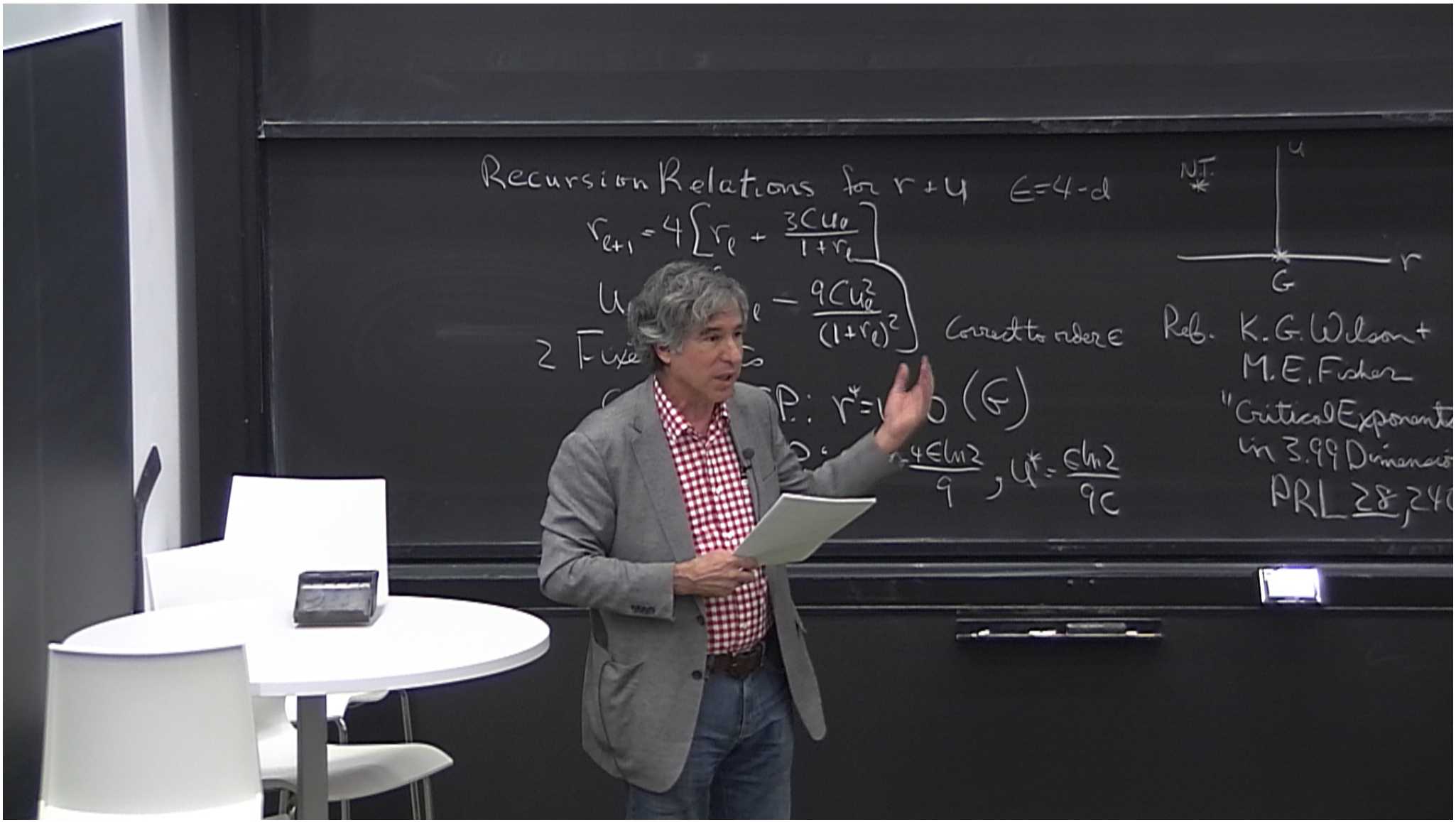
Title: Statistical Mechanics - Lecture 13

Date: Nov 28, 2012 10:30 AM

URL: http://pirsa.org/12110035

Abstract:


Recursion Relations for r + u ε=4-d
correct to order ε
Ref. K.G. Wilson +
M.E. Fisher
"Critical Exponents
in 3.99 Dimensio
PRL 28, 240
N.T.
u
r
G

Recursion Relations for r + u $\quad \epsilon = 4 - d$

$$r_{\ell+1} = 4\left[r_\ell + \frac{3Cu_\ell}{1+r_\ell}\right]$$

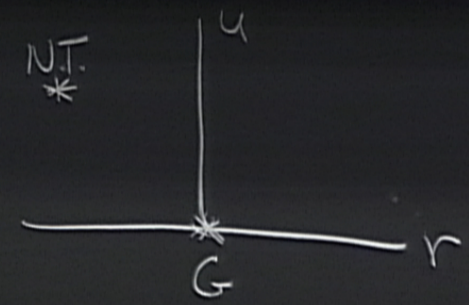
$$u_{\ell+1} = 2^{\epsilon}\left[u_\ell - \frac{9Cu_\ell^2}{(1+r_\ell)^2}\right]$$

Correct to order ϵ

2 Fixed Points

Gaussian F.P.: $r^* = u^* = 0$ (G)

Non-Trivial F.P.: $r^* = -\frac{4\epsilon \ln 2}{9}$, $u^* = \frac{\epsilon \ln 2}{9C}$



Ref. K.G. Wilson + M.E. Fisher "Critical Exponents in 3.99 Dimensions" PRL 28, 240 (1972).

Analyse flow in the r-u plane

Start with G

Near G

$$r' - \cancel{r^*}^{\,0} = \left.\frac{\partial r'}{\partial r}\right)_G (r - \cancel{r^*}^{\,0}) + \left.\frac{\partial r'}{\partial u}\right)_G (u - \cancel{u^*}^{\,0}) = T_{rr}\, r + T_{ru}\, u$$

(1972).

Analyse flow in the r-u plane

Start with G

Near G

$$r' - \cancel{r^*}^{\,0} = \left.\frac{\partial r'}{\partial r}\right)_G (r - \cancel{r^*}^{\,0}) + \left.\frac{\partial r'}{\partial u}\right)_G (u - \cancel{u^*}^{\,0}) = T_{rr} r + T_{ru} u$$

etc.

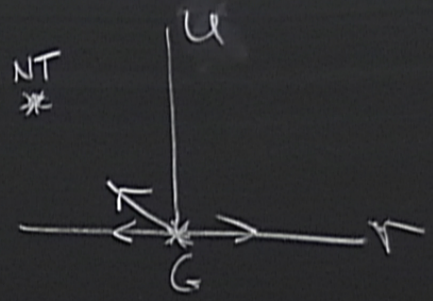
$T_{rr} = 4 \quad T_{ru} = 12C$

$T_{ur} = 0 \quad T_{uu} = \zeta^{\epsilon}$

Say that $\begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ then

$$T\begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} 4 & 12c \\ 0 & 2^{\epsilon} \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = 4\begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} r' \\ 0 \end{pmatrix} = 4\begin{pmatrix} r \\ 0 \end{pmatrix}$$



Similarly if $\begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} \frac{-12cx}{4-2^{\epsilon}} \\ x \end{pmatrix}$

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Linearize R.R. around N.T.

