Title: Penrose inequalities and the stability of non-uniform black strings

Date: Nov 01, 2012 01:00 PM

URL: http://pirsa.org/12110039

Abstract: In this talk I will first review static black holes in Kaluza-Klein theory. It is well-known that within this theory there exist black strings which are non-uniform along the Kaluza-Klein circle. Using numerical methods, I will explain how to construct (for the first time) non-uniform black strings in D>10, where D is the total number of spacetime dimensions. The stability of such black objects has not been discussed before, and in the last part of the talk I will explain how one can study the stability of non-uniform black strings using Penrose inequalities. This will lead to a new conjecture for the phase diagram of static Kaluza-Klein black holes.

PLAN OF THE TALK

- Static black holes in Kaluza-Klein (KK) theory
- Numerical construction of non-uniform black strings and the phase diagram of static black holes in KK theory
- Penrose inequalities and the stability of non-uniform black strings
- Summary and future directions

Static black holes in Kaluza-Klein theory

Consider vacuum General Relativity (GR) in D dimensions with compact Kaluza-Klein (KK) circle. We'll be interested in ***static*** black holes with SO(D-2) symmetry which are asymptotically KK:

$$ds^2 \approx -dt^2 + dr^2 + r^2 d\Omega^2_{D-3} + dz^2\,, \quad z \sim z + L$$

at large distances.

Simplest solution: Schwarzschild black string. Take the D-1 dimensional asymptotically flat Schw. black hole with a circle of length L:

$$ds^2 = ds^2_{\text{Schw}_{D-1}} + dz^2$$

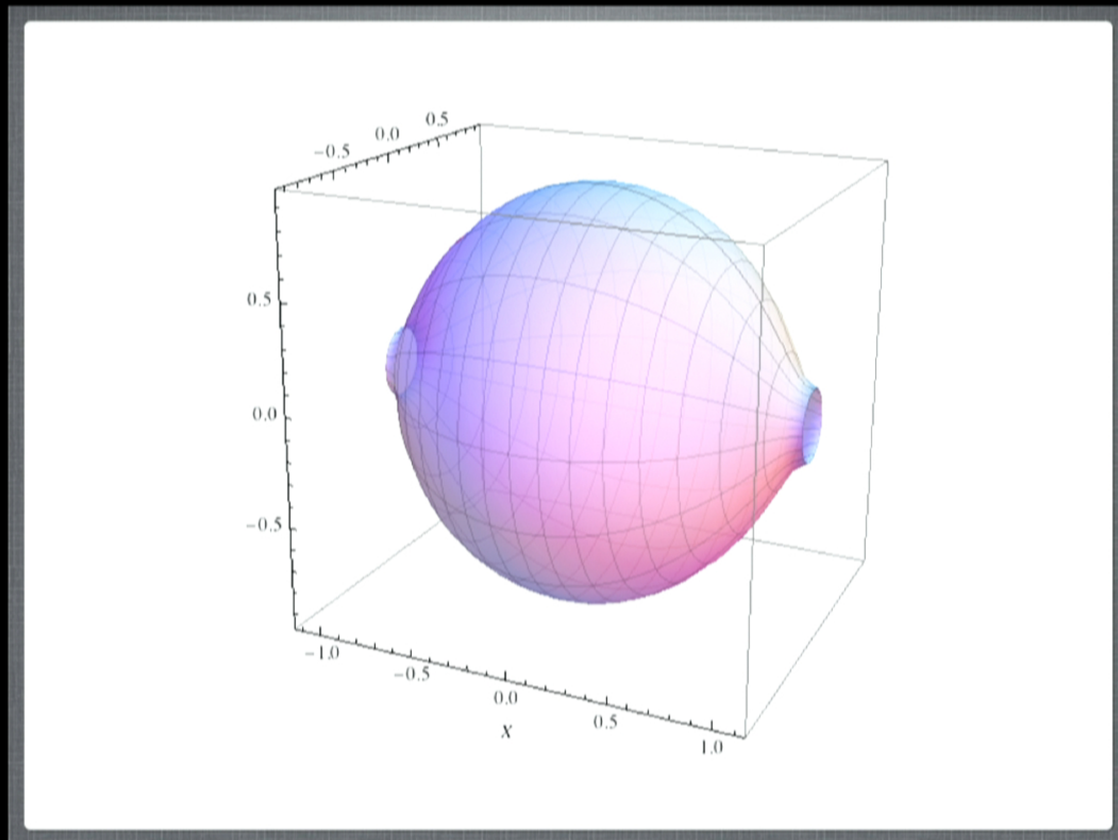
- Horizon topology: $S^1 \times S^{D-3}$
- Unstable under gravitational perturbations for r_+/L less than some critical number (which depends on D) [Gregory and Laflamme]
- These perturbations break the translational symmetry along the KK circle

