of a system is described by some CPTP superoperator acting on $\mathcal{B}(\mathcal{H}_d)$; and
4. A measurement is described by a POVM $\mathcal{M} = \{E_k\}$, where the outcome k occurs with probability $\mathrm{tr}(\rho E_k)$

Should we all convert to the church of the larger Hilbert space?

Summary

1. An isolated system is described by some $\rho \in \mathcal{B}(\mathcal{H}_d)$;
2. Composite systems are described by some $\rho \in \mathcal{B}(\bigotimes_n \mathcal{H}_{d_n})$;
3. The evolution of a system is described by some CPTP superoperator acting on $\mathcal{B}(\mathcal{H}_d)$; and
4. A measurement is described by a POVM $\mathcal{M} = \{E_k\}$, where the outcome k occurs with probability $\mathrm{tr}(\rho E_k)$

Should we all convert to the church of the larger Hilbert space?

Summary

1. An isolated system is described by some $\rho \in \mathcal{B}(\mathcal{H}_d)$;
2. Composite systems are described by some $\rho \in \mathcal{B}(\bigotimes_n \mathcal{H}_{d_n})$;
3. The evolution of a system is described by some CPTP superoperator acting on $\mathcal{B}(\mathcal{H}_d)$; and
4. A measurement is described by a POVM $\mathcal{M} = \{E_k\}$, where the outcome k occurs with probability $\mathrm{tr}(\rho E_k)$

Should we all convert to the church of the larger Hilbert space?

Summary

1. An isolated system is described by some $\rho \in \mathcal{B}(\mathcal{H}_d)$;
2. Composite systems are described by some $\rho \in \mathcal{B}\left(\bigotimes_n \mathcal{H}_{d_n}\right)$;
3. The evolution of a system is described by some CPTP superoperator acting on $\mathcal{B}(\mathcal{H}_d)$; and
4. A measurement is described by a POVM $\mathcal{M} = \{E_k\}$, where the outcome k occurs with probability $\mathrm{tr}(\rho E_k)$

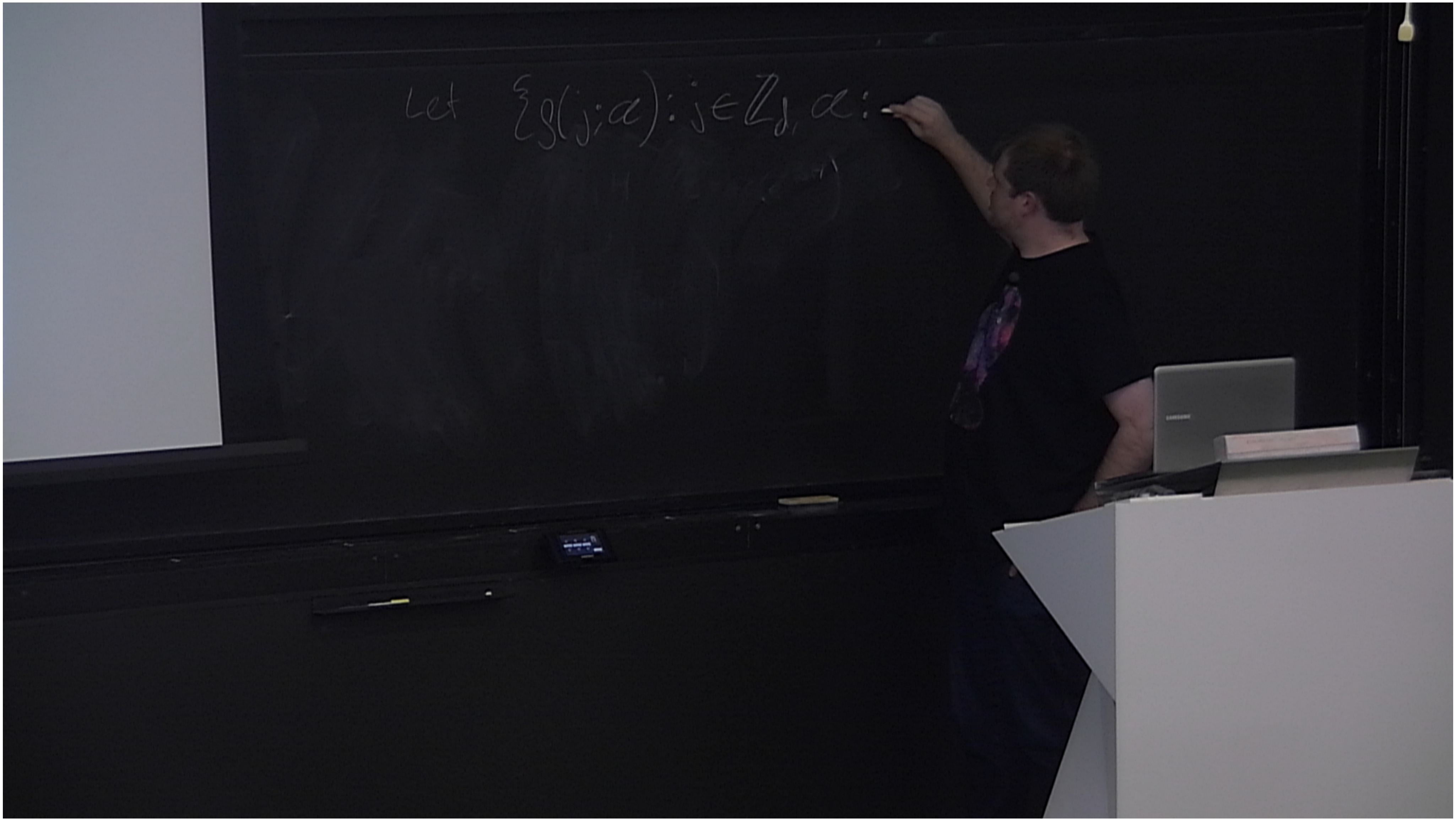
Should we all convert to the church of the larger Hilbert space?