$$u^* = \frac{\epsilon \ln 2}{4C}$$

$$T = \begin{bmatrix} 4 - \frac{12Cu^*}{(1+r^*)^2} & \frac{12C}{1+r^*} \\ 2\epsilon \frac{18Cu^{*2}}{(1+r^*)^2} & 2\epsilon\left(1 - \frac{18Cu^*}{(1+r^*)^2}\right) \end{bmatrix}$$

$$-2\epsilon \ln 2$$

$$(1+\epsilon \ln 2)(1-2\epsilon \ln 2)$$

$$= 1 - \epsilon \ln 2$$

$$= \begin{bmatrix} 4 - \frac{4}{3}\epsilon \ln 2 & 12C \\ 0 & 1 - \epsilon \ln 2 \end{bmatrix}$$

$$\frac{x + 18Cx - 2\epsilon \quad Cx}{4-2} \quad x \Bigg) = 2^{\epsilon}\begin{pmatrix} \frac{-12Cx}{4-2\epsilon} \\ x \end{pmatrix} = 2^{\epsilon}\begin{pmatrix} v \\ u \end{pmatrix}$$

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$$(1+\epsilon \ln 2)(1-2\epsilon \ln 2)$$

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$$= \begin{bmatrix} 4 - \frac{4}{3}\epsilon \ln 2 & 12C \\ 0 & 1 - \epsilon \ln 2 \end{bmatrix}$$

Eigenvalues are

$4 - \frac{4}{3}\epsilon \ln 2$ (relevant)

$1 - \epsilon \ln 2$ (irrelevant)

$$\begin{pmatrix} \frac{-\cancel{48Cx} + \cancel{48Cx} - 2^\epsilon 12Cx}{4 - 2^\epsilon} \\ 2^\epsilon x \end{pmatrix} = 2^\epsilon \begin{pmatrix} \frac{-12Cx}{4-2^\epsilon} \\ x \end{pmatrix} = 2^\epsilon \begin{pmatrix} v \\ u \end{pmatrix}$$

$-r$

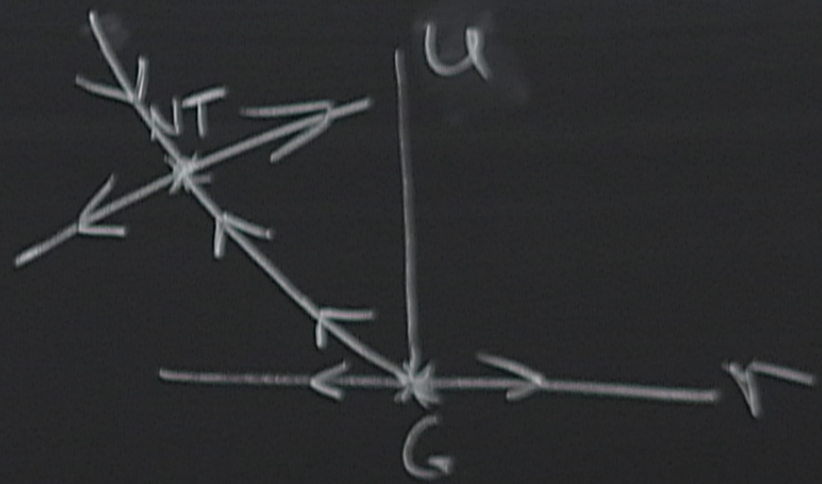
$$\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$ then

$$\begin{pmatrix} 12c & \\ & 2^{\epsilon} \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = 4\begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$+\begin{pmatrix} r \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} \frac{-12cx}{4-2^{\epsilon}} \\ x \end{pmatrix}$$

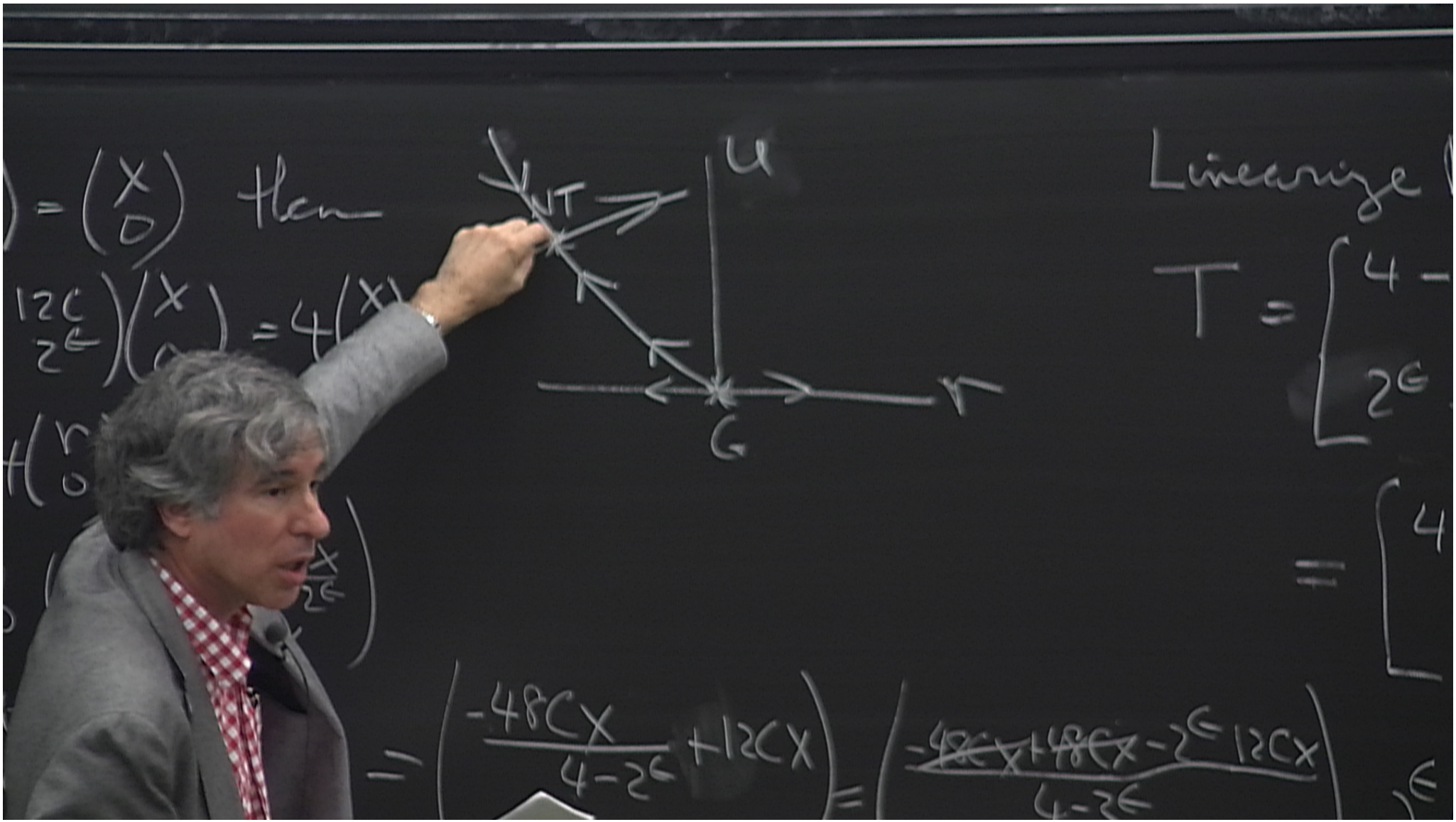
$$\begin{pmatrix} & 12c \\ 0 & 2^{\epsilon} \end{pmatrix}\begin{pmatrix} \frac{-12cx}{4-2^{\epsilon}} \\ \end{pmatrix} = \begin{pmatrix} \frac{-48cx}{4-2^{\epsilon}} + 12cx \end{pmatrix} = \begin{pmatrix} \frac{\cancel{-48cx} + \cancel{48cx} - 2^{\epsilon}\,12cx}{4-2^{\epsilon}} \end{pmatrix}$$