What is the endpoint of the GL instability?

One of the GL modes is time-independent but breaks the translational symmetry along the KK circle $\Rightarrow$ one can construct a linearised solution of the Einstein equations which is non-uniform along the KK circle

These black strings were subsequently constructed perturbatively at the non-linear level [Gubser] and fully non-perturbatively up to D=11 [Wiseman, Sorkin, Kunz et al.]

Up to D=13 it was observed that *perturbative* non-uniform black strings have less entropy than uniform black strings with the same mass, so they cannot be the endpoint of the GL instability. However, in D>13 they have greater entropy. [Sorkin]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X

It is useful to parametrise them in terms of the non-uniformity parameter:

$$\lambda = \frac{1}{2}\left(\frac{R_{max}}{R_{min}} - 1\right)$$

It is expected that non-uniform black strings exist for all $\lambda > 0$ and that at $\lambda = \infty$ they merge with a 1-parameter family of black holes which localised on the KK circle, with horizon topology S^{D-2}.

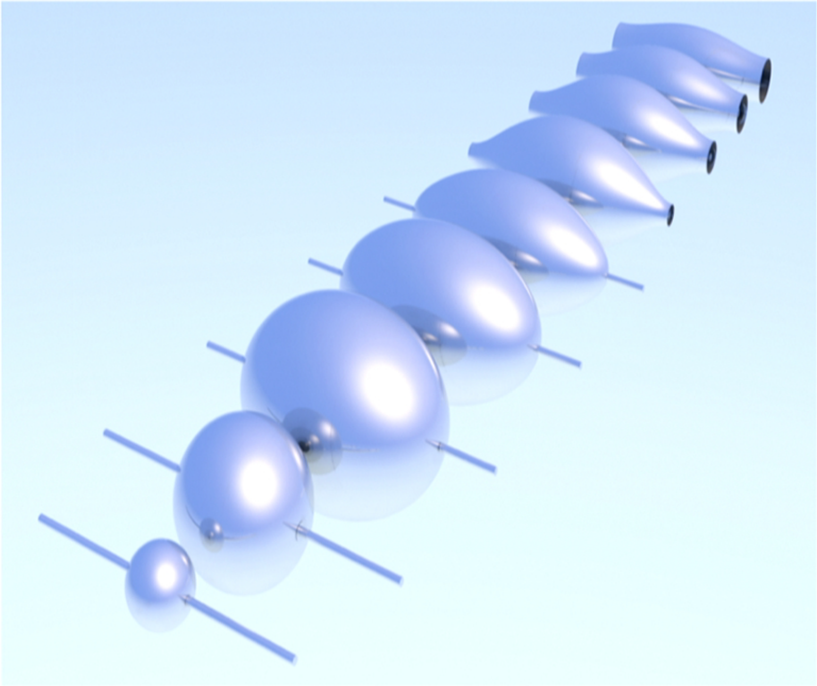

Image by Toby Wiseman

What about the stability of these objects? What are the physically relevant phases?

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Clearly small (compared to L) localised black holes should be stable under gravitational perturbations since AF 5d Schw. is stable.

The perturbative construction of non-uniform black strings shows an interesting change in behaviour at $D = 13$ [Sorkin].