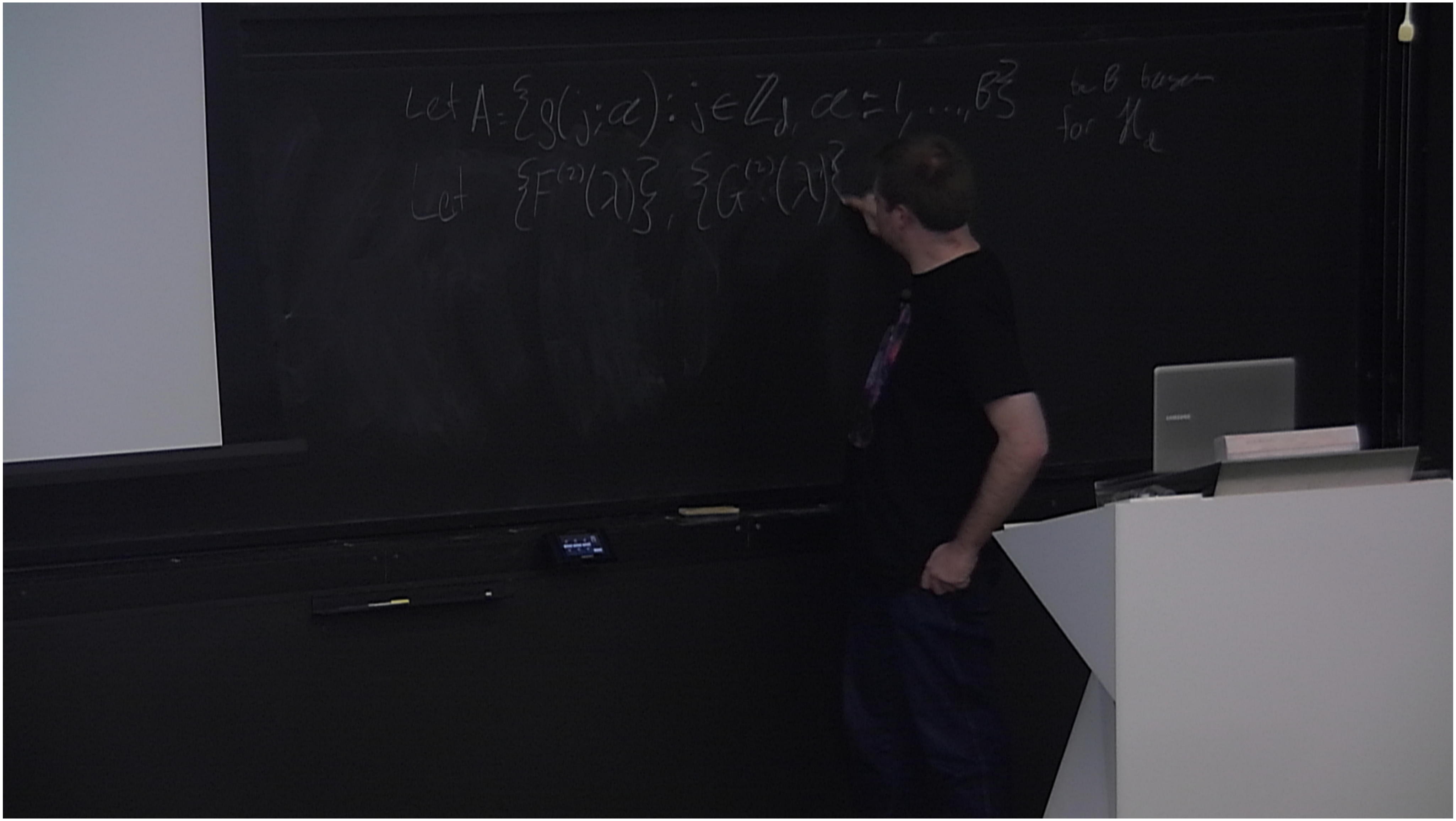
Other projects

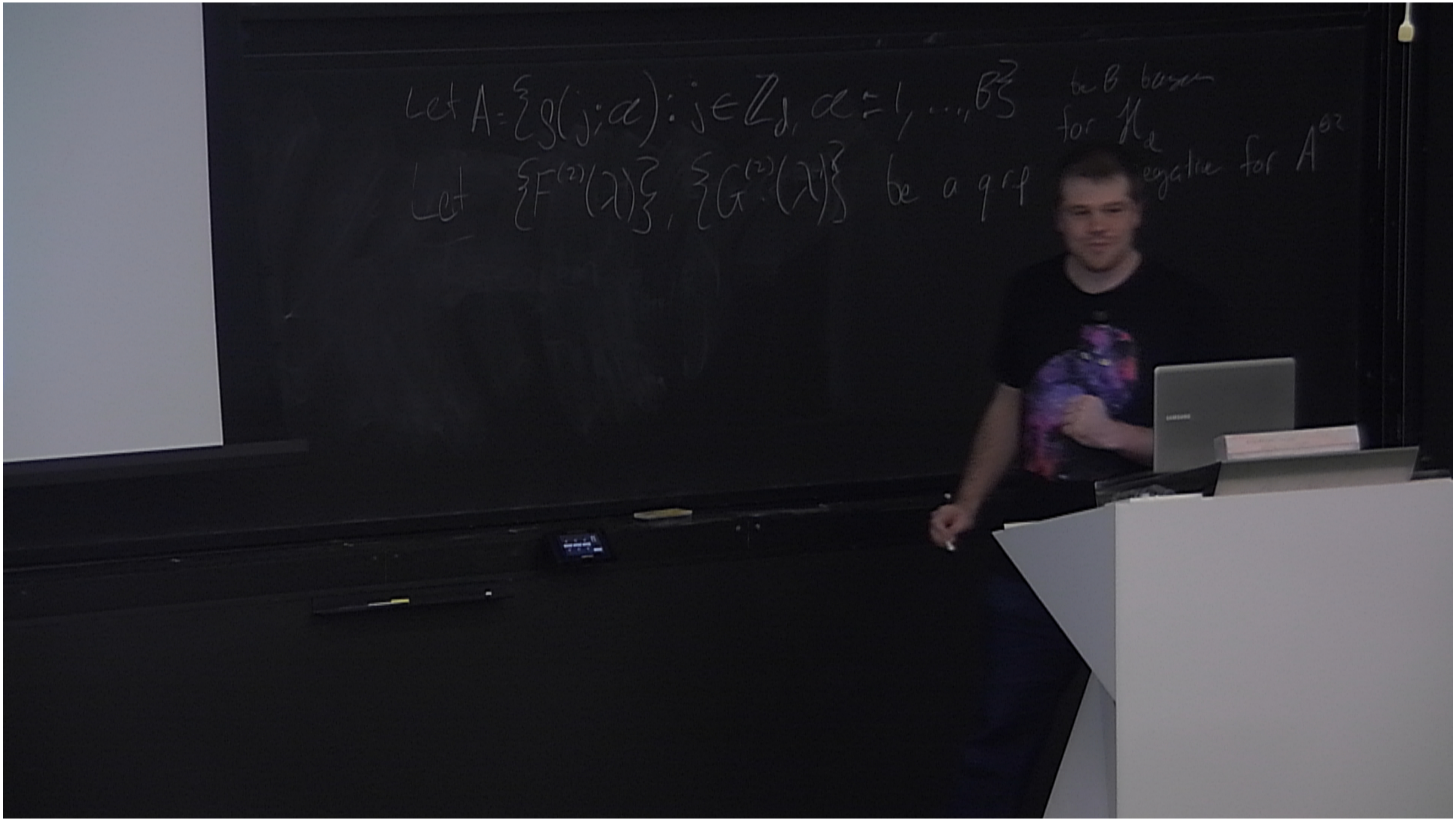
- IQP circuits, MBQC and nonlocality (with Matty Hoban, Dan Browne and Robert Raussendorf);
- Bell inequalities and MBC (with Matty Hoban and Dan Browne);
- Epistemic interpretations of quantum mechanics (lots of people);
- Alternative to Unruh-DeWitt monopole detector in QFT (with Nick Meniccuci);
- Quantum computation as a test of contextuality;
- CTCs and QM; and
- Reference frames and Bell inequalities.