Linearize

$$T = \begin{bmatrix} 4 - \\ 2^{\epsilon} \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ \end{bmatrix}$$


then
NT
u
r
G
Linearize
T =

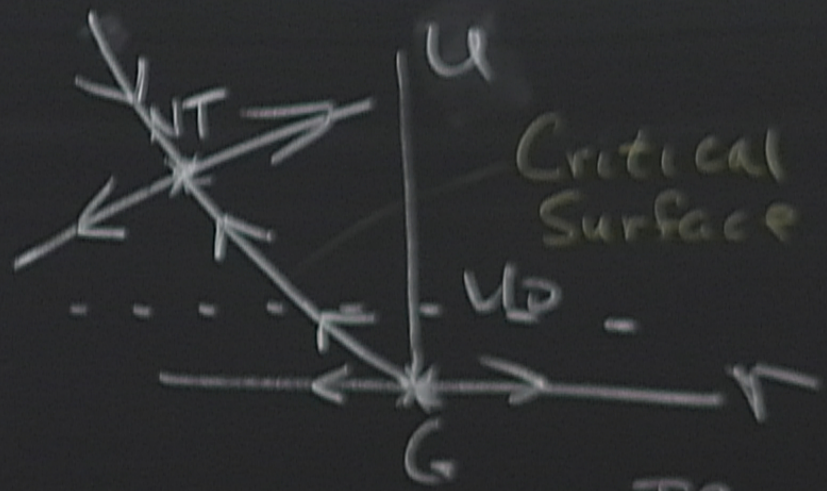
$\begin{pmatrix} x \\ 0 \end{pmatrix}$ then

$\begin{pmatrix} c \\ \epsilon \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = 4\begin{pmatrix} x \\ 0 \end{pmatrix}$

$\begin{pmatrix} r \\ 0 \end{pmatrix}$

$\begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} \frac{-12cx}{4-2\epsilon} \\ x \end{pmatrix}$

$\begin{pmatrix} 12c \\ 2\epsilon \end{pmatrix}\begin{pmatrix} \frac{-12cx}{4-2\epsilon} \end{pmatrix} = \begin{pmatrix} \frac{-48cx}{4-2\epsilon} + 12cx \end{pmatrix} = \begin{pmatrix} \frac{\cancel{-48cx} + \cancel{48cx} - 2\epsilon\, 12cx}{4-2\epsilon} \end{pmatrix}$



If starting Hamiltonian contains $u_0 > 0$

Linearize R.

$T = \begin{bmatrix} 4 - \frac{12}{(} \\ 2\epsilon \quad \frac{18}{(} \end{bmatrix}$

$= \begin{bmatrix} 4 - \end{bmatrix}$

Recursion Relations for r + u $\quad \epsilon = 4-d$

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Correct to order ϵ

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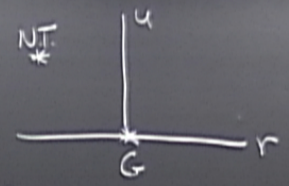

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Analyze flow in the r-u plane

Start with G

Near G

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etc.

$T_{rr} = 4 \quad T_{ru} = 12C$

$T_{ur} = 0 \quad T_{uu} = 2^\epsilon$

Eigenvalues are: 4 and 2^ϵ

Say that $\begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ then

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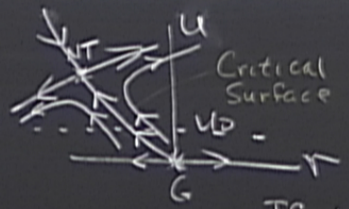

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If starting Hamiltonian contains $u_0 > 0$

Linearize R.R. around N.T.

$$T = \begin{bmatrix} 4 - \frac{12Cu^*}{(1+r^*)^2} & \frac{12C}{1+r^*} \\ 2^\epsilon \frac{18Cu^{*2}}{(1+r^*)^2} & 2^\epsilon\left(1 - \frac{18Cu^*}{(1+r^*)^2}\right) \end{bmatrix}$$

$u^* = \frac{\epsilon\ln 2}{9C}$

$-2\epsilon\ln 2$

$(1+\epsilon\ln 2)(1-2\epsilon\ln 2)$

$= 1 - \epsilon\ln 2$

$$= \begin{bmatrix} 4 - \frac{4}{3}\epsilon\ln 2 & 12C \\ 0 & 1 - \epsilon\ln 2 \end{bmatrix}$$