- For $D \leq 13$ infinitesimally non-uniform black strings have a *smaller* horizon area than uniform black strings with the same mass $\Rightarrow$ unstable?
- For $D > 13$, infinitesimally non-uniform black strings have a *greater* horizon area than uniform black strings with the same mass $\Rightarrow$ stable?

In the rest of this talk I will discuss that stability of non-uniform black strings with finite λ.

Numerical construction of non-uniform black strings in $D \geq 11$

THE METHOD

[Headrick, Kitchen and Wiseman]

We want to solve (numerically) the Einstein vacuum equations

$$R_{\mu\nu} = 0$$

for a spacetime $(\mathcal{M}, g)$ in D dimensions.

We are interested in two types of PDEs according to the nature of the problem:

- Find static/stationary spacetimes $\rightarrow$ elliptic problem
- Evolve dynamical spacetimes $\rightarrow$ hyperbolic problem

but the Einstein equations do not have a definite character (elliptic or hyperbolic) unless some kind of gauge-fixing is introduced.

Basic idea: implement the (generalised) harmonic gauge in the elliptic case.
→ Borrowed from the early Ricci flow literature [DeTurck]

Instead of considering the Einstein equations, we consider a modified version of it (the Harmonic Einstein equations) which is manifestly elliptic/hyperbolic:

$$R^H_{\mu\nu} = R_{\mu\nu} - \nabla_{(\mu}\xi_{\nu)} = 0 \qquad \xi^\mu = g^{\alpha\beta}(\Gamma^\mu_{\alpha\beta} - \bar{\Gamma}^\mu_{\alpha\beta})$$

where $\bar{\Gamma}$ is the Levi-Civita connection associated to a reference metric $\bar{g}$ on the manifold.

- $R^H_{\mu\nu} = 0$ is manifestly elliptic/hyperbolic: $R^H_{\mu\nu} \sim -\frac{1}{2}\, g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}$
- Harmonic gauge: $\xi^\mu = 0 \quad \Rightarrow \quad \Delta_g x^\mu = H^\mu = -g^{\alpha\beta}\bar{\Gamma}^\mu_{\alpha\beta}$
- Fully covariant

Ultimately we want to solve the Einstein equations, so how do we achieve this?

How do we achieve $\xi^\mu = 0$?

Hyperbolic case:

- Choosing $\xi^\mu = 0$ and $\partial_t \xi^\mu = 0$ on a Cauchy surface Σ ensures that the solutions to $R^H{}_{\mu\nu} = 0$ are Einstein!

NUMERICAL CONSTRUCTION OF NON-UNIFORM BLACK STRINGS IN *D* DIMENSIONS

Metric ansatz:

$$ds^2 = -4\,r_0^2\,\Delta\,y^2\,e^T\,dt^2 + \frac{r_0^2\,e^S}{f(y)^{\frac{2}{D-4}}}\,d\Omega^2_{(D-3)} + e^A\,dx^2 + \frac{4\,r_0^2\,\Delta\,e^B}{f(y)^{\frac{2(D-3)}{D-4}}}\,(dy + y\,f(y)\,F\,dx)^2$$

$$y^2 = 1 - \frac{r_0^{D-4}}{r^{D-4}} \qquad f(y) = 1 - y^2\,, \qquad \Delta = 1/(D-4)^2$$