Other projects

- IQP circuits, MBQC and nonlocality (with Matty Hoban, Dan Browne and Robert Raussendorf);
- Bell inequalities and MBC (with Matty Hoban and Dan Browne);
- Epistemic interpretations of quantum mechanics (lots of people);
- Alternative to Unruh-DeWitt monopole detector in QFT (with Nick Meniccuci);
- Quantum computation as a test of contextuality;
- CTCs and QM; and
- Reference frames and Bell inequalities.


Let {g(j;α): j∈Z_d, α:


Let A = { g(j; α) : j ∈ Z_d, α = 1, …, B }
for $\mathcal{H}_d$
Let { F^(λ)(λ) }, { G^(λ)(λ) }

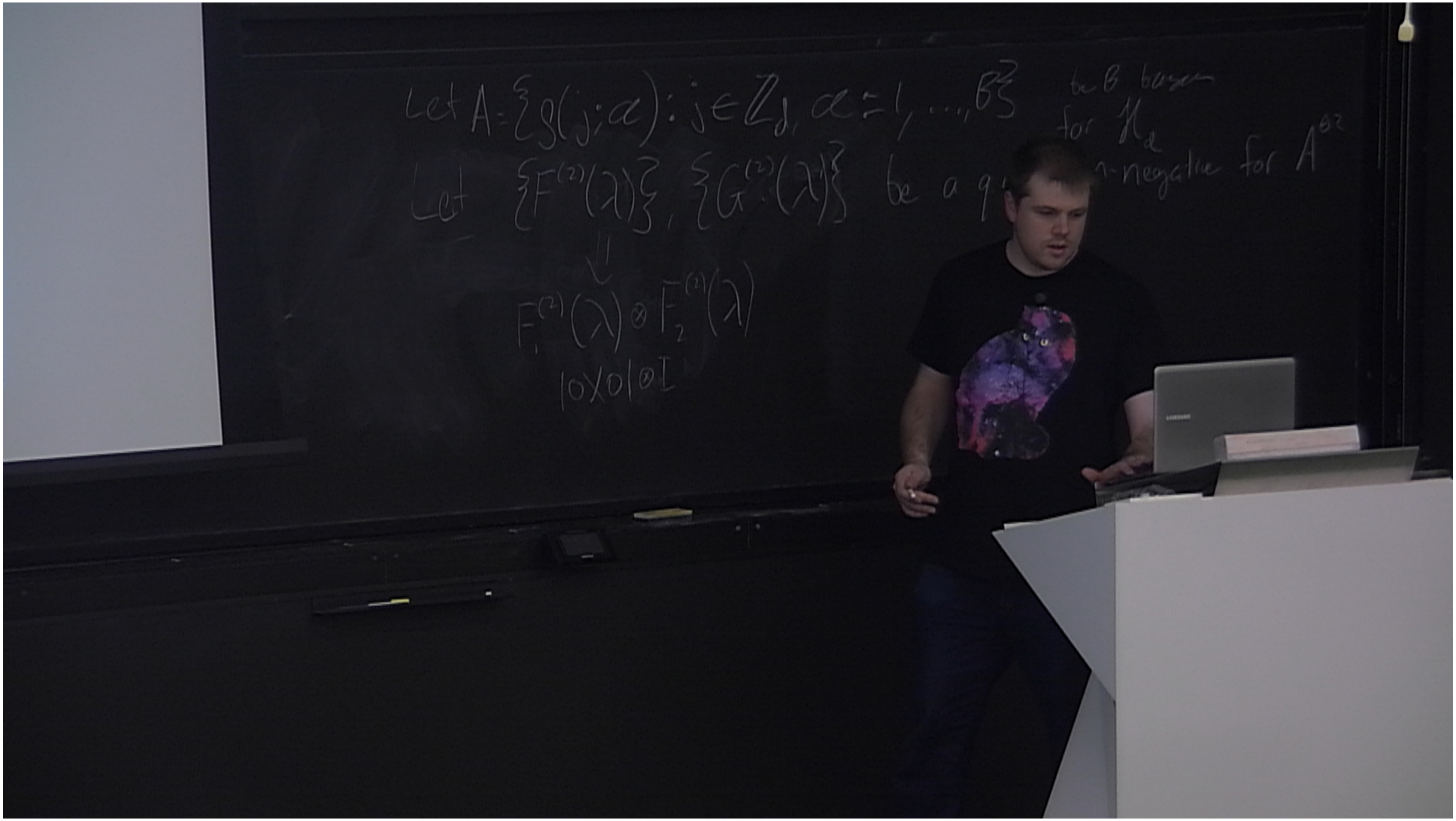

Let A = {g(j;α) : j ∈ Z_d, α = 1, ..., B} be B bases for H_d
Let {F^(2)(λ)}, {G^(2)(λ)} be a qrf
negative for A^⊗2

Non-negative qubit subtheories

Question: can we avoid these problems using a different quasiprobability representation for multiple systems?

To define a quasiprobability representation we need a phase space Λ_N for N systems and frames $\{F^{(N)}(\lambda)\}$ and $\{G^{(N)}(\lambda)\}$.

Theorem 3: any quasiprobability representation that is non-negative for a locally-tomographic subtheory gives a locally causal model for that subtheory.

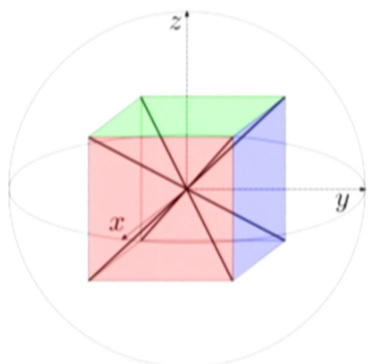


Let A = {g(j;α) : j ∈ Z_d, α = 1, …, B} be B bases for H_d
Let {F^(2)(λ)}, {G^(2)(λ)} be a q… …n-negative for A^⊗2
F_1^(2)(λ) ⊗ F_2^(2)(λ)
|0⟩⟨0| ⊗ I

Example: single-qubit stabilizer subtheory