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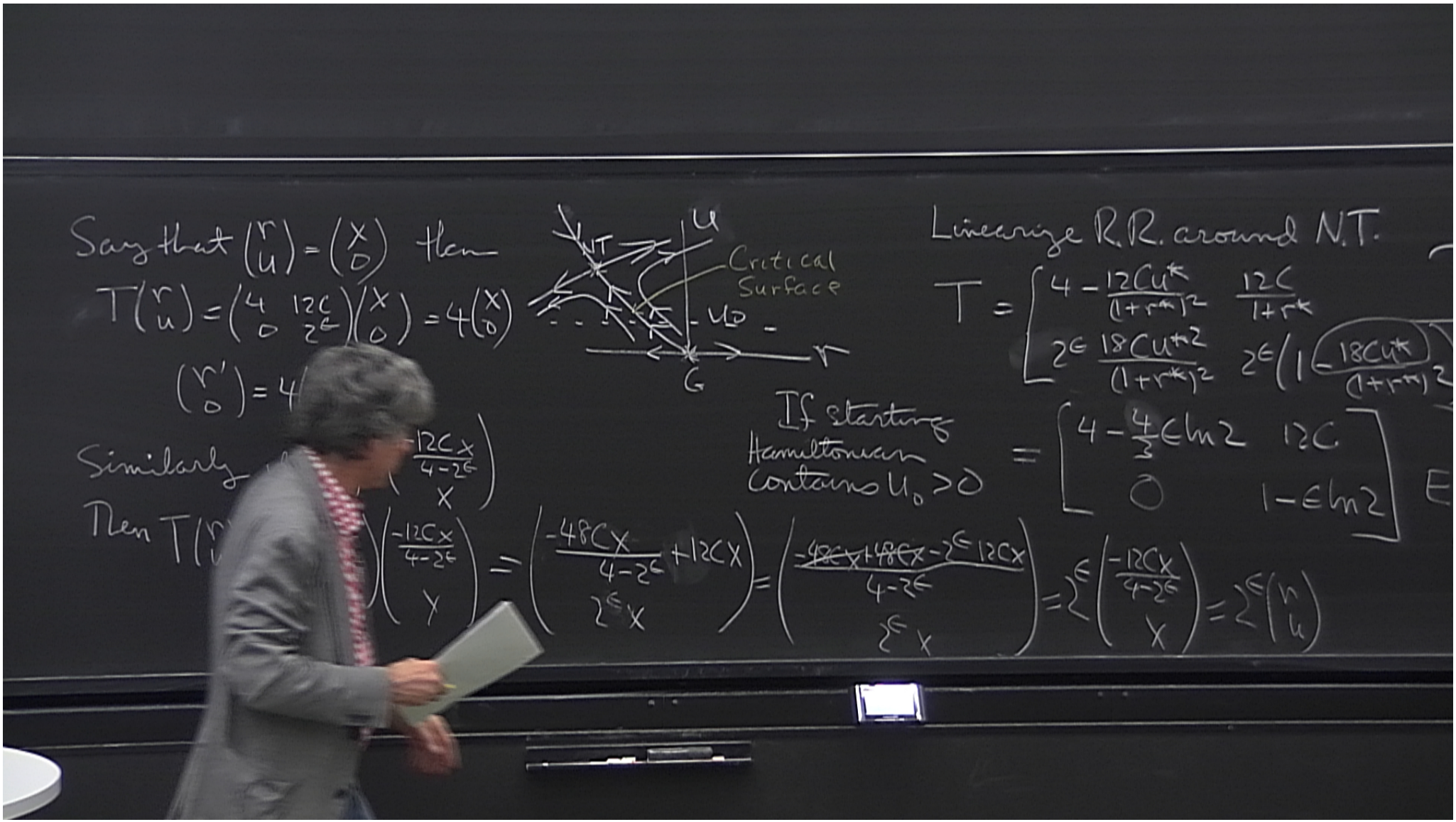
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$T\begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} 4 & 12C \\ 0 & 2^{\epsilon} \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = 4\begin{pmatrix} x \\ 0 \end{pmatrix}$
$\begin{pmatrix} r' \\ 0 \end{pmatrix} = 4$
Similarly
Then $T($
$\begin{pmatrix} -\frac{12Cx}{4-2^{\epsilon}} \\ x \end{pmatrix}$
$= \begin{pmatrix} \frac{-48Cx}{4-2^{\epsilon}} + 12Cx \\ 2^{\epsilon}x \end{pmatrix} = \begin{pmatrix} \frac{\cancel{-48Cx} + \cancel{48Cx} - 2^{\epsilon}12Cx}{4-2^{\epsilon}} \\ 2^{\epsilon}x \end{pmatrix} = 2^{\epsilon}\begin{pmatrix} \frac{-12Cx}{4-2^{\epsilon}} \\ x \end{pmatrix} = 2^{\epsilon}\begin{pmatrix} r \\ u \end{pmatrix}$
NT
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If starting Hamiltonian contains $u_0 > 0$
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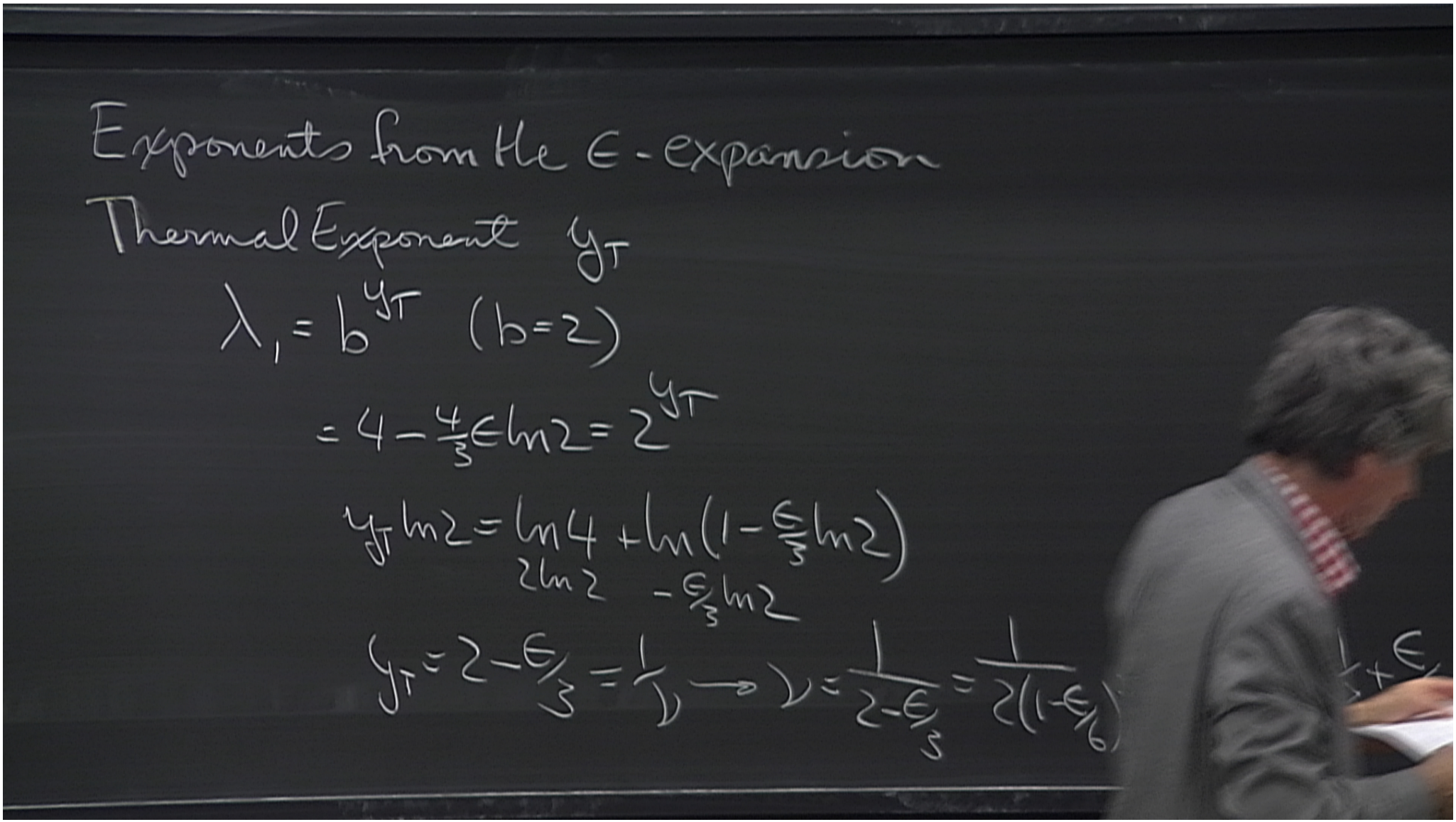

Exponents from the ϵ-expansion
Thermal Exponent y_T
$\lambda_1 = b^{y_T}$ $(b=2)$
$= 4 - \frac{4}{3}\epsilon \ln 2 = 2^{y_T}$
$y_T \ln 2 = \ln 4 + \ln(1 - \frac{\epsilon}{3}\ln 2)$
$2\ln 2$ $-\frac{\epsilon}{3}\ln 2$
$y_T = 2 - \epsilon/3 = \frac{1}{\nu} \rightarrow \nu = \frac{1}{2-\epsilon/3} = \frac{1}{2(1-\epsilon/6)}$

Exponents from the ϵ-expansion $\quad \nu = \frac{1}{2} + \frac{\epsilon}{12}$

Thermal Exponent y_T

$$\lambda_1 = b^{y_T} \quad (b=2)$$

$$= 4 - \frac{4}{3}\epsilon \ln 2 = 2^{y_T}$$

$$y_T \ln 2 = \underbrace{\ln 4}_{2\ln 2} + \underbrace{\ln\left(1 - \frac{\epsilon}{3}\ln 2\right)}_{-\frac{\epsilon}{3}\ln 2}$$

$$y_T = 2 - \epsilon/3 = \frac{1}{\nu} \rightarrow \nu = \frac{1}{2 - \epsilon/3} = \frac{1}{2(1-\epsilon/6)} = \frac{1}{2}\left(1 + \frac{\epsilon}{6}\right) = \frac{1}{2} + \frac{\epsilon}{12}$$