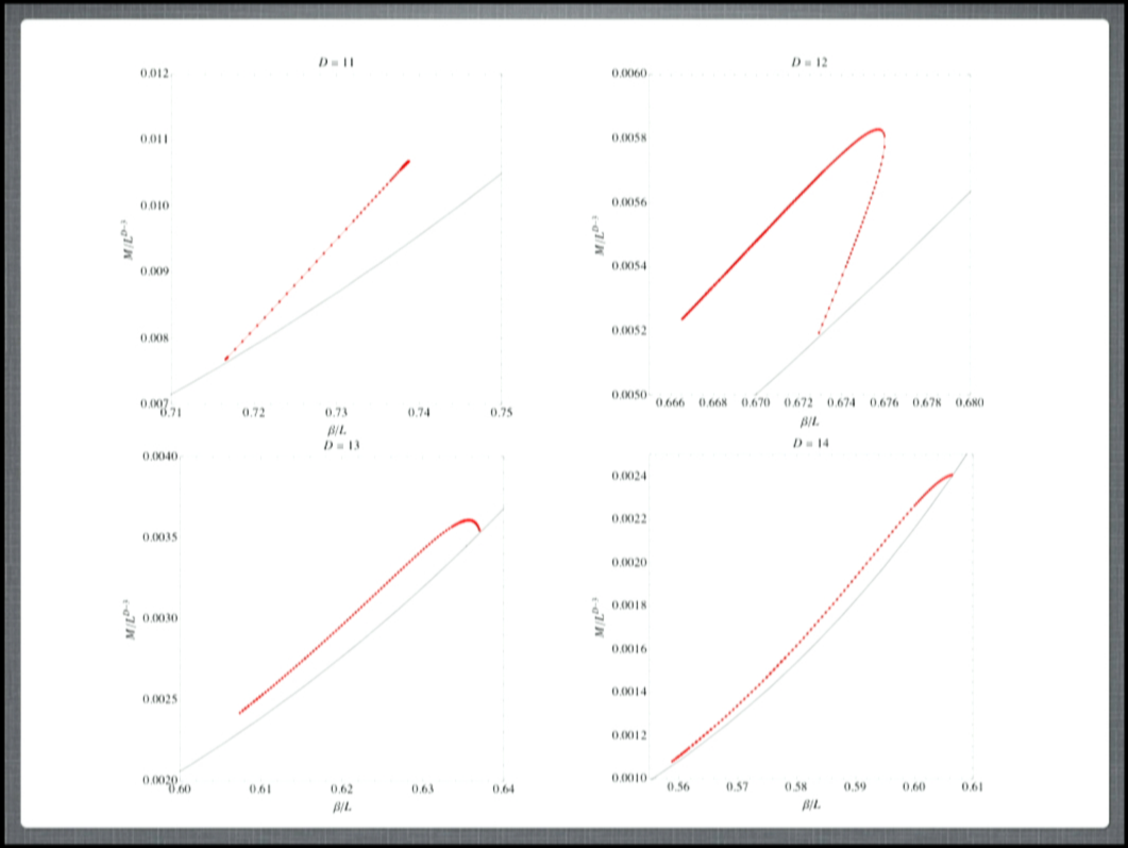
Boundary conditions:

- KK asymptotics: T = S = A = B = F = 0 at $y = 1$
- Regularity at the horizon ($y = 0$)
- Periodicity in x (with $x \sim x+L$, and $L = 2$)

⇒ No Ricci soliton can exist!

• Solve the Harmonic Einstein equations numerically using a spectral multi-domain method