How about Clifford transformations?

Any non-negative unitary permutes the $\{F(a,b)\}$.



If the CNOT is a non-negative transformation, then it must map $G_1(\lambda) \otimes G_1(\lambda)$ to some element of $\{G_1(\lambda)\}^{\otimes 2}$. However, the CNOT permutes signed tensor products of Pauli matrices and maps

$$\mathbf{11} + XX + YY + ZZ \to \mathbf{11} + X\mathbf{1} - XZ + \mathbf{1}Z \tag{38}$$

under conjugation. As the right-hand side cannot be written as

$$(\mathbf{1} + (-1)^{\alpha} X)(\mathbf{1} + (-1)^{\beta} Z) \tag{39}$$

for any value of α, β, the CNOT does not permute the elements of the frame $\{G(\lambda) \otimes G(\lambda')\}$ and so is not a non-negative transformation. □

In the above proof, we have used the following lemma, which gives the full set of frame operators that can be non-negative for the set of single qubit stabilizer states and measurements. As an aside, note that the vectors $\vec{r}$ correspond to the vertices of the cube that is dual to the octahedron whose vertices are the stabilizer states.

Lemma VI.2. *The only quasiprobability representations in which the single qubit stabilizer states are non-negative have frame operators of the form*

$$F(\lambda) = \frac{p(\lambda)}{2}(\mathbf{1} + \vec{r}(\lambda) \cdot \vec{\sigma})$$

$$G(\lambda) = \frac{1}{\ldots}(\mathbf{1} + \vec{r}(\lambda) \cdot \vec{\sigma}) \tag{40}$$