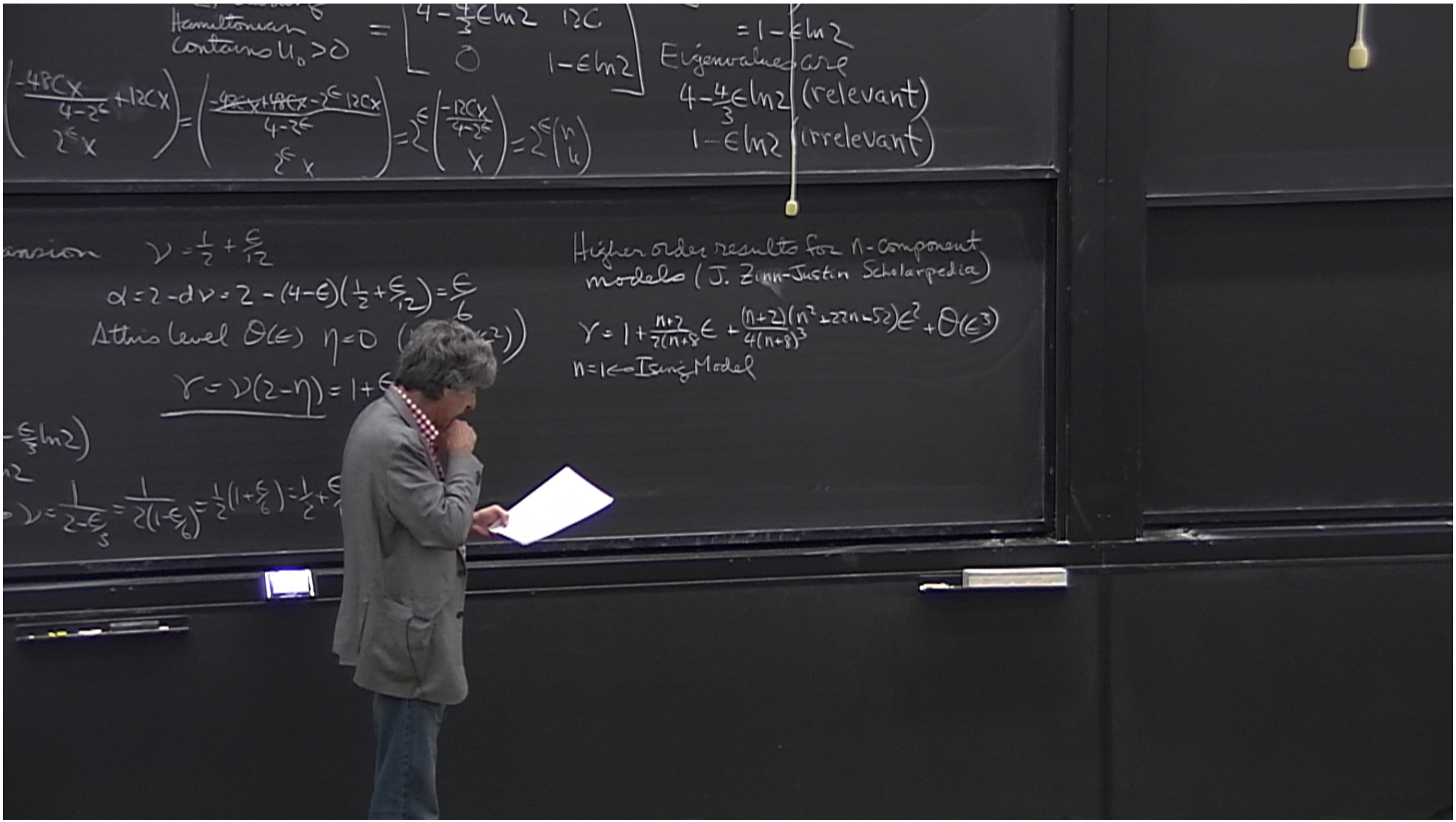
$$\alpha = 2 - d\nu = 2 - (4-\epsilon)\left(\frac{1}{2} + \frac{\epsilon}{12}\right) = \frac{\epsilon}{6}$$

At this level $\mathcal{O}(\epsilon)$ $\eta = 0$ (η is $\mathcal{O}(\epsilon^2$

$$\underline{\gamma = \nu(2-\eta)} = 1 + \epsilon/6$$

Higher order results for n-component models (J. Zinn-Justin Scholarpedia)

$$\gamma = 1 + \frac{n+2}{2(n+8)}\epsilon + \frac{(n+2)(n^2+22n-52)}{4(n+8)^3}\epsilon^2 + O(\epsilon^3)$$


Hamiltonian contains $u_0 > 0$
Eigenvalues are
$4-\frac{4}{3}\epsilon\ln 2$ (relevant)
$1-\epsilon\ln 2$ (irrelevant)
$\nu = \frac{1}{2} + \frac{\epsilon}{12}$
$\alpha = 2 - d\nu = 2 - (4-\epsilon)(\frac{1}{2}+\frac{\epsilon}{12}) = \frac{\epsilon}{6}$
At this level $\mathcal{O}(\epsilon)$ $\eta = 0$
$\gamma = \nu(2-\eta) = 1+\epsilon$
Higher order results for n-component models (J. Zinn-Justin Scholarpedia)
$\gamma = 1 + \frac{n+2}{2(n+8)}\epsilon + \frac{(n+2)(n^2+22n-52)}{4(n+8)^3}\epsilon^2 + \mathcal{O}(\epsilon^3)$
$n=1 \leftarrow$ Ising Model

Higher order results for n-component models (J. Zinn-Justin Scholarpedia)

$$\gamma = 1 + \frac{n+2}{2(n+8)}\epsilon + \frac{(n+2)(n^2+22n-52)}{4(n+8)^3}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

$n=1 \Leftarrow$ Ising Model

Has been calculated to order ϵ^5

$) = \frac{\epsilon}{6}$

is $\mathcal{O}(\epsilon^2)$)

Higher order results for n-component models (J. Zinn-Justin Scholarpedia)

$$\gamma(n) = 1 + \frac{n+2}{2(n+8)}\epsilon + \frac{(n+2)(n^2+22n-52)}{4(n+8)^3}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

$n=1 \leftarrow$ Ising Model

Has been calculated to order ϵ^5

$\gamma(1)$: 1.000, 1.666...,

For $\epsilon = 1$ ϵ^0

$(d=3)$

$) = \frac{\epsilon}{6}$

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Higher order results for n-component models (J. Zinn-Justin Scholarpedia)

$$\gamma(n) = 1 + \frac{n+2}{2(n+8)}\epsilon + \frac{(n+2)(n^2+22n-52)}{4(n+8)^3}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

$n=1 \leftrightarrow$ Ising Model

Has been calculated to order ϵ^5

$\gamma(1)$: 1.000, 1.666..., 1.2438, 1.1948, 1.3384, 0.8918

For $\epsilon=1$ $(d=3)$: ϵ^0, ϵ^1, ϵ^2, ϵ^3, ϵ^4, ϵ^5

$= \frac{\epsilon}{6}$

$+ \mathcal{O}(\epsilon^2)$)

Higher order results for n-component models (J. Zinn-Justin Scholarpedia)

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$\boxed{\text{Borel Summation}}$