• Vary r_0 (i.e., temperature) to move along the branch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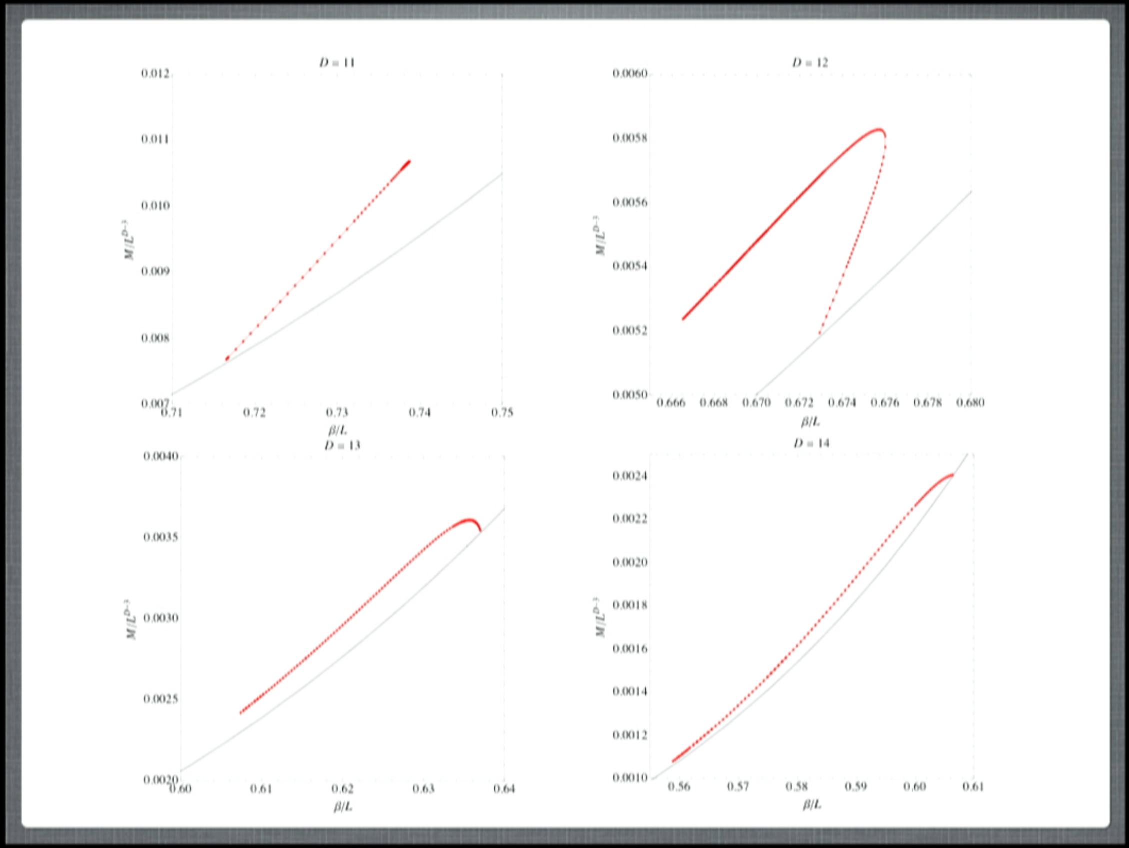
D = 12
D = 13
D = 14
M/L^{D-3}
β/L


D = 11
M/L^{D-3}
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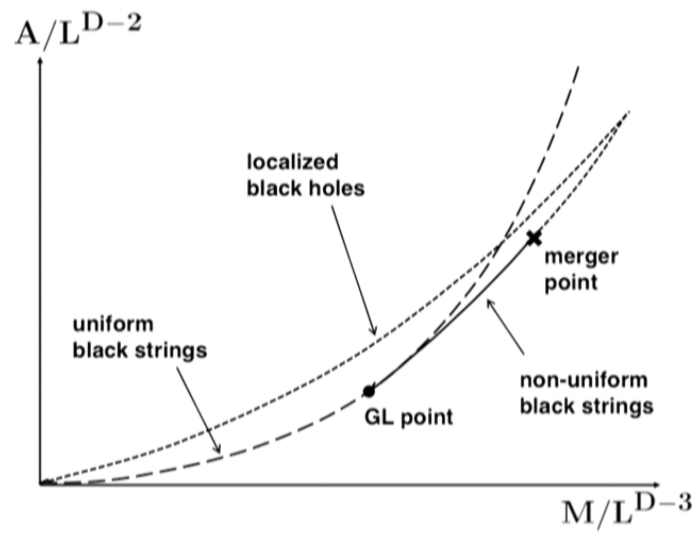
CONJECTURED PHASE DIAGRAM OF STATIC KK BLACK HOLES

$D \leq 11$



Under dynamical evolution, we have:

1. The area of the horizon increases: $\frac{dA_H}{dt} \geq 0$

2. In asymptotically flat or KK spacetimes, the Bondi energy must decrease:

$$\frac{dE}{dt} \leq 0$$

→ gravitational waves carry energy

Using these two general facts we can argue that a given black hole must be unstable as follows:

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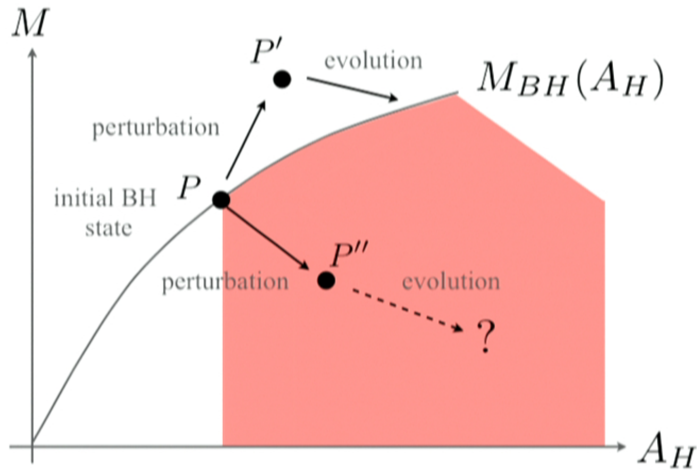
MAIN IDEA



If the initial data lies in the shaded region then under evolution the spacetime CANNOT settle down to the same sequence of stationary solutions

Therefore, constructing suitable initial data we can demonstrate that a given black hole spacetime is unstable

PENROSE INEQUALITIES

General form of the Penrose inequality [Penrose]:

$$A_{min} \leq A_{BH}(E)$$

A_{min} : greatest lower bound on the area of any surface that encloses the apparent horizon

E : ADM energy of the asymptotically flat/KK initial data

4d Penrose inequality: $A_{min} \leq 16\pi E^2$

- Assumes that the spacetime resulting from the initial data containing an apparent horizon settles down to the Kerr family
- Equality for Schwarzschild initial data
- Proven for time-symmetric initial data [Huisken and Ilmanen; Bray]

Local Penrose inequalities: consider initial data which is a “small” perturbation of a certain black hole spacetime [Gibbons]

If the initial data violates the inequality

$$A_{\min} \leq A_{BH}(E)$$

then the assumption that the spacetime settles down to a black hole belonging to the same family with a small variation of the physical parameters must be wrong

$\Rightarrow$ INSTABILITY

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Consider perturbations around a known black hole spacetime to obtain the "local" version of the Penrose inequality:

$$Q \equiv \ddot{M}(0) - \frac{T}{4}\,\ddot{A}_{app}(0) - \frac{1}{T\,c_L}\,\dot{M}(0)^2 \geq 0$$

• Q is the canonical energy of the perturbations [Hollands and Wald]

• The violation of this inequality is a necessary and sufficient condition for the existence of an instability [Hollands and Wald]

Remark 1.
For asymptotically flat rotating black holes one gets:

$$A_{min} \leq A_I \leq A_{BH}(M_F, J_{iF}) \leq A_{BH}(E, J_{iF})$$

→ not very useful because we know nothing about J_{iF}

Assuming that the perturbations preserve the N commuting rotational symmetries of the background spacetime, then the angular momenta will be conserved:

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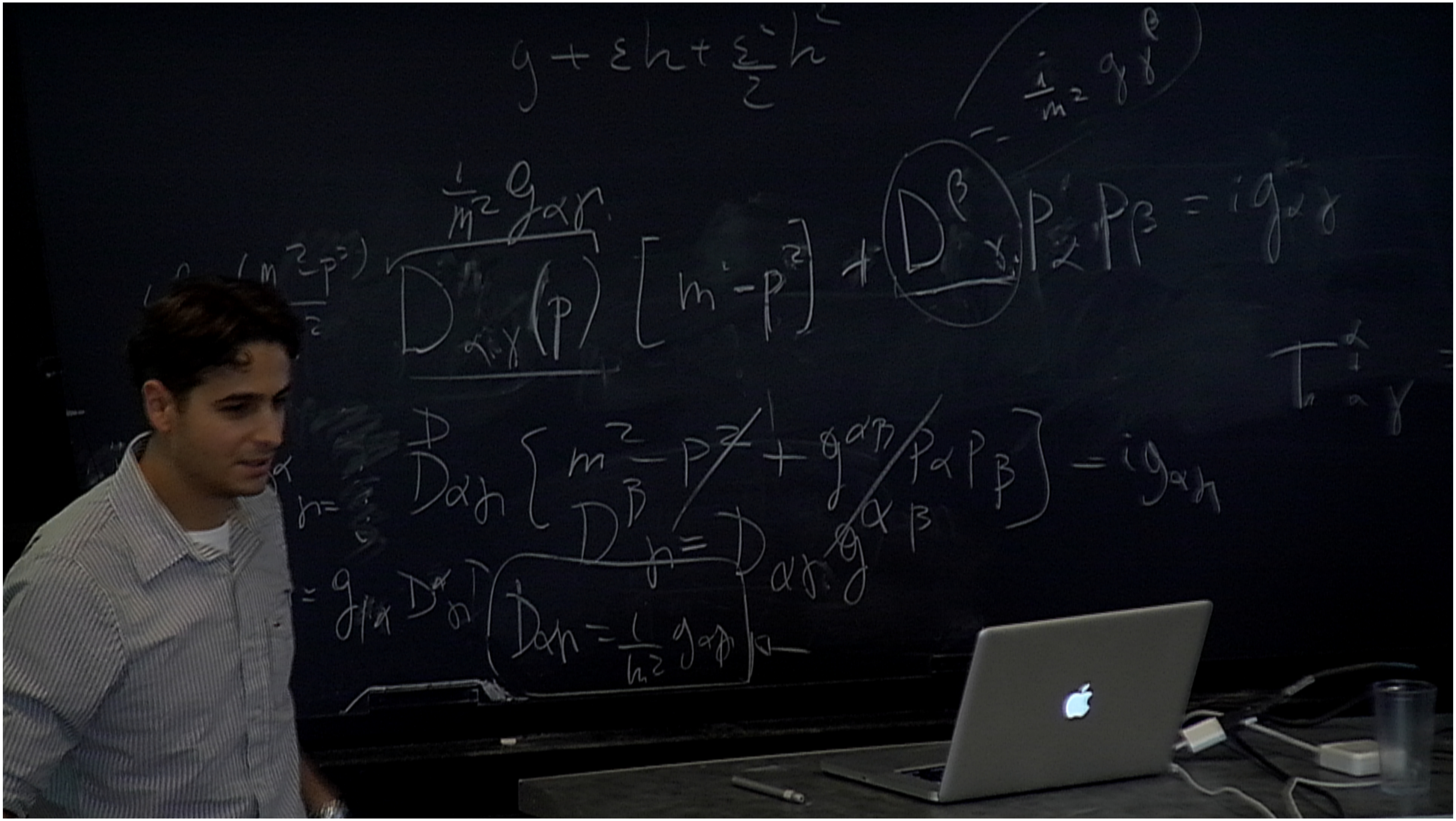


Our strategy for finding instabilities:

Construct initial data up to second order in perturbation theory and check whether it violates the local Penrose inequality

Initial data for the Einstein vacuum equations: $(\Sigma,\ \bar{h}_{ab},\ \bar{K}_{ab})$

Hamiltonian constraint: $\bar{R} + \bar{K}^2 - \bar{K}_{ab}\bar{K}^{ab} = 0$

Momentum constraint: $\bar{\nabla}_b \bar{K}^b{}_a - \bar{\nabla}_a \bar{K} = 0$

For time-symmetric initial data,

$$h_{ab} = \Psi^{4/(D-3)}\bar{h}_{ab}\,, \quad K_{ab} = \bar{K}_{ab} = 0$$

we to solve a single scalar equation:

$$\bar{\nabla}^2\Psi = 0$$

Expanding Ψ in terms of a small parameter,

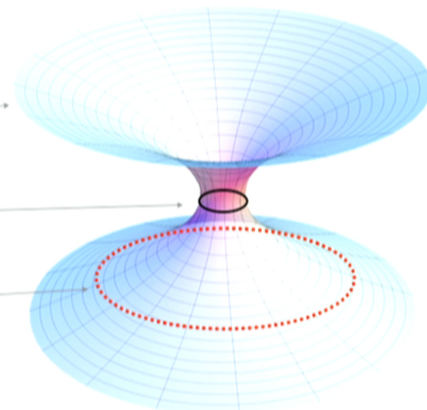
$$\Psi = 1 + \epsilon\,\dot{\Psi}(0) + \epsilon^2\,\ddot{\Psi}(0)/2 + \ldots$$

we can construct perturbatively initial data by solving 2 linear equations:

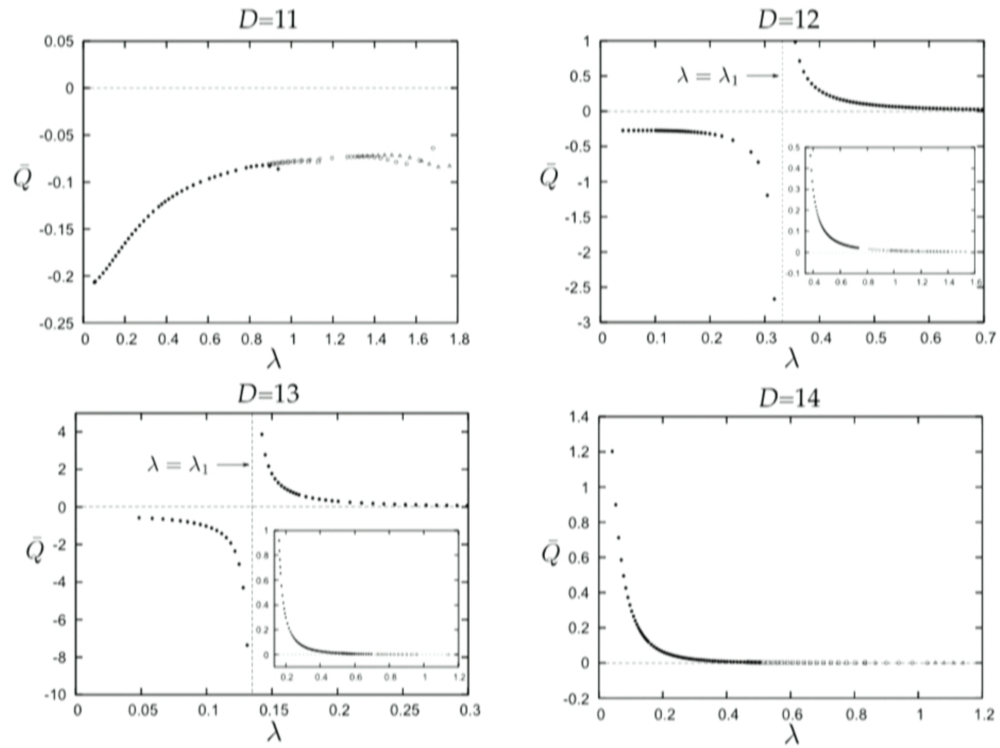
$$\bar{\nabla}^2\dot{\Psi} = \bar{\nabla}^2\ddot{\Psi} = 0$$

Boundary conditions:

- Asymptotically KK
- Bifurcation surface
- Prescribe Dirichlet
- Periodicity



RESULTS



SUMMARY

1. We have constructed non-uniform black strings with large values of λ in $D = 11, 12, 13, 14, 15$.
2. In $D = 12, 13$ there is a maximum of the mass along the non-uniform black string branch
3. In $D \leq 11$ non-uniform black strings with finite λ are unstable
4. In $D = 12, 13$ there exists a λ_1 such that for $\lambda > \lambda_1$ non-uniform black strings are stable for the class of initial data that we have considered.
5. In $D \geq 14$, all non-uniform black strings appear to be stable, at least for our class of initial data