$\gamma(1)$: 1.000, 1.666..., 1.2438, 1.1948, 1.3384, 0.8918

For $\epsilon=1$ $(d=3)$: ϵ^0, ϵ^1, ϵ^2, ϵ^3, ϵ^4, ϵ^5

Borel Summation of
ϵ-expansion gives
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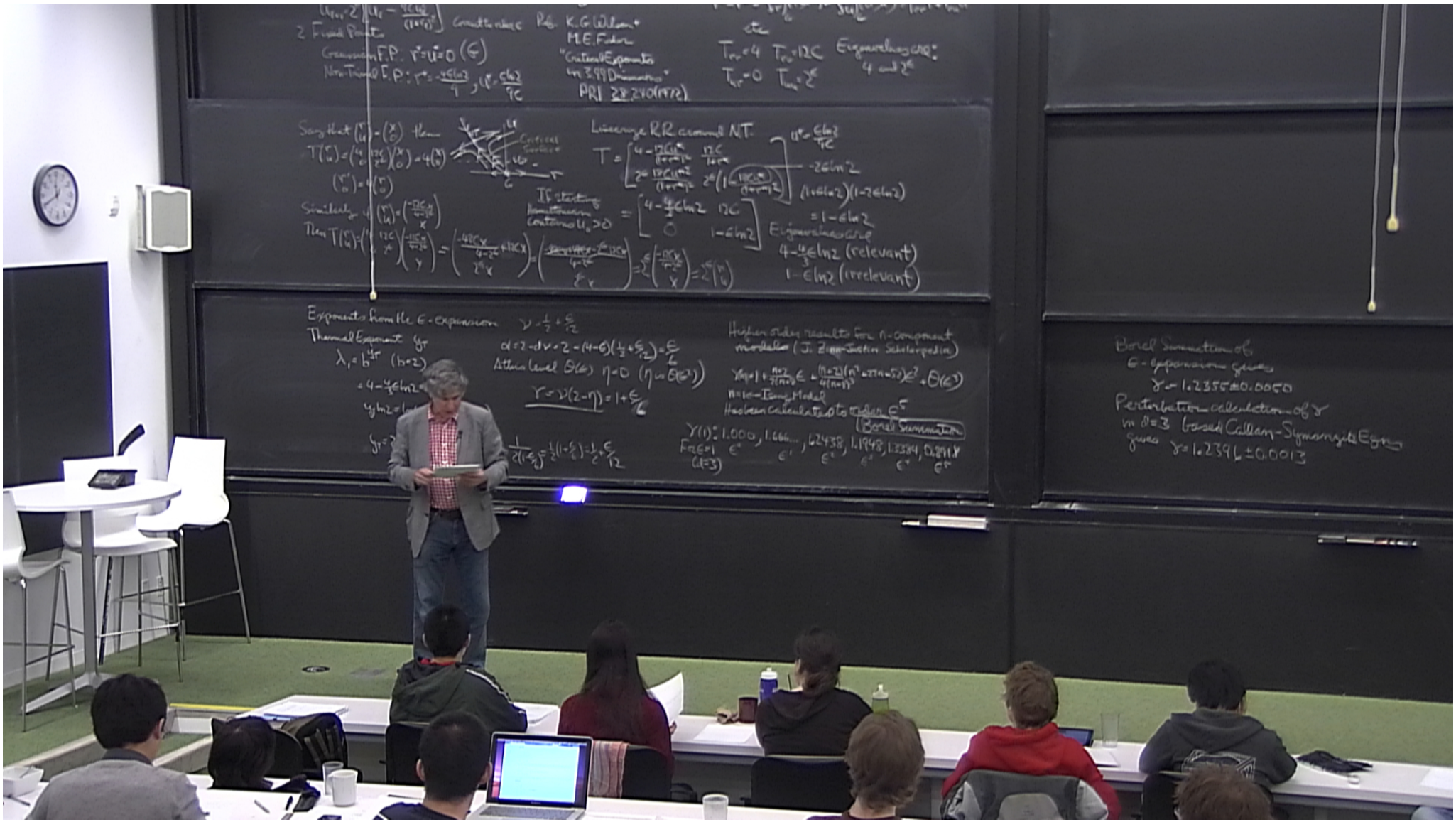
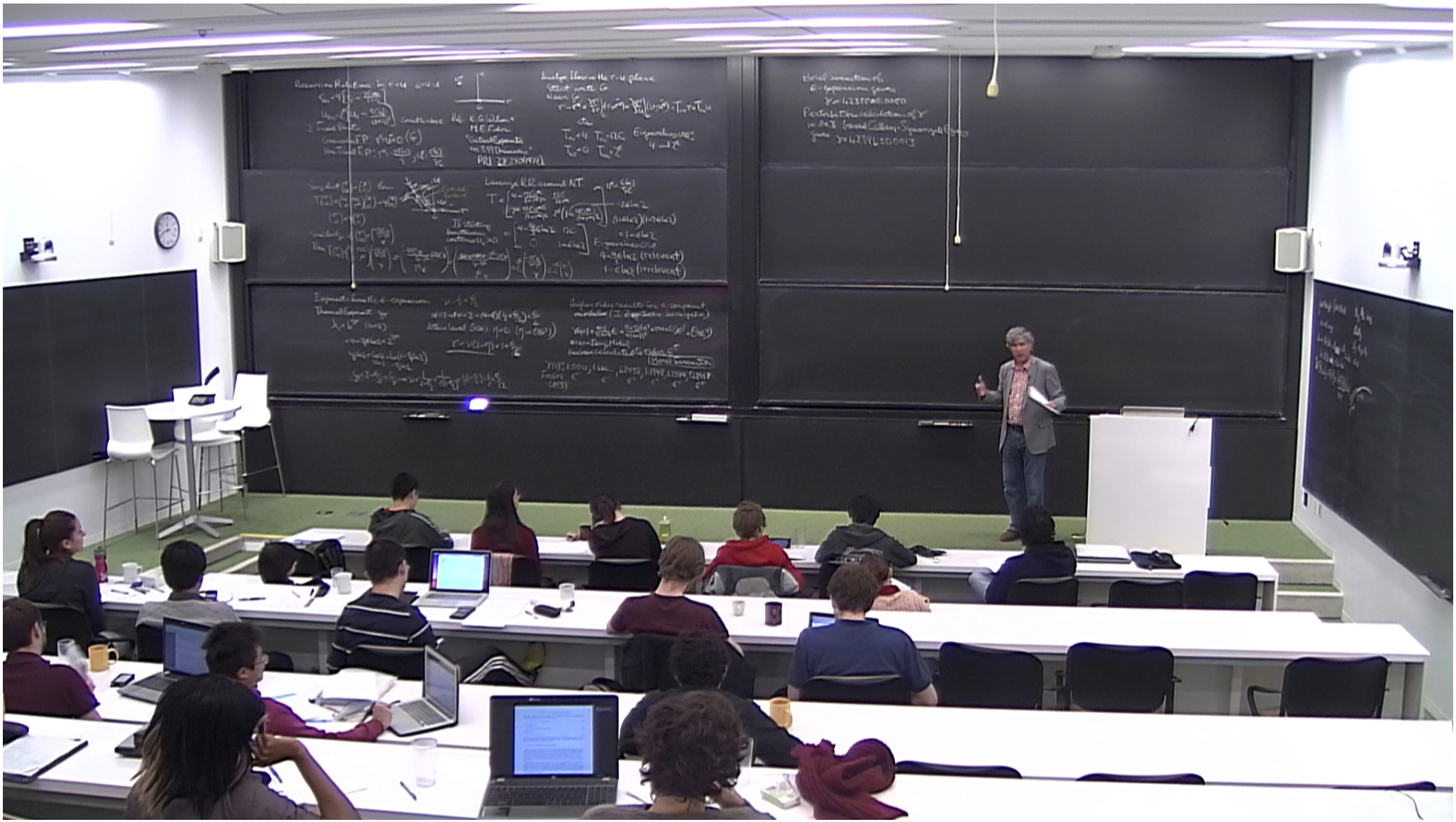
Perturbation calculation of γ
in $d=3$ based Callan-Symanzik Eqn
gives $\gamma = 1.2396 \pm 0.0013$


K.G. Wilson
M.E. Fisher
PRL 28, 240 (1972)
Critical Surface
Exponents from the ε-expansion
Thermal Exponent
Higher order results for n-component model (J. Zinn-Justin Scholarpedia)
Borel Summation of ε-expansion gives
γ = 1.2355 ± 0.0050
Perturbation calculation of γ in d=3 (Brezin Collam-Symanzik Eqns) gives γ = 1.2396 ± 0.0013



Corrections to Scaling

Ref. F. J. Wegner PRB 5, 4529 (1972)

Recall, singular part of free energy

$$f_s(\lambda^{y_T} t, \lambda^{y_H} H) = \lambda^{d} f_s(t, H)$$

implies that $f_s(t,H) = |t|^{2-\alpha} g\left(\frac{H}{|t|^{\Delta}}\right)$ where $2-\alpha = \frac{d}{y_T}$

$$\Delta = \frac{y_H}{y_T}$$

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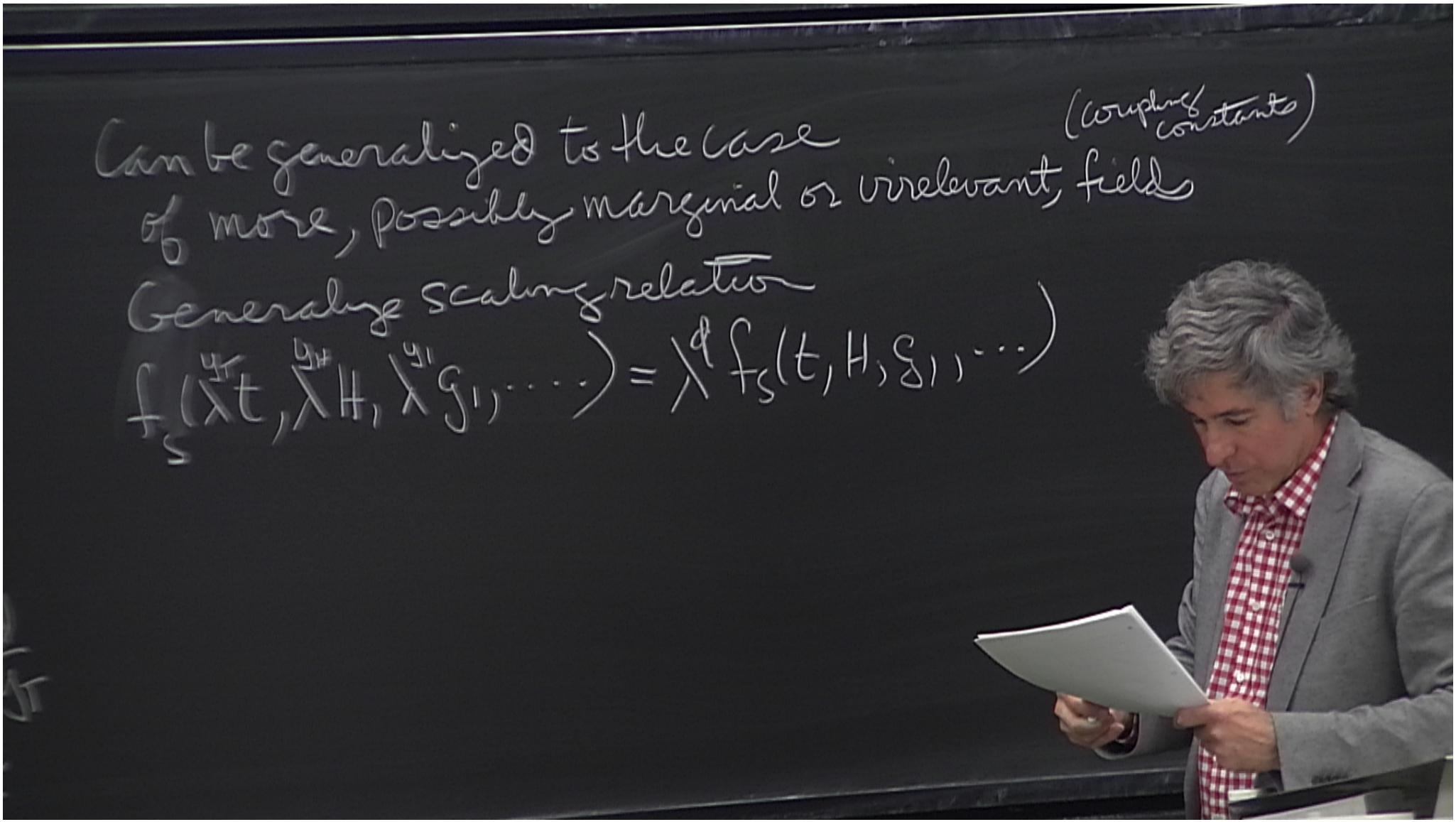
Can be generalized to the case (coupling constants)
of more, possibly marginal or irrelevant, fields


Can be generalized to the case
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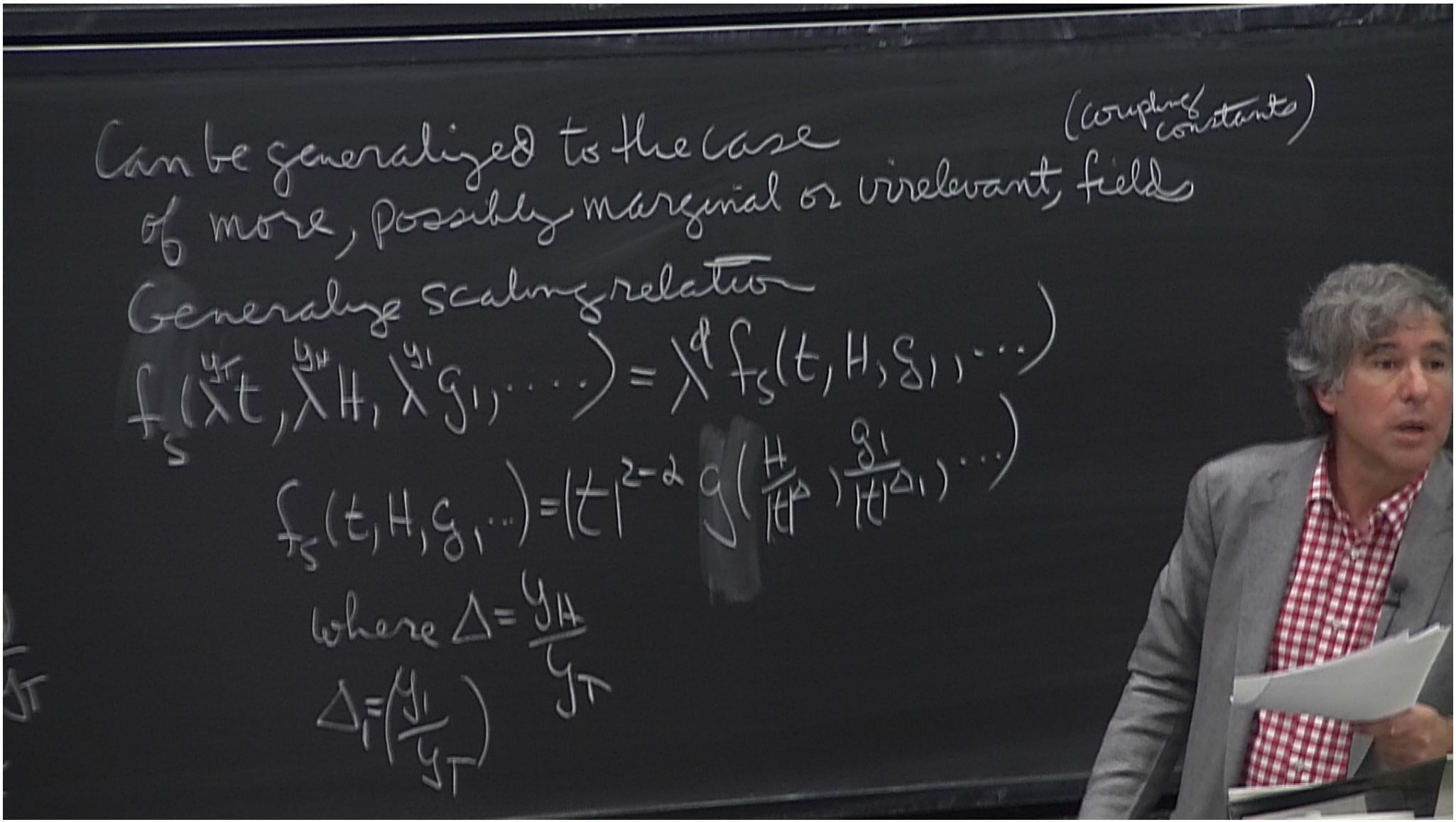
Generalize scaling relation
$f_s(\lambda^{y_T} t, \lambda^{y_H} H, \lambda^{y_1} g_1, \ldots) = \lambda^{d} f_s(t, H, g_1, \ldots)$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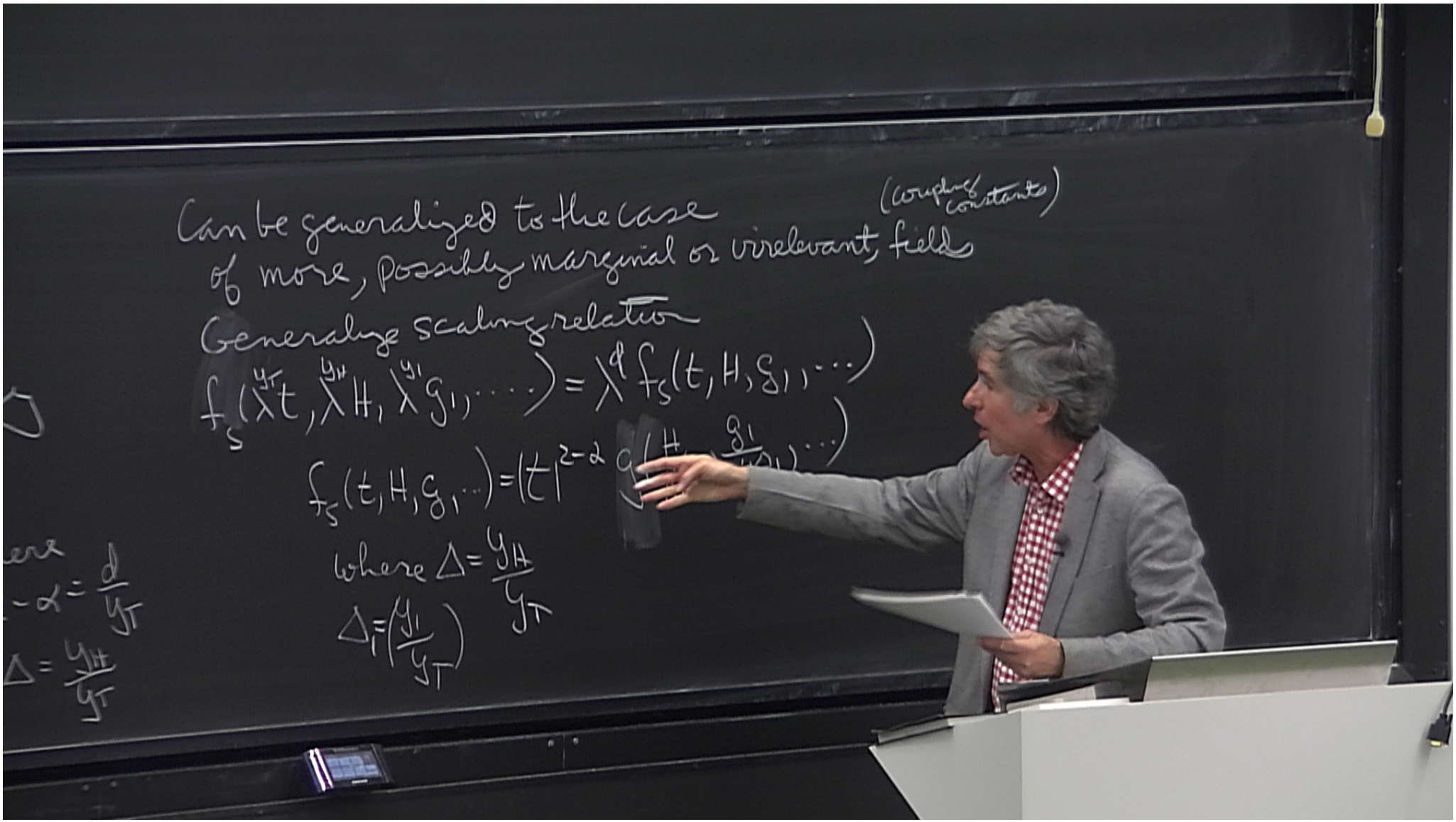
$f_s(t, H, g_1, \ldots) = |t|^{2-\alpha} g\left(\frac{H}{|t|^{\Delta}}, \frac{g_1}{|t|^{\Delta_1}}, \ldots\right)$
where $\Delta = \frac{y_H}{y_T}$
$\Delta_i = \left(\frac{y_i}{y_T}\right)$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Generalize scaling relation
$f_s(\lambda^{y_T} t, \lambda^{y_H} H, \lambda^{y_1} g_1, \ldots) = \lambda^{d} f_s(t, H, g_1, \ldots)$
$f_s(t, H, g_1, \ldots) = |t|^{2-\alpha} g(\frac{H}{|t|^{\Delta}}, \frac{g_1}{|t|^{\Delta_1}}, \ldots)$
where $\Delta = \frac{y_H}{y_T}$
$\Delta_i = \left(\frac{y_i}{y_T}\right)$
$-\alpha = \frac{d}{y_T}$
$\Delta = \frac{y_H}{y_T}$

An interesting irrelevant field is $u_0 - u^*$

What is Δ_u

$$\Delta_u = \frac{y_u}{y_T} = \frac{\ln(1-\epsilon \ln 2)}{\ln(4-\frac{4}{3}\epsilon \ln 2)} = \Delta_u = -\frac{\epsilon}{2} + O(\epsilon^2)$$

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$$\therefore \frac{u_* - u^*}{|t|^{-\epsilon/2}} \to 0 \text{ as } t \to 0$$

So expand

$$f_s(t, H, u - u^*) = |t|^{2-\alpha} f_s\left(\frac{H}{|t|^{\Delta}}, 0\right) + (u - u^*)|t|^{2-\alpha-\Delta_1} f_s^{(0,1)}\left(\frac{H}{|t|^{\Delta}}, 0\right) + \cdots$$

$$\Delta = \frac{y_H}{y_T} \qquad \Delta_1 = \left(\frac{y_1}{y_T}\right)$$

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$$m = \frac{\partial f}{\partial H} \longrightarrow m = |t|^{\beta} f_s^{(1,0)}(0,0) + (u_0 - u^*)|t|^{\beta-\Delta_1} f^{(1,1)}(0,0) + \cdots$$

$$+ O(\epsilon^2)$$

$$m = m_0 |t|^{\beta}(1 + a_m |t|^{-\Delta_1} + \cdots$$

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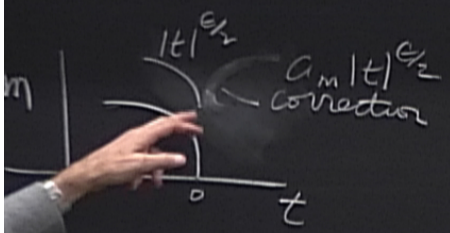
$$|t|^{-\alpha} f\left(\frac{H}{|t|^{\Delta}}\right) \qquad 2-\alpha = \frac{d}{y_T} \qquad \Delta = \frac{y_H}{y_T}$$

where $\Delta = \frac{y_H}{y_T}$

$$\Delta_i = \left(\frac{y_i}{y_T}\right)$$



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↑ Confluent singularity